Math 647 Final

Answer as many questions as you can! Make sure you state clearly any theorems from class that you use.

Part I. Definitions.

1. What is a *solvable* group?

   Let G be a group. Define $G^{(n)}$ inductively by setting $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ (the subgroup of $G^{(n)}$ generated by all commutators $[a, b] = aba^{-1}b^{-1}$ for $a, b \in G^{(n)}$). Then G is solvable if $G^{(n)} = \{1\}$ for some $n \geq 0$.

Part II. True or False. Justify your answers briefly.

1. The subgroup B of $GL_2(\mathbb{F}_3)$ consisting of all invertible upper triangular matrices is isomorphic to the alternating group A_4.

   False. Note the upper unitriangular matrices U form a normal subgroup of B of order 3. But A_4 has no normal subgroup of order 3: the only possibility would be the subgroup generated by a 3-cycle WLOG $\{1, (1\,2\,3), (1\,3\,2)\}$. But conjugating $(1\,2\,3)$ by $(1\,2)(3\,4)$ gives $(2\,1\,4)$ so it is not normal.

2. Suppose that $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$. If $G_1 \cong G_2$ and $N_1 \cong N_2$ then $G_1/N_1 \cong G_2/N_2$.

   False. For example take $G_1 = G_2 = \mathbb{Z}, N_1 = 2\mathbb{Z}, N_2 = 3\mathbb{Z}$.

3. All subgroups of the quaternion group Q_3 of order 8 are normal.

   True. Subgroups of order $1, 4$ or 8 are automatically normal. So we just need to think about subgroups of order 2. $Q_3 = \{1, i, j, k, -1, -i, -j, -k\}$. The only element of order 2 is -1. That is in the center so it is normal.

4. Let F be the free group on the set $X = \{a, b\}$. Let N be the normal subgroup of F generated by the element ab. Then, $F/N \cong \mathbb{Z}$.

   True. Define a map $F \to \mathbb{Z}, a \mapsto 1, b \mapsto -1$. It sends ab to 0, hence it induces a surjective map $f : F/N \to \mathbb{Z}$. Now since F/N is generated by the images of a and b, and the image of b is a^{-1}, F/N is generated by the element a. So we can define a surjective map $g : \mathbb{Z} \to F/N, n \mapsto a^n N$. This is a two-sided inverse to f.

5. If G is a primitive permutation group on a finite set X, then the action of G on X is 2-transitive.

   False. Take the symmetric group S_5 acting on the set X of 2-element subsets of $\{1, 2, 3, 4, 5\}$. The stabilizer of the 2-element subset $\{1, 2\}$ is $S_2 \times S_3$ which is a maximal subgroup of S_5. Hence it is a primitive permutation group. But it is not 2-transitive: for example there is no permutation taking $\{1, 2\}$ to $\{1, 2\}$ and $\{2, 3\}$ to $\{3, 4\}$.

Part III. Longer problems.

1. Use the Sylow theorems to show that there is no simple group of order 144.

   Solution. Note $144 = 3^2 2^4$. Consider n_3. It equals $1, 4$ or 16. If its 1 we have a normal Sylow 3-subgroup, done. If is 4 there is a non-trivial map from G to S_4 coming from conjugation action on Sylow 3-subgroups. It must have kernel, done. So $n_3 = 16$.

   Suppose every pair P, Q of Sylow 3-subgroups have trivial intersection. Then there are $16.8 = 128$ elements of order 3 or 9 in our group. This leaves room for just 16 more elements. So there must be only one Sylow 2-subgroup, necessarily normal. Done.

Hence there exist Sylow 3-subgroups P, Q with $|P \cap Q| = 3$. Let $N = N_G(P \cap Q)$. If it is all of G then G is not simple, done. It contains P and Q so it is of order > 9 and divisible by 9. But it cannot be 18 since then P and Q would be different normal Sylow 3-subgroups of N, contradicting the Sylow theorems. So it is of order 36 or 72. But then $[G : N] = 4$ or 2. The action of G on cosets of N gives a non-trivial map from G to S_2 or S_4 again and it has kernel...

2. Let p be an odd prime. Classify all finite groups of order $2p$ up to isomorphism.

   Solution. By the Sylow theorems, $n_p = 1$, hence there is a normal Sylow p subgroup P. If there is a normal Sylow 2 subgroup too, then G is the direct product $C_p \times C_2$. Otherwise, let x be an element of order 2. Then G is a non-abelian semidirect product $P \rtimes \langle x \rangle$. Pick a generator $y \in P$. Then, xyx^{-1} is of order p and does not equal y, hence it is y^a for some $1 < a < p$. But $x^2yx^{-2} = y = xy^ax^{-1} = (xyx^{-1})^a = (y^a)^a = y^{a^2}$. Hence $a^2 \equiv 1 \pmod{p}$. Hence $a = p-1$. Hence $xyx^{-1} = y^{-1}$. This shows that $G = \langle x, y \,|\, x^2 = y^p = 1, xyx^{-1} = y^{-1} \rangle$, so G is the dihedral group D_p.

3. If N is a non-trivial normal subgroup of a finite nilpotent group G, prove that $N \cap Z(G) \neq \{1\}$.

   Solution. Pick a prime p dividing $|N|$. Since G is nilpotent, it is the direct product of its Sylow subgroups. So $Z(G)$ is the direct product of the centers of its Sylow subgroups. Let N' be a Sylow p-subgroup of N, embed it into a Sylow p-subgroup G' of G. Note $N' = N \cap G'$ so N' is normal in G'. If we can show that $N' \cap Z(G') \neq \{1\}$, we'll be done since $N \geq N'$ and $Z(G') \leq Z(G)$. This reduces the problem to the case that G is a finite p-group.

   So now assume $|G| = p^a$ some prime p. Consider action of G on N by conjugation. The orbits of size 1 are precisely the elements of $Z(G) \cap N$. So the class equation implies $|G| \equiv |Z(G) \cap N| \pmod{p}$. Hence $Z(G) \cap N \neq \{1\}$.