Chapter 4

General theory of modules

4.1 Short exact sequences

With the example of modules over PIDs behind us, we turn to the case of modulesover a general ring R. So let R be a ring. For the sake of definiteness, all R-modules in this section will be *left* R-modules (though all the definitions make sense for right R-modules too).

An *exact sequence* of R-modules means a (finte or infinite) sequence

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \ldots$$

such that $\ker f_{i+1} = \operatorname{im} f_i$ for each i. If one just has that $\ker f_{i+1} \supseteq \operatorname{im} f_i$, i.e. that $f_{i+1} \circ f_i = 0$, then it is called a *complex* of R-modules. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow K \xrightarrow{i} M \xrightarrow{\pi} Q \longrightarrow 0.$$

So exactness in this special case means that i is injective, π is surjective and $\operatorname{im} i = \ker \pi$.

Observe that if we have a short exact sequence as above, then M contains the isomorphic copy $i(K)$ of K as a submodule, and the factor module $M/i(K)$ is isomorphic to Q. Conversely, given an R-module M and a submodule K, we define $i : K \hookrightarrow M$ to be the inclusion and let $\pi : M \to M/K = Q$ be the quotient map. Then we obtain a short exact sequence of R-modules as above. In other words, short exact sequences simply give a convenient notation for writing down *extensions* of a module Q by a module K. You should compare with section 1.7 where we discussed extensions of groups!

Split short exact sequences. *Let*

$$0 \longrightarrow K \xrightarrow{i} M \xrightarrow{\pi} Q \longrightarrow 0.$$

be a short exact sequence of R-modules. Then, the following properties are equivalent:

(1) *there exists an R-module homomorphism $\tau : Q \to M$ such that $\pi \circ \tau = \mathrm{id}_Q$;*

(2) *there exists an R-module homomorphism $j : M \to K$ such that $j \circ i = \mathrm{id}_K$;*

(3) *$i(K)$ has a complement in M, i.e. there exists a submodule $Q' \leq M$ with $M = i(K) \oplus Q'$ (note Q' is obviously isomorphic to Q).*

Proof. (1)⇒(3). Let $Q' = \tau(Q)$. Since $\pi \circ \tau = \mathrm{id}_Q$, τ is injective, hence an isomorphism between Q and its image Q'. We now claim that $M = i(K) \oplus Q'$. Take $m \in M$. Then, $m - \tau(\pi(m))$ is in the kernel of π hence the image of i. So, $m - \tau(\pi(m)) = i(k)$ for some $k \in K$, i.e. $m = i(k) + \tau(\pi(m))$ showing $M = i(K) + Q'$. Moreover, if $m \in i(K) \cap Q'$ then $\pi(m) = 0$ and $m = \tau(q)$ some $q \in Q$. Then, $0 = \pi(\tau(q)) = q$ so $m = \tau(q) = 0$. This shows $i(K) + Q'$ is direct.

(3)⇒(2). We have that $M = i(K) \oplus Q'$. Define $\bar{j} : M \to i(K)$ to be the projection along this direct sum. Set $j = i^{-1} \circ \bar{j}$, where $i^{-1} : i(K) \to K$ is the isomorphism. Then it is immediate that $j \circ i = \mathrm{id}_K$.

(2)⇒(1). Set $Q' = \ker j$. Then, $M = i(K) \oplus Q'$, which is proved in an entirely similar way to the proof of (1)⇒(3) above. Then, the restriction of π to Q' is an isomorphism between Q' and Q; let τ be its inverse, giving an isomorphism $\tau : Q \to Q'$. This does the job. □

We call a short exact sequence in which any of the equivalent conditions of the lemma hold a *split short exact sequence*. The maps τ and j in the lemma are called *splittings*, of π and i respectively. Note if we have a short exact sequence, then $M \cong Q \oplus K$, and the extension of Q by K encoded by the sequence is called a *split extension* of Q by K.

So, the split extensions of R-modules are exactly *direct sums*. You should compare this with section 1.7: there, the split extensions of groups were *semidirect products*. Of course, for *Abelian groups*, all semidirect products are just ordinary direct products so the two theories coincide for Abelian groups (= $\mathbb{Z}$-modules).

Exercise (the five lemma). Let

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

and

$$0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow B_3 \longrightarrow 0$$

be short exact sequences of R-modules. Suppose there are vertical maps $\gamma_i : A_i \to B_i$ for each $i = 1, 2, 3$ so that the resulting diagram commutes. Prove that:

(a) γ_1, γ_3 monomorphisms $\Rightarrow$ γ_2 is a monomorphism;

(b) γ_1, γ_3 epimorphisms $\Rightarrow$ γ_2 is an epimorphism;

(c) γ_1, γ_3 isomorphisms $\Rightarrow$ γ_2 is an isomorphism.

4.2 Projectives and injectives

Let R be a ring and consider the category R-$\mathfrak{mod}$ again. We now introduce the notion of a *projective R-module*, which should be viewed as a generalization of free modules.

For the definition, an R-module P is called *projective* if for every exact sequence

$$B \xrightarrow{\pi} C \longrightarrow 0$$

and every homomorphism $\gamma : P \to C$, there exists a homomorphism $\beta : P \to B$ such that $\gamma = \pi \circ \beta$. In words, P is projective if every homomorphism from P to a quotient of an R-module B *lifts* to a homomorphism from P to B itself.

4.2.1. **Lemma.** *Let P_i $(i \in I)$ be R-modules. Then, $P = \bigoplus_{i \in I} P_i$ is projective if and only if each P_i is projective.*

Proof. Let $\pi : B \twoheadrightarrow C$ be an epimorphism and $\iota_i : P_i \to P$ be the canonical inclusions.

Suppose each P_i is projective. Take a map $\gamma : P \to C$. Writing $\iota_i : P_i \to P$ for the canonical inclusion, we get for each i a map $\gamma_i = \gamma \circ \iota_i : P_i \to C$. Since P_i is projective, we can lift γ_i to a map $\beta_i : P_i \to B$ with $\gamma_i = \pi \circ \beta_i$ for each i. Then the universal property of coproducts gives a unique map $\beta : P \to B$ such that $\beta_i = \beta \circ \iota_i$. Then, $\pi \circ \beta \circ \iota_i = \pi \circ \beta_i = \gamma_i = \gamma \circ \iota_i$ for each i, hence $\pi \circ \beta = \gamma$ as required.

Conversely, if P is projective and we have a map $\gamma_i : P_i \to C$, the universal property of coproduct gives us a unique map $\gamma : P \to C$ such that $\gamma_i = \gamma \circ \iota_i$ and $\gamma \circ \iota_j = 0$ for all $j \neq i$. Then, there is a lift $\beta : P \to B$ with $\gamma = \pi \circ \beta$. Define $\beta_i = \beta \circ \iota_i$ to obtain the required lift of γ_i showing that P_i is projective. □

4.2.2. **Corollary.** *Free R-modules are projective.*

Proof. Let F be a free R-module. Then, F is a direct sum of copies of the regular R module ${}_RR$. So by the lemma, we just need to show that ${}_RR$ is projective. Suppose $\pi : B \twoheadrightarrow C$ and we have a map $\gamma :_R R \to C$. Choose $b \in B$ such that $\pi(b) = \gamma(1_R)$ and define an R-module homomorphism $\beta :_R R \to B$ by $r \mapsto rb$ for $r \in R$. This is a lift of γ. ◻

Characterization of projectives. *Let P be an R-module. The following properties are equivalent.*

(1) *P is projective.*

(2) *Every short exact sequence of the form*

$$0 \longrightarrow K \longrightarrow M \longrightarrow P \longrightarrow 0$$

ending in P is split.

(3) *P is isomorphic to a summand of a free R-module.*

Proof. (1)⇒(2). Take a short exact sequence

$$0 \to K \to M \xrightarrow{\pi} P \to 0.$$

If P is projective, the idenity map $P \to P$ lifts to a map $\tau : P \to M$ such that $\pi \circ \tau = \mathrm{id}_P$. This is a splitting, so the short exact sequence is split.

(2)⇒(3). By Theorem 3.3.1, there is an epimorphism $\pi : F \to P$ where F is free. Letting $K = \ker \pi$, we obtain a short exact sequence

$$0 \to K \to F \xrightarrow{\pi} P \to 0.$$

By assumption it splits. So, $F \cong K \oplus P'$ with $P' \cong P$. Hence, P is isomorphic to a summand of a free module.

(3)⇒(1). If $P \oplus K$ is free, it is projective by Corollary 4.2.2 hence P is projective by Lemma 4.2.1. ◻

Examples. Using the characterization, we can give some examples of projectives.

(1) The regular $\mathbb{Z}_6$-module $\mathbb{Z}_6$ is free hence projective. By the Chinese remainder theorem, $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$ as a $\mathbb{Z}_6$-module. Hence, both $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are projective $\mathbb{Z}_6$-modules. They are *not* free!!

