Optimal Domestic Processing of Exhaustible Resource Exports under Stock Uncertainty*

Ramesh C. Kumar

Department of Economics
University of Waterloo
200 University Avenue West
Waterloo, ON, N2L 3G1
Canada
e-mail: rkumar@uwaterloo.ca

Abstract

The ever-increasing integration of the world economy through international trade and investment does not appear to have diminished the attractiveness of greater domestic processing of exhaustible natural resource exports among policy-makers in developing countries. The existing analysis and results underpinning its economic rationale, however, pertain to the case when the size of the initial resource stock is known with perfect certainty. Expressing uncertainty via the hazard function, we examine the robustness of the existing results under stock uncertainty for a small open economy. In addition to fully extending the major existing results with minimal additional assumptions to the uncertainty scenario for a wide class of the iso-elastic utility function and continuous resource stock distributions with finite support, we obtain a complete qualitative characterization of the optimal program in terms of the extraction rate, level of domestic processing and capacity expansion.

Key Words: Exhaustible Resource, Domestic Processing, Uncertainty, Hazard Function
JEL Classification Codes: Q32, O13 and D80

* WEAI, 84th Annual Conference, Vancouver, BC, Canada, June 29-July 3, 2009. Not to be quoted without permission.

Optimal Domestic Processing of Exhaustible Resource Exports under Stock Uncertainty

1. Introduction

Notwithstanding the ever-increasing integration of the world economy through international trade and investment, development strategies that encourage exporting exhaustible natural resources in the processed rather than raw form continue to remain as attractive to economic policy-makers in developing countries as in the 1970s. Witness, for example, the statements of policy on minerals processing by China and India, or Saudi Arabia's resolve to develop greater capabilities in crude oil refining and other ancillary activities. Even the highly developed and globally well-integrated economies of Australia and Canada place restrictions on processing their minerals outside of their territories.[1] Nevertheless, the analytical literature on the subject has remained relatively sparse.

Long (1974), Vousden (1975), Kemp and Suzuki (1975), Aarrested (1979), Harris (1982) and Withagen (1985) are the earliest important analytical contributors, but with a clear focus on the optimal rates of resource extraction in the presence of domestic processing rather than on the degree of domestic processing. Kumar (1988, 1997) takes the broader, development policy perspective, posing three distinct questions: What, if any, are the determinants of domestic

---

[1] See http://www.finfacts.com/irelandbusinessnews/publish/article_100047805.shtml for recent announcement by the Saudi Arabia of two new refineries, each with a processing capacity of 400,000 barrels of oil per day, to increase the exports of refined products. See also U.S.-Saudi Business Council (2005) for a more direct statement of the policy. For China, see the "White paper on Resource Exploitation"and "China's Policy on Mineral Resource"respectively available on http://www.china.org.cn/c-white/20031223/3htm and, for India, Government of India (1993). Also, see Government of Ontario, Ministry of Natural Resources' publication "Mineral Exploration and Mining Development"at http://www.serviceontario/mining/english/MineDevelopment.htm, and Lloyd (1991).

processing? Is the usual presumption of the policy-makers in favor of enhanced levels of domestic processing of an exhaustible natural resource prior to exporting theoretically justified under free trade? If yes, what are the optimal levels of domestic processing and the associated optimal investment policy? Kumar (1988) explores the first two questions in the context of a small open economy and malleable capital, while Kumar (1997) focuses on the second and the third in the presence of capital stock adjustment costs, deriving the optimal time-profiles of the rate of extraction, the level of domestic processing and capacity expansion in the processing sector. Three principal findings emerge from this analysis. First, if capital is the only other (other than the natural resource) factor necessary for processing, a constant returns-to-scale technology in the processing sector gives rise to complete specialization in the export markets. Second, replacing constant with decreasing returns-to-scale or including adjustment costs that increase with the rate of capacity expansion imply – unlike the preceding result – a presumption in favor of a steadily increasing level of domestic processing, as measured by the proportion of extracted resource amount undergoing some processing prior to exporting. Third, if and when capacity expansion is deemed desirable, the optimal pattern of investment activity is of the front-end loading variety so that all of the capacity expansion takes place well before resource exhaustion.

These results and the underlying analysis pertain to the case of perfect certainty regarding the size of the initial resource stock. In what follows, we attempt to extend the analysis to the case when the size of the initial resource stock is uncertain, with a view to determining the robustness of the perfect certainty results.

We devote the next section to describing the economic setting and analyzing the case of malleable capital. In section 3, we widen the scope of our analysis by incorporating increasing adjustment costs. We conclude in section 4 by summarizing our findings.

2. Domestic Processing under Malleable capital

In keeping with Kumar (1988), we postulate a small open economy completely dependent upon natural resource exports that are made possible by the exploitation of a non-renewable, natural resource stock of uncertain size. Let the random variable $S > 0$ represent the initial size of the resource stock. As the economy extracts the resource, it may export the entire extracted amount, say $X(t)$, in its raw form at price P_1 or elect to further process a proportion $\mu(t)$ thereof prior to exporting at price P_2. We also assume that the economy does not possess the wherewithal necessary for further processing. It therefore imports (rents) the necessary plant and equipment $K(t)$ at the fixed price (real rental) r from the international capital goods (services) market to carry out the processing activity. After paying for its capital goods (services) imports, the economy uses the remaining export revenue to import a composite consumption good in the amount $C(t)$. If $U(C(t))$ denotes the instantaneous, strictly concave utility function with $U(0) = 0$, the basic planning problem faced by the economy to determine the socially desirable level of domestic processing may be specified as:

$$\underset{< X(t),\ \mu(t),\ K(t) >}{Maximize} \quad E[\int_0^\tau U(C(t))e^{-\delta t}dt] \tag{1}$$

subject to

$$C(t) = P_1(1-\mu(t))X(t) + P_2F(K(t), \mu(t)X(t)) - rK(t), \tag{2}$$

$$K(t) \geq 0, \quad C(t) \geq 0, \quad 0 \leq \mu(t) \leq 1, \tag{3}$$

$$\int_0^{\tau} X(t)dt \leq \bar{s}, \qquad X(t) \geq 0, \qquad S \sim g(s), \quad 0 < s \leq \bar{s}, \tag{4}$$

where all of the prices are time-independent and measured in units of the consumption good; $F(\ ,\ )$ is the production function describing processing sector technology, exhibiting positive but diminishing marginal products for both inputs; $g(s)$ is the probability distribution of S with $\bar{s}$ the maximum possible value of S; $\delta > 0$ is the constant discount or time-preference rate; and E stands for mathematical expectation, which is taken over the probability distribution of τ, a random variable denoting the uncertain resource exhaustion date.

If we assume that S is continuous and let $Q(t)$ and $G(s)$ denote respectively the cumulative extraction to date and the cumulative probability distribution of S, we may transform the preceding stochastic optimal control problem into a deterministic one as follows:

$$\underset{< X(t),\ \mu(t),\ K(t) >}{Maximize} \int_0^T U(C(t))\pi(Q(t)e^{-\delta t}dt \tag{5}$$

subject to

$$\dot{Q}(t) = X(t), \qquad X(t) \geq 0, \tag{6}$$

$$Q(0) = 0, \qquad Q(T) = \bar{s}; \qquad 0 < \bar{s} \leq \infty, \tag{7}$$

plus definition (2) and the non-negativity constraints (3). A dot over a variable denotes its time derivative. $\pi(Q(t)) = 1 - G(Q(t))$ is the survival rate for S, and T is the least upper bound of the support of the implied probability distribution of τ. Since τ is non-negative, $T \geq 0$, and is free to vary.[2]

Assuming that an optimal program exists[3], the necessary conditions for solving this deterministic control problem include, in addition to definition (2), state equation (6) and boundary conditions (7),

$$U'(C)\pi(Q)[P_1(1-\mu)X + P_2\mu F_2(K,\mu X]e^{-\delta t} + \lambda + \theta_4 = 0\,, \tag{8}$$

$$U'(C)\pi(Q)[-P_1 + P_2F_2(K,\mu X)]Xe^{-\delta t} + \theta_1 - \theta_2 = 0\,, \tag{9}$$

$$U'(C)\pi(Q)[P_2F_1(K,\mu X) - r]\,e^{-\delta t} + \theta_3 = 0\,, \tag{10}$$

$$\dot{\lambda} = -\,U(C)\pi'(Q)\,e^{-\delta t}, \tag{11}$$

$$\theta_1 \geq 0, \qquad \theta_1\mu = 0\,, \tag{12}$$

$$\theta_2 \geq 0, \quad \theta_2(1-\mu) = 0\,, \tag{13}$$

$$\theta_3 \geq 0, \qquad \theta_3 K = 0\,, \tag{14}$$

$$\theta_4 \geq 0, \qquad \theta_4 X = 0\,, \tag{15}$$

$$Lim_{t\to T}\ H(t) = Lim_{t\to T}\ [U(C)\,\pi(Q)\,e^{-\delta t} + \lambda X] = 0\,, \tag{16}$$

where λ is the co-state variable; θ's are the Lagrange multipliers; F_i (i = 1, 2) are the two partial derivatives of $F(\ ,\ )$; and H is the Hamiltonian. Also, for ease in notation, we have suppressed the time argument of the various functions. We continue to follow this practice in the rest of the

[2] See Kumar (2005), pp 409-11, for details.

