EXAM 2 Review

Math 111

WINTER 2004

Let f and g be functions.

- §3.4 Graphs and Transformations

Vertical shifts	$f(x)+b$	up b
Horizontal shifts	$f(x-a)$	right a
Vertical stretches	$df(x)$	stretch by d (vert)
Horizontal stretches	$f(\frac{1}{c}x)$	stretch by c (horz)
Vertical reflections	$-f(x)$	flip (vert)
Horizontal reflections	$f(-x)$	flip (hoz)
Combinations (order matters!)	$-df(-\frac{1}{c}(x-a))+b$	all

- §3.4A Symmetry

Even functions	$f(-x)=f(x)$	Reflection about y-axis
Odd functions	$f(-x)=-f(x)$	Reflection about origin

- §3.5 Operations on Functions

Sum	$(f+g)(x)=f(x)+g(x)$
Difference	$(f-g)(x)=f(x)-g(x)$
Product	$(f*g)(x)=f(x)*g(x)$
Quotient	$\frac{f}{g}(x)=\frac{f(x)}{g(x)}$, $(g(x)\neq 0)$
Composite	$(g\circ f)(x)=g(f(x))$
	$(f\circ g)(x)=f(g(x))$
Domain	Depends on original functions, as well as resultant simplified function rule

- §3.7 Inverse Functions

  1. Existence when:
     - A function is one-to-one
     - $a\neq b\Rightarrow f(a)\neq f(b)$
     - $f(a)=f(b)\Rightarrow a=b$
     - Horizontal line test: graph of function intersects horizontal line at most once.
  2. Definition: The function g is the inverse of f
     - Algebraically
       (a) if $g(y)=x$ exactly when $f(x)=y$
     - Graphically
       (a) if when (a,b) is on the graph of f, then (b,a) is on the graph of g.
  3. If f is the inverse of g, then g is the inverse to f. For all x in the domain:
     - $(g\circ f)(x)=g(f(x))=x$
     - $(f\circ g)(x)=f(g(x))=x$

4. Domain and ranges given that g and f are inverses of each other.
   - Domain of g is range of f.
   - Range of f is domain of f.
5. Notation: f^{-1} denotes the inverse of f.
6. Finding the inverse, given $f(x)$:
   - Algebraically
     (a) Examine $f(g(x))$.
     (b) Solve for $g(x)$.
   - Graphically
     (a) Flip $f(x)$ about the line $x = y$

- §4.1 Quadratic Functions

1. Rule:
   - Algebraically
     (a) $f(x) = ax^2 + bx + c$ for constants a, b, and c with $a \neq 0$
   - Graphically
     (a) Parabola
2. Complete the square to express f in standard vertex form: $f(x) = a(x - h)^2 + k$
3. Graphing without graph transformations:
   - Determine direction of parabola, let a be the leading coefficient.
     * $a > 0 \Rightarrow$ parabola opens up $\Leftrightarrow$ vertex is min
     * $a < 0 \Rightarrow$ parabola opens down $\Leftrightarrow$ vertex is max
   - Find the vertex given:
     * $f(x) = a(x - h)^2 + k$. Vertex is (h, k).
     * $f(x) = ax^2 + bx + c$. Vertex is $(\frac{-b}{2a}, f(\frac{-b}{2a}))$
   - Find the y-intercept

- §4.2 Polynomial Functions

1. Definitions: a polynomial in x: $a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$
   - The degree of the polynomial is n if $n \geq 0$ and is an integer.
   - The coefficients are the constants a_0, a_1, a_2, ..., a_{n-1} a_n
   - The leading coefficient is the coefficient of the highest power of x (a_n)
   - The constant term is the free standing coefficient $(a_0 = a_0 x^0)$.
2. Division algorithm for polynomials
   It is possible to divide a polynomial $f(x)$ by a non-zero polynomial $h(x)$ with smaller degree, such that

$$f(x) = h(x)q(x) + r(x)$$

where either $r(x) = 0$ or the degree of $r(x)$ is less then the degree of $h(x)$.
The remainder in polynomial division is 0 exactly when the divisor is a factor so was can write

$$f(x) = h(x)q(x)$$

3. Remainder theorem:
If the polynomial $f(x)$ is divided by $(x - c)$, then the remainder is $f(c)$.
4. Factor theorem:
The number c is a root exactly when $(x - c)$ is a factor of $f(x)$.
5. A polynomial of degree n has at most n roots.

- §4.4 Graphs of Polynomial functions

1. Graph of $f(x) = ax^n$

   $a > 0$ and n odd　　　　　$a < 0$ and n odd

   $a > 0$ and n even　　　　　$a < 0$ and n even

2. A function is continuous if the graph contains no gaps, jumps, or holes.
   - No polynomial contains any horizontal line segments or sharp edges in its graph.
3. End behavior: When $|x|$ is very large, the graph of a polynomial function resembles the graph of it's highest degree term.
   - Odd degree polynomial: one end goes up and the other goes down
   - Even degree polynomial: both ends go up or both end go down.
4. Root of multiplicity: A number c is a root of multiplicity k of a polynomial $f(x)$ if $(x - c)^k$ is a factor of $f(x)$
   - If k is odd, then the graph crosses the x-axis at $x = c$.
   - If k is even, then the graph touches but doesn't cross at $x = c$
5. Let n be the degree of a polynomial $f(x)$

n	$\geq$	number of roots
$n - 1$	$\geq$	number of extrema
$n - 2$	$\geq$	number of points of inflection