(2) Let R is a PID and M be a finitely generated R-module. We showed that M is free if and only if it is torsion-free. Now, if M is free then it is projective. Conversely, if it is projective, then it is a summand of a free module, so is torsion-free, so is free. Hence: *finitely generated modules over a PID are free if and only if they are projective.*

(3) On the other hand, consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. We have observed before that although $\mathbb{Q}$ is torsion-free, it is not free (allowed since it is not finitely generated). It is not projective either. Proof: suppose $\mathbb{Q} \oplus M$ is free for some $\mathbb{Z}$-module M. Then, it has a basis $\{f_i\}_{i \in I}$. Consider $1 \in \mathbb{Q} \subseteq \mathbb{Q} \oplus M$. Then, $1 = \sum_{i \in I} a_i f_i$ for integers a_i all but finitely many of which are zero. Let p be coprime to all the non-zero a_i. Write $1/p = \sum_{i \in I} b_i f_i$. Then, $1 = p(1/p) = \sum_{i \in I} p b_i f_i = \sum_{i \in I} a_i f_i$, hence since the f_i form a basis, we get that $pb_i = a_i$ for all i so $p | a_i$ for all i, a contradiction.

We turn now to discussing the dual notion of *injective modules.* An R-module I is called *injective* if for every exact sequence

$$0 \longrightarrow A \xrightarrow{\ i\ } B$$

and every R-module homomorphism $\alpha : A \to I$, there exists a homomorphism $\beta : B \to I$ such that $\alpha = \beta \circ i$. In words, I is injective if every homomorphism from a submodule of B to I extends to a homomorphism from B to I. In the category R-$\mathfrak{mod}$, the notion of injective is less useful that that of projective; but in many other subjects it is the notion of injective that is the dominant one.

4.2.3. **Lemma.** *Let I_j $(j \in J)$ be R-modules. Then, $I = \prod_{j \in J} I_j$ is injective if and only if each I_j is injective.*

Proof. This is exactly the same proof as Lemma 4.2.1 but carried out in the *opposite category* $(R\text{-}\mathfrak{mod})^{op}$! If you're not happy with this, you can easily dualize the proof of Lemma 4.2.1 yourself. □

Unfortunately at this point the development of injectives diverges from the development above of projectives. In short, there is no nice characterization of injectives analogous to "P is projective if and only if it is a summand of a free module". However, observe that Theorem 3.3.1 and Corollary 4.2.2 show that every R-module is a quotient of a projective module. There is a dual statement for injectives: every R-module is a submodule of an injective module. The proof takes a little longer.

Criterion for injectivity. *A left R-module I is injective if and only if for every left ideal L of R and every R-module homomorphism $f : L \to I$, there exists an extension of f to an R-module homomorphism $\bar{f} : R \to I$.*

Proof. Suppose I has the given property. Let $i : A \to B$ be any monomorphism and $\alpha : A \to I$. Let

$$\mathscr{H} = \{h : C \to I \mid i(A) \subseteq C \subseteq B, h \circ i = \alpha\}.$$

Note $\mathscr{H}$ is non-empty; partially order $\mathscr{H}$ so that $(h : C \to I) \leq (h' : C' \to I)$ if $C \subseteq C'$ and $h'|_C = h$.

To apply Zorn's lemma, take a chain $\{h_\omega : C_\omega \to I\}_{\omega \in \Omega}$. Set $C = \bigcup_{\omega \in \Omega} C_\omega$ and define $h : C \to I$ by $h(c) = h_\omega(c)$ for any $\omega \in \Omega$ for which $c \in C_\omega$. Then, $h : C \to I$ is an upper bound for the chain. Hence $\mathscr{H}$ has a maximal element, call it $h : C \to I$.

Now we claim that $C = B$, which will complete the proof. Well, suppose $C \neq B$ and pick $b \in B - C$. Let $L = \{r \in R \mid rb \in C\}$, a left ideal of R. Define a map $g : L \to I$ by $r \mapsto h(rb)$; this is an R-module homomorphism. So by assumption, there exists an R-module homomorphism $k : R \to I$ extending g. Now define $\bar{h} : C + Rb \to I$ so that $a + rb \mapsto h(a) + k(r)$. Then, this is a well-defined R-module homomorphism extending h, which contradicts the maximality of h. □

Divisible Abelian groups. Now we can describe the injective $\mathbb{Z}$-modules. Call an Abelian group *divisible* if for every $a \in A$ and every $n \in \mathbb{N}$, there exists an $x \in A$ such that $nx = a$. For example, $\mathbb{Q}$ is obviously divisible, but $\mathbb{Z}$ is not, nor is any finite Abelian group. We claim:

A is an injective $\mathbb{Z}$-module if and only if it is a divisible Abelian group.

Indeed, if A is injective, take $y \in A$ and $n \in \mathbb{N}$. Consider the exact sequence

$$0 \to n\mathbb{Z} \to \mathbb{Z}$$

Let $\alpha : n\mathbb{Z} \to A$ be the map $nz \mapsto yz$ Then there exists $\beta : \mathbb{Z} \to A$ extending α. Set $x = \beta(1)$. Then, $y = nx$ so A is divisible. Conversely, if G is divisible, use the criterion for injectivity and reverse the argument just given.

4.2.4. **Lemma.** *If A is an Abelian group, A can be embedded into a divisible Abelian group.*

Proof. Let $\pi : F \twoheadrightarrow A$ be an epimorphism, where F is free (Theorem 3.3.1) so that

$$F = \bigoplus_{i \in I} \mathbb{Z}_i,$$

a direct sum of copies of $\mathbb{Z}$. Embed each $\mathbb{Z}_i$ in a corresponding copy $\mathbb{Q}_i$ of $\mathbb{Q}$, to construct an embedding

$$i : F \hookrightarrow \bigoplus_{i \in I} \mathbb{Q}_i = \bar{F}.$$

Now, $A \cong F/\ker\pi \cong i(F)/i(\ker\pi)$. The right hand side is a submodule of $\bar{F}/i(\ker\pi)$, which is a quotient of a divisible Abelian group hence divisible. We thus obtain the required embedding of A into a divisible group. □

Now we can prove the main result:

Injective embedding theorem. *For any ring R, every (left) R-module M can be embedded into an injective R-module.*

Proof. In the first place, M is an Abelian group so by the lemma we can find a $\mathbb{Z}$-module monomorphism $f : M \hookrightarrow I$ where I is a divisible Abelian group. Consider the map

$$\bar{f} : \mathrm{Hom}_{\mathbb{Z}}(R, M) \to \mathrm{Hom}_{\mathbb{Z}}(R, I), \quad \theta \mapsto \bar{f}(\theta)$$

where $\bar{f}(\theta)(r) = f(\theta(r))$. If for any Abelian group we view $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ as a left R-module by the rule $(r\theta)(s) = \theta(sr)$, $r, s \in R, \theta : R \to A$, then $\bar{f}$ is an R-module homomorphism. Now consider

$$M \to \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_{\mathbb{Z}}(R, M) \xrightarrow{\bar{f}} \mathrm{Hom}_{\mathbb{Z}}(R, I).$$

The first map here is the map $m \mapsto \theta_m$, where $\theta_m(r) = rm$ for $r \in R$. The second map here is the obvious inclusion. The third map $\bar{f}$ is as constructed above. We obtain an R-module monomorphism $M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, I)$.

It just remains to check that the R-module $\mathrm{Hom}_{\mathbb{Z}}(R, I)$ is injective. We use the criterion for injectivity. So let $f : L \to \mathrm{Hom}_{\mathbb{Z}}(R, I)$ be a homomorphism, where L is a left ideal of R. Then, $\phi : a \mapsto (f(a))(1_R)$ is a $\mathbb{Z}$-module homomorphism $L \to I$. Since I is an injective $\mathbb{Z}$-module, there exists a $\mathbb{Z}$-module homomorphism $\bar{\phi} : R \to I$ extending ϕ. Now define

$$\bar{f} : R \to \mathrm{Hom}_{\mathbb{Z}}(R, I), \quad r \mapsto r\bar{\phi}.$$

Now you check that this is an R-module homomorphism extending f: for $r \in L$ and $s \in R$,

$$\bar{f}(r)(s) = (r\bar{\phi})(s) = (\bar{\phi})(sr) = \phi(sr) = f(sr)(1_R) = f(r)(s)$$

whence $\bar{f}(r) = f(r)$ for all $r \in L$. □

To illustrate the theorem, we can now prove a version of the characterization of projective modules above for injectives (but observe that there is now no case (3)!).

Characterization of injectives. *Let I be an R-module. The following properties are equivalent:*

(1) *I is injective.*

(2) *Every short exact sequence of the form*

$$0 \longrightarrow I \longrightarrow M \longrightarrow Q \longrightarrow 0$$

starting in I is split.

Proof. (1)⇒(2). Suppose I is injective and we are given a short exact sequence starting in I. Then, the identity map $I \to I$ extends to a map $j : M \to I$. This defines a splitting to show that the short exact sequence is split.

