Math 307
Homework Due Wednesday, April 29

1. For each of the following mathematical statements, write it out in words and then decide if it is true or false. Then write a mathematical statement expressing its negation (not by just putting a $\sim$ in front!) and write out the negation in words as well.

   In some cases I have given two equivalent forms of the same statement so that you can get used to going back and forth between the two.

   (a) $(\forall x)[x \in \mathbb{Z} \Rightarrow (\exists y)[y \in \mathbb{Z} \wedge x = y^2]]$, or equivalently, $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})[x = y^2]$.
   (b) $(\exists a)[a \in \mathbb{N} \wedge (\forall b)[[b \in \mathbb{N} \wedge a \neq b] \Rightarrow b > a]]$, or equivalently, $(\exists a \in \mathbb{N})(\forall b \in \mathbb{N})[a \neq b \Rightarrow b > a]$.
   (c) $(\forall a)(\forall b)[(a, b \in \mathbb{R} \wedge a < b) \Rightarrow (\exists c)[c \in \mathbb{R} \wedge [a < c \wedge c < b]]]$, or equivalently, $(\forall a, b \in \mathbb{R})[a < b \Rightarrow (\exists c \in \mathbb{R})[a < c \wedge c < b]]$.

2. The phrase "the integer x is a perfect square" means $x \in \mathbb{Z} \wedge (\exists y \in \mathbb{Z})[x = y^2]$. Keeping this in mind, write out mathematical statements—using only quantifiers and other math symbols, no English words—which say the same thing as the following sentences. Do not worry about whether the statements are true or false!

   (a) Every integer which is a perfect square is also a perfect cube.
   (b) There exists an integer p with the following properties: p is even and for every pair of integers a and b, if p divides ab then it must divide either a or b.
   (c) For all integers x, if x is congruent to three mod eight then there exists an integer y such that $x \cdot y$ is congruent to one mod eight.
   (d) Every nonzero integer is either a multiple of six or a multiple of seven.
   (e) Every nonzero element of $\mathbb{Z}_5$ has a multiplicative inverse.

3. Translate the following into a symbolic logic problem, then provide a proof:

   Given: If Smith wins the nomination, he will be happy, and if he is happy, he is not a good campaigner. But if he loses the nomination, he will lose the confidence of the party. He is not a good campaigner if he loses the confidence of the party. If he is not a good campaigner, then he should resign from the party. Either Smith wins the nomination or he loses it.

   Prove: Smith should resign from the party.

4. In class we have been doing what I at one point called "line proofs". These are less formal than logic proofs, in that you may condense several logical steps into one when you think it is clear enough; yet the proofs still keep their numbered line-by-line structure. Give line proofs for the following statements:

   (a) $(\forall a, b, k \in \mathbb{Z})[[k \geq 1 \wedge a|b] \Rightarrow a^k|b^k]$.
   (b) $(\forall a, b, c \in \mathbb{Z})[(a|b \wedge b|c) \Rightarrow a|c]$.
   (c) $(\forall x \in \mathbb{Z})[3|x^2 \Rightarrow 3|x]$.
   (Here's a hint for this last one. Start your proof as follows:

      (1) Assume $x \in \mathbb{Z}$.
      (2) Assume $3|x^2$.
      (3) Assume 3 does not divide x.
      (4) Then there exist $n, e \in \mathbb{Z}$ such that $x = 3n + e$ and e is either 1 or 2.

   Now derive a contradiction.
   (d) $(\forall x \in \mathbb{Z})[2|x \Rightarrow [x^4 \equiv_{32} 0 \vee x^4 \equiv_{32} 16]]$