1 The Concept of Pareto Efficiency.

Definition 1 *An allocation of goods, either input or output goods, is said to be* Pareto Efficient *if we cannot find a reallocation of those goods such that we can produce more of something (utility or output) without producing less of something else.*

Definition 2 *A reallocation of goods that allows more of something to be produced without the sacrifice of something else is said to be* Pareto Improving.

It follows immediately from the definitions above that a Pareto Efficient allocation is one where all Pareto improvements have been exhausted. Pareto efficiency may be thought of as a minimum requirement for a "good" allocation of societies resources, one where all the opportunities to get something for nothing have been exploited. Pareto efficiency does not involve value judgements about what goods are produced or who receives them.

1.1 Pareto Efficiency in a Two Person, Two Firm, Two Good, Two Input, Economy.

We shall now define Pareto Efficiency for an economy consisting of two consumers who consume two goods that are produced using two factors of production.

1.1.1 Efficiency in Consumption.

We shall assume that there are two consumers Al (A) and Boris (B) each of whom may consume quantities of two goods Vodka (X) and Caviar (Y), we assume the total quantities of the two good to be given by $\{X, Y\}$ of which $\{X^A, Y^A\}$ and $\{X^B, Y^B\}$ are enjoyed by Al and Boris respectively. We write the *initial endowments* of the two goods as

$$\begin{aligned} X &= X_0^A + X_0^B \\ Y &= Y_0^A + Y_0^B \end{aligned}$$

The utilities that the two individuals derive from their endowments are described by the utility functions

$$\begin{aligned} U^A &= U^A(X^A, Y^A) \\ U^B &= U^B(X^B, Y^B) \end{aligned}$$

we assume the utility functions to be increasing and concave.

For an allocation of X and Y between the two individuals to be Pareto Efficient it is required that we cannot raise one individuals utility without lowering the utility of another, expressed another way this involves the problem

$$\begin{aligned} &\underset{X^A,Y^A}{Max}\, U^A(X^A,Y^A) \\ &s.t\; U^B(X^B,Y^B) \geq \overline{U}^B \\ &X = X^A + X^B \\ &Y = Y^A + Y^B. \end{aligned}$$

Where $\overline{U}^B$ is the level of utility that B must realize. By substitution this problem reduces to

$$\begin{aligned} &\underset{X^A,Y^A}{Max}\, U^A(X^A,Y^A) \\ &s.t\; U^B(X-X^A,Y-Y^A) \geq \overline{U}^B \end{aligned}$$

Forming the Lagrangian we obtain

$$\underset{X^A,Y^A}{Max}\, U^A(X^A,Y^A) + \lambda\left[\overline{U}^B - U^B(X-X^A,Y-Y^A)\right]$$

where λ is the Lagrange Multiplier associated with the utility constraint. Now maximizing yields

$$\begin{aligned} \frac{\partial U^A}{\partial X^A} - \lambda \frac{\partial U^B}{\partial X^B}\frac{\partial X^B}{\partial X^A} &= 0 \\ \frac{\partial U^A}{\partial Y^A} - \lambda \frac{\partial U^B}{\partial Y^B}\frac{\partial Y^B}{\partial Y^A} &= 0 \end{aligned}$$

utilizing $\frac{\partial X^B}{\partial X^A} = \frac{\partial Y^B}{\partial Y^A} = -1$ and rearranging the expressions gives the condition for Pareto Efficiency in consumption

$$\frac{\frac{\partial U^A}{\partial X^A}}{\frac{\partial U^A}{\partial Y^A}} = \frac{\frac{\partial U^B}{\partial X^B}}{\frac{\partial U^B}{\partial Y^B}}$$

The left hand side (LHS) of this expression is the ratio of marginal utilities for the two good for individual A, the RHS is the ratio of marginal utilities for the two good for individual B, alternatively expressed these are the *marginal rates of substitution.* The condition may be reexpressed as

$$MRS^A_{XY} = MRS^B_{XY} \qquad \text{(Pareto Efficiency in Consumption.)}$$

Remark 3 *Recall from intermediate micro that the marginal rate of substitution between two good is equal to the slope of the indifference curves, thus the condition for Pareto Efficiency in consumption tells us that the indifference curves for the two individuals must have equal slope at a Pareto efficient allocation.*

The Contract Curve and Utility Transformation Frontier. The condition for Pareto Efficiency in consumption describes an efficient allocation of goods given that individual B is guaranteed some arbitrary level of utility $\overline{U}^B$. We can now think about varying this arbitrary utility level and examining the implications,.two things will occur

- For Pareto Efficient allocations as the utility of B rises (falls) the utility of A must necessarily fall (rise), hence a curve known as the *Utility Transformation Frontier* is traced out. Define the utility transformation frontier as $U^B = U^B(U^A)$.
- Too maintain efficiency in consumption as the utility levels of the two individuals vary it is necessary to reallocate goods between the two, this traces out a curve known as the *Contract Curve*.

1.1.2 Efficiency in Production.

Suppose now that the two goods $\{X, Y\}$ are each produced using two input goods capital and labor $\{K, L\}$ respectively. the inputs are allocated to the production of the outputs according to $\left\{K^X, L^X\right\}$ and $\left\{K^Y, L^Y\right\}$ where the initial endowments of the factors are given by

$$\begin{aligned} K &= K_0^X + K_0^Y \\ L &= L_0^X + L_0^Y \end{aligned}$$

The outputs of the two goods that may be derived from the inputs are given by the production technologies

$$\begin{aligned} X &= X(K^X, L^X) \\ Y &= Y(K^Y, L^Y) \end{aligned}$$

these functions are assumed to be increasing and concave.

For the production of the two goods to be Pareto Efficient we require that we cannot reallocate the inputs between the production of the two outputs such that more of one is produced without giving up some of the other. Alternatively expressed

$$\begin{gathered} \underset{K^X, L^X}{Max}\, X(K^X, L^X) \\ s.t\, Y(K^Y, L^Y) \geq \overline{Y} \\ K = K^X + K^Y \\ L = L^X + L^Y. \end{gathered}$$

where $\overline{Y}$ is the level of production of that good which must not be reduced. By substitution and using the method of Lagrange we get

$$\underset{K^X, L^X}{Max}\, X(K^X, L^X) + \mu\left[\overline{Y} - Y(K - K^X, L - L^X)\right]$$

where μ is the Lagrange multiplier associated with the production constraint $Y(K^Y, L^Y) \geq \overline{Y}$. Maximization yields first order conditions

$$\begin{aligned}\frac{\partial X}{\partial K^X} - \mu \frac{\partial Y}{\partial K^Y}\frac{\partial K^Y}{\partial K^X} &= 0 \\ \frac{\partial X}{\partial L^X} - \mu \frac{\partial Y}{\partial L^Y}\frac{\partial L^Y}{\partial L^X} &= 0\end{aligned}$$

using $\frac{\partial K^Y}{\partial K^X} = \frac{\partial L^Y}{\partial L^X} = -1$ and rearranging these expressions we get the condition for Pareto Efficiency in production

$$\frac{\frac{\partial X}{\partial K^X}}{\frac{\partial X}{\partial L^X}} = \frac{\frac{\partial Y}{\partial K^Y}}{\frac{\partial Y}{\partial L^Y}}$$

The left hand side (LHS) of this expression is the ratio of marginal products for the two inputs in the production of good X, the RHS of this expression is the ratio of marginal products for the two inputs in the production of good Y, alternatively expressed these are the *marginal rates of technical substitution.* The condition may be reexpressed as

$$MRTS_{KL}^X = MRTS_{KL}^Y \qquad \text{(Pareto Efficiency in Production.)}$$

Remark 4 *Recall from intermediate micro that the marginal rate of technical substitution between two inputs is equal to the slope of the isoquant, thus the condition for Pareto Efficiency in production tells us that the isoquants for the two goods must have equal slope at a Pareto efficient allocation.*

The Locus of Pareto Efficient Points in Production and the Transformation Frontier. The condition for Pareto Efficiency in consumption describes an efficient allocation inputs between the production of the two outputs given some arbitrary level of production $\overline{Y}$. We can now think about varying this arbitrary production level and examining the implications,.two things will occur

- For Pareto Efficient allocations as the production of Y rises (falls) the production of X must necessarily fall (rise), hence a curve known as the *Transformation Frontier* is traced out. The equation of the transformation frontier is called the *Transformation Function* and may be written

$$F(X, Y) = 0$$

Claim 5 *The slope of the transformation function is defined (positive) as the* Marginal Rate of Transformation *and can be show too be equal to* $\frac{\partial Y(K^Y, L^Y)}{\partial K^Y} / \frac{\partial X(K^X, L^X)}{\partial K^X}$ *or* $\frac{\partial Y(K^Y, L^Y)}{\partial L^Y} / \frac{\partial X(K^X, L^X)}{\partial L^X}$.

- Too maintain efficiency in production as the production of the two goods vary it is necessary to reallocate inputs between the two, this traces out a curve known as the *Locus of Pareto Efficient Points in Production.*

1.1.3 Efficiency in Production and Consumption.

Our first efficiency condition tell us how to efficiently allocate goods *once they are produced*, the second tells us how to efficiently produce *given* combinations of goods. We now need a condition that characterizes when the combination of goods produced is efficient vis-a-vis the combination of goods consumers wish to consume. For the combination of X and Y produced to be Pareto Efficient it is required that we cannot raise the individuals utilities by changing the output mix, expressed another way this involves the problem

$$\begin{gathered}\underset{X^A,Y^A,X^A,Y^A,X,Y}{Max} U^A(X^A,Y^A)\\ s.t\ U^B(X^B,Y^B)\geq \overline{U}^B\\ X=X^A+X^B\\ Y=Y^A+Y^B\\ F(X,Y)=0\end{gathered}$$

forming the Lagrangian gives us

$$\begin{gathered}\underset{X^A,Y^A,X^A,Y^A,X,Y}{Max} U^A(X^A,Y^A)+\lambda\left[\overline{U}^B-U^B(X^B,Y^B)\right]\\ +\mu_1\left[X-X^A-X^B\right]+\mu_2\left[Y-Y^A-Y^B\right]\\ +\mu_3 F(X,Y)\end{gathered}$$

the first order conditions are

$$\begin{aligned}\frac{\partial U^A}{\partial X^A}-\mu_1 &= 0,\ \frac{\partial U^A}{\partial Y^A}-\mu_2=0,\\ -\lambda\frac{\partial U^B}{\partial X^B}-\mu_1 &= 0,\ -\lambda\frac{\partial U^B}{\partial Y^B}-\mu_2=0,\\ \mu_1+\mu_3\frac{\partial F(X,Y)}{\partial X} &= 0,\ \mu_2+\mu_3\frac{\partial F(X,Y)}{\partial Y}=0\end{aligned}$$

simple algebra now reveals

$$\frac{\frac{\partial U^A}{\partial X^A}}{\frac{\partial U^A}{\partial Y^A}}=\frac{\frac{\partial U^B}{\partial X^B}}{\frac{\partial U^B}{\partial Y^B}}=\frac{\frac{\partial F(X,Y)}{\partial X}}{\frac{\partial F(X,Y)}{\partial Y}}$$

which tells us that the marginal rates of substitution must equal the slope of the transformation frontier or

$$MRS^A_{XY}=MRS^B_{XY}=MRT_{XY}$$

(Pareto Efficiency in Production and Consumption.)

