Math 246, Calculus I, Fall 2008, Professor Arkady Berenstein

FINAL EXAM

1	2	3	4	Total

NAME ______________________________
(PRINT THE FIRST AND THE LAST NAME)

ID NUMBER ______________ **SIGNATURE** ______________

PLEASE DO THE FOLLOWING 6 PROBLEMS. SHOW ALL WORK!

1. (20) Compute derivatives using **logarithmic** differentiation:

(a) $f(x) = \frac{(5x+3)^7(x+1)^3}{(3x+5)^{11}}$

$$\ln(y) = \ln\left(\frac{(5x+3)^7\cdot(x+1)^3}{(3x+5)^{11}}\right) = 7\ln(5x+3) + 3\ln(x+1) - 11\ln(3x+5)$$

$$\frac{y'}{y} = 7\cdot\frac{(5x+3)'}{5x+3} + 3\,\frac{(x+1)'}{x+1} - 11\cdot\frac{(3x+5)'}{3x+5}$$

$$y' = y\cdot\left(\frac{35}{5x+3} + \frac{3}{x+1} - \frac{33}{3x+5}\right) =$$

$$\frac{(5x+3)^7(x+1)^3}{(3x+5)^{11}}\left(\frac{35}{5x+3} + \frac{3}{x+1} - \frac{11}{3x+5}\right)$$

(b) $f(x) = x^{(x^3)}$

$$\ln(y) = \ln(x^{x^3}) = x^3\ln(x)$$

$$\frac{y'}{y} = (x^3\ln(x))' = (x^3)'\ln(x) + x^3(\ln(x))'$$

$$y' = y\cdot\left(3x^2\ln(x) + x^3\cdot\frac{1}{x}\right) = y\cdot x^2(3\ln(x)+1) =$$

$$= x^{x^3}\cdot x^2(3\ln(x)+1)$$

2. (15) Find the tangent line to the graph of $f(x) = \ln(3x-5) + \cos(x^3 - 4x)$ at $x = 2$.

$$f'(x) = (\ln(3x-5))' + (\cos(x^3-4x))' =$$

$$= \frac{(3x-5)'}{3x-5} + \left[-\sin(x^3-4x)(x^3-4x)'\right] =$$

$$= \frac{3}{3x-5} - 3x^2 \sin(x^3-4x)$$

$$f'(2) = \frac{3}{3\cdot 2-5} - 3\cdot 2^2 \sin(2^3 - 4\cdot 2) =$$

$$= 3 - 12\sin(0) = 3$$

$$f(2) = \ln(3\cdot 2-1) + \cos(2^3-4\cdot 2) =$$

$$= \ln(1) + \cos(0) = 1$$

$$y - 1 = 3(x-2)$$

$$y - 1 = 3x - 6$$

$$\boxed{y = 3x - 5}$$

3. (15) Assume that $x^3(1-y)+y^3=2x+3$.

(a) Find the general formula for $\frac{dy}{dx}$. USE IMPLICIT DIFFERENTIATION

$$(x^3(1-y))' + (y^3)' = (2x+3)'$$

$$(x^3)'(1-y) + x^3(1-y)' + 3y^2y' = 2$$

$$3x^2(1-y) - x^3y' + 3y^2y' = 2$$

$$y'(3y^2-x^3) = 2-3x^2(1-y)$$

$$y' = \frac{2-3x^2(1-y)}{3y^2-x^3}$$

(b) Compute $\frac{dy}{dx}$ when $y=1$.

SUBSTITUTE $y=1$ INTO THE ORIGINAL EQUATION IN ORDER TO FIND x.

$$x^3(1-1)+1^3 = 2x+3$$

$$2x+3=1$$

$$2x=-2$$

$$x=-1.$$

$$y' = \frac{2-3(-1)^2(1-1)}{3\cdot 1^2-(-1)^3} = \frac{2}{4} = \boxed{\frac{1}{2}}$$

4. (10) Suppose that $f(x) = \frac{1}{x}$, $1 \le x \le 4$.