(2)⇒(1). Let I be a module with the given property. Apply the injective embedding theorem to find an embedding $i : I \hookrightarrow M$ with M injective. The short exact sequence splits, so $M = I \oplus Q'$ for some R-module Q'. In other words, $I \oplus Q' \cong I \times Q'$ is injective, so I is too, applying Lemma 4.2.3. □

4.3 Semisimple modules

We have already mentioned briefly that an R-module M is called *simple* or *irreducible* if it is non-zero and has no submodules other that M itself and the zero submodule (0). We call M *semisimple* if M can be decomposed as a direct sum of simple R-modules. Note the zero module counts as a semisimple modules (being the direct sum of zero simple R-modules).

4.3.1. **Lemma.** *Let M be an R-module with $M = \sum_{i \in I} S_i$ where the S_i are simple submodules of M. If N is any submodule of M, there exists a subset $J \subseteq I$ such that*

$$M = N \oplus \bigoplus_{j \in J} S_j.$$

In particular (taking $N = (0)$) $M = \bigoplus_{k \in K} S_k$ for some subset $K \subseteq I$.

Proof. Let $\mathscr{F}$ be the collection of all subsets J of I such that $N + \sum_{j \in J} S_j$ is direct. To apply Zorn's lemma, we check that every chain $(J_\omega)_{\omega \in \Omega}$ in $\mathscr{F}$ has an upper bound. Indeed, consider $J = \bigcup_{\omega \in \Omega} J_\omega$; we need to verify that $J \in \mathscr{F}$. Well, if $N + \sum_{j \in J} S_j$ is not direct, then there exists $n \in N$ and non-zero $s_{j_k} \in S_{j_k}$ for $j_1, \dots, j_r \in J$ such that $n + s_{j_1} + \cdots + s_{j_r} = 0$. But then for sufficiently large $\omega \in \Omega$, all j_k lie in J_ω so that $N + \sum_{j \in J_\omega} S_j$ is not direct either, a contradiction.

So by Zorn's lemma, we can pick a maximal element $J \in \mathscr{F}$. For this J, $N + \sum_{j \in J} S_j$ is direct. It just remains to show that $P = N + \sum_{j \in J} S_j$ is actually equal to M. Well, if not, we can find $i \in I$ such that $P + S_i$ is strictly larger than P. So, $P \cap S_i \neq S_i$, hence $P \cap S_i = (0)$ as S_i is simple. So $P + S_i$ is direct, which contradicts the maximality of J. □

Recall that a submodule N of an R-module M has a *complement* if there is an R-submodule C of M such that $M = N \oplus C$.

Characterization of semisimple modules. *For an R-module M, the following are equivalent:*
(1) *M is semisimple;*
(2) *M is a sum of simple modules;*
(3) *every R-submodule of M is complemented;*
(4) *every short exact sequence*

$$0 \longrightarrow K \longrightarrow M \longrightarrow Q \longrightarrow 0$$

with M in the middle is split.

Proof. (1)⇒(2). Obvious.

(2)⇒(3). Follows by Lemma 4.3.1.

(3)⇒(1). Let S be the sum of all the simple submodules of M. We claim that $S = M$. Well, if not, then by assumption, $M = S \oplus T$ for some non-zero submodule T. Let N be a non-zero cyclic submodule of T, and choose (by Zorn's lemma) as maximal submodule N' of N. Then, N' has a complement in M by assumption, say $M = N' \oplus U$. Then,

$$N = N' \oplus (U \cap N)$$

so N' has a complement in N. Now, $U \cap N \cong N/N'$ is simple so lies in S. But by construction, $U \cap N$ lies in T so cannot lie in S. This gives the contradiction needed.

(3)⇔(4). This is immediate from the definition (3) of split short exact sequence in section 4.1. □

There is a basic but very important warning to be made at this point. Suppose $M = \bigoplus_{i \in I} S_i$ is a semisimple R-module and $N \leq M$ is a submodule. By Lemma 4.3.1, $M = N \oplus \bigoplus_{j \in J} S_j$ for some subset $J \subseteq I$. Hence, $N \cong M / \bigoplus_{j \in J} S_j$ so

$$N \cong \bigoplus_{i \in I - J} S_i.$$

But you will *not* in general be able to write

$$N = \bigoplus_{i \in I - J} S_i$$

here. You should appreciate the subtle but fundamental difference between the preceeding two formulae! Note the argument just given showed in particular that any submodule of a semisimple module is itself semisimple. It is also obvious from (2) of the characterization that every quotient module of a semisimple module is semisimple. So:

4.3.2. **Corollary.** *Every submodule and every quotient module of a semisimple module is itself semisimple.*

We conclude this discussion of semisimple modules with an almost trivial, but nevertheless critical, example of semisimple modules: Suppose that D is a division ring. Then, ${}_D D$ is a simple left D-module (because obviously D has no non-zero proper left ideals). It follows immediately that all non-zero cyclic D-modules are simple. Now let M be any left D-module. Then, for each $0 \neq m \in M$, the cyclic submodule Dm is simple. Since $M = \sum_{0 \neq m \in M} Dm$, we get from the characterization that M is semisimple. Thus:

All modules over a division ring are semisimple.

Of course the fundamental property of vector spaces – that every vector space has a basis – is a special case of this argument.

Exercise. The following conditions on a ring R are equivalent:
(1) *every* left R-module is projective;
(2) *every* left R-module is semisimple;
(3) *every* left R-module is injective.
If moreover every left R-module is free, then conditions (1)–(3) hold, but the converse need not!

4.4 Tensor products

Suppose initially that F is a *field*, and let V, W be vector spaces over F. Fix bases $(v_i)_{i \in I}$ for V and $(w_j)_{j \in J}$ for W. Define $V \otimes W$ to be the F-vector space with basis

$$\{v_i \otimes w_j \mid i \in I, j \in J\}.$$

So if V and W are both finite dimensional,

$$\dim V \otimes W = \dim V \dim W.$$

Now we define a map $\iota : V \times W \to V \otimes W$ by

$$\iota\left(\sum_{i \in I} a_i v_i, \sum_{j \in J} b_j w_j\right) = \sum_{i \in I} \sum_{j \in I} a_i b_j v_i \otimes w_j$$

for arbitrary coefficients a_i and b_j in F (all but finitely many of each being zero).

The map $\iota : V \times W \to V \otimes W$ is *bilinear*, meaning that

$$\iota(cv + c'v', w) = c\iota(v, w) + c'\iota(v', w), \quad \iota(v, cw + c'w') = c\iota(v, w) + c'\iota(v, w')$$

for all $v, v' \in V, w, w' \in W, c, c' \in F$.

For $v \in V, w \in W$, we denote $\iota(v, w) \in V \otimes W$ instead by $v \otimes w$, and call such an element of the vector space $V \otimes W$ a *pure tensor*. It is crucial to bear in mind that *not every vector in* $V \otimes W$

can be represented as a pure tensor – for instance, consider $v_1 \otimes w_1 + v_2 \otimes w_2$!. But certainly, every vector in $V \otimes W$ can be written as a sum of finitely many pure tensors, that is, the pure tensors *generate $V \otimes W$ as an Abelian group.*

The vector space $V \otimes W$, together with the bilinear map $\iota : V \times W \to V \otimes W$, has the crucial property:

> *given an F-vector space U and a bilinear map $f : V \times W \to U$, there exists a unique linear map $\bar{f} : V \otimes W \to U$ such that $f = \bar{f} \circ \iota$.*

Indeed, there is no choice but to define $\bar{f}(v_i \otimes w_j) = f(v_i, w_j)$ on the basis elements $v_i \otimes w_j$ of $V \otimes W$ and then extend to all of $V \otimes W$ linearly. The resulting linear map $\bar{f}$ clearly satisfies the property $f = \bar{f} \circ \iota$. This universal property is the key to finding the right generalization of our tensor product $V \otimes W$ of two vector spaces to arbitrary modules over arbitrary rings...

But first, we need to first introduce the notion of a *bimodule*. Let R and S be rings. An *R, S-bimodule* means an Abelian group M that has the additional structure both of a left R-module and of a right S-module, such that

$$(rm)s = r(ms)$$

for all $r \in R, m \in M, s \in S$. So the actions of R and S on M commute with each other. When we need to stress that M is an R, S-bimodule, we write ${}_RM_S$. There is an obvious notion of homomorphism of R, S-bimodules: a map that is a homomorphism both as a left R-module and as a right S-module.

For example, any left R-module M is automatically an $R, \mathbb{Z}$-bimodule, the right $\mathbb{Z}$-action being the one uniquely determined by the Abelian group structure of M. Similarly, any right R-module is automatically a $\mathbb{Z}, R$-bimodule. On the other hand, if R is *commutative* and M is a left R-module, we can view M as a right R-module in the standard way by defining $mr := rm$ for $r \in R, m \in M$. This makes M into an R, R-bimodule: to check for example that the left and right actions commute you have that

$$(r_1m)r_2 = r_2(r_1m) = (r_2r_1)m = (r_1r_2)m = r_1(r_2m) = r_1(mr_2).$$

Similarly, any right R-module can be viewed as an R, R-module in this standard way.