1.2 Efficiency and Social Welfare.

If we assume that social welfare is derived from the utilities of the individuals in the economy, then we may write a social welfare function as

$$W = W(U^A, U^B)$$

social indifference curves are then simply defined by

$$W(U^A, U^B) = \text{Constant.}$$

totally differentiating we find that

$$\frac{\partial W}{\partial U^A} dU^A + \frac{\partial W}{\partial U^B} dU^B = 0$$

Hence the societal indifference curves have slope

$$\frac{dU^B}{dU^A} = -\frac{\frac{\partial W}{\partial U^A}}{\frac{\partial W}{\partial U^B}} < 0$$

To maximize social welfare we need to solve the problem

$$\underset{U^A}{Max}\; W(U^A, U^B)$$
$$s.t.\; U^B = U^B(U^A)$$

or

$$\underset{U^A}{Max}\; W(U^A, U^B(U^A))$$

This tells us that a social welfare optimum must occur where

$$\frac{\partial W}{\partial U^A} + \frac{\partial W}{\partial U^B} \frac{\partial U^B(U^A)}{\partial U^A} = 0$$

rearranging this yields

$$-\frac{\frac{\partial W}{\partial U^A}}{\frac{\partial W}{\partial U^B}} = \frac{\partial U^B(U^A)}{\partial U^A}.$$

The social optimum thus occurs where the slope of the societal indifference curve equals the slope of the utility transformation frontier.

Remark 6 *It follows immediately that a social welfare optimum must be Pareto Efficient.*

1.3 The Fundamental Theorem of Welfare Economics.

Theorem 7 *A perfectly competitive market economy achieves a Pareto efficient allocation.*

This powerful theorem tells us that a competitive economy does not waste any resources, the minimum requirement we might want satisfied by any system of resource allocation. It does *not* tell us that a competitive equilibrium will be a social welfare optimum. To demonstrate that the theorem is true and understand why we need to show that a market system satisfies the three Pareto conditions

- Efficiency in consumption $MRS^A_{XY} = MRS^B_{XY}$.
- Efficiency in production $MRTS^X_{KL} = MRTS^Y_{KL}$.
- Efficiency in production and consumption $MRS^A_{XY} = MRT_{XY}$.

1.3.1 Perfect Competition and Efficiency in Consumption.

Suppose that the markets for the two goods $\{X, Y\}$ are perfectly competitive, then each consumer is a price taker for each good. We assume that there are given incomes $\left\{I^A, I^B\right\}$or equivalently initial endowments such that $I^A = P^X X^A_0 + P^Y Y^A_0$, $I^B = P^X X^B_0 + P^Y Y^B_0$. The consumer's utility maximization problems are

$$\underset{X^A,Y^A}{Max}\, U^A(X^A, Y^A)$$
$$s.t\; P^X X^A + P^Y Y^A = I^A = P^X X^A_0 + P^Y Y^A_0$$

and

$$\underset{X^B,Y^B}{Max}\, U^B(X^B, Y^B)$$
$$s.t\; P^X X^B + P^Y Y^B = I^B = P^X X^B_0 + P^Y Y^B_0$$

utility maximization requires

$$\underset{X^A,Y^A}{Max}\, U^A(X^A, Y^A) + \theta^A \left[I^A - P^X X^A - P^Y Y^A\right]$$

and

$$\underset{X^B,Y^B}{Max}\, U^B(X^B, Y^B) + \theta^B \left[I^B - P^X X^B - P^Y Y^B\right]$$

with first order conditions for the two problems

$$\begin{aligned}
\frac{\partial U^A}{\partial X^A} - \theta P^X &= 0, \\
\frac{\partial U^A}{\partial Y^A} - \theta P^Y &= 0, \\
\frac{\partial U^B}{\partial X^B} - \theta P^X &= 0, \\
\frac{\partial U^B}{\partial Y^B} - \theta P^Y &= 0
\end{aligned}$$

which reduce to

$$\frac{\frac{\partial U^A}{\partial X^A}}{\frac{\partial U^A}{\partial Y^A}} = \frac{P^X}{P^Y} = \frac{\frac{\partial U^B}{\partial X^B}}{\frac{\partial U^B}{\partial Y^B}}$$

or

$$MRS^A_{XY} = \frac{P^X}{P^Y} = MRS^B_{XY}.$$

This tells us that Pareto efficiency is achieved on a competitive market via the coordinating role played by market prices.

1.3.2 Offer Curves.

From the budget constraints we have

$$\begin{aligned} P^X(X^A - X_0^A) + P^Y(Y^A - Y_0^A) &= 0, \\ P^X\left(X^B - X_0^B\right) + P^Y\left(Y^B - Y_0^B\right) &= 0. \end{aligned}$$

Since prices are non-negative this implies that each consumer must be a supplier of one good and demander of the other i.e. if $X^A - X_0^A > 0$ then $Y^A - Y_0^A < 0$, and if trade is to take place $X^B - X_0^B < 0$ and $Y^B - Y_0^B > 0$.

Now suppose we perform a *normalization* and express both good in terms of units of good X so that $P^X = 1$, we now have

$$\begin{aligned} X^A &= X_0^A - P^Y(Y^A - Y_0^A), \\ X^B &= X_0^B - P^Y(Y^B - Y_0^B) \end{aligned}$$

substituting these two expression into the first order conditions gives

$$P^Y \frac{\partial U^A(X_0^A - P^Y(Y^A - Y_0^A), Y^A)}{\partial X^A} = \frac{\partial U^A(X_0^A - P^Y(Y^A - Y_0^A), Y^A)}{\partial Y^A}$$

and

$$P^Y \frac{\partial U^B(X_0^B - P^Y(Y^B - Y_0^B), Y^B)}{\partial X^B} = \frac{\partial U^B(X_0^B - P^Y(Y^B - Y_0^B), Y^B)}{\partial Y^B}$$

which implicitly defines relationships between P^Y and Y^A for given $\left\{Y_0^A, X_0^A\right\}$ and P^Y and Y^B for given $\left\{Y_0^B, X_0^B\right\}$which may be written

$$\begin{aligned} Y^A &= Y^A(P^Y \mid Y_0^A, X_0^A) \\ Y^B &= Y^B(P^Y \mid Y_0^B, X_0^B) \end{aligned}$$

the budget constraints now immediately imply that we may write

$$\begin{aligned} X^A &= X^A(P^Y \mid Y_0^A, X_0^A) \\ X^B &= X^B(P^Y \mid Y_0^B, X_0^B) \end{aligned}$$

Thus individual A offers (to buy or sell)

$$Y^A(P^Y \mid Y_0^A, X_0^A) - Y_0^A$$

and in return asks for

$$X^A(P^Y \mid Y_0^A, X_0^A) - X_0^A$$

thus the offer curve of A is defined by

$$X^A(P^Y \mid Y_0^A, X_0^A) - X_0^A = -P^Y \left[Y^A(P^Y \mid Y_0^A, X_0^A) - Y_0^A\right] \tag{A's Offer Curve.}$$

Similarly individual B offers (to buy or sell)

$$Y^B(P^Y \mid Y_0^B, X_0^B) - Y_0^B$$

and in return asks for

$$X^B(P^Y \mid Y_0^B, X_0^B) - X_0^B$$

thus B's offer curve is defined by

$$X^B(P^Y \mid Y_0^B, X_0^B) - X_0^B = -P^Y \left[Y^B(P^Y \mid Y_0^B, X_0^B) - Y_0^B\right] \tag{B's Offer Curve.}$$

1.3.3 Market Equilibrium.

Ensures supply equals demand on both markets or

$$\begin{aligned} Y^A(P^Y \quad &\mid \quad Y_0^A, X_0^A) - Y_0^A + Y^B(P^Y \mid Y_0^B, X_0^B) - Y_0^B = 0, \\ X^A(P^Y \quad &\mid \quad Y_0^A, X_0^A) - X_0^A + X^B(P^Y \mid Y_0^B, X_0^B) - X_0^B = 0 \end{aligned}$$

This is simply the condition that the market equilibrium occurs where the offer curves cross. But since each offer curve is only a rewriting of the consumers first order conditions that involves $MRS_{XY} = \frac{P^X}{P^Y}$, then the point at which the offer curves cross must involve $MRS_{XY}^A = \frac{P^X}{P^Y} = MRS_{XY}^B$ and must be Pareto Efficient.

1.3.4 Perfect Competition and Efficiency in Production.

We now examine the behavior of two firms each of which produces one of the two output goods, X and Y, using the two input goods, K and L. Each input is assumed to trade in a competitive input market at the prices P^K and P^L. The problem faced by each firm is to minimize the cost of producing a given level of output.

$$\underset{K^X,L^X}{Min}\ P^K K^X + P^L L^X$$
$$s.t.\ X(K^X, L^X) = \overline{X}$$

$$\underset{K^Y,L^Y}{Min}\ P^K K^Y + P^L L^Y$$
$$s.t.\ Y(K^Y, L^Y) = \overline{Y}$$

forming the two Lagrangians gives us

$$\underset{K^X,L^X}{Min}\ P^K K^X + P^L L^X + \gamma^X \left[\overline{X} - X(K^X, L^X)\right]$$

and

$$\underset{K^Y,L^Y}{Min}\ P^K K^Y + P^L L^Y + \gamma^Y \left[\overline{Y} - Y(K^Y, L^Y)\right]$$

the four first order conditions are

$$\begin{aligned}
P^K - \gamma^X \frac{\partial X(K^X, L^X)}{\partial K^X} &= 0, \\
P^L - \gamma^X \frac{\partial X(K^X, L^X)}{\partial L^X} &= 0, \\
P^K - \gamma^Y \frac{\partial Y(K^Y, L^Y)}{\partial K^Y} &= 0, \\
P^L - \gamma^Y \frac{\partial Y(K^Y, L^Y)}{\partial L^Y} &= 0
\end{aligned}$$

dividing the first condition by the second and the third by the fourth yields

$$\frac{\frac{\partial X(K^X,L^X)}{\partial K^X}}{\frac{\partial X(K^X,L^X)}{\partial L^X}} = \frac{P^K}{P^L} = \frac{\frac{\partial Y(K^Y,L^Y)}{\partial K^Y}}{\frac{\partial Y(K^Y,L^Y)}{\partial L^Y}}$$

or

$$MRTS^X_{KL} = \frac{P^K}{P^L} = MRTS^Y_{KL}$$

Hence competitive cost minimizing firms achieve a Pareto Efficient allocation in production.

