Math 307

Homework Due Wednesday, May 27

Prove each of the following statements by induction:

1. For all $n \in \mathbb{N}$, if $n \geq 1$ then $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.
2. For all $n > 4$, $2^n > n^2$.
3. For all $n \geq 2$, $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$.
4. For all $n \geq 2$, $\left(\frac{2^2-1}{2^2}\right) \cdot \left(\frac{3^2-1}{3^2}\right) \cdots \left(\frac{n^2-1}{n^2}\right) = \frac{n+1}{2n}$.
5. For all $n \geq 2$, $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} < \frac{n^2}{n+1}$.
6. Imagine that you have an infinite supply of \$3 bills and \$5 bills. Prove that for all $n \geq 8$, you can make a combination of bills exactly totalling n dollars. Use strong induction, and for the base case check $n = 8$, $n = 9$, and $n = 10$ by hand.
7. Imagine that you have a method for taking a function f and producing a new function Δf. You should think of this as like the process of "differentiation" in calculus. Imagine that you know the following rules:

   (1) $\Delta(x) = 1$ (here "x" means the function $f(x) = x$)

   (2) $\Delta(f + g) = (\Delta f) + (\Delta g)$

   (3) $\Delta(c \cdot f) = c \cdot (\Delta f)$ for any $c \in \mathbb{R}$.

   (4) $\Delta(f \cdot g) = (\Delta f) \cdot g + f \cdot (\Delta g) + (\Delta f) \cdot (\Delta g)$

   Again, the first three rules should be very reminiscent of the differentiation process. The last rule is similar to the product rule for derivatives, except that there is a weird term tacked on to the end.

   (a) Determine $\Delta(x^2)$, $\Delta(x^3)$, and $\Delta(x^4)$. Determine $\Delta(3x^2 + 2x)$.

   (b) Prove by induction that for all $n \geq 1$, $\Delta(x^n) = \sum_{k=0}^{n-1} \binom{n}{k} x^k$.

   [Hint: If you have a sum $\sum_{k=0}^{r} a_k$, this can always be rewritten as $\sum_{k=1}^{r+1} a_{k-1}$. You will need to use this trick at some point.]