MATH 241, LECTURE 14

0.1. **Practice with the quotient rule.** Remember our mnemonic, dwarf-friendly form of the quotient rule as we apply it a few times.

Example 1. *Take the derivatives of:*

- $\frac{1}{x-2}$.
- $\frac{\sqrt{x}}{x^2+2x-4}$.
- $\frac{x-1}{x+1}$

Example 2. *There are* $35-t$ *cases of a flu after* t *months in a town of* $5000+t^2$ *people. What is the percentage of people with flu? What is the derivative of this percentage?*

The derivative of percentage should not be confused with the notion of *percentage rate of change*. If $f(x)$ is measuring a quantity, the percentage rate of change of that quantity is $\frac{f'(x)}{f(x)} \times 100\%$. In practical terms, the derivative of percentage can be volatile, while the percent rate change is more of a "big picture" number.

1. Composite functions

The rule we will use most frequently in computing derivatives will be the *chain rule*, which will allow us to compute derivatives of complicated functions from simple functions which make them up. Complicated functions made by repeated applications of simple ones are known as composite functions.

Composite functions are everywhere. When we study the function $\sqrt{x+1}$, we naturally think of adding x and 1 first and then taking the square root of that. Without knowing it, we are breaking down a function as a composite. Understanding composite functions can be tricky for many reasons:

- Composing two functions is so natural that you've done it without thinking about it.
- It makes no sense to take the composite of two numbers so unlike addition of functions we cannot rely on previous intuition.
- While composition is simple conceptually, the ways to express it can be confusing.
- In some important ways, composition behaves quite differently from addition and multiplication of functions.

Definition 3. *The composite of the functions* $f(x)$ *and* $g(x)$ *is the function whose value at* x *is* $f(g(x))$.

Informally, one is plugging g into f.

Example 4.

- $f(x)=x^2$, $g(x)=x+1$.
- $f(x)=x+1$, $g(x)=x^2$.
- *The taxes you pay are a function of your salary. Your salary can be a function of variables such as seniority. Therefore, one can view the taxes you pay as a function of variables such as seniority.*
- *The temperature in an oven as it warms up is* $T(t)=75+30t$. *The density of air changes with temperature according to* $\rho=\frac{289}{\frac{T}{2}+257}$. *Therefore, one can determine the density of air in the oven as a function of time as it is warming up.*
- *If* $f(x)=x^2-x$ *what is* $f(x+h)$*?*

Question 5. *What happens when you compose any* $f(x)$ *with* $i(x)=x$*?*

The function $i(x) = x$ behaves for composition of functions like 0 does for addition of numbers and 1 does for multiplication of numbers. We sometimes call it the identity function.

Definition 6. *Two functions $f(x)$ and $g(x)$ are inverse to one another if $f(g(x))$ and $g(f(x))$ are both the function $i(x) = x$.*

Informally, functions are inverse if they "undo each other". You have seen inverse functions before.

Sometimes composition of functions is denoted $f \circ g$, which makes it look even more akin to multiplication.

2. The chain rule

We get the most flexibility in making new functions from old by composing them. The chain rule governs the derivative of the composite of functions. We will also see that it has to do with decomposing a rate of change as a product of more basic rates. As with the product rule, the first guess one would make as to what the rule should be is not correct.

Example 7. *Given that the derivative of u^2 is $2u$, we would think that the derivative of $(3x+2)^2$ should be $2(3x+2) = 6x+4$. But if we expand $(3x+2)^2$ we get $9x^2 + 12x + 4$, whose derivative is $18x + 12$, exactly three times what our guess was. Where does that three come from?*

Conceptually, it helps to think of a practical example when one wants to understand the rate of change of a composite quantity.

Suppose a car gets 25 miles per gallon, or equivalently 0.04 gallons per mile. Suppose that we are driving along at 60 miles per hour. How much gallons of gas are burned per hour?

$$\left(0.04\frac{\text{gallons}}{\text{mile}}\right) \times \left(60\frac{\text{miles}}{\text{hour}}\right) = 2.4\frac{\text{gallons}}{\text{hour}}$$

Moral of the story: to find a composite rate of change you need to multiply the constituent rates of change.

Theorem 8 (Chain Rule)**.** *The derivative of a composition of two functions is given by*

$$(f(g(x)))' = f'(g(x)) \times g'(x).$$

Example 9. *Find the derivatives of*

- $(3x^2 - 4)^2$
- $(-2x + \sqrt{5})^{100}$
- $(\frac{3x+5}{3x-5})^3$