1.3.5 Perfect Competition and Efficiency in Production and Consumption.

Consider now that each firm may produce both goods if it wishes and sell each at the competitive market prices $P^X = 1$ and P^Y. Since each firm is problem is identical we may represent the problem with the analysis of a single firm

$$\begin{gathered} Max\ X + P^Y Y - P^K K - P^L L \\ s.t.\ X = X(K^X, L^X) \\ Y = Y(K^Y, L^Y) \\ K = K^X + K^Y \\ L = L^X + L^Y \end{gathered}$$

by substitution this reduces to

$$\underset{K^X, K^Y, L^X, L^Y}{Max}\ X(K^X, L^X) + P^Y Y(K^Y, L^Y) - P^K \left(K^X + K^Y\right) - P^L \left(L^X + L^Y\right)$$

the first order conditions to this problem are

$$\begin{aligned} \frac{\partial X(K^X, L^X)}{\partial K^X} - P^K &= 0, \\ \frac{\partial X(K^X, L^X)}{\partial L^X} - P^L &= 0, \\ P^Y \frac{\partial Y(K^Y, L^Y)}{\partial K^Y} - P^K &= 0, \\ P^Y \frac{\partial Y(K^Y, L^Y)}{\partial L^Y} - P^L &= 0 \end{aligned}$$

rearranging and dividing the third condition by the first and the fourth by the second gives

$$\begin{aligned} \frac{P^Y \frac{\partial Y(K^Y, L^Y)}{\partial K^Y}}{\frac{\partial X(K^X, L^X)}{\partial K^X}} &= 1 \\ \frac{P^Y \frac{\partial Y(K^Y, L^Y)}{\partial L^Y}}{\frac{\partial X(K^X, L^X)}{\partial L^X}} &= 1 \end{aligned}$$

or

$$\frac{\frac{\partial Y(K^Y, L^Y)}{\partial K^Y}}{\frac{\partial X(K^X, L^X)}{\partial K^X}} = \frac{\frac{\partial Y(K^Y, L^Y)}{\partial L^Y}}{\frac{\partial X(K^X, L^X)}{\partial L^X}} = \frac{1}{P^Y}$$

alternatively expressed

$$MRT_{XY} = \frac{1}{P^Y}$$

but we already know

$$MRS^A_{XY} = MRS^B_{XY} = \frac{1}{P^Y}$$

so

$$MRS^A_{XY} = MRS^B_{XY} = MRT_{XY}$$

the firms produce the Pareto efficient mix of outputs.

2 Externalities.

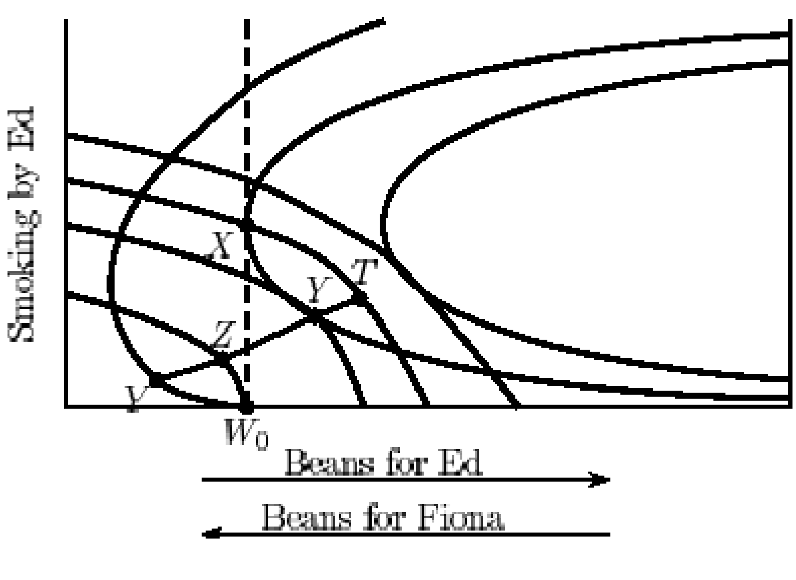
Definition 8 *An externality is an effect that one economic agent has on another over which the effected agent does not fully consent.*

In a sense an externality is a gain or loss that one agent imposes on another. This may be a gain or loss between consumers, firms, regions, countries or some combination of these. Since the imposition of an externality does not typically involve taking fully into account the preferences of the effected party, it is often the case that the external effect is inefficiently supplied. An inefficiency of course implies the possibility of realizing a Pareto improvement for the agents involved if they engage in voluntary trade.

2.0.6 Example: Bergstrom's "Smoking Box".

2 goods, beans and smoke. Ed likes beans and smoking, Fiona also likes beans but suffers a negative externality from Ed's smoking. The initial allocation of beans is given by W_0.In the absence of restrictions on smoking the outcome would be at X.

Figure 5.1: A One-Sided Externality



A shift away from X to any point on the line YT would be a pareto improvement.

2.1 Sources of Externalities.

2.1.1 Externalities as Missing Markets.

In the framework of a competitive market economy we might view the presence of an externality as synonymous with the absence of a market. Consider two consumers one of who plays music at high volume, but only derives a small benefit from doing do, the other has a headache and would greatly benefit from quiet. Why doesn't the headache victim pay the music player to turn it off? Why is there no market for this good? We know that if a market for the good were established the first fundamental theorem of welfare economics would apply and the market outcome would be Pareto efficient.

2.1.2 Externalities as the Absence of Property Rights.

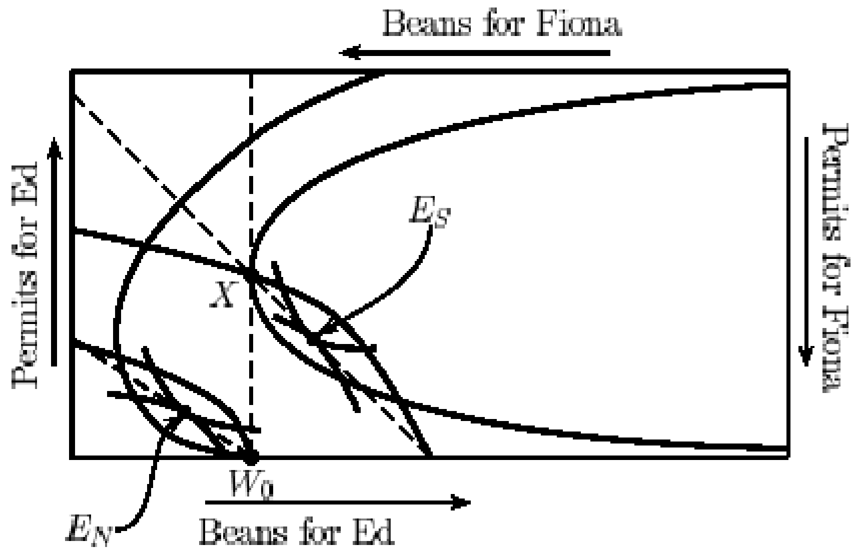
Following from the realization that externalities may be viewed as arising from the absence of markets we necessarily ask why such markets do not exist. property rights over the good in question are not clearly defined, if this is the case it is not clear if anyone has the option of placing the good up for sale. Consider the loud music example, does the headache sufferer have the right to silence of the music lover the right to his music. If the headache sufferer has the rights to silence then the music lover must pay him to listen to his tunes. If the music lover has a right to listen to his tunes, then the headache sufferer must pay him for silence.

2.1.3 The Reciprocal Nature of Externalities.

There is always potential for problems in establishing property rights as externalities are by definition reciprocal in nature. One economic agent benefits when the actions of one agent generate a positive or negative effect on another the initial action benefits the actor, the consequences either positive or negative effect the other agent. Does the first agent have the right to the action, or the second the right to deny the action? Both desire the property rights as they have value. Property rights are wealth!

Bergstrom's "Smoking Box" example revisited. Here smoking permits may be created but the outcome depends on who receives the initial property rights. If Ed receives then we start at X and Fiona must pay him not to smoke. The resultant equilibrium is at E_S. If Fiona receives the property right Ed must pay her for the right to smoke and the resultant equilibrium will be at E_N. Clearly it is better to receive the property rights.

Figure 5.3: A Market for Smoking Permits



2.1.4 The Costs of Establishing or Maintaining Property Rights.

Even if there is agreement that the property rights to a good should belong to a given agent it may still be costly to establish or maintain them. Consider an area of public grazing land, if private property rights are not established each rancher does not consider the effects on other ranchers of his cattle grazing the "commons". Degradation of the commons effects all ranchers and is thus a negative externality. If the commons is divided between all the ranches such that each gets a private ranch, then fences must be built to prevent cattle straying between the areas. But if there are many small ranches building the necessary fences may be too costly and the property right may not be effectively established.

2.1.5 Property Rights may not be Enough: Non-Convexities.

It is possible to show that even if

- property rights are well established
- a competitive market equilibrium potentially exists
- in the competitive equilibrium price ratios between all goods satisfy the marginalist conditions for efficiency