Now we can introduce the notion of tensor product of modules in the general case. Let R, S, T be rings and suppose ${}_RM_S$ and ${}_SN_T$ are bimodules over the rings as indicated. If U is any R, T-bimodule, we call a map

$$f : {}_RM_S \times {}_SN_T \to {}_RU_T$$

a *balanced map* if the following properties hold:

(1) $f(m + m', n) = f(m, n) + f(m', n)$;

(2) $f(m, n + n') = f(m, n) + f(m, n')$;

(3) $f(ms, n) = f(m, sn)$;

(4) $f(rm, n) = rf(m, n)$;

(5) $f(m, nt) = f(m, n)t$;

for all $r \in R, s \in S, t \in T, m, m' \in M, n, n' \in N$. You should think of this definition of balanced as a generalization of the notion of "bilinear".

Now we define a *tensor product* of ${}_RM_S$ and ${}_SN_T$ to be an R, T-bimodule

$$M \otimes_S N = {}_RM_S \otimes_S {}_SN_T$$

together with a balanced map

$$\iota : M \times N \to M \otimes_S N$$

with the property that for any other R, T-bimodule ${}_RU_T$ and any other balanced map $f : M\times N \to U$, there exists a unique R, T-bimodule homomorphism $\bar{f} : M\otimes_S N \to U$ such that $f = \bar{f} \circ \iota$. As usual with universal properties, if such a bimodule $M \otimes_S N$ and map ι exists, it is unique up to canonical isomorphism. So we will always just call it *the* tensor product of M and N over S. But still, we need to prove existence:

Existence of tensor products. *For any R, S-bimodule ${}_RM_S$ and any S, T-bimodule ${}_SN_T$, the tensor product $\iota : M \times N \to M \otimes_S N$ exists.*

Proof. Let F be the free Abelian group on the *set* $M \times N$ (this could be absolutely huge!). Before doing anything, we need to make F into an R, T-bimodule. Given $r \in R$ define a map $l_r : M \times N \to F$ by $l_r(m, n) = (rm, n) \in F$. The universal property of F gives that this extends uniquely to a homomorphism $\bar{l}_r : F \to F$ of Abelian groups. This map $\bar{l}_r$ determines left multiplication by r on F, making F into a left R-module. Similarly, we make F into a right T-module, hence an R, T-bimodule.

Now let K be the Abelian subgroup of F generated by all elements
(1) $(m + m', n) - (m, n) - (m', n)$;
(2) $(m, n + n') - (m, n) - (m, n')$;
(3) $(ms, n) - (m, sn)$
for all $m, m' \in M, n, n' \in N, s \in S$. Consider the quotient Abelian group $M \otimes_S N := F/K$. Note that K is a left R-submodule of F, as well as a right S-submodule (it is a sub-R, S-bimodule!). Hence the R, S-bimodule structure on F induces a well-defined R, S-bimodule structure on $M \otimes_S N = F/K$. So $M \otimes_S N$ is an R, T-bimodule. Writing $m \otimes n$ for the image of the basis element (m, n) of F in $M \otimes_S N$ under the quotient map, we obtain a map

$$\iota : M \times N \to M \otimes_S N, \quad (m, n) \mapsto m \otimes n.$$

This map is balanced!

Now we check that ι and the R, T-bimodule $M \otimes_S N$ we have constructed really do satisfy the universal property. Let U be an R, T-bimodule and $f : M \times N \to U$ be a balanced map. By the universal property of free module, f extends uniquely to an Abelian group homomorphism $\bar{f} : F \to U$, which is automatically an R, T-bimodule map. Since f is balanced, all generators of K are annihilated by $\bar{f}$. So $\bar{f}$ factors through the quotient F/K to induce the unique $f' : M \otimes N \to U$ with $f = f' \circ \iota$ as required. □

Remarks. (i) Our construction of ${}_RM_S \otimes_S {}_SN_T$ really did not involve the left R- or the right T-module structures. In other words, if we viewed M just as a $\mathbb{Z}, S$-bimodule and N just as an $S, \mathbb{Z}$-bimodule, the resulting tensor product ${}_{\mathbb{Z}}M_S \otimes_S {}_SN_{\mathbb{Z}}$ would have been the *same* as ${}_RM_S \otimes_S {}_SN_T$. The only point of including the left and right module structures was to check that they are preserved throughout the construction.

(ii) If R is a commutative ring and M, N are (left, say) R-modules, we have remarked earlier that there is a standard way to view M and N as R, R-bimodules. Then, the tensor product $M \otimes_R N$ makes sense and is an R, R-bimodule. For instance, let F be a field and V, W be F-vector spaces. Then $V \otimes_F W$ is an F-vector space in this way. Moreover, it is canonically isomorphic to the basis-dependent construction of $V \otimes W$ in terms of bases given at the beginning of the section, since that was shown to also satisfy the universal property of tensor product.

We now give some examples of how to apply the universal property to prove things about tensor products.

Examples. (1) Let R be a commutative ring and M, N be R-modules viewed as R, R-bimodules in the standard way. Then, $M \otimes_R N \cong N \otimes_R M$. Proof. Define a map $M \times N \to N \otimes_R M$ by $(m, n) \mapsto n \otimes m$. This is balanced, hence by the universal property it induces a unique R-module map $M \otimes_R N \to N \otimes_R M$ with $m \otimes n \mapsto n \otimes m$. Similarly, there is a unique R-module map $N \otimes_R M \to M \otimes_R N$ with $n \otimes m \mapsto m \otimes n$. The two are inverse to each other, hence $M \otimes_R N \cong N \otimes_R M$.

(2) If F is a field and V, W are finite dimensional F-vector spaces, then

$$\mathrm{Hom}_F(V,W) \cong V^* \otimes W.$$

Proof. Define a map $\theta : V^* \times W \to \mathrm{Hom}_F(V,W)$ by $(f,w) \mapsto \theta_{f,w}$ where $\theta_{f,w}(v) = f(v)w$ for all $v \in V$. This is bilinear so induces a unique map $\bar{\theta} : V^* \otimes W \to \mathrm{Hom}_F(V,W)$.

I claim this map $\bar{\theta}$ is bijective. Well, pick bases $v_1, \dots, v_n$ for V and $w_1, \dots, w_m$ for W. Define $f_{i,j} : V \to W$ by letting $f_{i,j}(\sum_{s=1}^n a_s v_s) = a_i w_j$. Then, the $f_{i,j}$ form a basis for $\mathrm{Hom}_F(V,W)$ (matrices!). Also, if $f_1, \dots, f_n$ denotes the basis for V^* dual to $v_1, \dots, v_n$, the $f_i \otimes w_j$ form a basis for $V^* \otimes W$. Now our map θ satisfies $\theta_{f_i,w_j} = f_{i,j}$. Hence, $\bar{\theta}$ maps one basis bijectively to the other. So $\bar{\theta}$ is an isomorphism.

(3) If ${}_RM_S$, ${}_SN_T$ and ${}_TP_U$ are three bimodules, then

$$({}_RM_S \otimes_S {}_SN_T) \otimes_T {}_TP_U \cong_R M_S \otimes_S ({}_SN_T \otimes_T {}_TP_U)$$

as R,U-modules. Proof. We would like just to define the isomorphism between them on generators by $(m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)$. The whole problem is to show that there really is such an R,U-bimodule homomorphism! Of course, we need to use the universal property to prove this!

So first, fix $p \in P$ and define a map $f_p : M \times N \to M \otimes (N \otimes P)$ by $f_p(m,n) = m \otimes (n \otimes p)$. This is balanced, so induces a unique $\bar{f_p} : M \otimes N \to M \otimes (N \otimes P)$. Now define a map $(M \otimes N) \times P \to M \otimes (N \otimes P)$ by $(u,p) \mapsto \bar{f_p}(u)$. Again, this is balanced, so induces a unique map $(M \otimes N) \otimes P \to M \otimes (N \otimes P)$, which satisfies $(m \otimes n) \otimes p = f_p(m,n) = m \otimes (n \otimes p)$ on generators, as we wanted originally!

(4) For any ring R and any left R-module M,

$$R \otimes_R M \cong M.$$

Proof. Define a map $R \times M \to M$ by $(r,m) \mapsto rm$. This is balanced, so induces a unique R-module map $R \otimes_R M \to M$ such that $r \otimes m \mapsto rm$ on generators. The inverse is the map $m \in M$ to $1_R \otimes m \in R \otimes_R M$.

(5) $\otimes$ commutes with arbitrary direct sums. This means that given R,S-bimodules M_i $(i \in I)$ and S,T-bimodules N_j $(j \in J)$,

$$\left(\bigoplus_{i \in I} M_i\right) \otimes_S \left(\bigoplus_{j \in J} N_j\right) \cong \bigoplus_{i \in I, j \in J} M_i \otimes_S N_j$$

as R,T-bimodules. Indeed, one defines a map

$$\left(\bigoplus_{i \in I} M_i\right) \times \left(\bigoplus_{j \in J} N_j\right) \cong \bigoplus_{i \in I, j \in J} M_i \otimes_S N_j$$

by

$$\left(\sum_{i \in I} m_i, \sum_{j \in J} n_j\right) \mapsto \sum_{i \in I, j \in J} m_i \otimes n_j.$$

This is balanced, and so the universal property induces the required isomorphism. (How do you construct the inverse to prove this?)

