MATH 241, LECTURE 10

1. Practice with using rules for derivatives

Right now we have three "short-cut" rules for taking derivatives (we will soon learn more - enough to take the derivative of any function made from our basic ones).

Theorem 1. (1) *If $g(x) = cf(x)$ then $\frac{d}{dx}g(x) = c\frac{d}{dx}f(x)$.*
(2) $\frac{d}{dx}\{f(x) + g(x)\} = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.
(3) $\frac{d}{dx}x^n = nx^{n-1}$.

After some practice, using these will become second nature.

Example 2. *Find the derivatives of each of the following functions, stating which rules are being used at each step.*

- x^{10000}
- $\frac{1}{x^3}$.
- $2x^{\frac{7}{4}} - \sqrt{\frac{5}{x}}$.
- $3x^{500} - 5.734 + x^{-500}$.

Our rules are far from being enough to take the derivatives of many familiar functions.

Example 3. *For each of the following functions, either take the derivative or say that it is not possible to take the derivative with the rules we have.*

- $(x+1)^2$. *(What about $(x+1)^{100}$?)*
- $\sqrt{x+1}$.
- $5^x + x^{2.1}$.
- $\frac{x}{\sqrt[3]{x}}$.

2. Limits

Before we apply the derivative and learn more rules for calculating it, we stop to tie up one loose end, namely the "$\lim_{h\to 0}$" in the definition of derivative.

Limits are one of the most difficult notions in mathematics to make precise. It took the best minds in mathematics about two-hundred years (after Newton and Leibniz founded the calculus) to formulate them correctly. They throw many a math major for a loop.

Fortunately, it is possible to understand them intuitively, which suffices for a first taste of calculus. So, we will not develop the precise treatment for this class (you can take MA 315 if you are interested).

2.1. Limits at infinity.

Often in physical and social sciences one is interested in long-term behavior. The limit of a function as a variable tends to infinity captures this notion within mathematics.

Definition 4. *(Intuitive) We say the limit of $f(x)$ as x tends to (positive) infinity is L if the values $f(x)$ are always arbitrarily close to L once x is large enough. Notationally, we say $\lim_{x\to\infty} f(x) = L$. If there is no L for which this is true, we say that the limit does not exist.*

The phrase "arbitrarily close" indicates that we can arbitrarily choose a tolerance for how far $f(x)$ can be away from L, as long as x is large enough. In other words, if we wait long enough $f(x)$ will be as close as we want to L for all x after some point.

If we plot $f(x)$ on our calculator, we see that for large values of x it would look like a constant function, with value L.

Example 5.
- *Graphical examples.*
- $\lim_{x\to\infty}(2 - \frac{3}{x}) = 2$
- *If* $f(x) = -5$ *for all* x, $\lim_{x\to\infty} f(x) = -5$.
- $\lim_{x\to\infty} x^2$ *does not exist as a finite number. We say it is* $+\infty$.
- *The population of Absurdia has been measured in year* t *by* $P(t) = \frac{3t^2}{2t^2+4} \times 10^6$. *In the long term its expected population is* 1.5 *million.*
- *The limit of* $W(t)$, *which measures Oprah Winfrey's weight at time* t, *does not exist.*

2.2. **Limits at finite points.**

Definition 6. *(Intuitive) We say the limit of* $f(x)$ *as* x *tends to* c *is* L *if the values* $f(x)$ *are always arbitrarily close to* L *once* x *is close enough to* c. *Notationally, we say* $\lim_{x\to c} f(x) = L$. *If there is no* L *for which this is true, we say that the limit does not exist.*

Limits at finite points are conceptually trickier. In order to get the logic straight sometimes one has to pretend that the function does not exist at that point and use the values at nearby points to, if possible, come up with a single possible value which fits. This can be clearly understood graphically.

Example 7.
- *Some graphical examples.*
- $\lim_{x\to 3}(x^2 + 1) = 10$.
- $\lim_{x\to 5} \frac{x+3}{5-x}$ *does not exist.*
- $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$. *(Graph it!)*
- $\lim_{x\to 0} \frac{x}{|x|}$ *does not exist, which is easier to see when this function is rewritten.*
- $\lim_{h\to 0} \frac{3h+h^2}{h} = 3$.

There are many rules, highlighted in the book, which make it easier to take limits, many of which we have already used in these examples. For example "the limit of the sum is the sum of the limits".