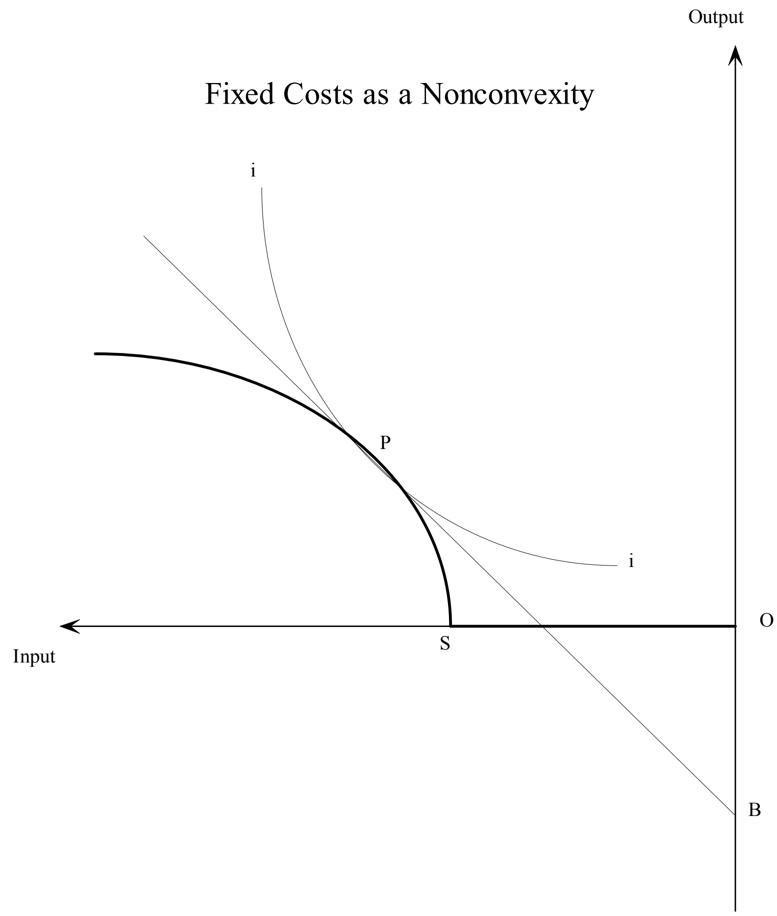
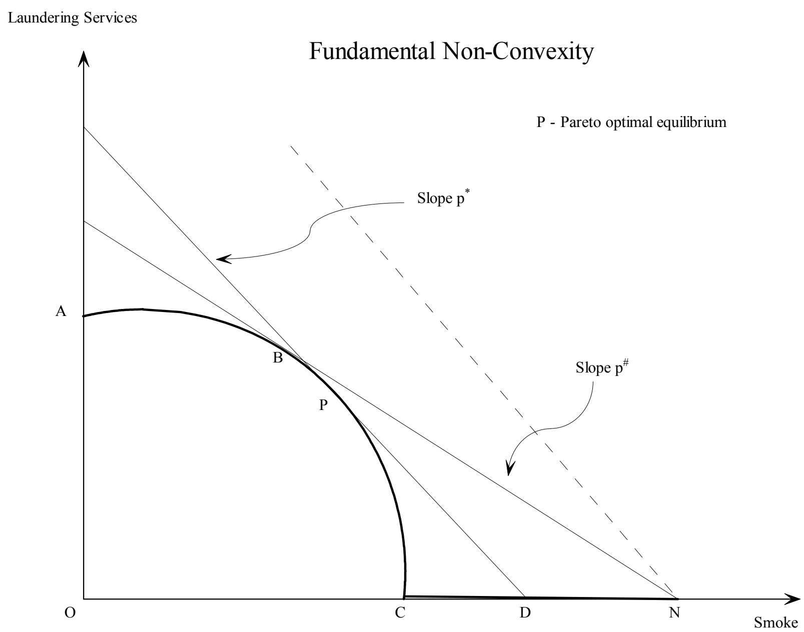
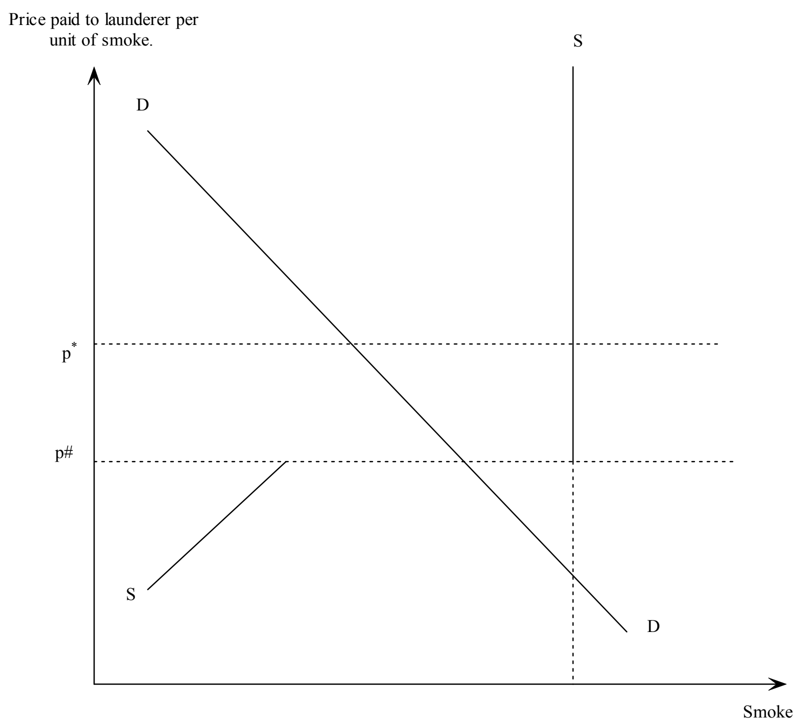


Figure 1:

It may be the case that in the presence of an externality some agents will find it in their private best interests to close down some markets. This may be because

1. Fixed costs make the production of some goods unprofitable.
2. Even if production of some goods is profitable, if they have externalities attached to them it might be more profitable to sell off the rights to the externalities and cease production.


Laundering Services
Fundamental Non-Convexity
P - Pareto optimal equilibrium
Slope p*
Slope p#
A
B
P
O
C
D
N
Smoke


Price paid to launderer per unit of smoke.
S
D
p*
p#
S
D
Smoke

2.2 Solutions to Externality Problems

2.2.1 Pigouvian Solutions.

Pigouvian solutions rely on "pricing" the externality via a tax on the externality generating activity. The agent generating the externality is then induced to "internalize " it as the tax makes him face the full social cost of his actions/decisions.

Example. Suppose there are two firms one produces banjos and the other produces books. Readers of books are adversely effected by the sound of bad banjo playing, and thus both read less and purchase fewer books. The reading of books does not effect the enjoyment or sales of banjos except through standard market mechanisms. Let the profits of the firm that makes banjos be written

$$\pi_1 = rx - c(x)$$

where r is the competitive market price of banjos, x the number of banjos produced, $c(x)$ is an increasing convex cost function. For the firm that produces books profits are given by

$$\pi_2 = \Pi - e(x)$$

where Π is a constant and $e(x)$ is an increasing convex function that captures the negative effect of banjo sales on the book producers profits. (Note this is Varian's simple model with the addition of a constant).

In the absence of any corrective measures the banjo manufacturer chooses to

$$\underset{x}{Max} rx - c(x)$$

the first order condition to which defines the privately optimal action $\widetilde{x}$ as satisfying

$$r - c'(\widetilde{x}) = 0$$

Efficiency requires the banjo producer maximizes profit taking into account the full costs of his actions or

$$\underset{x}{Max} rx - c(x) - e(x)$$

the first order condition to which defines the optimal action x^*as satisfying

$$r - c'(x^*) - e'(x^*) = 0$$

The Pigouvian solution then requires setting a pigouvian tax p such that $p = e'(x^*)$. the banjo manufacturers profit maximization problem becomes

$$\underset{x}{Max} rx - c(x) - px$$

and we immediately get

$$r - c'(x^*) - p = 0 = r - c'(x^*) - e'(x^*)$$

and efficiency is obtained.

Problems with the Pigouvian Solution. To set $p = e'(x^*)$ it is necessary that the government know $e'(x^*)$. This is difficult for the government to know,

1. The government may not be able to observe the effect $e(x)$.
2. Even if $e(x)$ has been observed it has only been seen at the point $\widetilde{x}$. It needs to know the slope of the function at x^*.

This strongly suggests that Pigouvian taxes while they are observed in the real economy are an imprecise way of dealing with externalities.

2.2.2 Coasian Solutions.

The classic Coasian solution to an externality problem simply involves establishing property rights and then letting the agents concerned bargain (a market is a special case of bargaining where both sides make take-it-or-leave-it offers at the going market price). The Coase theorem then goes on to say that the result will be efficient whatever the allocation of property rights. To see that this work consider our banjo/books example from above. We shall assume that after property rights are established the agents bargain and reach agreement according to the Nash bargaining solution.

Remark 9 *The Nash bargaining solution involves the two bargainers maximizing the product of their joint surplus over and above that which they could receive in the absence of agreement. It can be justified via a bargaining process of alternating concessions where each agent concedes to the other until further concessions on their part are more costly than concessions by the other agent.*

1. Property rights allocated to the banjo producer.

Here the outcome (*threat points*) in the absence of agreement is

$$\widetilde{\pi}_1 = r\widetilde{x} - c(\widetilde{x})$$

and

$$\widetilde{\pi}_2 = \Pi - e(\widetilde{x})$$

the Nash bargaining solution involves

$$\begin{aligned} & \underset{x,A}{Max}(\pi_1 - \widetilde{\pi}_1)(\pi_2 - \widetilde{\pi}_2) \\ = \; & [rx - c(x) + A - r\widetilde{x} + c(\widetilde{x})]\,[\Pi - e(x) - A - \Pi + e(\widetilde{x})] \\ = \; & [rx - c(x) + A - r\widetilde{x} + c(\widetilde{x})]\,[e(\widetilde{x}) - e(x) - A] \end{aligned}$$

where A is a *side payment* agreed between the two bargainers. The first order conditions are

$$[r - c'(x)]\,[e(\widetilde{x}) - e(x) - A] - e'(x)\,[rx - c(x) + A - r\widetilde{x} + c(\widetilde{x})] = 0 \quad (1)$$

$$[e(\widetilde{x}) - e(x) - A] - [rx - c(x) + A - r\widetilde{x} + c(\widetilde{x})] = 0 \quad (2)$$

from (2) we have

$$[e(\widetilde{x}) - e(x) - A] = [rx - c(x) + A - r\widetilde{x} + c(\widetilde{x})] \tag{3}$$

substituting (3) into (1) gives

$$[r - c'(x) - e'(x)]\,[e(\widetilde{x}) - e(x) - A] = 0$$

dividing both sides by $[e(\widetilde{x}) - e(x) - A]$ gives

$$r - c'(x) - e'(x) = 0$$

hence we have efficiency when the property rights are allocated to the banjo producer.

2. Property rights allocated to the book producer.

If the book producer were allocated the property rights then prior to any bargaining the banjo producer would be denied the right to produce hence the outcome in the absence of agreement is given by

$$\overline{\pi}_1 = 0$$

and

$$\overline{\pi}_2 = \Pi$$

the Nash bargaining solution now involves

$$\begin{aligned} & \underset{x,B}{Max}(\pi_1 - \overline{\pi}_1)(\pi_2 - \overline{\pi}_2) \\ = \;& [rx - c(x) - B]\,[\Pi - e(x) + B - \Pi] \\ = \;& [rx - c(x) - B]\,[B - e(x)] \end{aligned}$$

notice that the side payment B is now a payment from the banjo producer to the book producer. The first order conditions are

$$[r - c'(x)]\,[B - e(x)] - e'(x)\,[rx - c(x) - B] = 0 \tag{4}$$

$$[rx - c(x) - B] - [B - e(x)] = 0 \tag{5}$$

from (5) we have

$$[rx - c(x) - B] = [B - e(x)] \tag{6}$$

substituting (6) into (4) gives

$$[r - c'(x) - e'(x)]\,[B - e(x)] = 0$$

dividing both sides by $[B - e(x)]$ gives

$$r - c'(x) - e'(x) = 0$$

hence we have efficiency when the property rights are allocated to the book producer.

Problems with Coasian Solutions. As we previously noted the problems with Coasian solutions are twofold.

1. Everyone will want to obtain the initial property rights so establishing them is not a trivial problem.
2. In some cases, perhaps due to non-convexities, the market solution will be a corner solution rather than the interior pareto efficient solution.

2.2.3 Varian's Solution - Compensation Mechanisms.

Suppose we are in situation where a government does not have the information necessary to design pigouvian taxes, nor is it able (perhaps due to political constraints) to adopt a Coasian solution. Varian suggests that if the agents involved in the externality problem have full information concerning its causes and effects then the government may exploit this to achieve efficiency.

