MATH 241, LECTURE 15

1. Lots of practice with the chain rule

Just as synonymous words can be more or less useful in different contexts, different mathematical notation for the same concept can be helpful in different situations. (For example, have you ever tried to multiply large numbers using Roman numerals?)

We have seen a few different ways to name the derivative, $f'(x)$, f', $\frac{d}{dx}f$, and now closely related to the last if we say $y = f(x)$ then a name for the derivative is $\frac{dy}{dx}$.

Now suppose y is a function of u, and in turn u is a function of x, so that in composite y is a function of x.

The chain rule states that

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

The benefit of this formulation is that it can be easier to remember because it seems like it should be true by some sort of cancellation (and in fact cancellation is an important part of its proof).

The confusing part of this formulation is that the roles of functions and variables seems mixed up, especially for u. Ultimately, this mixing of roles is beneficial.

Example 1. *If $y = u^2 - \sqrt{u}$ and $u = \frac{1}{x+1}$, express y as a function of x and find $\frac{dy}{dx}$.*
Repeat the exercise if the dependence of y on u does not change, but $u = \frac{1}{\sqrt{x}+1}$.

Example 2. *Differentiate the functions*

- $\sqrt{\frac{x-1}{2x+5}}$
- $\frac{375}{(x^3-27\sqrt{x}+\frac{3}{x^2})^3}$
- $(x-2)^4 + (3\sqrt{x}-2)^3$.
- $(\frac{3x+5}{3x-5})^3$

1.1. The general power rule.

If we combine the rule $\frac{d}{dx}[x^n] = nx^{n-1}$ with the chain rule, we find a time-saving rule which codifies how we have been taking derivatives of functions of the form $[h(x)]^n$.

Theorem 3. *The derivative of $(h(x))^n$ is*

$$n\,(h(x))^{n-1}\,h'(x).$$

Example 4. *Find the derivatives of $(3x^2-4)^2$, $f(x) = \sqrt{x^3-2x+577}$, and $(-2x+\sqrt{5})^{100}$.*

2. The derivative of exponential functions

Looking at the definition of the derivative of a^x, we have

$$\lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = \lim_{h\to 0} \frac{a^x[x^h-1]}{h}.$$

But notice that a^x has no dependence on h so we can pull it out of the limit and get $a^x \lim_{h\to 0} \frac{a^h-1}{h}$. This last limit is going to be some constant, independent of x. It turns out to be equal to $\ln(a)$!

Theorem 5. *The derivative of $a^x = \ln(a)a^x$. In particular $\frac{d}{dx}e^x = e^x$.*

It is remarkable that the derivative of e^x is itself. This is a special property of exponentiation and the number e.

Example 6. *Find the derivatives of*

- 10^x
- e^{x^2}
- $\frac{2^x}{x}$

3. The derivative of the natural logarithm function

Recall from the properties of logarithm functions that $\log_a(x) = \log_a e \times \ln x$. We differentiate $\ln x$, since all other logarithm functions differ from it by a constant.

Theorem 7. *The derivative of* $\ln(x)$ *is* $\frac{1}{x}$.

We will learn why this is once we learn about *implicit differentiation*.

Note that this fills in a spot which has been missing on the list of derivatives. In general the derivative of $\frac{1}{n+1}x^{n+1} = x^n$. But this does not work for $n = -1$. But in the function $\ln(x)$ has derivative x^{-1}.

Example 8. *Find the derivative of the function* $f(x) = x\ln(x)$.