Math 307

Homework Due Wednesday, May 6

For the logic proofs we do from now on, I suggest trying to do them without using any non-obvious tautologies—that is, without using tautologies other than the five or six that are so common you know them by heart.

1. Given $[\sim T \wedge P] \Rightarrow \sim Q$, $[\sim T \vee P] \Rightarrow R$, and $\sim S$, prove $P \Rightarrow \big[[Q \vee [R \Rightarrow S]] \Rightarrow T\big]$.

2. Given $[Q \wedge S] \Rightarrow R$ and $\sim S \Rightarrow T$, prove $[P \Rightarrow Q] \Rightarrow [\sim T \Rightarrow [\sim P \vee R]]$.

3. For sets A and X, define $X - A = \{x \,|\, x \in X \wedge x \notin A\}$. Give line proofs for each of the following statements:

   (a) $X - (A \cap B) = (X - A) \cup (X - B)$.

   (b) $X - (A \cup B) = (X - A) \cap (X - B)$.

   (c) $A \cap (X - B) = (A \cap X) - (A \cap B)$.

4. Here is a fact about prime numbers. If $p \in \mathbb{N}$ is a prime, then $(\forall a, b \in \mathbb{N})[p|ab \Rightarrow [p|a \vee p|b]]$. Accept this as a fact that you can use whenever you want.

   For any $n \in \mathbb{N}$, let $M_n = \{x \in \mathbb{N} \,|\, x \equiv_n 0\}$; that is, M_n is the set of all multiples of n.

   (a) Prove or disprove: $M_2 \cap M_3 = M_6$.

   (b) Prove or disprove: $M_2 \cup M_3 = \mathbb{N}$.

   [If you choose to prove the claim, give a line proof. To disprove a claim, explain which subset direction you are disproving and give a specific element which disproves it.]

5. Read about functions on pages 10–15 of the textbook (or any math textbook for that matter). If f is a function from S to T and $A \subseteq S$, define $f(A) = \{x \,|\, (\exists y \in A)[x = f(y)]\}$. This is called the **image of** A **under** f.

   (a) Give an example of a function $f\colon \mathbb{N} \to \mathbb{N}$ where $f(M_2) = \mathbb{N}$ (notation as in problem 4).

   (b) Suppose $f\colon S \to T$ and $A \subseteq S$, $B \subseteq S$. For each of the following, either prove it (by giving a line proof) or disprove it by giving a single example showing it to be false.

      (i) $f(A \cap B) = f(A) \cap f(B)$

      (ii) $f(A \cup B) = f(A) \cup f(B)$

6. If $f\colon S \to T$ and $A \subseteq T$, define $I_f(A) = \{x \in S \,|\, f(x) \in A\}$. This is called the **inverse image of** A **under** f.

   (a) Give an example of a function $f\colon \mathbb{N} \to \mathbb{N}$ where $I_f(M_2) = \mathbb{N}$.

   (b) Suppose $f\colon S \to T$ and $A \subseteq T$, $B \subseteq T$. For each of the following, either prove it (by giving a line proof) or disprove it by giving a single example showing it to be false.

      (i) $I_f(A \cap B) = I_f(A) \cap I_f(B)$

      (ii) $I_f(A \cup B) = I_f(A) \cup I_f(B)$

      (iii) $f(I_f(A)) = A$

      (iv) For each $C \subseteq S$, $I_f(f(C)) = C$.

7. You have 500 seven-ounce weights, 500 eleven-ounce weights, and a balance scale. Someone brings you a bar of gold which he claims weighs 500 ounces. Can you determine whether he is lying or not? Explain how. Secondly, what would happen if you instead had six-ounce weights and nine-ounce weights; could you determine whether he was lying? Explain.