We want to explain that the isomorphisms just constructed in these examples are *natural*! Roughly speaking, this means that they were defined without resorting to anything specific like fixed bases or generators. (In example (2) we did pick bases but only *after* we had constructed the map in a choice-independent way.) So we now need to understand how $\otimes$ is a *functor*, and then will be able to reinterpret isomorphisms like in the examples above as *natural isomorphisms* between functors. The word *natural* in mathematics should always indicate that there is some underlying natural isomorphism between functors – you should never use the too-often-overused word "natural" for anything else!

To start with, fix an R, S-bimodule ${}_RB_S$. Then, given any right R-module A_R, we have explained that

$$A_R \otimes_R {}_RB_S$$

is a right S-module. Now suppose that we are given a morphism $f : A_R \to A'_R$ between two right R-modules. We want to define a morphism

$$f \otimes \mathrm{id}_B : A_R \otimes_R {}_RB_S \to A'_R \otimes_R {}_RB_S$$

of right S-modules. To do this, start with the map $A \times B \to A' \otimes_R B$ defined by $(a, b) \mapsto f(a) \otimes b$. This is balanced, since f is a right R-module homomorphism. Hence by the universal property it induces a unique S-module homomorphism $f \otimes \mathrm{id}_B : A \otimes_R B \to A' \otimes_R B$ as required, satisfying $(f \otimes \mathrm{id}_B)(a \otimes b) = f(a) \otimes b$ for all $a \in A, b \in B$.

In this way, we obtain an (additive) functor

$$? \otimes_R {}_RB_S : \mathfrak{mod}\text{-}R \to \mathfrak{mod}\text{-}S.$$

You could of course do all the same arguments on the other side, to construct instead an additive functor

$${}_RB_S \otimes_S ? : S\text{-}\mathfrak{mod} \to R\text{-}\mathfrak{mod},$$

sending an object ${}_SC$ to the left R-module $B \otimes_S C$, and a morphism $f :_S C \to_S C'$ to the morphism

$$\mathrm{id}_B \otimes f :_R B_S \otimes_S {}_SC \to_R B_S \otimes_S {}_SC'.$$

Indeed, more generally still, you can think of

$$? \otimes_R ?$$

as a functor from $\mathfrak{mod}\text{-}R \times R\text{-}\mathfrak{mod}$ to $\mathfrak{ab}$. Here, $\mathfrak{mod}\text{-}R \times R\text{-}\mathfrak{mod}$ denotes the *product* of the categories $\mathfrak{mod}\text{-}R$ and $R\text{-}\mathfrak{mod}$. This is a fairly obvious notion: the objects in a product of two categories are simply pairs (A, B) of objects in each of the categories, while morphisms are pairs of morphisms in each of the categories. So $? \otimes_R ?$ maps an object

$$(A_R, {}_RB) \in \mathfrak{mod}\text{-}R \times R\text{-}\mathfrak{mod}$$

to their tensor product

$$A_R \otimes_R {}_RB$$

which is an Abelian group. Given morphisms $f : A_R \to A'_R$ and $g :_R B \to_R B'$ (i.e. a morphism (f, g) in the category $\mathfrak{mod}\text{-}R \times R\text{-}\mathfrak{mod}$), the functor $? \otimes ?$ maps (f, g) to the morphism

$$f \otimes g : A_R \otimes_R {}_RB \to A'_R \otimes_R {}_RB'$$

of Abelian groups. This is defined to be the map with $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$ (of course, to check that there really is such a well-defined map you need to appeal to the universal property).

Okay, so now we understand that $\otimes$ is a functor, it is now possible to make precise what we meant when we said that the isomorphisms constructed in the examples (1)–(5) above are *natural* isomorphisms. For instance:

Exercise (associativity of tensor product). Let R, S be rings. We have the tensor functors

$$?\otimes_R ? : \mathfrak{mod}\text{-}R \times R\text{-}\mathfrak{mod} \to \mathfrak{ab}$$

and

$$?\otimes_S ? : \mathfrak{mod}\text{-}S \times S\text{-}\mathfrak{mod} \to \mathfrak{ab}.$$

Composing them in two different ways gives two different functors $F = (?\otimes_R ?)\otimes_S ?$ and $G = ? \otimes_R (?\otimes_S ?)$ from $\mathfrak{mod}\text{-}R \times R\text{-}\mathfrak{mod}\text{-}S \times S\text{-}\mathfrak{mod}$ to $\mathfrak{ab}$.

(i) Explain what it means to say that the functors F and G are naturally isomorphic.

(ii) Now use the universal property of tensor product to prove that F and G are isomorphic functors.

4.5 Adjoint functors

So now we have introduced the tensor functors. The other important functors that arise in studying module categories are the *hom functors*. (Actually, we already made use of hom functors in the proof of the injective embedding theorem in section 4.2 – which should make more sense by the end of the present discussion!)

Let R be a ring and ${}_RB, {}_RC$ be left R-modules. We have already observed that

$$\mathrm{Hom}_R({}_RB, {}_RC)$$

is an Abelian group (R-$\mathfrak{mod}$ is an additive category). If ${}_RB$ has the additional structure of an R, S-bimodule for some new ring S, there ought to be some additional structure on

$$\mathrm{Hom}_R({}_RB_S, {}_RC).$$

Indeed, it is a *left* S-module. The left action of $s \in S$ on $f : B \to C$ is defined so that $sf : B \to C$ is the map with $(sf)(b) = f(bs)$ for all $b \in B$. You check associativity, for instance, with

$$((ss')f)(b) = f(b(ss')) = f((bs)s') = (s'f)(bs) = (s(s'f))(b).$$

Thus, the *right S-module structure on the first argument B leads to left S-module structure on* $\mathrm{Hom}_R(B, C)$. Instead, suppose that ${}_RC$ has the additional structure of an R, T-bimodule for some new ring T. Then this time,

$$\mathrm{Hom}_R({}_RB, {}_RC_T)$$

has additional structure of a *right* T-module. The right action of $t \in T$ on $f : B \to C$ is defined by $(ft)(b) = f(b)t$ for all $b \in B$. Thus, the *right T-module structure on the second argument C leads to right T-module structure on* $\mathrm{Hom}_R(B, C)$. Putting both cases together, if B is an R, S-bimodule and C is an R, T-bimodule, then

$$\mathrm{Hom}_R({}_RB_S, {}_RC_T)$$

is actually an S, T-bimodule. You could also consider this all for *right* R-modules instead. Then, the Abelian group

$$\mathrm{Hom}_R({}_SB_R, {}_TC_R)$$

is actually a T, S-bimodule: the left S-module structure on the first argument B leads to right S-modue structure on the hom space, while the left T-module structure on the second argument C leads to left T-module structure overall.

Example. Let C be an R, T-bimodule. Consider the space

$$\mathrm{Hom}_R({}_RR_R, {}_RC_T)$$

of homomorphisms of left R-modules. We have explained above how to view this as an R, T-bimodule. We claim that in fact,

$$\mathrm{Hom}_R({}_RR_R, {}_RC_T) \cong_R C_T$$

as an R, T-bimodule. Indeed, the isomorphism is given in the forward direction by the map Ev where $Ev(f) = f(1_R)$ – thus, Ev is "evaluation at 1_R". Let us check that evaluation really is an R, T-bimodule homomorphism:

$$Ev(rft) = (rft)(1_R) = f(1_R r)t = f(r1_R)t = rf(1_R)t = rEv(f)t$$

for every $f \in \mathrm{Hom}_R(R, C)$, $r \in R$ and $t \in T$, as required. It remains to see that Ev is bijective, which we do by exhibiting an inverse map $\gamma : C \to \mathrm{Hom}_R(R, C)$. Given $c \in C$, define $\gamma(c) : R \to C$ by $\gamma(c)(r) = rc$. You check $\gamma(c)$ really is an R-module homomorphism from ${}_RR$ to ${}_RC$. Then,

$$Ev(\gamma(c)) = \gamma(c)(1_R) = c$$

and

$$\gamma(Ev(f))(r) = \gamma(f(1_R))(r) = rf(1_R) = f(r1_R) = f(r),$$

showing that γ and Ev are two-sided inverses to each other.

Now we explain how $\mathrm{Hom}_R({}_RB_S, ?)$ is an (additive) functor from the category of left R-modules to the category of left S-modules. We have already explained how given a left R-module ${}_RC$,

$$\mathrm{Hom}_R({}_RB_S, {}_RC)$$

is a left S-module. So it remains to see that if $f :_R C \to_R C'$ is a morphism of left R-modules, we can define a morphism

$$\mathrm{Hom}_R(B, f) : \mathrm{Hom}_R({}_RB_S, {}_RC) \to \mathrm{Hom}_R({}_RB_S, {}_RC')$$

so that $\theta : B \to C$ maps to $f \circ \theta : B \to C'$. Thus, $\mathrm{Hom}_R({}_RB_S, ?)$ is a (covariant) functor from the category of left R-modules to the category of left S-modules.