- §4.5 Rational Functions

  1. "Big-Little" Principle: $\frac{1}{big} = little$ and $\frac{1}{little} = big$, where *big* and *little* refer to the distance from 0.
  2. Domain: Let $f(x) = \frac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials, the domain of $f(x)$ is the set of real numbers that are not roots of $h(x)$.
  3. Asymptotes: A part of a graph of a function that gets arbitrarily close to a value but never reaches it. Asymptotes can be vertical or horizontal.
     - Horizontal: Let $f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_0}$ be a *reduced* rational function.
       * If $n = m$, there is a horizontal asymptote at $y = \frac{a_n}{b_m}$
       * If $n < m$, there is a horizontal asymptote at $y = 0$.
       * If $n > m$, there is no horizontal asymptote.
     - Vertical: Let $f(x) = \frac{g(x)}{h(x)}$ be a *reduced* rational function. There is a vertical asymptote when $h(x) = 0$ and $g(x) \neq 0$
  4. Intercepts: Let $f(x) = \frac{g(x)}{h(x)}$
     - X-intercepts occur when $g(x) = 0$ and $h(x) \neq 0$
     - Y-intercepts occur at $f(0)$
  5. Holes vs Vertical Asymptotes: In the rational function $f(x) = \frac{h(x)}{g(x)}$, if d is a root of both $g(x)$ and $h(x)$ and the multiplicity is greater or equal in the numerator compared to the denominator, then $f(x)$ has a hole at $x = d$.
  6. Graphing a rational function:
     - Factor the numerator and the denominator.
     - Note the domain.
     - Cancel out like terms
     - Find any vertical asymptotes and mark them on the graph.
     - Find any horizontal asymptotes and mark them on the graph.
     - Find and graph the x and y intercepts
     - Determine graph behavior at intervals.
     - Sketch the graph.
     - Make appropriate holes in your graph by recalling the domain.

- §4.6A Absolute Value Inequalities

  1. Properties: For $k > 0$ and x any real number
     - $|x| \leq k \Leftrightarrow -k \leq x \leq k$
     - $k \leq |x| \Leftrightarrow x \leq -k$ or $k \leq x$

2. Solving
   - Replace the inequality with an '='
   - Solve absolute value as done in §1.2A
   - Check values in resulting intervals to see which one(s) to keep
     or
   - Use above property to solve directly
   - *Remember* to switch signs when multiplying or dividing by a negative!

- §5.1 Radicals and Rational Exponents

1. Definition: Let c be a real number and n be a positive integer, the nth rood of c

$$\sqrt[n]{c} = c^{\frac{1}{n}}$$

   - is the solution of $x^n = c$ when n is odd
   - is the nonnegative solution of $x^n = c$ when n is even

2. Definition: Let c be a real number and $\frac{r}{s}$ be a rational number with positive denominator, then

$$c^{\frac{r}{s}} = (c^r)^{\frac{1}{s}} = (c^{\frac{1}{s}})^r$$

3. Exponent Laws: Let c, and d be real numbers and r and s be positive integers.
   $c^r c^s = c^{r+s}$ $\qquad$ $\frac{c^r}{c^s} = c^{r-s}, c \neq 0$
   $(c^r)^s = c^{rs}$ $\qquad$ $(cd)^r = c^r d^r$
   $(\frac{c}{d})^r = \frac{c^r}{d^r}, d \neq 0$ $\qquad$ $c^{-r} = \frac{1}{c^r}, c \neq 0$

4. Rationalize numerator/denominator
   - multiply by $1 = \frac{\sqrt{a}}{\sqrt{a}}$ *or*
   - multiply by fancy 1, $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}$ or $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

- §5.2 Exponential Functions

1. Graph of $f(x) = a^x$
   - When $a > 1$

     * above y-axis

* y-intercept is 1
    * $f(x)$ increasing
    * negative x-axis is the horiz. asymptote
    * larger base a makes graph rise more steeply ot the right
  - When $0 < a < 1$
    * above y-axis
    * y-intercept is 1
    * $f(x)$ decreasing
    * positive x-axis is the horizontal asymptote
    * closer base a is to 0, the more steeply the graph falls to the right

2. exponential growth: $f(t) = Pa^t$ $(a > 1)$ or $f(t) = Pe^{rt}$ $(r > 0)$, where P is the intial amount, a is the growth factor, and r is the rate of growth
3. exponential decay: $f(t) = Pa^t$ $(0 < a < 1)$ or $f(t) = Pe^{rt}$ $(r < 0)$
4. radioactive decay: $M(t) = c(0.5)^{\frac{t}{h}}$ is the mass at time t where c is the intial mass, and h is the half-life

- §5.2A Compound interest and the number e

1. compound interest: $A(t) = P(1+r)^t$, where A is the amount at time t, P is the principal (initial amount), and r is the rate
2. compounded more frequently $A_n(t) = P\left(1 + \frac{r}{n}\right)^{nt}$, where n is the number of times compounded per year
3. as $n \to \infty$, $\left(1 + \frac{1}{n}\right)^n \to e$, where $e \approx 2.718281828459045\ldots$.
4. $A(t) = Pe^{rt}$ gives the balance after time t when compounded continuously at a rate of r