The mechanism consists of two stages

Announcement Stage: Each firm simultaneously announces a pigouvian tax. Firm 1's announcement is p_1, firm two announces p_2.

Choice Stage: Regulator enforces the tax schedules announced by the two firms according the next set of equations, the firms then choose their output levels.

$$\Pi_1 = rx - c(x) - p_2 x - \alpha_1 (p_1 - p_2)^2$$
$$\Pi_2 = \Pi + p_1 x - e(x)$$

The model is solved for the *subgame perfect* equilibrium. That is we solve for the last stage first, then solve for the first stage given how the last stage depends on the first. Consider first the choice stage.

Firm 1 maximizes

$$\underset{x}{Max}\ \Pi_1 = rx - c(x) - p_2 x - \alpha_1 (p_1 - p_2)^2$$

with first order condition

$$r = c'(x) + p_2$$

notice that this implies

Firm 2 is passive in the choice stage.$x = x(p_2)$ with $x'(p_2) < 0$.

Next consider the announcement stage. Notice that for any p_2 that firm 1 expects firm 2 to announce it simply chooses

$$p_1 = p_2$$

Now firm 2 knows that p_2 has an indirect effect on its own profits via $x(p_2)$ so it chooses p_2 to maximize

$$\Pi_2 = \Pi + p_1 x(p_2) - e(x(p_2))$$

so the first order condition is

$$\left[p_1 - e'(x)\right] x'\left(p_2\right) = 0$$

Since $x'\left(p_2\right) \neq 0 \Rightarrow p_1 = e'(x)$ since $p_1 = p_2 \Rightarrow r = c'(x) + p_2 = c'(x) + p_1 = c'(x) + e'(x)$ which is the condition for efficiency.

Remark 10 *The intuition here is that firm 1 always wishes to match firm 2's announcement. Firm 2 can manipulate x via its announcement p_2, hence it will always choose p_2 so that $p_1 = e'(x)$.*

Problems with Varian's mechanism. Given the firms have full information why doesn't firm 2 figure out that firm 1 always plays $p_1 = p_2$ and manipulate its own profits via

$$\Pi_2 = \Pi + p_2 x(p_2) - e(x(p_2))$$

this is a deviation from Nash behavior, but it might be appropriate.

2.2.4 The Ellis - van den Nouweland Mechanism.

Suppose the two firms of our previous example are owned by $N_1$and N_2 shareholders respectively, then in the Nash equilibrium each share earns

$$\frac{1}{N_1}\left[r\widetilde{x} - c(\widetilde{x})\right]$$

and

$$\frac{1}{N_2}\left[\Pi - e(\widetilde{x})\right].$$

Suppose that the property rights system is reformed such the $S = N_1 + N_2$ shares are now acceptable as claims on the profits of either (but only one) firm. Let s_1 be the shares that make claims on firm 1 and $S - s_1$ make claim on firm 2. It follows that the S shares will be allocated such that they earn the same return everywhere, so

$$\left(\frac{1}{s_1}\right)\left[rx - c(x)\right] = \left(\frac{1}{S - s_1}\right)\left[\Pi - e(x)\right]$$

which may be rewritten

$$\left(S - s_1\right)\left[rx - c(x)\right] = \left(s_1\right)\left[\Pi - e(x)\right]$$

This may be rewritten

$$s_1 = \frac{S\left[rx - c(x)\right]}{\Pi - e(x) + rx - c(x)}$$

Firm 1 now maximizes

$$\underset{x}{Max}\left(\frac{1}{s_1}\right)\left[rx - c(x)\right]$$

subject to

$$s_1 = \frac{S\left[rx - c(x)\right]}{\Pi - e(x) + rx - c(x)}$$

substituting into the objective function from the constraint gives

$$\begin{aligned} \underset{x}{Max} \left(\frac{1}{s_1}\right)\left[rx - c(x)\right] &= \left(\frac{\Pi - e(x) + rx - c(x)}{S\left[rx - c(x)\right]}\right)\left[rx - c(x)\right] \\ &= \left(\frac{\Pi - e(x) + rx - c(x)}{S}\right) \end{aligned}$$

the first order condition to which is

$$\frac{1}{S}\left[x - c'(x) - e'(x)\right] = 0 \Rightarrow x - c'(x) - e'(x) = 0$$

and we have efficiency.

Remark 11 *The intuition is that each firm knows profits will be arbitraged, so the only way they can maximize their own profits per share is to maximize total profits, but this requires they internalize the externality.*

Problems with the Ellis van den Nouweland mechanism. It only works if you can set up the share system as described.

3 Public Choice.

Public choice is the study of situations that require *collective action* where a good or service provision level need to be jointly agreed upon. Obvious examples include provision of public goods such as legal systems, police, defence, pollution abatement, and the like. The need for collective action stems from the standard *prisoners dilemma* problem..

		Player #1.	
		No-Cooperation	Cooperation
Player #2	No-Cooperation	5,5	7,2
	Cooperation	2,7	6,6

We see immediately that no-cooperation is a *dominant strategy*, hence no-cooperation is both a *dominant strategy equilibrium* and the unique *Nash equilibrium*. There is a need for some form of collective action to achieve cooperation. The way in which societies often determine whether or not to engage in collective action is to put the option up to a vote thus we investigate voting theory. There are essentially two strands to this area, unanimous voting rules and majority rules.

3.1 Unanimity Rules.

Unanimity rules require every voter agree on a decision. To understand these rules Mueller's diagram is very useful

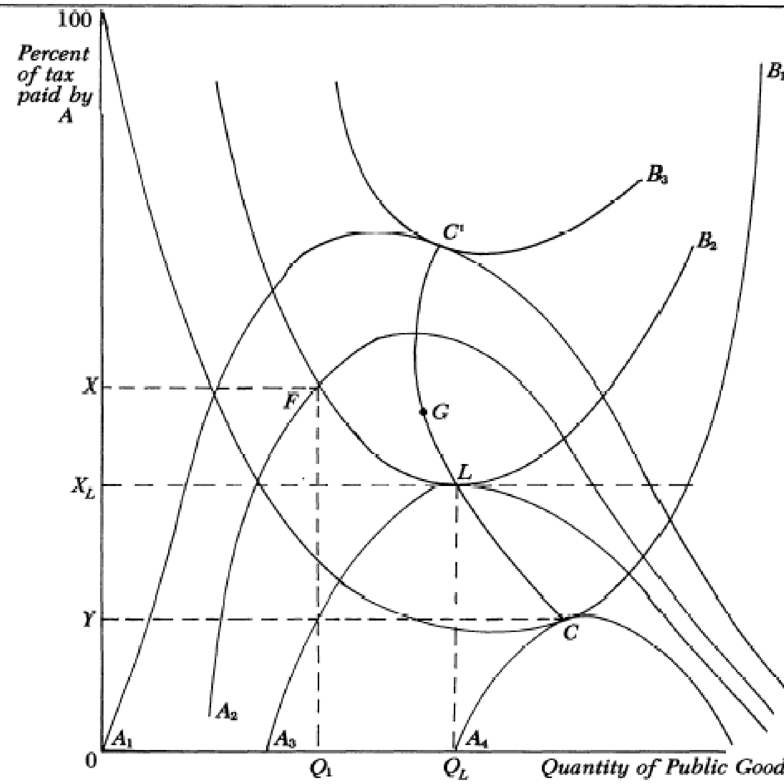


Figure 1.

On the diagram two individuals A and B who vote over the levels of public good provision when faced by different shares of the costs. The vertical axis represents the share of the cost of the provision of a public good borne by individual A, 100-A is the share borne by B, the horizontal axis gives the quantity of the public good.. We begin our analysis by assuming that F is the status quo, this is clearly inefficient at it des not lie on the contract curve CC'. Next consider voting equilibria.

3.1.1 Pairwise Voting.

Suppose that a government were to hold a sequence of votes between the status quo and a proposed alternative. Any point proposed that lies in the lens originating from F would be preferred to F and receive the votes of all the voters. The new point would become the status quo, a new lens would emerge and a new vote held, again the status quo would be unanimously beaten . The process would continue until no new point could be found that would unanimously defeat the status quo. This would be a pairwise voting equilibrium, and would be pareto efficient. Any point on the contract curve within the original lens represents a potential outcome. It follows that where the equilibrium finally occurs lies partially at the discretion of whoever proposes the alternatives. Indeed as we can demonstrate while the outcome is an equilibrium under unanimity, the individuals would possibly still each prefer a different level of public good provision given their tax shares.

3.1.2 Lindahl Voting.

With Lindahl voting the participants are faced with a personalized tax share X, each then votes on the quantity of the public good they desire. If they vote for the same level of provision there is unanimity, if not then a new set of taxes is proposed and they vote again. On the diagram there is a unique Lindahl equilibrium at L

Problems

1. Pairwise voting does not lead to unique outcome.
2. Both pairwise and Lindahl mechanisms might be subject to strategic manipulation.
3. If voting is costly there is a free rider problem, why bother to vote. This gets worse as the number of participants increases. Each individuals vote faces a smaller chance of being crucial and thus it takes less to discourage them from voting.

3.2 Majority Voting.

In the real world a large number of different voting procedures are adopted to make decisions, these each involve some form of majority voting. Quite commonly the options are considered pairwise with the winner determined by a simple majority. As can be seen from the following examples this potentially leads to problems

3.2.1 Single and Multiple Peakedness: Pairwise voting with a simple majority.

Consider the following preferences

		Options		
		A	B	C
Players	Fred (Brian)	1	2	3
Rankings	Brian	3	1	2
	Melissa (Brian)	3	2	1

these preferences are single peaked and lead to the simple result that B beats both A and C and is a clear winner. Preferences here are single peaked.

Now suppose

		Options		
		A	B	C
Players	Fred (Brian)	1	2	3
Rankings	Brian	2	3	1
	Melissa (Brian)	3	1	2

Now preferences are not single peaked (Brian is causing trouble again!!). A beats B, C beats A, B beats C!![1] This leads to

1. Voting cycles - no clear result obtains.
2. Agenda manipulation - the sequence of voting determines the outcome.

Conclusion - We might want to consider other forms of majority voting. One conclusion that is often reached when problems of this type arise is that the problem is that individuals are unable to express the strength of their preferences. That a candidate receives a lot of second place votes counts for very little. The following voting procedure "solves" this problem.

3.2.2 Borda's Rule.

Each candidate picks up points from their position in the ranking of each voter. If a candidate is ranked last 0 points are received, last but one 1 point is received and so on. Consider the following example

	Rankings			
	bca	acb	cba	abc
Number of individuals	7	7	6	1

Borda scores are as follows
a gets $(7 \cdot 2) + (1 \cdot 2) = 16$
b gets $(7 \cdot 2) + (6 \cdot 1) + (1 \cdot 1) = 21$
c gets $(7 \cdot 1) + (7 \cdot 1) + (6 \cdot 2) = 26$

Hence c wins according to the Borda rule. Notice that under a standard one vote plurality rule the votes are b=7, a=8, and c=6 and a wins!!

