Math 307
Homework Due Wednesday, May 20

1. Given $R \vee S$, $\sim P$, and $P \Leftrightarrow Q$, prove that $[[Q \vee \sim R] \Rightarrow S]$.
2. Consider the function $f \colon \mathbb{Z}_7 \to \mathbb{Z}_7$ given by $f(x) = x^3 + 1$. Answer the following questions:
   (a) Is f one-to-one? Explain why or why not.
   (b) Determine $f(S)$ where $S = \{0, 2, 4, 6\}$.
   (c) If $A = \{1, 2, 3, 4\}$ and $B = \{0, 4, 5, 6\}$, determine $I_f(A)$ and $I_f(B)$. Also determine $I_f(A \cap B)$.
3. In each part, prove the indicated statement by induction:
   (a) $(\forall n \in \mathbb{N})[\frac{(2n)!}{n! \cdot 2^n}$ is an odd number$]$
   (b) $(1 + \frac{1}{2})^n > 1 + \frac{n}{2}$ for all $n \geq 2$.
   (c) $1^3 + 2^3 + \cdots + n^3 = [\frac{n(n+1)}{2}]^2$ for all $n \geq 1$.
4. Let $a_n = 1 - 2 + 3 - 4 + \cdots + (-1)^{n+1} n$. Prove by induction that $a_{2n} = -n$ for all $n \geq 1$.
5. Consider the sequence given by $a_n = \sqrt{2 + a_{n-1}}$ and $a_0 = 2$. Prove by induction that $a_n \leq 2$ for all $n \geq 0$.
6. Consider the sequence given by $a_n = 2a_{n-1} + 4a_{n-2}$ and initial conditions $a_0 = 0$, $a_1 = 3$. Prove that $3 | a_n$ for all $n \geq 0$.
7. Suppose $f \colon \mathbb{Z} \to \mathbb{Z}$ is a function with the property that $(\forall x, y \in \mathbb{Z})[f(x + y) = f(x) + f(y)]$.
   (a) Prove by induction that $(\forall n \in \mathbb{N})[n \geq 1 \Rightarrow (\forall x \in \mathbb{Z})[f(nx) = n \cdot f(x)]]$.
   (b) Give a line proof that $(\forall k \in \mathbb{N})[f(M_k) \subseteq M_k]$.
8. If $f \colon S \to T$ and $g \colon T \to U$, then there is a function denoted $(g \circ f) \colon S \to U$ called the **composition** of g and f. It is defined by the formula

   $$(g \circ f)(x) = g(f(x)).$$

   You can read about compositions on pages 85–87 of your book.
   (a) If $f \colon \mathbb{Z} \to \mathbb{Z}$ is given by $f(x) = x^2 - 1$ and $g \colon \mathbb{Z} \to \mathbb{Z}$ is given by $g(x) = 3x + 2$, determine $(g \circ f)(0)$ and $(g \circ f)(2)$. Determine an algebraic formula for $(g \circ f)(x)$ for any integer x.
   (b) Suppose $f \colon S \to T$ and $g \colon T \to U$, and $X \subseteq U$. Give a line proof that $I_{g \circ f}(U) = I_g(I_f(U))$.
   (c) Again suppose that $f \colon S \to T$ and $g \colon T \to U$. If $A \subseteq S$, give a line proof that $(g \circ f)(A) = g(f(A))$.