You should not be surprised now that $\mathrm{Hom}_R(?, {}_RB_S)$ is also a functor from the category of left R-modules, but this time to the category of right S-modules, because each

$$\mathrm{Hom}_R({}_RA, {}_RB_S)$$

is a right S-module. Given a morphism $f :_R A \to_R A'$ of left R-modules, we define

$$\mathrm{Hom}_R(f, B) : \mathrm{Hom}_R({}_RA', {}_RB_S) \to \mathrm{Hom}_R({}_RA, {}_RB_S)$$

to be the right S-module homomorphism sending $\theta : A' \to B$ to the map $\theta \circ f : A \to B$. Thus, $\mathrm{Hom}_R(?, B)$ is a functor – but this time it is a *contravariant functor* because the direction of the morphism $f : A \to A'$ was reversed to give the morphism $\mathrm{Hom}_R(f, B) : \mathrm{Hom}_R(A', B) \to \mathrm{Hom}_R(A, B)$. Putting both constructions together, we obtain a functor

$$\mathrm{Hom}_R(?, ?) : (R\text{-}\mathfrak{mod})^{\mathrm{op}} \times R\text{-}\mathfrak{mod} \to \mathfrak{ab}.$$

Thus, the functor $\mathrm{Hom}_R(?, ?)$ is contravariant in the first argument and covariant in the second. This is a little more complicated than the tensor functor $? \otimes_R ?$ which was covariant in both arguments.

So now we have introduced both the tensor and the hom functors. The connection between the two comes from the following fundamental theorem:

Adjointness of tensor and hom. *Given rings R, S and modules $A_R, {}_R B_S, C_S$, there is a natural isomorphism*

$$\tau : \mathrm{Hom}_S(A_R \otimes_R {}_R B_S, C_S) \xrightarrow{\sim} \mathrm{Hom}_R(A_R, \mathrm{Hom}_S({}_R B_S, C_S))$$

of Abelian groups.

Proof. Given $f : A \otimes_R B \to C$, define $\tau f : A \to \mathrm{Hom}_S(B, C)$ so that $(\tau f)(a)$ is the function

$$b \mapsto f(a \otimes b)$$

for each $b \in B$.

There are now many things to check! For instance, we need to see that each $(\tau f)(a)$ is a right S-module morphism: $(\tau f)(a)(bs) = f(a \otimes bs) = f(a \otimes b)s = (\tau f)(a)(b)s$. Then we need to see that $\tau f : A \to \mathrm{Hom}_S(B, C)$ is a right R-module homomorphism, so that τf is really an element of $\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$. Well, $(\tau f)(ar)(b) = f(ar \otimes b) = f(a \otimes rb) = (\tau f)(a)(rb) = (((\tau f)(a)r)(b)$ hence $(\tau f)(ar) = ((\tau f)(a))r$ as required. Finally we need to see that τ is a bijection. For this, we construct a two-sided inverse

$$\sigma : \mathrm{Hom}_R(A_R, \mathrm{Hom}_S({}_R B_S, C_S)) \to \mathrm{Hom}_S(A_R \otimes_R {}_R B_S, C_S).$$

To define σ, take $f : A \to \mathrm{Hom}_S(B, C)$. Define a map $A \times B \to C$ by $(a, b) \mapsto (f(a))(b)$. This is balanced, so induces by the universal property of tensor product a unique map $\sigma(f) : A \otimes_R B \to C$. This defines σ, and now you check that it is the two-sided inverse to τ. □

The statement of the theorem used the word "natural". I want to explain what this means precisely shortly, but first let us make an abstract categorical definition because I think in this case the abstraction helps to understand the significance of the theorem just proved.

Let $\mathscr{A}, \mathscr{B}$ be two categories and $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be functors. Then, (F, G) is called an *adjoint pair* if for each pair of objects $X \in \mathscr{A}$ and $Y \in \mathscr{B}$, there is a natural bijection

$$\mathrm{Hom}_{\mathscr{B}}(FX, Y) \cong \mathrm{Hom}_{\mathscr{A}}(X, GY).$$

The crucial word here is "natural". To interpret its meaning, we need to view

$$\mathrm{Hom}_{\mathscr{B}}(F?, ?) : \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathfrak{sets}$$

as a *functor*. Indeed, we understand how to plug an object in $\mathscr{A}$ into the first ? and an object in $\mathscr{B}$ into the second ? to get a set at the end. Moreover, we understand how to plug morphisms in too to get set maps (bearing in mind that $\mathrm{Hom}_{\mathscr{B}}(F?, ?)$ is contravariant in the first argument). Similarly,

$$\mathrm{Hom}_{\mathscr{A}}(?, G?) : \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathfrak{sets}$$

is a functor. Then, saying that (F, G) is an adjoint pair really means that there is an isomorphism of functors

$$\mathrm{Hom}_{\mathscr{B}}(F?, ?) \cong \mathrm{Hom}_{\mathscr{A}}(?, G?).$$

Again, this means that for each pair of objects $(X, Y) \in \mathscr{A} \times \mathscr{B}$ there exists a bijection (an isomorphism of sets!)

$$\eta_{X,Y} : \mathrm{Hom}_{\mathscr{B}}(FX, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{A}}(X, GY)$$

such that for arrows $f : X' \to X$ in $\mathscr{A}$ and $g : Y \to Y'$ in $\mathscr{B}$, we have that

$$\mathrm{Hom}_{\mathscr{A}}(f, Gg) \circ \eta_{X,Y} = \eta_{X',Y'} \circ \mathrm{Hom}_{\mathscr{B}}(Ff, g)$$

as maps from $\mathrm{Hom}_{\mathscr{B}}(FX, Y)$ to $\mathrm{Hom}_{\mathscr{A}}(X', GY')$ (draw the diagram!).

We will go back in a moment to the above theorem ("adjointness of tensor and hom"). But let's start with some *easier* examples of adjoint pairs of functors...

Examples of adjoint pairs. (1) Let $i : \mathfrak{ab} \to \mathfrak{groups}$ be the "inclusion" functor. Let $\alpha : \mathfrak{groups} \to \mathfrak{ab}$ be the "Abelianization" functor. Thus, $\alpha(G) = G/G'$, and if $f : G \to H$ is a group homomorphism, $\alpha f : G/G' \to H/H'$ is induced by the map $\pi \circ f : G \to H/H'$ factored through G' (H/H' is Abelian so $\pi \circ f$ maps G' to $\{1\}$). Then, there is a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{groups}}(G, iA) \cong \mathrm{Hom}_{\mathfrak{ab}}(G/G', A)$$

so that $(\mathfrak{ab}, i)$ is an adjoint pair of functors. One calls the left hand functor $\mathfrak{ab}$ the *left* adjoint and the right hand functor i the *right* adjoint in the pair.

(2) Let $G : \mathfrak{sets} \to \mathfrak{groups}$ be the functor sending a set X to the free group on X. Let $F : \mathfrak{groups} \to \mathfrak{sets}$ be the forgetful functor. Then, the universal property of free groups gives us a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{groups}}(GX, H) \cong \mathrm{Hom}_{\mathfrak{sets}}(X, FH)$$

so again (G, F) is an adjoint pair of functors. In fact, most definitions by universal property can be interpreted in terms of an adjoint pair of functors in this way.

(3) Now consider tensor and hom. The theorem proved above gives that for rings R, S and modules $A_R, {}_R B_S, C_S$, there is a natural isomorphism

$$\tau : \mathrm{Hom}_S(A_R \otimes_R {}_R B_S, C_S) \to \mathrm{Hom}_R(A_R, \mathrm{Hom}_S({}_R B_S, C_S)).$$

This can be reinterpreted as saying that $(? \otimes_R {}_R B_S, \mathrm{Hom}_S({}_R B_S, ?))$ *is an adjoint pair of functors.* Informally, "tensor is left adjoint to hom" or "hom is right adjoint to tensor".

Now we have the basic notion of an adjoint pair of functors, I am going just to *state without proof* some of the basic (but remarkably useful) properties which follow by general nonsense just from the additive adjoint pair assumption. I will state these properties just in the case of an adjoint pair of functors between module categories. The correct setting really for these results is that of an additive adjoint pair of functors between arbitrary *Abelian categories*. But this material is taking us far enough as it is...

We need one more definition. Let $F : \mathfrak{mod}\text{-}R \to \mathfrak{mod}\text{-}S$ be a (covariant) functor, for rings R, S. Then, F is called *left exact* if it is additive and moreover exactness of any sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

in $\mathfrak{mod}\text{-}R$ implies exactness of the sequence

$$0 \longrightarrow FA \xrightarrow{F\alpha} FB \xrightarrow{F\beta} FC$$

in $\mathfrak{mod}\text{-}S$. Similarly, F is called *right exact* if it is additive and exactness of any sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

in $\mathfrak{mod}\text{-}R$ implies exactness of the sequence

$$FA \xrightarrow{F\alpha} FB \xrightarrow{F\beta} FC \longrightarrow 0$$

in $\mathfrak{mod}\text{-}S$. Finally, F is called *exact* if it is both left and right exact. It is an exercise to check that F being exact as just defined is *equivalent* to saying that F is additive and exactness of any sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

in $\mathfrak{mod}\text{-}R$ implies exactness of the sequence

$$0 \longrightarrow FA \xrightarrow{F\alpha} FB \xrightarrow{F\beta} FC \longrightarrow 0$$

in $\mathfrak{mod}\text{-}S$.