[3] In the presence of $\pi(Q(t))$ in the objective functional, the strict concavity of $U(C(t))$may not be sufficient for existence. In what follows, however, it becomes obvious that a solution does indeed exist and is unique.

paper unless the possibility of confusion dictates otherwise.

Some features of the optimal program are immediately obvious from the necessary conditions: (*i*) Since the natural resource must be necessary for processing in that $F(K, 0) = 0$, $X > 0$ ($\theta_4 = 0$) along the optimal program, except possibly at T; (*ii*) $K > 0$ if only if $\mu > 0$, and is imported until P_2F_1 (the marginal revenue product) is equal to r (the real rental on capital); (*iii*) $\theta_1 > 0$ and $\theta_2 > 0$ cannot hold simultaneously so that there exist at most three, possibly repeatable, phases of domestic processing: no domestic processing ($\mu = 0$, or $\theta_1 > 0$ and $\theta_2 = 0$), partial domestic processing ($0 < \mu < 1$, or $\theta_1 = 0$ and $\theta_2 = 0$) and full domestic processing ($\mu = 1$, or $\theta_1 = 0$ and $\theta_2 > 0$).

We examine next the optimal program by characterizing the phases the economy may experience and their sequence. As in the case of perfect certainty, the nature of processing sector technology plays a crucial role.

Proposition 1: *If* $F(K, \mu X)$ *is linear homogenous, the phase* $0 < \mu < 1$ *is in general not part of the optimal program.*

Proof: As $U'(C) > 0$ always and $\pi(Q) > 0$ except at the terminal date T, there does not exist a $0 < \mu < 1$ such that (9) and (10) are satisfied simultaneously for arbitrary, positive X and K unless P_1, P_2 and r are of very special magnitudes.■

While necessary condition (9) determines whether the economy of proposition 1 at all engages in domestic processing, (8)-(11) together describe the time-paths of extraction, consumption, and capital imports. It is easy to check that regardless of which of the two possible phases the economy is in,

$$\frac{\dot{C}}{C} = \frac{\dot{X}}{X} = \frac{1}{\varepsilon(C)}\left(\frac{U(C)}{U'(C)C} - 1\right)h(Q)X - \frac{\delta}{\varepsilon(C)}, \tag{17}$$

$$\frac{\dot{\lambda}}{\lambda} = -\frac{U(C)}{U^{\prime}(C)C}h(Q)X, \tag{18}$$

where $\varepsilon(C) = -U^{\prime\prime}(C)C/U^{\prime}(C)$ is the consumption elasticity of marginal utility (the inter-temporal elasticity of substitution), and $h(Q) = -\pi^{\prime}(Q)/\pi(Q)$ is the hazard function of S.[4] Solving (17) and (18) is fundamental to characterizing the optimal program. However, closed form solutions do not seem possible without further describing the nature of the utility and hazard functions. Proposition 2 below extends the essential message of proposition 2 in Kumar (1988) to the case of stock uncertainty for a wide class of the iso-elastic utility function and any continuous initial resource stock probability distribution with finite support.

Proposition 2: *If* (*a*) *$F(K, \mu X)$ is linear homogenous,* (*b*)*$U(C)$ is iso-elastic with $0 < \varepsilon < 1$, and* (*c*) $0<\bar{s}<\infty$, (*i*) *the economy is in the* $\mu = 0$ ($\mu = 1$) *phase accordingly as* $-P_1+P_2F_2(K(X),\ X)<0\ \ (\geq 0)$, *where $K(X)$ is such that* $P_2F_1(K(X),\ X)=r$. (*ii*) *Optimal* $T = \infty$ and, along the optimal program, $\dot{C}/C=\dot{X}/X<0$ and $K=\dot{K}=0$ when $\mu = 0$*, and* $\dot{C}/C=\dot{X}/X=\dot{K}/K<0$ *when* $\mu = 1$.

Proof: (*i*) In view of (*a*) and proposition1, this part of the proposition is immediately obvious from (9) and (10). (*ii*) (17) ensures $\dot{C}/C=\dot{X}/X$ in both phases. Next, so long as $\mu = 0$, $K=\dot{K}=0$ is the obvious optimal choice, for otherwise C is unnecessarily lower and H is not maximized. If, however, $\mu = 1$, (*a*) and (10) ensure $\dot{X}/X=\dot{K}/K$. Now, adapting (17) to the special case of the

[4] To derive (17) and (18) for the no processing phase, we make use of the fact that $\mu = 0 \rightarrow C = P_1 X$. Similarly, for $\mu = 1$, $P_2 F_1 = r$ and $C = P_2 F_2 X$ such that $x = X/K$, F_1 and F_2 are all constants.

stipulated iso-elastic utility function yields $dX/dQ|_{\dot{X}=0} = -h'(Q)X/h(Q) < 0$ such that, along the $\dot{X}=0$ contour, terminal $X = 0$.[5] In the light of (*c*), therefore, there exist at most three possibilities for optimal X as depicted in Figure 1: $\dot{X}>0$ throughout with $X(T) > 0$ and optimal $T < \infty$; $\dot{X}>0$ after at most a finite period with $X(T) > 0$ and optimal $T < \infty$; and $\dot{X}<0$ throughout with $X(T) = 0$ but optimal T indeterminate. Now, the integration of (18) for a constant ε yields $\lambda(t) = \lambda(0)\pi(Q)^{-\frac{1}{1-\varepsilon}}$; whence $\lim_{t\to T} \lambda(t) = 0$. Consequently, (8) and (11) imply

$Lim_{t\to T}\ U'(C(t)) = Lim_{t\to T}\ U(C(t)/[C(t) + \delta/h(Q(t)] = U(C(T))/C(T)$, which may hold only if $C(T) = X(T) = 0$. This precludes the first two possibilities. Finally, integrating $\varepsilon(C)\dot{C}/C$ as stipulated in (17) for the case of iso-elastic utility yields $Lim_{t\to T}U'(C(t)) = U'(0) = U'(C(0))\,Lim_{t\to T}e^{\int_0^T (\delta - \frac{1}{1-\varepsilon}h(Q)X)dt}$.

Given that $\dot{C}/C = \dot{X}/X < 0$ under the third option, the integrand in the preceding expression for $U'(0)$ is positive. Whence it follows that $C(T) = X(T) = 0$ only if $T = \infty$. As this is also consistent with the transversality condition (16), we may conclude that optimal $\dot{X}/X < 0$ forever in either of the two phases, thereby completing the proof.[6]■

---

[5] Follows from the observations that for $0 < \bar{s} < \infty$, $h'(Q) > 0$ and $Lim_{Q\to\bar{s}}h(Q) = \infty$. See Kumar (2005), p 413, for the rationale.

[6] Below, we repeatedly utilize this method of proof, or a variant thereof, for identifying the optimal program. It essentially involves three steps: (*i*) establishing the existence and nature of the stationary contour for X, (*ii*) delineating qualitatively the possible programs in the phase plane, and (*iii*) demonstrating that one and only one program is consistent with the transversality

We allow next for a decreasing returns-to-scale processing technology, a typical characterization of the Dasgupta-Heal economy [7]. The result below replicates Kumar (1988), proposition 3, for the utility function and resource stock distributions of proposition 2.

