Math 307
Homework Due Wednesday, May 13

1. Given $Q \Rightarrow R$, prove $[P \Rightarrow T] \Rightarrow [(Q \vee \sim T) \Rightarrow (\sim P \vee R)]$.

2. In each case, use mathematical notation to write the negation of the given statement, in such a way that no quantifier is immediately preceded by a negation sign. For parts (a)–(d), decide which is true: the given statement or its negation (and explain why).
   (a) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[x + y = 0]$
   (b) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[x + y = 0]$
   (c) $(\exists x, y \in \mathbb{R})[x^2 + y^2 = -1]$
   (d) $(\forall x \in \mathbb{R})[x > 0 \Rightarrow (\forall y, z \in \mathbb{R})[(y > 0 \wedge z > 0 \wedge y^2 = x \wedge z^2 = x) \Rightarrow y = z]]$
   (e) $(\forall a, b \in \mathbb{Z})[a^2 \equiv_3 b^2 \Rightarrow (a \equiv_3 b \vee a \equiv_3 -b)]$
   (f) $(\forall \epsilon \in \mathbb{R})[\epsilon > 0 \Rightarrow (\exists \delta \in \mathbb{R})[0 < \delta \wedge (\forall x \in \mathbb{R})[1 - \delta < x < 1 + \delta \Rightarrow |f(x) - 5| < \epsilon]]]$
   (g) $(\forall a, b \in \mathbb{R})(\exists c \in \mathbb{R})[a < c < b \wedge f'(c) = \frac{f(b)-f(a)}{b-a}]$.

   Part (g) is a statement which is true for differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$, and it is a well-known theorem taught in every calculus class. What is the common name of this theorem?

3. Give a line proof that $(\forall x \in \mathbb{N})[2 \not| x \Rightarrow 8 | x^2 - 1]$. [Hint: Any odd number has the form $2n + 1$ for some $n \in \mathbb{Z}$.]

4. Let $f\colon S \to T$ be a function. Give line proofs of each of the following:
   (a) If $X \subseteq T$, then $S - I_f(X) = I_f(T - X)$.
   (b) If $A \subseteq S$ and $B \subseteq S$ then $f(A) - f(B) \subseteq f(A - B)$.
   (c) If $X \subseteq T$ and $Y \subseteq T$ then $I_f(X) - I_f(Y) \subseteq I_f(X - Y)$.

5. A function $f\colon S \to T$ is called **one-to-one** if it has the property that $(\forall a, b \in S)[f(a) = f(b) \Rightarrow a = b]$.
   (a) Write down, in mathematical notation, what it means for a function to **not** be one-to-one.
   (b) Give an example of a function that is one-to-one, and give an example of a function that is not one-to-one.
   (c) Suppose that $f$ is one-to-one and $A \subseteq S$ and $B \subseteq S$. Prove that $f(A \cap B) = f(A) \cap f(B)$.

6. Give line proofs for each of the following.
   (a) If $C \subseteq A$ and $D \subseteq B$ then $D - A \subseteq B - C$.
   (b) If $A \cup B \subseteq C \cup D$, $A \cap B = \emptyset$, and $C \subseteq A$, then $B \subseteq D$.

7. (a) A cafeteria has 6 selections for main entree, 10 vegetable selections, 5 dessert selections. If a dinner consists of one entree, two vegetables, and one dessert, how many possible dinner combinations are there?
   (b) You have a group of ten people and you need to distribute five prizes, whose amounts are \$1000, \$1000, \$2000, \$2000, and \$6000. How many ways are there to distribute the prizes?

8. You have eight children: five girls and three boys. In each case below, find the number of ways they could line up subject to the given conditions:
   (a) The three boys are first in line.

(b) The five girls are all together.

(c) The three boys are all together.

Now suppose that we only want to line up five of the children.

(d) How many ways are there to do this if we want three girls and two boys?

(e) How many ways have three girls, two boys, and the first in line is a girl?

[Note: Parts (b) and (c) are a little more challenging than the other parts.]

9. In each of the following parts, determine the indicated coefficients (and explain why your answers are correct). Your answers should involve binomial coefficients, and perhaps other numbers as well.

   (a) The coefficient of $x^{10}y^4z^8$ in $(2x+3y+4z)^{22}$.

   (b) The coefficients of $x^{34}$ and $x^{47}$ in $(x+\frac{1}{x})^{100}$.

   (c) The coefficients of $x^{161}$ and $x^{111}$ in $(x^2-\frac{3}{x})^{100}$.

   (d) The coefficient of $x^{24}y^{110}$ in $(\frac{y^3}{x}+2x-5x^2)^{100}$.