(a) Find the slope of the secant line connecting the points $(1, 1)$ and $(4, \frac{1}{4})$.

$$m = \frac{\frac{1}{4} - 1}{4 - 1} = \frac{\frac{1-4}{4}}{3} = \frac{\left(\frac{-3}{4}\right)}{3} = -\frac{1}{4}.$$

(b) Find a number c in the interval $(1, 4)$ such that $f'(c)$ equals the slope of the secant line you computed in (a).

$$f'(x) = (x^{-1})' = -x^{-2}$$

$$f'(c) = -c^{-2}$$

SET THE EQUATION

$$-c^{-2} = -\frac{1}{4}$$

$$c^{-2} = \frac{1}{4}$$

$$c^2 = 4$$

$$\boxed{c = 2.}$$

5. (20) Consider the function $f(x) = \frac{e^x}{x}$ for $x \neq 0$.

(a) Find the vertical and horizontal **asymptotes** of the graph of $f(x)$ (hint: compute limits $\lim_{x \to 0} f(x)$ and $\lim_{x \to -\infty} f(x)$).

$$\lim_{x \to 0^+} \frac{e^x}{x} = \frac{e^0}{0^+} = \frac{1}{0^+} = +\infty$$

$$\lim_{x \to 0^-} \frac{e^x}{x} = \frac{1}{0^-} = -\infty$$

← VERTICAL ASYMPTOTE AT $x = 0$.

$$\lim_{x \to -\infty} \frac{e^x}{x} = \frac{e^{-\infty}}{-\infty} = 0^-$$

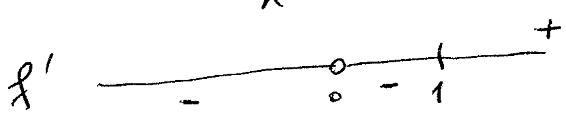
← HORIZONTAL ASYMPTOTE $y = 0$.

(b) Find the intervals where the $f(x)$ is increasing and the intervals where $f(x)$ is decreasing.

$$f'(x) = \left(\frac{e^x}{x}\right)' = \frac{(e^x)' \cdot x - e^x \cdot (x)'}{x^2} =$$

$$= \frac{e^x \cdot x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$$

CRITICAL POINTS: $x = 1$



FUNCTION DECREASES FOR $x < 0$
FUNCTION DECREASES FOR $0 < x < 1$
FUNCTION INCREASES FOR $x > 1$.

(c) Find the maxima and minima of $f(x)$.

THE ONLY CRITICAL POINT $x = 1$ BRINGS LOCAL MINIMUM $(1, e)$.

THIS MINIMUM IS NOT GLOBAL.

NO MAXIMA.

$f(-1) = (-1)^2(1-(-1)) =$
$= 2$

$f(1) = x^2(1-1) = 0$

6. (20) Consider the function $f(x) = x^2(1-x)$ on the interval $-1 \leq x \leq 1$.
(a) Find the maxima and minima of $f(x)$.

CRITICAL POINTS FIRST:

$$f'(x) = \left(x^2(1-x)\right)' = (x^2)'(1-x) + x^2(1-x)' =$$
$$= 2x(1-x) + x^2(-1) =$$
$$x(2(1-x) + x\cdot(-1)) = x(2-3x)$$

$f'(x) = 0$ WHEN $x = 0$ OR $x = \frac{2}{3}$.

f': $-$ | 0 | $+$ | $\frac{2}{3}$ | $-$

$f(0) = 0$ $\quad f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2\left(1-\frac{2}{3}\right) = \frac{4}{27}$

POINT $(0,0)$ GIVES A LOCAL MINIMUM ← ALSO GLOBAL

POINT $\left(\frac{2}{3}, \frac{4}{27}\right)$ GIVES a LOCAL MAXIMUM

END POINTS:

$(-1,2)$ ← GLOBAL MAX

$(1,0)$ ← GLOBAL MIN

(b) Find the interval where the graph of $f(x)$ is concave up.

COMPUTE $f''(x)$

$$f''(x) = (x(2-3x))' = (2x-3x^2)' = 2-6x$$

$f''(x) = 0$ FOR $x = \frac{2}{6} = \frac{1}{3}$.

$f''(x) > 0$ FOR $x < \frac{1}{3}$

ANSWER: GRAPH OF $f(x)$ IS CONCAVE UP PRECISELY WHEN $\boxed{-1 < x < \frac{1}{3}}$