Proposition 3: *If* (a) $F(K, \mu X)$ *exhibits decreasing returns-to-scale and* $Lim_{K\to 0}F_1 = Lim_{\mu X\to 0}\ F_2 = \infty$, ($b$) $U(C)$ *is iso-elastic with* $0 < \varepsilon < 1$, *and* (c) $0<\bar{s}<\infty$, (i) *the economy is in the* $0 < \mu < 1$ ($\mu = 1$) *phase accordingly as* $-P_1 + P_2F_2(K(X),\ X)<0\ \ (\geq 0)$, *where* $K(X)$ *is such that* $P_2F_1(K(X),\ X) = \boldsymbol{r}$. ($ii$) *The* $0 < \mu < 1$ phase, *if observed*, *lasts for at most a finite period of time and is necessarily followed by the* $\mu = 1$ *phase*, *which lasts forever.* (iii) *Along the optimal program,* $\dot{X}/X<0$, $\dot{C}/C<0$ $\ \dot{\mu}/\mu \geq 0$ and $\dot{K}/K \leq 0$.

Proof: (i) Let F_{ij} denote the partial derivative of F_i with respect to the j-th argument ($i,\ j$ =1, 2). In view of the condition on marginal products, the existence of $K(X) > 0$ such that $P_2F_1(K(X),\ X) = \boldsymbol{r}$ and $K'(X) = -F_{12}/F_{11} > 0$ is trivially assured for arbitrary $X > 0$. Let $-P_1 + P_2F_2(K(X),\ X)<0$. Differentiating $-P_1 + P_2F_2(K(X),\ X)$ with respect to X and substituting for $K'(X)$ yields $P_2(-F_{21}F_{12}/F_{11} + F_{22})<0$ because $F(\ ,\ )$ exhibits decreasing returns-to-scale. Consequently, the restriction (a) on F_2 implies that there always exists a $0 < \mu <1$ such that $P_2F_1(K(\mu X),\ \mu X) = \boldsymbol{r}$ and $-P_1 + P_2F_2(K(\mu X),\ \mu X) = 0$ simultaneously. If, however, $-P_1 + P_2F_2(K(X),\ X) \geq 0$, the necessary condition (9) implies that $\mu = 1$ is the obvious optimal choice from the start. (ii) If $0 < \mu <1$,

condition and thereby optimal. For a more comprehensive treatment of the method in the context of a strictly concave utility function, see Kumar (2005), pp 417-18.

[7] Dasgupta, Eastwood and Heal (1978).

$P_2F_2(K(\mu X),\ \mu X)$ is constant. Whence $(-F_{21}F_{12}/F_{11}+F_{22})(\dot{\mu}/\mu+\dot{X}/X)=0$ and $\dot{\mu}/\mu=-\dot{X}/X$ throughout the phase. This in turn implies $\dot{K}/K=0$ throughout the phase. As a result, $C=P_1(1-\mu)X+P_2F(K(\mu X),\mu X)-rK(\mu X)=P_1X+N$ with N a constant. Hence $\dot{C}/C=(P_1X/C)\dot{X}/X$.

Next differentiating (8) with respect to t and substituting for $\dot{C}/C$, we obtain $\frac{\varepsilon P_1X}{C}\frac{\dot{X}}{X}=(\frac{1}{1-\varepsilon}\frac{C}{P_1X}-1)h(Q)X-\delta$ such that $dX/dQ_{|\dot{X}=0}=\frac{1-\varepsilon}{\varepsilon}(\frac{1}{1-\varepsilon}\frac{C}{P_1X}-1)h'(Q)X/h(Q)\leq 0$. This implies that there exist once again at most three possibilities for optimal X: (*i*) optimal X rises continuously; (*ii*) optimal X rises continuously after declining for a finite period of time; and (*iii*) optimal X declines continuously throughout the phase. The use of essentially the same argument as that employed in the proof of part (*ii*) of the preceding proposition ensures that optimal $\dot{X}/X<0$ throughout the phase, with a terminal $X=0$ and $T=\infty$ if exhaustion takes place in the phase. As $\dot{\mu}/\mu=-\dot{X}/X>0$, this in turn ensures $\mu=1$ in finite time before optimal $X=0$. When $\mu=1$, however, $C=P_2F(K,\ X)-rK$ such that, along the optimal program, $\dot{C}/C=(P_2F_2X/C)\dot{X}/X$. Once again, differentiating (9) with respect to t and replacing $\dot{C}/C$ with the expression just derived yield[8]

$$[\varepsilon\frac{C}{P_2F_2X}-\frac{F_{11}F_{22}-F_{12}F_{21}}{F_{11}}\frac{X}{F_2}]\frac{\dot{X}}{X}=(\frac{1}{1-\varepsilon}\frac{C}{P_2F_2X}-1)h(Q)X-\delta \text{ and that}$$

[8] Follows from $C_2>P_2F_2X$ and decreasing returns-to-scale.

$$\frac{dX}{dQ}\Big|_{\dot{X}=0} = -\left(\frac{1}{1-\varepsilon}\frac{C}{P_2F_2X} - 1\right)h'(Q)X/\left[\left(\frac{\varepsilon}{1-\varepsilon} - \frac{1}{1-\varepsilon}\frac{C}{P_2F_2}\frac{F_{11}F_{22}-F_{12}F_{21}}{F_{11}F_2}\right]h(Q) \le 0$$

Following, yet again, essentially the same logic as before, we may conclude that optimal $\dot{X}/X<0$ and that the resource stock is exhausted asymptotically. (*iii*) In view of the preceding, it suffices to note that when $\mu = 1, \dot{K}<0$ because $K'(X) > 0$.■

3. Domestic Processing in the Presence of Capital Stock Adjustment Costs

In the preceding section, the economy is precluded from engaging in any domestic capital formation. We now do away with this restriction and stipulate instead that the economy possesses a certain initial capability or capacity in the form of plant and equipment stock to further process the resource. Therefore, along with choosing the level of domestic processing in each time-period, it also decides on the extent to which it should add, through imports, to its current capacity for processing. In keeping with Kumar (1997), we also stipulate that adjusting capacity entails real costs which are additional to those of importing plant and equipment. If adjustment costs, say C_A, are directly related to the level of gross investment *I,* the altered planning problem becomes:

$$\underset{< X(t),\ \mu(t),\ I(t) >}{Maximize} \int_0^T U(C(t))\pi(Q(t)e^{-\delta t}dt \tag{19}$$

subject to

$$C = P_1(1-\mu)X + P_2F(K,\mu X) - P_I I - C_A(I), \tag{20}$$

$$\dot{K} = I, \qquad I \geq 0, \quad K(0) > 0, \tag{21}$$

plus constraints (3), state equation (6) and boundary conditions (7), where P_I is the constant import price of investment goods and C_A (I) is adjustment costs *a la* Gould (1968) such that

$$C_A(I) > 0, \quad C_A^{\prime}(I) > 0, \quad C_A^{\prime\prime}(I) > 0, \quad I > 0 \text{ and } C_A(0) = C_A^{\prime}(0) = 0. \tag{22}$$

Upon appropriately defining the Hamiltonian and the Lagrangean for the problem, the necessary conditions corresponding to (8)-(16) turn out to be:

$$U^{\prime}(C)\pi(Q)[P_1(1-\mu) + P_2\mu F_2(K, \mu X]e^{-\delta t} + \lambda_1 + \theta_4 = 0 \tag{23}$$

$$U^{\prime}(C)\pi(Q)[-P_1 + P_2F_2(K, \mu X)]Xe^{-\delta t} + \theta_1 - \theta_2 = 0, \tag{24}$$

$$U^{\prime}(C)\pi(Q)[P_I + C_A^{\prime}(I)]e^{-\delta t} - \lambda_2 - \theta_3 = 0, \tag{25}$$

$$\dot{\lambda}_1 = -U(C)\pi^{\prime}(Q)e^{-\delta t} \tag{26}$$

$$\dot{\lambda}_2 = -U^{\prime}(C)\pi(Q)P_2F_1e^{-\delta t} \tag{27}$$

$$\theta_1 \geq 0, \qquad \theta_1\mu = 0, \tag{28}$$

$$\theta_2 \geq 0, \qquad \theta_2(1-\mu) = 0, \tag{29}$$

$$\theta_3 \geq 0, \qquad \theta_3 I = 0, \tag{30}$$

$$\theta_4 \geq 0, \qquad \theta_4 X = 0, \tag{31}$$

$$Lim_{t \to T}\ H(t) = Lim_{t \to T}\,[U(C)\pi(Q)e^{-\delta t} + \lambda_1 X + \lambda_2 I] = 0. \tag{32}$$

State equations (6) and (21), boundary conditions (7) and definition (20) are the other necessary conditions.