A Problem with Borda's Rule - The Independence of Irrelevant Alternatives (IIA). Consider the following

	Number of votes for each set of preferences					
	30	1	29	10	10	1
Preferences	Brian	Brian	Ghandi	Ghandi	Reid	Reid
over	Ghandi	Reid	Brian	Reid	Brian	Ghandi
candidates	Reid	Ghandi	Reid	Brian	Ghandi	Brian

[1] Thanks Brian.

according too the Borda rule
Brian gets (30.2) + (1.2) + (29.1) + (10.1) = 101
Ghandi gets (30.2) + (29.2) + (10.2) + (1.1) = 139
Reid gets (1.1) + (10.1) + (10.2) + (1.1) = 32

According to the Borda rule the correct ordering of candidates is Ghandi, Brian, Reid, but if we take a pairwise comparison we get
Brian vs Ghandi - Brian wins 41-40,
Brian vs Reid - Brian wins 60-21.

Problem: *Brian beats them both head to head!.* The key to the problem involves the *Independence of Irrelevant Alternatives* which states that in making comparisons between any two options only those options should matter, everything else should be treated as irrelevant. In our example Reid is considered inferior to both Brian and Ghandi, so why should the choice between them depend in any way on Reid?

The Problem's Bigger that it Might Appear. Not only does the *Independence of Irrelevant Alternatives* cause problems for Borda's rule it causes problems for *any scoring rule*. If a scoring rule involves a sequence of real numbers such that $s_1 > s_2 > .... > s_n$ where higher ranked alternatives receive higher scores. Then we can show that the scoring rule will always give poor results.

Consider our example again but let $s_1 = 8, s_2 = 1, s_3 = 0$. According to this new scoring rule
Brian gets (30.8) + (1.2) + (29.1) + (10.1) = 281
Ghandi gets (30.8) + (29.8) + (10.8) + (1.1) = 553
Reid gets (1.1) + (10.1) + (10.8) + (1.1) = 92.
Again Ghandi beats Brian according to the scoring rule but head to head we get the same as before
Brian vs Ghandi - Brian wins 41-40,
Brian vs Reid - Brian wins 60-21.

Why the Independence of Irrelevant Alternatives?

1. If it is not present the outcome can be manipulated by introducing extraneous alternatives. In our example we see that Reid is a "No Hoper" but under a scoring rule he can change the outcome by entering the contest.

2. From a practical point of view it allows decisions to be made over a restricted range of choices, we don't have to consider every alternative. It is thus quick and cheap.

3.2.3 Arrow's Impossibility Theorem.

Unfortunately Arrow has shown that when there are more than two alternatives available every reasonable decision rule sometimes violates the IIA condition. First lets define a reasonable decision rule by giving some properties that we think one should possess. Suppose we have two individuals Andy and Ghandi, who choose between three options A,B, and C. We require that any ranking of the options A, B, C satisfy the following axioms

1. Completeness: Either $A \succ B$, $B \succ A$, or AIB.
2. Transitivity: If $A \succ B$ and $B \succ C$ then $A \succ C$.
3. If $A \succ B$ for both Andy and Ghandi then the ranking must rank A ahead of B.
4. IIA: If $A \succ B$ and $B \succ C$ then these do not change simply because some new option D appears.
5. The ranking should be derived from the preferences of the individuals.
6. No dictatorship: No one individuals preferences may determine societies preferences.

Suppose now for our two individuals we have the following preferences
Andy: $A \succ_a B \succ_a C$.
Ghandi: $C \succ_g A \succ_g B$.
We shall show that we cannot obtain a social ranking of these options and not violate one of Arrow's axioms.

- $B \succ_a C$ and $C \succ_g B$ it must be the case socially that CIB or we would violate axiom 6 and have dictatorship.
- Since $A \succ_a B$ and $A \succ_g B$ it must be the case by axiom 3 that for a social preference ranking $A \succ B$.
- So applying axiom 2. transitivity we get $A \succ BIC \Rightarrow A \succ C$ but this violates the non-dictatorship axiom 6.

Conclusion: We have to give up an axiom.

3.2.4 Condorcet Winners.

The idea behind Condorcet's approach is that there is a best outcome to any voting situation and that individuals will on average know what that best out come is. Consider the following situation Two candidates, George and Al, run for political office, each promises to build a new road across the country, each claims to be able to organize the project more efficiently that the other. For simplicity we shall assume there are two voters the famous Brian twins, we now that each Brian is able to correctly identify the best candidate 60% of the time. We observe that both Brians vote for George who then obtains a simple majority (50%+1 here is 2, and note that this also implies for Al to win requires both Brians vote for him.). We now ask what is the likelihood that the Brians voted correctly and the correct winner is chosen by majority voting. We do this by first computing conditional probabilities

1. The probability that both Brians choose George given that George is indeed the best candidate.

$$P(\text{Brians 1 and 2 choose George}) = (0.6)\,(0.6) = 0.36$$

2. The probability that both Brians choose Al given that George is indeed the best candidate.

$$P(\text{Brians 1 and 2 choose Al}) = (0.4)\,(0.4) = 0.16$$

So the choice of George is $\frac{0.36}{0.16} = 2.25$ more likely when George is the best choice. This is called the likelihood ratio. Indeed it can be shown that with a large population (many Brians) it is only required that the probability that each individual makes the correct decision be slightly over 50% for the odds of the correct individual winning to be come very large (approach 1 as the size of the population goes to infinity).

Three of More Alternatives. Suppose the objective is to reduce traffic congestion in Eugene, the options are

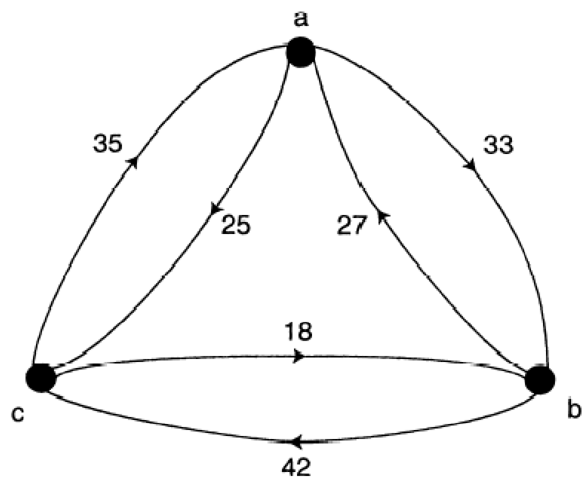
a. Supply more busses.

b. Build more roads.

c. Make it more difficult to obtain a driving licence, by requiring greater testing.

The question then is which of these alternatives is the most effective per dollar. The votes of the voters are given by the following graph.

A Vote Graph



Each vertex on the graph represents a number of votes, for example the arrow from $a \to b$ represents 33 votes for a over b, while b has 27 votes over a. To calculate the pairwise support for the ranking *abc* we compute $a \to b$ plus $b \to c$ plus $a \to c$ so $33 + 42 + 25 = 100$.

Hence

abc	100	bca	104
acb	76	cab	86
bac	94	cba	80

Using his probabilistic method Condorcet showed that the ranking that is most likely to be correct is the one with maximal pairwise support. In this case *bca* this solution is known as *Condorcet's rule of three.* To see that this is true let's compute the relative likelihoods that each ranking is correct. Let $p > 1/2$ be the probability the an individual voter chooses correctly.

1. *abc* - $p^{33}(1-p)^{27} \times p^{42}(1-p)^{18} \times p^{25}(1-p)^{35} = p^{100}(1-p)^{80}$
2. *bca* - $p^{42}(1-p)^{18} \times p^{35}(1-p)^{25} \times p^{27}(1-p)^{33} = p^{104}(1-p)^{76}$
3. *acb* - $p^{25}(1-p)^{35} \times p^{18}(1-p)^{42} \times p^{33}(1-p)^{27} = p^{76}(1-p)^{104}$
4. *cab* - $p^{35}(1-p)^{25} \times p^{33}(1-p)^{27} \times p^{18}(1-p)^{42} = p^{86}(1-p)^{94}$
5. *bac* - $p^{27}(1-p)^{33} \times p^{25}(1-p)^{35} \times p^{42}(1-p)^{18} = p^{94}(1-p)^{86}$
6. *cba* - $p^{18}(1-p)^{42} \times p^{27}(1-p)^{33} \times p^{35}(1-p)^{25} = p^{80}(1-p)^{100}$

Since $p > 1/2$ it immediately follows that the ranking is $bca \succ abc \succ bac \succ cab \succ cba \succ acb$.

Condorcet and Arrow's axioms. By definition Condorcet's rule does not satisfy all of Arrow's axioms (that has been shown to be impossible) but how close does it come? Surprisingly quite close. It satisfies a weakened version of the IIA called the local independence of irrelevant alternatives. LIIA. Consider the following

30	1	29	10	10	1
Kevin	Kevin	Ty	Ty	Reid	Reid
Ty	Reid	Kevin	Reid	Kevin	Ty
Reid	Ty	Reid	Kevin	Ty	Kevin

clearly the real contest here is between Kevin and Ty, Reid has far fewer first place votes and many more last place votes. Notice that Kevin and Ty are not separated by any other candidate in the rankings, they are in the same interval. The LIIA asks only that IIA holds within that interval. That is the ranking *within* the interval should be invariant to what happens *outside* that interval. It can be shown that Condorcet's method has this property.

4 The Economics of the Family.

4.1 The Rotten Kid Theorem.

The rotten kid theorem is an example of a two stage mechanism that has implications way beyond the original interpretation given it by originator Gary Becker.

Theorem 12 *If a family consists of an altruistic head who transfers income to the other family members who are selfish, then each member of the family will seek to maximize family income. That is each will internalize all externalities between them and act efficiently.*

To see how the theory works consider the following. Suppose that each family member i consumes a single consumption good X_i. All family members except the head maximize their own selfish consumption, the head is altruistic and cares only about the utility of the other family members. The heads utility is written

$$U(X_1, ..., X_n)$$

the family members utilities are

$$V(X_i) \;\; i = 1, .., n$$

and let I_i be the income of family member i The families budget constraint must satisfy (all prices are assumed to be unity)

$$\sum_i X_i = \sum_i I_i$$

If we now assume all goods X_i are normal goods (that is the head consumes more of each as income increases) then each family members utility will be increasing in total family income , and each will have an incentive to maximize total family income. This proves the theorem.

If we substitute Federal Government or European Parliament for altruistic head, and member states for family members we see just how powerful this theorem might be.

4.1.1 Problems with the Rotten Kid Theorem.

Lazy Rotten Kids. Asymmetric Information.