Properties of adjoint pairs. *Let* $F : \mathfrak{mod}\text{-}R \to \mathfrak{mod}\text{-}S$ *and* $G : \mathfrak{mod}\text{-}S \to \mathfrak{mod}\text{-}R$ *be functors such that* (F, G) *is an adjoint pair. Then:*

(0) *[Maclane[1], p.83] Any other functor* G' *which is right adjoint to* F *is isomorphic to* G*; any other functor* F' *which is left adjoint to* G *is isomorphic to* F*;*

(1) *[Maclane, p.83 and p.193]* F *and* G *are automatically additive functors, and the natural bijection*

$$\mathrm{Hom}_S(FM, N) \cong \mathrm{Hom}_R(M, GN)$$

arising from the adjunction is necessarily an isomorphism as Abelian groups.

(2) *[Rotman[2], p.39]* F *is right exact;* G *is left exact;*

(3) *[Rotman,p.47,p.55]* F *commutes with arbitrary (not necessarily finite) direct sums;* G *commutes with arbitrary direct products;*

(4) *[Exercise!] If* F *is in fact exact (not just right exact), then* G *sends injective* S*-modules to injective* R*-modules; if* G *is in fact exact (not just left exact), then* F *sends projective* R*-modules to projective* S*-modules.*

Now let us consider the consequences of these properties in the case which interests us: the adjoint pair $(? \otimes_R {}_RB_S, \mathrm{Hom}_S({}_RB_S, ?))$. We get for any rings R, S that:
(1) For any left R-module ${}_RB$, the functor $\mathrm{Hom}_R({}_RB, ?)$ is left exact;
(2) For any right S-module B_S, the functor $B_S \otimes_S ?$ is right exact;
(3) $\mathrm{Hom}_R({}_RB, ?)$ commutes with arbitrary direct products;
(4) $B_S \otimes_S ?$ commutes with arbitrary direct sums.

Remark. Let me note that we have worked here with left modules. We could just as well have worked with right modules, using the adjoint pair $(? \otimes_R {}_RB_S, \mathrm{Hom}_S({}_RB_S, ?))$. Then we would have deduced:
(1)′ For any right S-module B_S, the functor $\mathrm{Hom}_S(B_S, ?)$ is left exact;
(2)′ For any right R-module ${}_RB$, the functor $? \otimes_R {}_RB$ is right exact;
(3)′ $\mathrm{Hom}_S(B_S, ?)$ commutes with arbitrary direct products;
(4)′ $? \otimes_R {}_RB$ commutes with arbitrary direct sums.

Now it is reasonable to ask the following questions:
(Q1) When is the functor $\mathrm{Hom}_R({}_RB, ?)$ exact (not just left exact)?
(Q2) When is the functor $B_S \otimes_S ?$ exact (not just right exact)?
We can easily answer (Q1):

4.5.1. **Lemma.** *The functor* $\mathrm{Hom}_R({}_RB, ?)$ *is exact if and only if* ${}_RB$ *is a projective left* R*-module.*

Proof. Take a short exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{f} Q \longrightarrow 0.$$

Since $\mathrm{Hom}_R({}_RB, ?)$ is always left exact, we just need to ask when

$$\mathrm{Hom}_R(B, M) \xrightarrow{\hat{f}} \mathrm{Hom}_R(B, Q)$$

is surjective, where $\hat{f}$ is defined by $(\hat{f})(\theta) = f \circ \theta$. Saying $\hat{f}$ is surjective is equivalent to saying that for every homomorphism $\phi : B \to Q$, there exists a homomorphism $\theta : B \to M$ such that $\phi = f \circ \theta$. In other words, every homomorphism from B to Q *lifts* through f to a homomorphism from B to M. This is exactly the definition of what it means for B to be a projective R-module! □

On the other hand, we do not know an answer to question (Q2) at present. So instead, let us *define* a right S-module B_S to be *flat* if the functor $B_S \otimes_S ?$ is exact (not just right exact). This introduces another very important sort of modules.

[1] The reference is to Maclane's book "Categories for the working mathematician"
[2] Rotman's book "Introduction to homological algebra"

Example. $\mathbb{Z}_n$ *is not a flat* $\mathbb{Z}$*-module.* Proof. Take the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z}$$

where f is the map determined by multiplication by n. Applying the functor $\mathbb{Z}_n \otimes_{\mathbb{Z}} ?$, we get the sequence

$$0 \longrightarrow \mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\bar{f}} \mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}$$

where the map $\bar{f}$ is multiplication by n. In other words $\bar{f}$ is zero!!! So $\bar{f}$ is no longer injective, and our sequence is no longer exact. So $\mathbb{Z}_n$ is not flat.

We do at least have:

4.5.2. **Lemma.** *If B_S is a projective S-module, then it is flat.*

Sketch. We need to show that $B_S \otimes_S ?$ sends injective homomorphisms to injective homomorphisms. Since B_S is projective, it is a summand of $\bigoplus_{i \in I} S_S$ for some I. So it suffices to show that

$$\left(\bigoplus_{i \in I} S_S \right) \otimes_S ?$$

sends injective homomorphisms to injective homomorphisms. In turn, since $\otimes$ commutes with direct sums, this reduces to checking just that

$$S_S \otimes_S ?$$

sends injective homomorphisms to injective homomorphisms. But this is clear, since $S_S \otimes_S {}_S M \cong_S M$ for any left S-module ${}_S M$. □

But *the converse to the lemma is false in general.* For example, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module but is not projective.

4.6 Morita equivalence

Throughout the section, let R and S be rings. We want to try to compare R and S by comparing their module categories.

4.6.1. **Lemma.** *Let F and G be functors from $\mathfrak{mod}$-R to $\mathfrak{mod}$-S which are right exact and commute with arbitrary direct sums. Suppose that $\eta : F \to G$ is a natural transformation of functors such that $\eta_R : FR \to GR$ is an isomorphism of right S-modules. Then, η is an isomorphism of functors.*

Sketch. Take $M \in \mathfrak{mod}$-R. There exists a surjection $S_1 \xrightarrow{f} M$ where S_1 is a direct sum of copies of R. Again, there exists a surjection $S_2 \xrightarrow{g} \ker f$ where S_2 is a direct sum of copies of R. Putting together, we have constructed an exact sequence

$$S_2 \xrightarrow{g} S_1 \xrightarrow{f} M \longrightarrow 0$$

Now apply the right exact functors F and G in turn to obtain two exact sequences

$$FS_2 \longrightarrow FS_1 \longrightarrow FM \longrightarrow 0$$

and

$$GS_2 \longrightarrow GS_1 \longrightarrow GM \longrightarrow 0.$$

Moreover, the natural transformation η gives us vertical maps $\eta_2 : FS_2 \to GS_2$, $\eta_1 : FS_1 \to GS_1$ and $\eta_M : FM \to GM$ so that the resulting diagram commutes. We want to show that η_M is an isomorphism. Using the five lemma, it suffices to show that both η_1 and η_2 are isomorphisms.

Well,

$$FS_2 = F(\bigoplus R) \cong \bigoplus FR$$

since F commutes with arbitrary direct sums. Similarly,

$$GS_2 = G(\bigoplus R) \cong \bigoplus GR.$$

By the assumption η_R is an isomorphism between $FR \cong GR$. Now using the fact that η is a natural transformation, you get that η_2 is an isomorphism between FS_2 and FS_1. Similarly, η_1 is an isomorphism. $\square$

4.6.2. **Lemma.** *A functor $F : \mathfrak{mod}\text{-}R \to \mathfrak{mod}\text{-}S$ has a right adjoint if and only if $F \cong ? \otimes_R {}_RP_S$ for some R, S-bimodule ${}_RP_S$.*

Proof. ($\Leftarrow$). The functor $? \otimes_R {}_RP_S$ has a right adjoint, namely, $\mathrm{Hom}_S({}_RP_S, ?)$, by the adjointness of tensor and hom.

($\Rightarrow$). Suppose $F : \mathfrak{mod}\text{-}R \to \mathfrak{mod}\text{-}S$ has a right adjoint $G : \mathfrak{mod}\text{-}S \to \mathfrak{mod}\text{-}R$, i.e. that (F, G) is an adjoint pair. Then, we get from the general properties of adjoint pairs that F is right exact and commutes with arbitrary direct sums.