As in the malleable capital case, some general features of the optimal program are immediately obvious from the necessary conditions. First, $X > 0$ ($\theta_4 = 0$) along the optimal program, except possibly at T as natural resource must be necessary for processing. Second, conditions (24), (28) and (29) together imply that the economy may once again experience at most three, possibly repeatable, phases of domestic processing, identified in exactly the same manner as before. Third, (25) suggests that, unlike the previous case, positive capital formation or capacity expansion is neither ensured nor ruled out a priori even in phases of positive domestic processing. Fourth, (24) and (25) together indicate that the constant returns-to-scale technology may no longer preclude incomplete specialization in domestic processing. In the rest of this section we examine the nature of the optimal program.

The no processing phase is the easiest to characterize as evidenced by the result below.

Proposition 4: (*i*) *The economy is in the* $\mu = 0$ *phase only so long as* $-P_1 + P_2 F_2(K,0) < 0$. (*ii*) *If in addition,* (*a*) *$U(C)$ is iso-elastic with* $0 < \varepsilon < 1$ *and* (*b*) $0 < \bar{s} < \infty$, *optimal* $\dot{X}/X = \dot{C}/C < 0, \dot{I} = I = 0$, *and the phase lasts for ever.* (*iii*) *If, however,* (*c*) $Lim_{\mu X \to 0}\ F_2 \geq P_1/P_2$, $\mu = 0$ *is never optimal.*

Proof: (*i*) Trivially obvious from (24). (*ii*) Since $\mu = 0$ implies zero output in the processing sector, $I = 0$ is the obvious optimal choice, for otherwise C would be smaller and the Hamiltonian will not be maximized. Moreover, with $\mu = I = 0$, $C = P_1 X$ and $\dot{C}/C = \dot{X}/X$. Now, differentiating (23) and substituting for $\dot{C}/C$ yield $\dot{X}/X = (1-\varepsilon)^{-1} h(Q)X - \delta/\varepsilon$, which is(17) adapted for the iso-elastic utility function. Next, combining (23) and (26) yields $\dot{\lambda}_1/\lambda_1 = -(1-\varepsilon)^{-1} h(Q)X$, the counterpart of(18) for our special case of iso-elastic utility. As the latter implies $\lambda_1(T) = 0$, we may argue in the manner of proposition 2 that optimal $C(T) = X(T) = 0$ as well, with the result that optimal $T = \infty$ and $\dot{X}/X < 0$.

Finally, as there is never any switch into a domestic processing phase, I remains zero from start to finish. (*iii*) Trivial, given the restriction, because for arbitrarily given $K > 0$ and $X > 0$, there always exists a $0 < \mu < 1$ such that $-P_1 + P_2 F_2(K, \mu X) = 0$.■

Characterizing the domestic processing phases is more involved as the economy may enter these with either a zero or a positive investment level. Preserving analytical tractability in the context demands progressively greater specificity of the processing technology as well as additional restrictive assumptions regarding the nature of investment activity as evidenced by the following results.

Proposition 5: (*i*) *If* (*a*) $Lim_{\mu X \to 0}\ F_2 \geq P_1/P_2$, *the economy is in the* $0 < \mu < 1$ *phase only so long as* $-P_1 + P_2 F_2(K, X) < 0$. (*ii*) *If in addition* (*b*) $F(\ ,\ )$ *is linear homogeneous,* (*c*) $U(C)$ *is iso-elastic with* $0 < \varepsilon < 1$, (*d*) $0 < \bar{s} < \infty$ *and* (*e*) $I = 0$, *optimal* $\dot{C}/C < 0,\ \dot{\mu}/\mu = -\dot{X}/X > 0$ *such that after at most a finite period of time* $\mu = 1$ *before resource exhaustion.*

Proof: (*i*) Trivial in view of (24) and condition (a). (*ii*) With $I = 0$, and $F(\ ,\ )$ linear homogenous, $C = P_1(1-\mu)X + P_2 F(K,\ \mu X) = P_1 X + P_2 F_1 K$ with $P_2 F_1 K$, a constant such that $\dot{C}/C = [P_1 X/(P_1 X + P_2 F_1 K)]\dot{X}/X$. Next, time-differentiation of (23) and subsequent substitution for $\dot{C}/C$ yield $\frac{\dot{X}}{X} = \frac{1}{\varepsilon}\frac{C}{P_1 X}[(\frac{1}{1-\varepsilon}\frac{C}{P_1 X} - 1)h(Q)X - \delta]$ with $\frac{dX}{dQ}|_{\dot{X}=0} = -\frac{1-\varepsilon}{\varepsilon}\frac{(\frac{1}{1-\varepsilon}\frac{C}{P_1 X} - 1)h'(Q)}{h(Q)} \leq 0$ as

$C > P_1 X$. Finally, proceeding in the manner of the proof of proposition 2, we may establish that optimal X and C decline asymptotically towards zero.[9] The linear homogeneity of $F(K, \mu X)$ and the constancy of K also imply $\dot{X}/X = -\dot{\mu}/\mu$, ensuring that $\mu = 1$ in finite time before resource exhaustion.■

Proposition 6: *If* (*a*) $F(\ ,\ )$ *is linear homogeneous Cobb-Douglas*, (*b*)$U(C)$ *is iso-elastic with* $0 < \varepsilon < 1$, (*c*) $0 < \bar{s} < \infty$, (*d*) $0 < \mu < 1$, (*e*) $I > 0$, and (*f*) $P_2F_1/(P_I + C_A^{\prime}) > \delta$, $C_A^{\prime\prime}I/(P_I + C_A^{\prime}) > (1 - S_X)/S_X$ and $P_2F_1K > (P_I + C_A^{\prime})I$, where S_X is the competitive output share of natural resource in the processing sector, $\dot{x}/x = \dot{X}/X - \dot{K}/K < 0$ and $\dot{\mu}/\mu > 0$ such that $\mu = 1$ *in finite time before resource exhaustion.*

Proof: From (A3) in part I of the appendix,

$$\frac{P_1X}{C}\frac{\dot{X}}{X} = \frac{1}{\varepsilon}[(\frac{1}{1-\varepsilon}\frac{C}{P_1X} - 1) - \delta]\,h(Q)X - \frac{P_2F_1I}{C} + \frac{(P_I + C_A^{\prime})I}{C}\frac{P_I + C_A^{\prime}}{C_A^{\prime\prime}I}[\frac{1}{1-\varepsilon}\frac{C}{P_1X}h(Q)X - \frac{P_2F_1}{P_I + C_A^{\prime}}]$$

such that $\frac{1}{1-\varepsilon}h(Q)X - \frac{\delta}{\varepsilon} - \frac{P_2F_1}{P_IS_X} < \frac{P_1X}{C}\frac{\dot{X}}{X} < \frac{1+\varepsilon}{(1-\varepsilon)\varepsilon}\frac{1}{S_X}h(Q)X - \frac{\delta}{\varepsilon}$. Consequently, the $\dot{X}/X = 0$ contour is bounded by $\frac{1}{1-\varepsilon}h(Q)X - \frac{\delta}{\varepsilon} - \frac{P_2F_1}{P_IS_X} = 0$ from above and by $\frac{1+\varepsilon}{(1-\varepsilon)}\frac{1}{S_X}h(Q)X - \delta = 0$