The rotten kid theorem fails to apply when there is asymmetric information. Let each family members income be a function $I_i(Y_i)$ and let the parent act as before transferring income to the children so as to maximize utility subject the family budget constraint. Then a selfish child will have insufficient incentive to work as she receives only a share of the income she generates but incurs the full disutility of working.

The Parent's Utility Depends on the Children's Utility.

Suppose instead the parent can observe effort, and his utility depend on their utilities rather than their consumption. We can see from the following example that problems again arise. Let the head of household have two children, Bart and Lisa, each child has the utility function

$$U_i = X_i\left(1 - Y_i\right)$$

which is maximized subject to

$$X_i = wY_i + T$$

where T is the transfer from the parent, so

$$U_i = \left(wY_i + T\right)\left(1 - Y_i\right)$$

with FOC

$$\frac{\partial U_i}{\partial Y_i} = w\left(1 - Y_i\right) - \left(wY_i + T\right) = 0$$

so

$$Y_i = \frac{w - T}{2w}$$

We immediately see that individual effort is decreasing in the transfer the child receives from the parent.

$$\frac{\partial Y_i}{\partial T} = -\frac{1}{2w} < 0$$

Hence the incentive problem is clearly exacerbated by the fact that the parent can observe effort.

The Case of the Controversial Night Light. In this example the general applicability of the Rotten Kid Theorem is examined by introducing a public good into the model. Suppose an altruistic husband gives gifts to a selfish wife, but also likes to read in bed. The wife likes the gifts, but dislikes the night light to husband uses to read by. Now suppose an electrician stops by the house in the husbands absence and offers to discretely disconnect the night light for the wife. According to the RKT the wife should decline the offer since reducing the husbands utility will effectively reduce his "full income" and hence the gifts that he gives to her. Following Bergstrom we shall show that she has the light disconnected!!

Let X_h, X_w be the consumption of a private good consumed by the husband and wife. Let Y be the number of hours the husband reads in bed (the night light is on). The preferences of the husband and wife are given by

$$U_h = X_h(Y + 1)\left(U_w\right)^a$$

where $0 < a < 1$ and

$$U_w = X_w e^{-Y}$$

Hence the altruistic husband maximizes

$$U_h = X_h(Y+1)\left(X_w e^{-Y}\right)^a = X_h X_w^a (Y+1)e^{-aY}$$

subject to the constraint

$$X_h + X_w = I$$

forming the Lagrangian we get

$$Max\ X_h X_w^a (Y+1)e^{-aY} + \lambda\left[I - X_h - X_w\right]$$

with FOC

$$\begin{aligned}
\frac{\partial \ell}{\partial X_h} &= X_w^a (Y+1)e^{-aY} - \lambda = 0 \\
\frac{\partial \ell}{\partial X_w} &= a X_h X_w^{a-1}(Y+1)e^{-aY} - \lambda = 0 \\
\frac{\partial \ell}{\partial Y} &= X_h X_w^a e^{-aY} - a X_h X_w^a (Y+1)e^{-aY} = 0
\end{aligned}$$

so rearranging and dividing the first FOC by the second yields

$$X_w = aX_h$$

substituting this into the budget constraint yields.

$$X_h + aX_h = I$$

so

$$\begin{aligned}
X_h &= \frac{I}{1+a} \\
X_w &= \frac{aI}{1+a}
\end{aligned}$$

From the third FOC we have

$$1 - a(Y+1) = 0$$

So

$$Y = \frac{1-a}{a}$$

which is efficient, *but*, since X_w is independent of Y *the wife has the night light turned off.*

5 The Theory of Marriage.

Based on two principles.

1. Marriage is voluntary (at least between parents) so it must represent a pareto improvement that can be analyzed using standard preference theory.

2. Since men and women compete for mates a market for marriages can be seen to exist.

5.1 The Gains from Marriage.

Consider two individuals Ken and Barbie who must decide whether to marry or not. Their marriage is voluntary so will only take place if they both benefit. Either as singles or a family Ken and Barbie engage in household production, that is they combine their time and consumption goods to produce commodities they consume. For example, time plus a car plus a swimming costume may be combined to produce a day at the beach, or, time plus food may be combined to produce a meal.

5.1.1 Ken and Barbie are Married.

Assume that household commodities may be combined into a single household good denoted Z, which is produced using time t_j, market goods x_i and v is non-labor income.

$$Z = f(x_1, .., x_m, t_1, ..., t_n)$$

A household budget constraint is then written

$$\sum_i p_i x_i = \sum_j w_j l_j + v$$

where w_j is the wage earned by the jth household member and l_j is the labor they sell on the market sector. Each individual time constraint may be written

$$t_j + l_j = T$$

where T is total time (24hrs). Combining the two constraints we get the *full income* constraint.

$$\sum_i p_i x_i + \sum_j w_j t_j = \sum_j w_j T + v = S$$

which is the maximum money achievable by the household. Note that this can be spent on t_j time spent on non-market activities.

We assume that an increase in Z makes no family member worse off, hence each will cooperate to maximize Z. We can now analyze the household optimization problem using the standard tools of consumer theory. The usual conditions equating marginal rates of substitution to price ratios then characterize the optimum. For the Ken and Barbie this translates into

$$\frac{MP_{t_k}}{MP_{t_b}} = \frac{\frac{\partial Z}{\partial t_k}}{\frac{\partial Z}{\partial t_b}} = \frac{w_k}{w_b}$$

$$\frac{MP_{x_i}}{MP_{t_b}} = \frac{p_i}{w_b}$$

If $w_k > w_b$ and $MP_{t_b} \geq MP_{t_k}$ when $t_f = t_b$ then more time would be allocated to the market sector by Ken than Barbie. Barbie would specialize in non-market production ($l_b = 0$) if $\frac{w_k}{w_b}$ or $\frac{MP_{t_b}}{MP_{t_k}}$ are sufficiently large.

5.1.2 Ken and Barbie are Single.

If they are single the problem is the same except for Ken $T_b = 0$, and for Barbie $T_k = 0$. We write their maximal outputs when single as x

$$Z_{k0}, Z_{0b}$$

where time is allocated optimally between market and non-market activities according to the same principles as for the family.

5.1.3 Are Ken and Barbie about to make a Terrible Mistake?

Our favorite dolls decide to marry!! Are they making a terrible mistake. Let's take a look.

Write m_k and m_b as the incomes our two plastic lovers enjoy in marriage, it follows that a necessary condition (but not sufficient ....as we shall see later Barbie and Ken are playing the field) fro them to marry is

$$\begin{aligned} m_k &\geq Z_{k0} \\ m_b &\geq Z_{0b} \end{aligned}$$

If follow that marriage will only occur if

$$m_k + m_b = Z_{mf} \geq Z_{k0} + Z_{0b}$$

one the obvious question is when will this condition hold. When can the two together earn more that two singles. Clearly if the time of one spouse is a perfect substitute for the time of the other then there is no gain from marriage. Each single is half a marriage. I on the other hand Ken and Barbie time contributions are not perfect substitutes for each other then marriage is a good idea. This can easily be seen if

$$Z = f(x_1, .., x_m, t_1, ..., t_n) = x^{\alpha} t_k^{\beta} t_b^{\gamma}$$

here clearly

$$Z_{k0} = Z_{0b} = 0 \; if \; t_k = 0 \text{ or } t_b = 0$$

5.1.4 So when is it more likely that Marriage is a good idea for Ken and Barbie?

1. The greater are the complementaries between the two spouses time contributions. E.g. the more important are children.

2. The greater is property income.

3. The greater are wage rate (typically).

4. The more different are w_b and w_k.

5.2 The Marriage Market.

5.2.1 Optimal Sorting.

Obviously individuals don't just marry the first partner for whom $m_k + m_b = Z_{kb} \geq Z_{k0} + Z_{0b}$, this may not be their best option. They wish to find the best partner. Suppose now that both Ken and Barbie have many (n) potential mates. How did they come to choose each other? They considered the following payoff matrix.

	F_1	......	F_n	
M_1	Z_{11}	......	Z_{1n}	Z_{10}
	......	......	......	
M_n	Z_{n1}	.....	Z_{nn}	Z_{n0}
	Z_{01}		Z_{0n}	$\times$

where M =male, F =female. Each individual has $n+1$ opportunities including remaining single. Further there are $n!$ ways to sort the individuals into pairs. Total output over all marriages for any giving sorting may be written

$$Z^h = \sum_{i \in M,, j \in F} Z_{ij} \ h = 1, ..., n!$$

number a sorting such that total output is maximized and lies on the diagonal and write

$$Z^* = \underset{h}{Max}\ Z^h$$

Now total output is divided between the mates so that

$$m_{ij} + f_{ij} = Z_{ij}\ \forall ij$$

If each mate chooses the partner that maximizes their income then optimal sorting will be pareto efficient. No individual can break away from the sorting and find a different mate and a division of their output such that they both prefer this to their current sorting. Surprisingly this end up maximizing the *total combined output* from all marriages

Example 13 *Suppose there are two males and two females who share the payoff matrix*

	F_1	F_2
M_1	8	4
M_2	9	7

We see that the maximum output from any marriage is $\{F_1, M_2\} = 9$, *but this leaves the other marriage to be* $\{F_2, M_1\} = 4$,*giving overall output* $\{F_1, M_2\} + \{F_2, M_1\} = 4 + 9 = 13$. *This is not an optimal sorting since* $\{F_1, M_1\} + \{F_2, M_2\} = 8 + 7 = 15$. *Suppose* $m_{11} = 3$, $f_{11} = 5$, $m_{22} = 5$, $f_{22} = 2$. *Then* M_2 *and* F_1 *have no incentive to marry since* $m_{22} + f_{11} = 5 + 5 = 10 > 9$, *and neither do* M_1 *and* F_2 *since* $m_{11} + f_{22} = 3 + 2 = 5 > 4$. *Thus the players will choose the optimal sorting.*

This can literally be thought of as a market, where one spouse offers the other a wage f_{ij} and receives the residual profits $m_{ij} = Z_{ij} - f_{ij}$, the spouse with the best match will be able bid highest, thus the sorting will be overall profit maximizing just like any other market.

5.2.2 Assortive Mating.

This involves sorting on a trait, that is do similar or dissimilar individuals mate? Becker's analysis tells us that depends on which maximizes household commodity output.

Assume that males differ in the characteristic A_m, while females differ on the characteristic A_f, and that each trait has a monotonically increasing effect on the value of any marriage. That is

$$\frac{\partial Z_{ij}\left(A_m, A_f\right)}{\partial A_m} > 0, \frac{\partial Z_{ij}\left(A_m, A_f\right)}{\partial A_f} > 0$$

1. Dissimilar Individuals Marry - If increasing both A_m and A_f adds more to output than the sum of the separate additions (increasing returns to the traits).

2. Similar Individuals Marry - If increasing both A_m and A_f adds less to output than the sum of the separate additions (decreasing returns to the traits).

