MATH 241, LECTURE 11

1. Limits at finite points

Definition 1. *(Intuitive) We say the limit of $f(x)$ as x tends to c is L if the values $f(x)$ are always arbitrarily close to L once x is close enough to c. Notationally, we say $\lim_{x\to c} f(x) = L$. If there is no L for which this is true, we say that the limit does not exist.*

Limits at finite points are conceptually tricky. Sometimes one has to pretend that the function does not exist at that point and use the values at nearby points to try to come up with a single possible value which fits with the rest. This can be clearly understood graphically.

Example 2. • *Some graphical examples.*

- $\lim_{x\to 3}(x^2+1) = 10$.
- $\lim_{x\to 5} \frac{x+3}{5-x}$ *does not exist.*
- $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$. *(Graph it!)*
- $\lim_{x\to 0} \frac{x}{|x|}$ *does not exist, which is easier to see when this function is rewritten.*
- $\lim_{x\to -1} \frac{x^2+3x+2}{x+1} = 1$.
- $\lim_{h\to 0} \frac{3h+h^2}{h} = 3$.

2. Continuity

We cover the concept of continuity only to reinforce our understanding of limits.

Definition 3. *If the limit $\lim_{x\to c} f(x)$ exists, $f(c)$ is defined, and these two quantities are equal we say that f is continuous at $x = c$.*

Intuitively, functions which are continuous everywhere are those which can be graphed without picking up the pencil.

Question 4. *In the examples above, which functions are continuous?*

Continuity is a property enjoyed by almost all basic functions at almost all of their values.

2.1. **Bonus topic: consequences of continuity.**

Continuity is such a fundamental property that there is a whole field of mathematics, called topology, devoted to understanding its consequences.

Theorem 5. *(The Intermediate Value Theorem) If $f(x)$ is a continuous function with $f(0) > 0$ and $f(1) < 0$ then there is some x in between 0 and 1 with $f(x) = 0$.*

The Intermediate Value Theorem is clear from pictures illustrating it. It is also easy to think of concrete examples such as the function $f(x) = 1 - 3x$.

In many variables, continuity has surprising consequences:

- Whenever you stir a cup of coffee, there is some molecule which ends up where it began.
- There are always two opposite points on the Earth with the exact same temperature and pressure (or the same population density and elevation or...).
- You can never comb the hair on a sphere without there being a part.

3. Some rules for calculating limits

The one way we have to calculate $\lim_{x\to c} f(x)$ is to plug in values for x which are close to c and see if the values of $f(x)$ approach one number. But there are some rules (which get established in Math 261 or 315) which make it much easier to compute some limits.

- If $f(x)$ is a polynomial, logarithmic or exponential function then it is continuous wherever it is defined. So $\lim_{x\to c} f(x) = f(c)$.
- $\lim_{x\to\infty} \frac{A}{x^k} = 0$ for $k > 0$ any A (no matter how large!).
- The limit as $x \to c$ of a sum of functions is the sum of their limits. Similarly for products, as well as quotients provided the limit of the denominator is not zero (unfortunately for you, in most problems it will be zero).
- A great trick for trying to compute $\lim_{x\to c} \frac{f(x)}{g(x)}$ where both the numerator and denominator go to zero or infinity is to multiply both the numerator and denominator by the same factor to get a finite limit.

Example 6.
- $\lim_{x\to 0} \frac{x(x+2)}{x^2}$
- $\lim_{x\to\infty} \frac{x^2-2x+4}{3x^2+1}$
- $\lim_{x\to\infty} \frac{x^2-2x+4}{3x^3+1}$

4. Procedure for computing limits

To do a limit problem go through the following steps.

(1) Substitute the limiting value into the function.
(2) (Optional) Make a table of values of the function near the limiting value (which for limits at positive or negative infinity means substituting large positive or negative numbers).
(3) (Optional) Graph the function near the limiting value.
(4) If the result of substitution is a finite number (including zero) you're done - the limit is that value. We may also deduce this when the graph indicates the function is continuous, or by plugging in values. We must use words to explain the deduction with these methods.

    Example: $\lim_{h\to 0} \frac{h^2-3}{h+1} = \frac{0-3}{0+1} = -3$.
(5) If this resulting expression is a finite number divided by infinity, then the limit is zero.
    Example: $\lim_{s\to\infty} \frac{45}{s^2+s}$ substitutes to $\frac{45}{\infty}$, so the limit is zero since 45 divided by a large number approaches zero.
(6) If the resulting expression is a finite number divided by zero, then the limit is positive or negative infinity. If we are in a setting where the limit must be finite, we would say it does not exist.

    Example: $\lim_{x\to 3} \frac{x^2+4}{x^2-9} = +\infty$ since the denominator goes to zero while the numerator approaches 13, and 13 divided by a really small number is a really big number.
(7) If the resulting expression is infinity, possibly over a finite number, then the limit is positive or negative infinity. Here in all of the methods we must justify by an observation that the function gets larger.

    Example: $\lim_{x\to +\infty} x^2 + 2x$ is $+\infty$ since $x^2 + 2x > x$ so the function gets larger as x does.
(8) If the resulting expression is zero over zero, then sometimes one can find a common factor, divide, and then get a well-behaved limit. Here the table and graphing methods can point us to the right answer, which is then best justified by the division method (though the table method is acceptable).

Example: $\lim_{x\to 3} \frac{x^2-x-6}{x-3}$ looks like $\frac{0}{0}$ so we check and find that $x^2 - x - 6 = (x-3)(x+2)$ so that the limit is the same as $\lim_{x\to 3}(x+2) = 5$.

(9) If the resulting expression is infinity over infinity, sometimse one can multiply the numerator and denominator by the same factor to get a well-behaved limit. Here the table and graphing methods are discouraged, since it is hard to know how large values need to be to approach the correct value.

Example: $\lim_{s\to+\infty} \frac{2s^2+1}{s^3-3s}$ looks like $\frac{\infty}{\infty}$. After trying a few things, we multiply both the numerator and denominator by $\frac{1}{s^2}$. We get $\lim_{s\to\infty} \frac{2+\frac{1}{s^2}}{s-\frac{3}{s}}$ which when we plug in looks like $\frac{2}{\infty}$, so the limit is zero.

5. Using limits to understand rules for derivatives

Our development of limits lets us fill in some details for our rules for taking derivatives.

For one example, to take the derivative of $f(x) + g(x)$ we consider

$$\begin{aligned}
&\lim_{h\to 0} \frac{[f(x+h)+g(x+h)] - [f(x)+g(x)]}{h}, \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}
\end{aligned}$$

For another example, we revisit the famous computation of the derivative of $f(x) = x^n$. The key is to understand a general formula for $(x+h)^n$, which starts out $x^n + n \cdot x^{n-1} + \cdots$.