[9] In demonstrating that terminal $X(T) = 0$, the only noteworthy difference from the proof of proposition 2 lies in the limiting value of $U'(C(T))$, which turns out to be greater than, rather than equal to, $U(C(T))/C(T)$.

from below as depicted in Figure 2. Next, arguing in the manner of proposition 2, we find that there now exist as many as four different options for optimal X if resource exhaustion were to occur within the phase: it continuously rises from start to finish; it continuously rises after at most a finite period of time; it continuously declines after at most a finite period of time; and it continuously declines from start to finish. As before, we may immediately preclude the first two options from further consideration as they imply resource exhaustion with a positive $X(T)$ over a finite T. Although the remaining options are consistent with the requirement of zero terminal extraction over $T = \infty$, the third option carries the possibility of $\dot{\mu}/\mu < 0$ (and reversion into the no processing phase)during periods of increasing X. In the light of the linear homogeneity of the processing technology, this is possible only if $\dot{x}/x > 0$ during such periods. In view of the first inequality in (f)and the expression for $\dot{X}/X$, this is possible only if $1/[(1-\varepsilon)S_X]\,h(Q)X - \delta > 0$.That is, optimal X must lie entirely above the contour $1/[(1-\varepsilon)S_X]\,h(Q)X - \delta = 0$. As the said contour is entirely located in the region between $\dot{X}/X = 0$ and its upper limiting contour $\dfrac{1}{1-\varepsilon}h(Q)X - \dfrac{\delta}{\varepsilon} - \dfrac{P_2F_1}{P_IS_X} = 0$ as depicted in Figure 2, it implies a contradiction[10]. Consequently, $\dot{x}/x < 0$ throughout under the third option. As this is also true of the fourth option as well, we may conclude that optimal $\dot{x}/x < 0$ throughout the phase. By implication, $\dot{\mu}/\mu > 0$ always and μ=1 in finite time.■

Corollary1: *Under conditions stipulated in proposition* 6*,* $\dot{X}/X < 0$, $\dot{C}/C < 0$, $\dot{I}/I < 0$ *throughout the*

[10] Follows from the easily made observation that X implied by the said contour is strictly less than that implied by the upper limiting contour.

$0 < \mu < 1$ *phase such that I may become zero before the switch into the* $\mu = 1$ *phase.*[11]

Proof: Follows trivially from the observation that optimal program lies entirely in the region below the contour $1/[(1-\varepsilon)S_X]\, h(Q)X - \delta = 0$ and the expressions for the three rates of change detailed in part I of the appendix.■

We turn next to characterizing the $\mu = 1$ phase. Once again, we must consider two possibilities: $I = 0$ throughout the phase and $I > 0$ sometime during the phase.

Proposition 7:(*i*) *The economy is in the* $\mu = 1$ *phase as long as* $-P_1 + P_2 F_2(K,\ X) \geq 0$. (*ii*) *If in addition*, (*a*) *F*(,) *is linear homogeneous Cobb-Douglas,* (*b*) *U*(*C*) *is iso-elastic with* $0 < \varepsilon < 1$, (*c*) $0 < \bar{s} < \infty$ *and* (*d*) $I = 0$ *throughout the phase*, *optimal* $\dot{X}/X < 0$ *and* $\dot{C}/C < 0$*, asymptotically exhausting the resource.*

Proof: (*i*) Trivial in view of condition (24). (*ii*) With $\mu = 1$ and $I = 0$ in (20), (*a*) implies $\dot{C}/C = (P_2 F_2 X/C)\dot{X}/X$. Next, differentiating (23) with respect to *t* and substituting for $\dot{C}/C$ yield

$$\left(\frac{\varepsilon P_2 F_2 X}{C} - \frac{F_{22}X}{F_2}\right)\frac{\dot{X}}{X} = \left[\frac{1}{(1-\varepsilon)S_X} - 1\right]h(Q)X - \delta \text{ such that } \frac{dX}{dQ}\Big|_{\dot{X}=0} = -\frac{h'(Q)X}{h(Q)} \leq 0 \text{ with terminal } X = 0.$$

Consequently, we may once again establish that optimal $\dot{X}/X < 0$ and $\dot{C}/C < 0$ with optimal $X(T) = C(T) = 0$ and $T = \infty$ if resource exhaustion takes place in the phase.■

Proposition 8: *If* (*a*) *F*(,) *is linear homogeneous Cobb-Douglas*, (*b*) *U*(*c*) *is iso-elastic with* $0 < \varepsilon < 1$, (*c*) $0 < \bar{s} < \infty$, (*d*) $\mu = 1$, (*e*) $I > 0$, (*f*) $P_2 F_1/(P_I + C_A') > \delta$, $C_A'' I/(P_I + C_A') > 1$ *and* $P_2 F_1 K > (P_I + C_A')I$,

[11] The result effectively rules out the third option in the preceding proposition as the optimal program.

(*g*) $1 - C/P_2F < 1/(1+\varepsilon)$, $\dot{x}/x < 0$ *and the* $\mu = 1$ *phase lasts forever.*

Proof: We proceed in the manner of the proof of proposition 6. As detailed in part II of the appendix, $\dot{X}/X = 0$ contour is bounded by $[(1+\varepsilon)/(1-\varepsilon)S_X - 1]\,h(Q)X - \delta = 0$ from below and

$[\varepsilon/(1-\varepsilon)]\,h(Q)X - \delta - 2\varepsilon(1-S_X)/S_x = 0$ from above in the manner of Figure 2 , with appropriate

changes in contour labels. Consequently, once again optimal $\dot{X}/X < 0$ after at most a finite period, asymptotically exhausting the resource if $\mu = 1$ is the terminal phase. Moreover, just as in the case of proposition 6, we may argue that $\dot{x}/x > 0$ is possible only if optimal X lies in the region above the contour $[1/(1-\varepsilon)S_X - 1]\,h(Q)X - \delta = 0$, which, of necessity, is confined entirely to the region between $\dot{X}/X = 0$ and the upper limiting contour. Since, as depicted in the diagram, optimal X never enters this region, it follows that $\dot{x}/x < 0$ along the optimal program so that a switch into another phase is never optimal.■

Corollary 2: *Under conditions stipulated in proposition* 8, *optimal* $\dot{X}/X < 0$, $\dot{C}/C < 0$, *and* $\dot{I}/I < 0$ *throughout the* $\mu = 1$ *phase such that* $I = 0$ *before resource exhaustion.*

Proof: From (A 15) in part III of the appendix,

$\dot{X}/X \leq -\left(\dfrac{1}{(1-\varepsilon)}\dfrac{C}{P_2F_2X}h(Q)X - \delta\right)\left[\dfrac{\varepsilon(P_I + C_A{}')I}{C} + \dfrac{C_A{}''I}{P_I + C_A{}'}\right]/\Delta$ such that $\dot{X}/X < 0$ provided

$\dfrac{1}{(1-\varepsilon)}\dfrac{C}{P_2F_2X}h(Q)X - \delta < 0$. As the optimal program lies entirely in the region below

$[1/(1-\varepsilon)S_X - 1]\,h(Q)X - \delta = 0$, it is obvious that condition is always satisfied, for $C > P_1X = P_2F_2X$ and $0 < S_X < 1$. Next, the equation system (A4) in part II of the appendix implies that $\dot{I}/I < 0$ provided

$\frac{1}{(1-\varepsilon)}\frac{C}{P_2F_2X}h(Q)X - \frac{P_2F_1}{P_I + C_A^{\,\prime}} - S_K\frac{\dot{x}}{x} < 0$. As from part IV of the appendix

$\frac{\dot{x}}{x} < -\varepsilon[\frac{1}{(1-\varepsilon)}\frac{C}{P_2F_2X}h(Q)X - \frac{P_2F_1}{P_I + C_A^{\,\prime}}]$, a straightforward substitution in the preceding inequality

completes the proof. A similar procedure can be utilized to show that $\dot{C}/C < 0$ as well.[12] Finally, we note that the expression for $\dot{I}/I$ also implies that it will become zero before resource exhaustion.■

4. Findings and Concluding Remarks

Above, we have reconsidered the issue of the social desirability of domestic processing of exhaustible, natural resource exports in a small open economy with a view to determining if the perfect certainty results due to Kumar (1988, 1997) extend to the case of uncertain resource stock.