Mathematically this states that positive or negative assortive mating will occur as

$$\frac{\partial^2 Z_{ij}\left(A_m, A_f\right)}{\partial A_m \partial A_f} \gtreqless 0$$

Assortive of likes is optimal when the traits are complements and assortive of unlikes is optimal when the traits are substitutes.

Example 14 *Consider the payoff matrix*

	A_1	A_2
A_1	Z_{11}	Z_{12}
A_2	Z_{21}	Z_{22}

, with $A_2 > A_1$

If $Z_{22} - Z_{12} > Z_{21} - Z_{11}$, *and if* $\frac{\partial^2 Z_{ij}(A_m, A_f)}{\partial A_m \partial A_f} > 0$, *then* $Z_{11} + Z_{22} > Z_{12} + Z_{22}$, *and a positive correlation between* A_m *and* A_f *maximizes total output.*

6 Crime and Punishment.

We do not assume that criminals are any different from other members of society, they are criminals because this is where their comparative advantage lies. They are either very good at being criminals or very bad at being anything else. Crime may be analyzed using standard the tools of economic analysis. Criminals supply crime and the criminal justice system demands it!!

6.1 A Model of Crime.

6.1.1 Damages.

Let O be the number of criminal offences committed, these offences harm society according to the function $H(O)$ with $H'(O) > 0$ and $H''(O) > 0$ hence the harm from crime increases at an increasing rate. The perpetrators of crimes gain from them according to the function $G(O)$ with $G'(O) > 0$ and $G''(O) < 0$, thus there are positive but diminishing returns to criminal activity. The net damage to society from O offences is given by

$$D(O) \equiv H(O) - G(O)$$

it is assumed that

$$D'(O) \equiv H'(O) - G'(O) > 0 \; \forall O \geq O_a$$

and we have

$$D''(O) \equiv H''(O) - G''(O) > 0$$

so the net damage due to criminal activity is an increasing function of the level of that activity at least once the number of offences exceeds the threshold O_a.

6.1.2 The Cost of Apprehension and Conviction.

Denote as A the level of activity in detecting and prosecuting offenders, this includes both the costs of the police and the judicial system and is written $C(A)$ we assume $C'(A) > 0$. One measure of A is the number of offences convicted, thus if p is the frequency of conviction then

$$A \simeq pO$$

we may thus write

$$\begin{aligned} C_p &= \frac{\partial C(pO)}{\partial p} = C'O > 0 \\ C_O &= \frac{\partial C(pO)}{\partial O} = C'p > 0 \end{aligned}$$

Costs are increasing in both the number of offences and the probability of any offence being convicted. Further

$$\begin{aligned} C_{pp} &= C''O^2 > 0 \\ C_{OO} &= C''p^2 > 0 \\ C_{pO} &= C_{Op} = C''pO + C' > 0 \end{aligned}$$

6.1.3 The Supply of Offences.

The number of offenses committed depends primarily on the probability of conviction and the penalty incurred if convicted or

$$O = O(p, f)$$

with $O_p < 0$, and $O_f < 0$, that is criminals are deterred by higher penalties or being caught and convicted.

6.1.4 The Social Costs of Punishment.

The cost of punishing offenders often effects others, and may have positive or negative effects.

1. Fines may be used to the benefit of others.
2. Imprisonment requires the use of societies resources.

We thus write the social cost of punishment as

$$f' = bf$$

whether $b \lesseqqgtr 1$ depends on the particular circumstances.

6.1.5 The Social Optimum.

The welfare to society from crime is measured by the loss function which is assumed to be of the linear form

$$L(D, C, bf, O) = D(O) + C(p, O) + bpfO$$

Since for society as a whole p is the frequency of conviction, so pO is the number of convictions. To maximize welfare society can set the fines f and choose the probability of successful conviction (by applying resources). Recall that $O = O(p, f)$ so the first order conditions for a social optimum are

$$\begin{aligned} \frac{\partial L}{\partial f} &= D'O_f + C_O O_f + bpfO_f + bpO = 0 \\ \frac{\partial L}{\partial p} &= D'O_p + C_O O_p + C_p + bpfO_p + bfO = 0 \end{aligned}$$

dividing the expressions by O_f and O_p respectively gives

$$\begin{aligned} D' + C_O + bpf + \frac{bpO}{O_f} &= 0 \\ D' + C_O + \frac{C_p}{O_p} + bpf + \frac{bfO}{O_p} &= 0 \end{aligned}$$

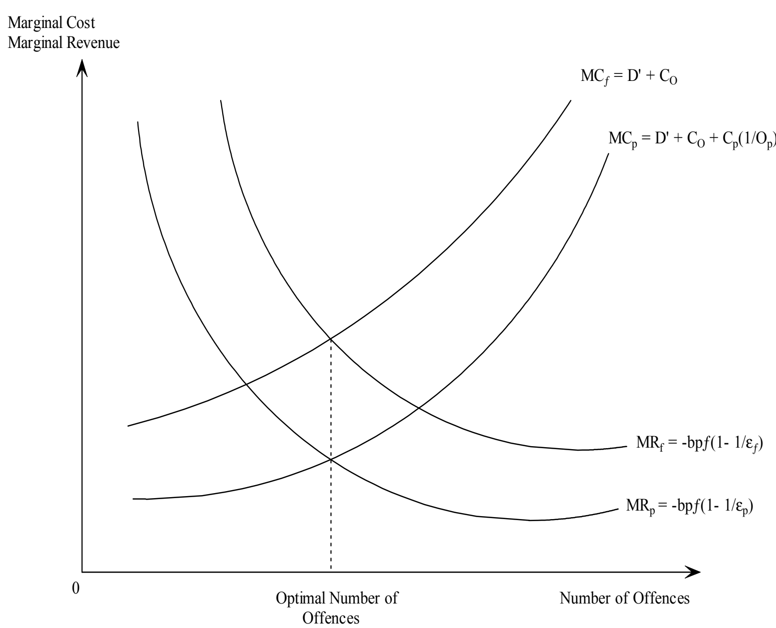
Now if we define the elasticities to commit offences with respect to fines and the probability of conviction as

$$\begin{aligned} \varepsilon_f &= -\frac{f}{O} O_f \\ \varepsilon_p &= -\frac{p}{O} O_p \end{aligned}$$

then we can rewrite the optimality conditions as

$$\begin{aligned} D' + C_O &= -bpf \left(1 - \frac{1}{\varepsilon_f}\right) \\ D' + C_O + \frac{C_p}{O_p} &= -bpf \left(1 - \frac{1}{\varepsilon_p}\right) \end{aligned}$$

the left hand side of each of these expressions represents the marginal cost of using the respective policy variable f or p, the right hand side is the marginal "revenue" (which can be positive or negative dependant on the magnitudes of the elasticities). Taken together these two sets of curves represent the social welfare optimum as seen in the following figure

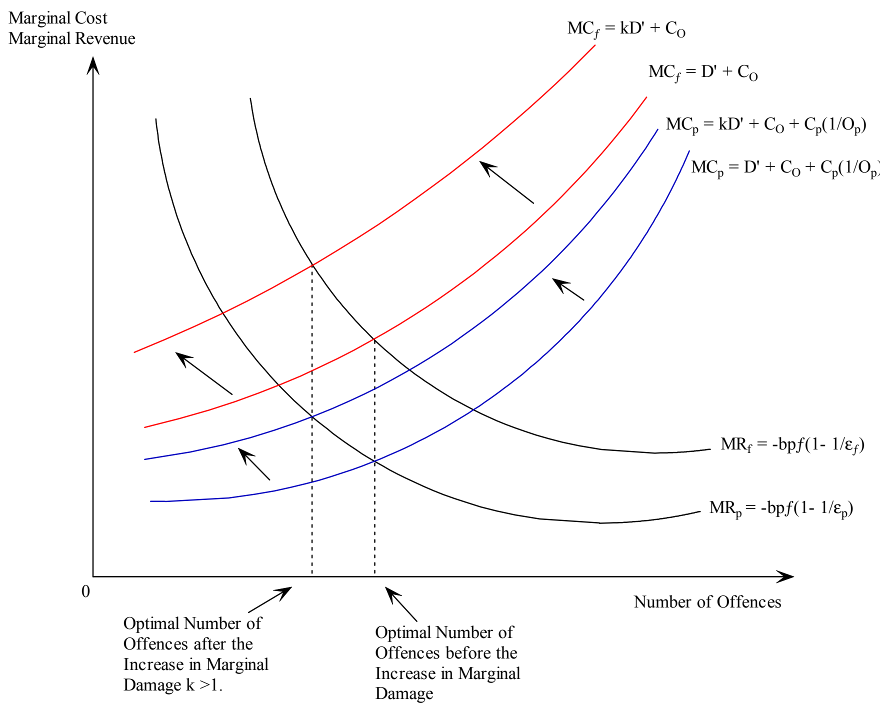


The Socially Optimal Number of Offenses.

6.1.6 Changes in the Optimal Number of Offences.

An Increase in Marginal Damages (D').

Suppose that the marginal damage done to society for any given number of crimes increases. This might be because society becomes less tolerant of crime over time, or perhaps because the type of crimes represented by the term offences changes, murders and rapes replace jay walking and speeding in the aggregate crime statistics. This shifts both marginal cost curves up as represented in the following diagram



A Rise in the Marginal Social Damage of Crime.

Hence we see that

1. There is an increase in optimal size of fines.
2. There is an increase in the optimal frequency of conviction
3. There is an decrease in the optimal number of offences.

An Increase in the Marginal Social Cost of Apprehension and Conviction (C_O).

These have the same effects as an increase in marginal damages. That fines should rise is perhaps obvious. Convictions also rise as the increased cost makes it more important to deter offences.

An Increase in the Marginal Social Cost of Apprehension and Conviction (C_p).

This increases the cost of using p to deter offences, its effects are partially offset by an increase in f hence the optimal number of offences rises.

Cost Increases that Raise both C_O and C_p

Typically have an ambiguous effect on the optimal number of offences. Examples might be

1. Increased salaries for the police.
2. Improved technology for the detection of crime (reduces both components of cost).

A Decrease in the Elasticity $\varepsilon_{f.}$

So that criminal activity becomes less sensitive to the fines used to deter it.

1. Increases the optimal number of offences.
2. Decreases the optimal f.
3. Increases the optimal p but not by enough to offset the effects of the change in f.

A Decrease in the Elasticity $\varepsilon_{p.}$

Criminals are less deterred by the probability of being caught and convicted.

1. Increases the optimal number of offences.
2. Decreases the optimal p.
3. Increases the optimal f but not by enough to offset the effects of the change in p.

A Decrease in both Elasticities

Criminals are simply less easy to deter.

1. Increases the optimal number of offences.
2. Decreases the optimal f.
3. Decreases the optimal p..

An Increase in b the Social Cost of Punishment.

If it is more socially costly to punish offenders then it is desirable to adjust p and f to increase the optimal number of offences. Either p or f or both must fall. It can be shown that the optimal value of p falls and the optimal value of f increases but only enough to have a partially offsetting effect on O.