Let $P_S = F(R_R)$, a right S-module. It is even an R, S-bimodule. Indeed, given $r \in R$, left multiplication by r defines a right R-module homomorphism $\lambda_r : R_R \to R_R$. So applying the functor F, we obtain a right S-module homomorphism

$$F\lambda_r : P_S \to P_S.$$

Now define the action of $r \in R$ on P_S by

$$rp = (F\lambda_r)(p)$$

for each $p \in P$. This makes P_S into an R, S-bimodule ${}_RP_S$. We know by the general theory of adjoint pairs that the functor $? \otimes_R {}_RP_S$ is right exact and commutes with arbitrary direct sums.

Now we claim that $F \cong ? \otimes_R {}_RP_S$, which will complete the proof. First, we construct a natural transformation between the functors. Given $M_R \in \mathfrak{mod}\text{-}R$, we define a map

$$\eta_M \in \mathrm{Hom}_R(M, \mathrm{Hom}_S(P, FM))$$

by the composite

$$M_R \xrightarrow{\sim} \mathrm{Hom}_R(R_R, M_R) \longrightarrow \mathrm{Hom}_S(FR, FM) = \mathrm{Hom}_S(P, FM).$$

Here, the first isomorphism is the canonical one and the second homomorphism is the map induced by the functor F. Now, by adjointness of tensor and hom, there is a canonical isomorphism

$$\mathrm{Hom}_R(M, \mathrm{Hom}_S(P, FM)) \cong \mathrm{Hom}_S(M \otimes_R P, FM)$$

so we can view η_M instead as a map $M \otimes_R P \to FM$. Then, η defines a natural transformation from the functor $? \otimes_R P$ to the functor $F?$.

Now to show that η is an isomorphism of functors it suffices applying Lemma 4.6.1 just to see that $\eta_R : R \otimes_R P \to FR = P$ is an isomorphism. But by definition η_R is just the canonical isorphism between $R \otimes_R P$ and P. $\square$

Recall that two categories $\mathscr{A}$ and $\mathscr{B}$ are equivalent if there exist (covariant) functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ such that

$$F \circ G \cong \mathrm{Id}_{\mathscr{B}} \qquad \text{and} \qquad G \circ F \cong \mathrm{Id}_{\mathscr{A}}\,.$$

Now we obtain the main result of the section:

Morita theorem (weak version). *Let R and S be rings. The following properties are equivalent:*

(i) *The categories* $\mathfrak{mod}$-R *and* $\mathfrak{mod}$-S *are equivalent.*

(ii) *The categories* R-$\mathfrak{mod}$ *and* S-$\mathfrak{mod}$ *are equivalent.*

(iii) *There exists bimodules* ${}_RP_S$ *and* ${}_SQ_R$ *with*

$$ {}_RP_S \otimes_S {}_SQ_R \cong {}_RR_R, \qquad {}_SQ_R \otimes_R {}_RP_S \cong {}_SS_S $$

as bimodules.

Proof. I just prove the equivalence of (i) and (iii), the equivalence of (ii) and (iii) being similar.

(iii)⇐(i). Let $F = ? \otimes_R {}_RP_S$ and $G = ? \otimes_S {}_SQ_R$. Then,

$$ F \circ G = (? \otimes_S {}_SQ_R) \otimes_R {}_RP_S \cong ? \otimes_S ({}_SQ_R \otimes_R {}_RP_S) \cong ? \otimes_S {}_SS_S \cong \text{id}, $$

using associativity of the tensor product functor. Similarly, $G \circ F \cong \text{id}$. Hence, the categories are equivalent.

(i)⇒(iii). If the categories are equivalent, then there exist functors $F : \mathfrak{mod}\text{-}R \to \mathfrak{mod}\text{-}S$ and $G : \mathfrak{mod}\text{-}S \to \mathfrak{mod}\text{-}R$ with $F \circ G \cong \text{Id}$ and $G \circ F \cong \text{Id}$. I claim that G is right adjoint to F, and similarly F is right adjoint to G.

Before proving the claim, let us see how the theorem follows. Well, applying Lemma 4.6.2 twice, we get bimodules ${}_RP_S$ and ${}_SQ_R$ with $F \cong ? \otimes_R {}_RP_S$ and $G \cong ? \otimes_S {}_SQ_R$. Then, since $FG \cong \text{Id}$, applying to the module ${}_SS_S$ we get that

$$ {}_SS_S \otimes_S {}_SQ_R \otimes_R {}_RP_S \cong_S S_S $$

hence ${}_SQ_R \otimes_R {}_RP_S \cong {}_SS_S$ and similarly ${}_RP_S \otimes_S {}_SQ_R \cong {}_RR_R$.

It remains to prove the claim, that G is right adjoint to F. The functor G induces a natural transformation between the functors

$$ \eta : \text{Hom}_S(F?, ?) \to \text{Hom}_R(?, G?), $$

where for a given R-module M and an S-module N, the map $\eta_{M,N}$ defining the natural transformation η comes from

$$ \text{Hom}_S(FM, N) \to \text{Hom}_R(GFM, GN) \cong \text{Hom}_R(M, GN) $$

defined simply by $\theta \mapsto G\theta$. To see that the map $\eta_{M,N}$ is in fact an isomorphism (which is all that remains to prove that G is right adjoint to F) note that F induces a map $\text{Hom}_R(M, GN) \to \text{Hom}_S(FM, FGN) \cong \text{Hom}_S(FM, N)$. Since $GF \cong \text{Id}$ and $FG \cong \text{Id}$, this map induced by F is a two-sided inverse to the map $\eta_{M,N}$ induced by G, proving that $\eta_{M,N}$ is bijective. □

Now call the rings R and S *Morita equivalent* if any of the equivalent properties in the Morita theorem hold.

Example. This is really the crucial example! Let R be any ring and $S = M_n(R)$, the ring of $n \times n$ matrices with entries in R. I claim that R and S are Morita equivalent rings.

Proof. Let ${}_RP_S$ be the R, S-bimodule consisting of all row vectors of the form $(a_1 \dots a_n)$ for $a_i \in R$. It is a left R-module by multiplication, and a right S-module by matrix multiplication. Similarly, let ${}_SQ_R$ be the S, R-bimodule consisting of all column vectors under matrix multiplication.

I claim that

$$ {}_SQ_R \otimes_R {}_RP_S \cong_S S_S $$

and

$$ {}_RP_S \otimes_S {}_SQ_R \cong_R R_R $$

as bimodules. Indeed, the isomorphisms in each case are just defined by multiplication. So in the first case, you take the map $Q \times P \to S$ given by sending a pair (c, r) (where c is a column vector and r is a row vector) to their product (which is an $n \times n$ matrix). This map is balanced so extends to a unique S, S-bimodule homomorphism as stated. Similarly in the second case, the map is induced by the universal property of tensor by the map $P \times Q \to R$ given by multiplication, this time the row vector times the column vector which gives a 1×1 matrix! In each case, it remains to check that the maps defined are indeed bijective. I leave this as an exercise.

Hence by the Morita theorem, R and S are Morita equivalent rings.

Having defined the relation "Morita equivalence" on the category of rings, it is reasonable to ask what sort of ring theoretic properties are preserved by Morita equivalences. In other words, what are the *Morita invariants* of rings. I give two examples, the first with proof the second without. Hopefully, these are enough to convince you that Morita equivalence – something defined entirely in terms of the modules of a ring not the ring itself – is a useful thing to consider.

First example of a Morita invariant property. *If R and S are Morita equivalent rings, then the lattice of two-sided ideals of R is isomorphic to the lattice of two-sided ideals of S.*

Sketch. By the Morita theorem, there exist bimodules ${}_RP_S$ and ${}_SQ_R$ such that the functors

$$? \otimes_R {}_RP_S \qquad \text{and} \qquad ? \otimes_S {}_SQ_R$$

give the mutually inverse equivalences. Applying $? \otimes_R {}_RP_S$ to a two-sided ideal in ${}_RR_R$ gives an R, S-subbimodule of $R \otimes_R P = P$. Conversely applying $? \otimes_S {}_SQ_R$ to an R, S-subbimodule of P gives a two-sided ideal in R. Since the two functors are inverse to each other, you deduce using naturality that they induce a lattice isomorphism between the lattice of two-sided ideals of R and the lattice of R, S-subbimodules of P.

Now repeat the argument instead with the functors

$${}_RP_S \otimes_S ? \qquad \text{and} \qquad {}_SQ_R \otimes_R ?$$

to deduce this time that there is a lattice isomorphism between the lattice of two-sided ideals of S and the lattice of R, S-subbimodules of P. Combining the conclusion in each of these two paragraphs completes the proof. □

Second example of a Morita invariant property. You can also show – with not too much extra work – that if R and S are Morita equivalent rings, then the *centers* of the rings R and S are isomorphic. For instance, since R and $M_n(R)$ are isomorphic, the centers of these rings are isomorphic – in this special case this is easy to prove directly!

In particular, two *commutative rings* R, S are Morita equivalent if and only R and S are in fact already isomorphic rings themselves. This shows that for commutative rings, *everything about the ring is encoded in the category of modules over the ring.* Perhaps this convinces you of the value of studying modules as an indirect way to get at the structure of a ring.