The first two of the three results under examination concern the commonly held presumption among policy makers in favour of a steadily rising level of domestic processing under free trade. Kumar (1988) has argued that such a presumption is not always justified, for if capital is a malleable input such as a flow of imported services, alterable at will, a constant returns-to-scale resource processing technology implies complete specialization in the export markets. Kumar(1997), on the

[12] See part IV of the appendix.

other hand, has demonstrated that the presumption does become justified when capital is a stock and capacity expansion is viewed as a non-reversible activity entailing adjustment costs. In proposition 1 and proposition 5(*i*) respectively we have been able to fully extend, without any additional assumptions, the two results to the case of a continuously distributed uncertain but ultimately finite resource stock.

The remaining result under examination relates to the pattern of capacity expansion in the context of non-malleable capital. Kumar (1997) has shown that if and when capacity expansion in the processing sector is deemed desirable, the optimal investment for capacity expansion declines continuously over time, coming to a stop well before resource exhaustion. Through propositions 6 and 8 and corollaries 1 and 2, we have also been able to confirm this front-end loading pattern of capacity expansion under stock uncertainty, though only under somewhat restrictive circumstance: uncertain but ultimately finite resource stock, Cobb-Douglas processing technology, iso-elastic utility and additional assumptions regarding the nature of investment activity for capacity expansion. While Cobb-Douglas production function and iso-elastic utility are the essential work-horses of natural resource economics, the additional assumptions may demand further justification.

Stipulated as conditions (*f*) and (*g*) in propositions 6 and 8, the additional assumptions comprise four inequalities. The first inequality in (*f*), which replicates (27) in Kumar (1997), simply ensures that investment is profitable on the margin during capacity expansion.

The next inequality in (*f*) stipulates that the investment elasticity of marginal adjustment costs is greater than the larger of $(1 - S_X)/S_X$ and unity, rather than being merely positive as posited in the certainty scenario. Other things remaining equal, more rapidly rising capital stock adjustment costs will generate relatively lower levels of investment. Moreover, in the exhaustible resource literature, stock uncertainty invariably implies a more conservative extraction profile. As less extraction must, of necessity, entail less processing and processing capacity, the stipulation is neither implausible nor inconsistent.

The last of the inequalities in (*f*) places an upper bound on the optimal rate of capital formation or capacity expansion by requiring that expansion of processing capacity must be entirely financed out of capital's share of the processing sector output. In the absence of any capacity expansion constraints, the restriction is intuitively not unrealistic. In any case, the alternative is worse, for, in the reverse situation, the model implies a positive lower bound on investment levels if and when capacity. expansion does take place.

The fourth inequality, stipulated only in proposition 8 as condition (*g*), places an upper limit on the social rate of savings (capital formation), determined by the inter-temporal elasticity of substitution. As the social savings rate is assumed (the third inequality above) to be never greater than the competitive output share of capital, the restriction will be always satisfied provided the share also does not exceed the stipulated limit. Notwithstanding the observation that similar restrictions on relative shares have been common place in neo-classical growth theory, it is easily verified that known estimates of capital's relative output share in the mining and mineral industry do satisfy the criterion for all reasonable values of the elasticity parameter.[13]

In addition to the results discussed above, we have also derived in the two preceding sections a number of other results that tend to duplicate the other main results in Kumar (1988, 1997). In particular, propositions 2 and 3 in section 2 are the uncertainty counterparts of propositions 2 and 3 in Kumar (1988). Similarly, corollaries 1 and 2 represent an attempt to duplicate the characterization of

[13] See footnote 15 in Kumar (1997).

the optimal program detailed in section 3 of Kumar (1997) in terms of the optimal rates of extraction, the levels of domestic processing and capacity expansion.

Finally, as regards the nature of the optimal program, the monotonic and continuously declining nature of the optimal rate of extraction is especially noteworthy in the light of Kemp's observation in his pioneering contribution.[14] As also explained in Kumar (2005), this is a direct result of the continuously increasing hazard function associated with an uncertain but ultimately finite resource stock. As a continuously declining extraction rate is fundamental to obtaining rising levels of domestic processing and capacity expansion of the front-end loading variety, one wonders if the uncertainty results presented here are themselves robust under different characterizations of the hazard function, say, for example, those associated with resource stock distributions with unbounded support. It is also worth pointing out that the analytical framework used is one of partial equilibrium. Would the introduction of a non-traded goods sector significantly change the results? These obviously make good material for further research.

---

[14] See, Kemp (1976).

References

Aarrested, J. (1979), "Resource Extraction, Financial Transactions and Consumption in an Open Economy", *Scandinavian Journal of Economics*, vol. 81, pp. 552-565.

Dasgupta, P. S., R. Eastwood and G. Heal (1978),"Resource Management in a Trading Economy", *Quarterly Journal of Economics*, vol. 92, pp. 297-306.

Gould, J. P. (1968), "Adjustment Costs in the Theory of the Investment of the Firm", *Review of Economic Studies*, vol. 35, pp. 47-55.

Government of India (1993), *National Mineral Policy*, Ministry of Mines, New Delhi. Also available on http://mines.nic.in/nmp.html .

Kumar, R. C. (1988), "On Optimal Domestic processing of Exhaustible Natural resource Exports", *Journal of Environmental Economics and Management*, vol.13, pp. 341-354.

___________ (1997), "On Optimal Capacity Expansion for Domestic Processing of an Exhaustible, Natural Resource", *Journal of Environmental Economics and Management*, vol. 32, pp. 154-169.

___________ (2005), "How to Eat a Cake of Unknown Size: A Reconsideration", *Journal of Environmental Economics and Management*, vol. 50, pp. 408-421.

Harris, R. (1981), "Trade and Depletable Resource: the Small Open Economy", *Canadian Journal of Economics*, vol. 14, pp. 649-664.

Kemp, M. C. (1976), "How to Eat a Cake of Unknown Size", in M. C. Kemp (ed.), *Three topics in the Theory of International Trade*, Amsterdam: North Holland.

Kemp, M. C. And H. Suzuki (1975), "International Trade with a Wasting but Possibly Replenishable Resource", *International Economic Review*, vol. 16, pp. 712-728.

Long , N. V. (1974), “International Borrowing for Resource Extraction”, *International Economic Review*, vol. 15, pp. 168-183.

Lloyd, P. J. (1991),”Domestic Processing, Value Added and Globalization”, in the *Economic Dynamics of the Australian Industry*, pp. 251-271.

U.S.-Saudi Arabia Business Council (2005), “*The Oil and gas sector in the Kingdom of Saudi Arabia*”, http://www.us-Saudi-business.org/oil%20gas20%2report%2005.pdf .

Vousden, N. (1974), “International trade and Exhaustible Resources, a theoretical Model”, *International Economic Review*, vol. 15, pp. 149-167.

Withagen, C. (1985), *Economic Theory and International Trade in Natural Exhaustible Resources*, Berlin: Springer Verlag.

Appendix

I. The time-differentiation of (23) with substitution from (26) yields

$$\dot{C}/C = \frac{1}{\varepsilon}\left(\frac{1}{1-\varepsilon}\frac{C}{P_1 X} - 1\right)h(Q)X - \frac{\delta}{\varepsilon}. \tag{A1}$$

Similarly, combining (23) and (25), time-differentiating the resultant equation and substituting from (26) and (27) yield

$$\frac{C_A^{\prime\prime} I}{P_I + C_A^{\prime}}\frac{\dot{I}}{I} = \frac{1}{1-\varepsilon}\frac{C}{P_1 X}h(Q)X - \frac{P_2 F_1}{P_I + C_A^{\prime}}. \tag{A2}$$

Next, (20) and (A1) - (A2) together yield

$$\begin{aligned}\frac{P_1 X}{C}\frac{\dot{X}}{X} &= \frac{1}{\varepsilon}\left[\left(\frac{1}{1-\varepsilon}\frac{C}{P_1 X} - 1\right)h(Q)X - \delta\right] - \frac{P_2 F_1 I}{C} \\ &+ \frac{(P_I + C_A^{\prime})I}{C}\frac{P_I + C_A^{\prime}}{C_A^{\prime\prime} I}\left(\frac{1}{1-\varepsilon}\frac{C}{P_1 X}h(Q)X - \frac{P_2 F_1}{P_I + C_A^{\prime}}\right)\end{aligned} \tag{A3}$$

Also, (*d*) must imply $C > P_1 X > P_1 \mu X = P_2 F_2 \mu X$. Whence $P_2 F_1 K > (P_I I + C_A)$. Combining these with the third inequality in (*f*) ensures in turn $P_2 F_1 K > (P_I + C_A^{\prime})I > (P_I I + C_A) > P_I I$ as well as

$\frac{(P_I + C_A)I}{C} < \frac{F_1 K}{F_2 \mu X} = \frac{1 - S_X}{S_X}$. When joined with the second inequality in (*f*), the last of the derived

relations yields $\frac{(P_I + C_A^{\prime})I}{C}\frac{P_I + C_A^{\prime}}{C_A^{\prime\prime} I} < 1$.

Whence, minor manipulations of (A3) in the light of the preceding inequalities generate the two bounds for $\frac{P_1 X}{C}\frac{\dot{X}}{X}$.

II. Time-differentiating (20), (23) and (25) with $\mu = 1$ and substituting from (26) and (27) yield the system:

$$\begin{bmatrix} \varepsilon & S_K & 0 \\ 1 & \dfrac{-P_2F_2X}{C} & \dfrac{(P_I+C_A{}'')I}{C} \\ 0 & S_K & \dfrac{C_A{}''I}{P_I+C_A{}'} \end{bmatrix} \begin{bmatrix} \dot{C}/C \\ \dot{X}/X \\ \dot{I}/I \end{bmatrix} = \begin{bmatrix} (\dfrac{1}{1-\varepsilon}\dfrac{C}{P_2F_2X}-1)h(Q)X-\delta+S_K\dfrac{\dot{K}}{K} \\ \dfrac{P_2F_1K}{C}\dfrac{\dot{K}}{K} \\ \dfrac{1}{1-\varepsilon}\dfrac{C}{P_2F_2X}h(Q)X-\dfrac{P_2F_1}{P_I+C_A{}'}+S_K\dfrac{\dot{K}}{K} \end{bmatrix} \tag{A4}$$

Whence

$$\begin{aligned}\frac{\dot{C}}{C}=\frac{1}{\Delta}\Bigg[&\left((\frac{1}{1-\varepsilon}\frac{C}{P_2F_2X}-1)h(Q)X-\delta+S_K\frac{\dot{K}}{K}\right)\left(-\frac{P_2F_2X}{C}\frac{C_A{}''I}{P_I+C_A{}'}-S_K\frac{(P_I+C_A{}')I}{C}\right)\\ &+S_K\left(\frac{(P_I+C_A{}')I}{C}(\frac{1}{1-\varepsilon}\frac{C}{P_2F_2X}h(Q)X-\frac{P_2F_1}{P_I+C_A{}'}+S_K\frac{\dot{K}}{K})\right)-\frac{P_2F_1K}{C}\frac{\dot{K}}{K}\frac{C_A{}''I}{P_I+C_A{}'}\Bigg]\end{aligned} \tag{A5}$$

$$\begin{aligned}\frac{\dot{I}}{I}=\frac{\varepsilon}{\Delta}\Bigg[&\frac{-P_2F_2X}{C}\left(\frac{1}{1-\varepsilon}\frac{C}{P_2F_2X}h(Q)X-\frac{P_2F_1}{P_I+C_A{}'}+S_K\frac{\dot{K}}{K}+S_K\frac{P_2F_1K}{P_2F_2X}\frac{\dot{K}}{K}\right)\\ &-S_K\left(h(Q)X-\delta+\frac{P_2F_1K}{P_I+C_A{}'}\right)\Bigg]\end{aligned} \tag{A6}$$

$$\begin{aligned}\frac{\dot{X}}{X}=\frac{1}{\Delta}\Bigg[&\varepsilon\left(\frac{P_2F_1K}{C}\frac{\dot{K}}{K}\frac{C_A{}''I}{P_I+C_A{}'}-\left(\frac{1}{1-\varepsilon}\frac{C}{P_2F_2X}h(Q)X-\frac{P_2F_1}{P_I+C_A{}'}+S_K\frac{\dot{K}}{K}\right)\frac{(P_I+C_A{}')I}{C}\right)\\ &-\left((\frac{1}{1-\varepsilon}\frac{C}{P_2F_2X}-1)h(Q)X-\delta+S_K\frac{\dot{K}}{K}\right)\frac{C_A{}''I}{P_I+C_A{}'}\Bigg]\end{aligned} \tag{A7}$$

where

$$\Delta = \varepsilon \left[-\frac{P_2F_2X}{C}\frac{C_A{}''I}{P_I+C_A{}'} - S_K\frac{(P_I+C_A{}')I}{C} \right] - S_K\frac{C_A{}''I}{P_I+C_A{}'} < 0 \tag{A8}$$

Consequently,

$$\frac{\dot{x}}{x} = \frac{\dot{X}}{X} - \frac{\dot{K}}{K} = \frac{1}{\Delta}\left[\varepsilon\left(\frac{P_2F}{C}\frac{\dot{K}}{K} - \left(\frac{1}{1-\varepsilon}\frac{C}{P_2F_2X}h(Q)X - \frac{P_2F_1}{P_I+C_A{}'} \right)\right)\frac{(P_I+C_A{}')I}{C} \right. \\ \left. - \left((\frac{1}{1-\varepsilon}\frac{C}{P_2F_2X} - 1)h(Q)X - \delta \right)\frac{C_A{}''I}{P_I+c_A{}'} \right] \tag{A9}$$

Next, in view of condition (*f*) in the proposition, A(7) implies

$$\frac{\dot{X}}{X} \leq -\frac{1}{\Delta}\left[\left(\frac{\varepsilon}{1-\varepsilon}\frac{C}{P_2F_2X}h(Q)X + (\frac{1}{1-\varepsilon}\frac{C}{P_2F_2X} - 1)h(Q)X - \delta \right)\frac{C_A{}''I}{P_I+C_A} \right. \\ \left. - \left(\varepsilon\frac{P_2F_1K}{C} - \varepsilon S_K - S_K \right)\frac{C_A{}''I}{P_I+C_A{}'}\frac{\dot{K}}{K} \right]$$

Whence

$$\frac{\dot{X}}{X} \leq -\frac{1}{\Delta}\left[(\frac{1+\varepsilon}{1-\varepsilon}\frac{1}{S_X} - 1)h(Q)X - \delta \right]\frac{c_A{}''I}{P_I+C_A{}'} \tag{A10}$$

because $C > P_2F_2X$ and condition (*g*) ensures that the expression in the second set of large parentheses in the preceding inequality is positive. Further, in view of the first inequality in condition (*f*), (A7) and (A8) also imply

$$\frac{\dot{X}}{X} > \frac{1}{\Delta}\left[\varepsilon\frac{P_2F_1I}{C}\frac{C_A{}''I}{P_I+C_A{}'} + \varepsilon\frac{P_2F_1}{P_I+C_A{}'}\frac{(P_I+C_A{}')I}{C} - \left((\frac{1}{1-\varepsilon}\frac{C}{P_{F_2}X} - 1)h(Q)X - \delta\right)\frac{C_A{}''I}{P_I+C_A{}'}\right] \tag{A11}$$

Moreover,

$$\frac{\dot{X}}{X} > \frac{1}{\Delta}\left[\varepsilon\frac{P_2F_1I}{C}\frac{C_A{}''I}{P_I+C_A{}'} + \varepsilon\frac{P_2F_1I}{C}\frac{C_A{}''I}{P_I+C_A{}'} - \left((\frac{1}{1-\varepsilon}\frac{C}{P_{F_2}X} - 1)h(Q)X - \delta\right)\frac{C_A{}''I}{P_I+C_A{}'}\right] \tag{A1 2}$$

because $C_A{}''I/(P_I+C_A{}')>1$, whereby

$$\frac{\dot{X}}{X} > -\frac{1}{\Delta}\left[\frac{\varepsilon}{1-\varepsilon}h(Q)X - \delta - 2\varepsilon\frac{P_2F_1K}{P_2F_2X}\right]\frac{c_A{}''I}{P_I+C_A{}'} \tag{A13}$$

as $C>P_2F_2X$ always and $I<K$. A(10) and A(13) then furnish respectively the lower and upper limiting contours for $\dot{X}/X=0$.

III. In view of the first inequality in (*f*), substituting δ for $P_2F_1/(P_I+C_A{}')$, (A7) yields

$$\begin{aligned}\frac{\dot{X}}{X} \le \frac{1}{\Delta}\Bigg[& -\left(\frac{1}{\varepsilon}\frac{C}{P_{F_2}X}h(Q)X - \delta\right)\left(\varepsilon\frac{(P_I+C_A{}')I}{C} + \frac{C_A{}''I}{P_I+C_A{}'}\right) \\ & + \left(\varepsilon\frac{P_2F_1K}{C}\frac{C_A{}''I}{P_I+C_A{}'} - \varepsilon S_K\frac{(P_I+C_A{}')I}{C} - S_K\frac{C_A{}''I}{P_I+C_A{}'}\right)\frac{\dot{K}}{K}\Bigg]\end{aligned} \tag{A14}$$

Now, the remaining two inequalities in (*f*) and condition (*g*) together ensure that the expression contained in the third set of large parentheses is positive such that

$$\frac{\dot{X}}{X} \le -\frac{1}{\Delta}\left(\frac{1}{\varepsilon}\frac{C}{P_{F_2}X}h(Q)X - \delta\right)\left(\varepsilon\frac{(P_I+C_A{}')I}{C} + \frac{C_A{}''I}{P_I+C_A{}'}\right) \tag{A15}$$

IV, If $\frac{1}{(1-\varepsilon)}\frac{C}{P_2F_2X}h(Q)X - \delta < 0$, (A 7) implies $\frac{\dot{x}}{x} < -\frac{\varepsilon}{\Delta}[\frac{1}{(1-\varepsilon)}\frac{C}{P_2F_2X}h(Q)X - \frac{P_2F_1}{P_I+C_A{}'}]\frac{(P_I+C_A{}')I}{C}$.

Substituting for Δ from (A8), it then follows that $\frac{\dot{x}}{x} < -\varepsilon[\frac{1}{(1-\varepsilon)} \frac{C}{P_2 F_2 X} h(Q) X - \frac{P_2 F_1}{P_I + C_A{}'}]$.

Similarly, (A 7) also implies that $\frac{\dot{x}}{x} < -\frac{1}{\Delta}[(\frac{1}{(1-\varepsilon)} \frac{C}{P_2 F_2 X} - 1) h(Q) X - \delta] \frac{C_A{}'' I}{P_I + C_A{}'}$. Whence, once again substituting for Δ from (A 8) yields $\frac{\dot{x}}{x} < [(\frac{1}{(1-\varepsilon)} \frac{C}{P_2 F_2 X} - 1) h(Q) X - \delta]$.

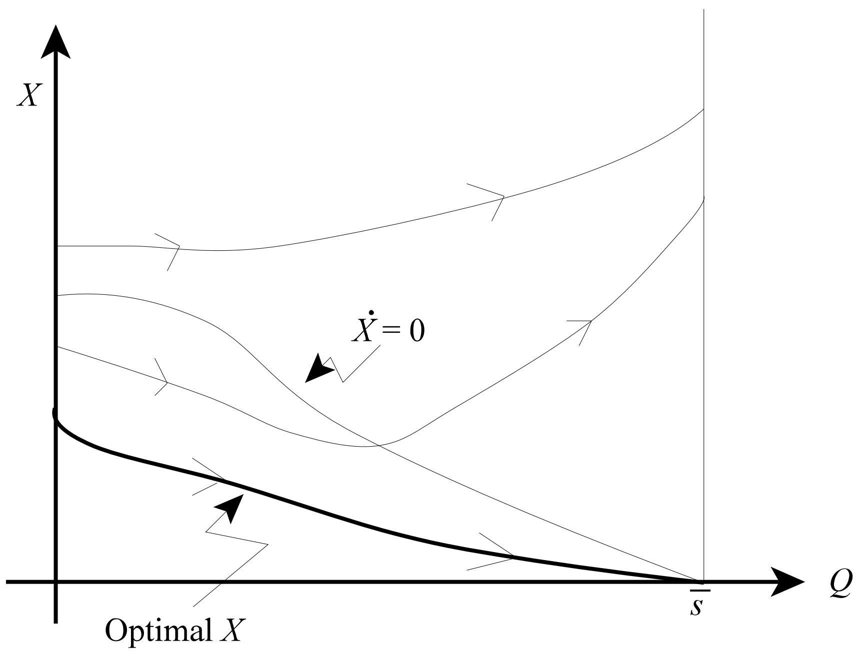
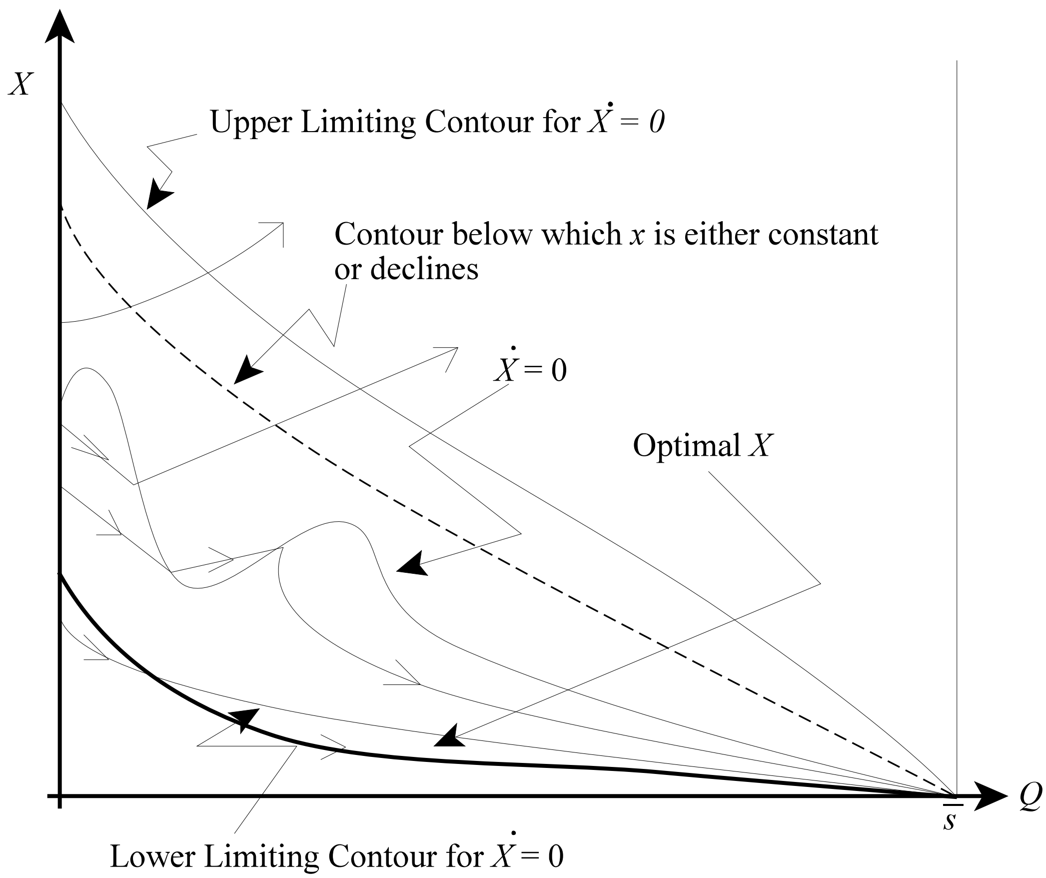


Figure 1



X
Upper Limiting Contour for $\dot{X} = 0$
Contour below which x is either constant or declines
$\dot{X} = 0$
Optimal X
Q
$\bar{s}$
Lower Limiting Contour for $\dot{X} = 0$

Figure 2