MATH 241, LECTURE 6

1. Logarithmic functions

Many common functions arise through “undoing” basic functions (more formally, we say they are inverses of basic functions). For example, subtraction was invented as the inverse of addition, division as the inverse of multiplication, and the square root as the inverse of the squaring function.

Definition 1. *The inverse of the exponential function a^x is called the logarithm function (with a base of a) denoted $\log_a(x)$. By this definition, $\log_a a^x = x$ and $a^{\log_a x} = x$. The logarithm with a base of e is called the natural log function and is denoted $\ln(x)$.*

Example 2.
- $\log_{10} 10000 = 6$ *because* $10^6 = 100000$.
- $\log_2 \frac{1}{8} = -3$ *because* $2^{-3} = \frac{1}{8}$.
- $\ln \sqrt{e} = \frac{1}{2}$ *because* $e^{\frac{1}{2}} = \sqrt{e}$.

The properties of the logarithm function follow from those of the exponential function.

- $\log_a(xy) = \log_a x + \log_a y$ because $a^{x+y} = a^x a^y$.
- $\log_a x^y = y \log_a x$ because $(a^x)^y = a^{xy}$.
- $\log_b x = \log_b a \cdot \log_a x$ follows from the previous property.

In other words, logarithms turn multiplication to addition and turn exponentiation to multiplication. Also note that the last property says that logarithms with different bases are related by multiplication by a constant.

Logarithms are handy in problems involving exponentials.

Example 3. *A radioactive material decays at a rate of 0.5% per year. How long would it take for half of the material to decay?*

Example 4. *How long would it take for \$1000 invested at 7% to become \$1500? How long would it take the balance to double?*

2. Growth and graphs of exponentials and logarithms

Exponential functions grow (or decay) very quickly. Logarithmic functions grow extremely slowly. For example, we may look at how fast each of these functions “go from 1 to 100:” x; x^2; 3^x; $\log_{10} x$.

Technology permitting, we may elaborate on this by looking at graphs of these and other functions.

3. Average rate of change

A fundamental philosophical truth is that everything changes. In physics, the change in position is known as velocity or speed. In economics, the change in price is known as inflation. In business, the change in costs is sometimes known as trend. In mathematics, the change in values of a function is known as the derivative. But to understand the derivative, which will measure “instantaneous” change, you need to to first be comfortable with “average” change over some intervals.

3.1. **Calculating average and instantaneous speed.** If you travel 200 miles in four hours, what is your average speed? What about 75 miles in one and a half hours?

$$\text{Average speed} = \frac{\text{distance travelled}}{\text{total time}}$$

Questions to ponder: Do these calculations necessarily mean you went 50mph for four hours? How does your speedometer calculate how fast you are going at the moment?

Next, what if, instead of giving you total distance travelled, you need to calculate from a position function which describes where you are?

Example 5. *A ball which is dropped from the top of the Tower of Pisa has travelled down $16t^2$ feet after t seconds. What is its average speed over the first three seconds? over the first five seconds? between the second and fifth second?*

Question 6. *How could we calculate exactly how fast the ball is going after two seconds?*

3.2. **Calculating slopes of secant lines to a curve.** Next we look at what at first appears to be unrelated to dropping a ball.

Definition 7. *A secant line goes through two points on the graph of the function. In symbols, it is a line through $(a, f(a))$ and $(b, f(b))$ for some a and b.*

Example 8. *Find the secant lines to the graph of $f(x) = 16x^2$ through the points with: $a = 0, b = 3$, $a = 0, b = 5$, $a = 2, b = 5$.*

What do you notice about this and the previous problem?

3.3. **General average rate of change.**

Definition 9. *In general, the average rate of change of some function $f(x)$ as x varies between values a and b is*

$$\frac{f(b) - f(a)}{b - a}.$$

This can be computed in any way that f is presented, through a formula, through a graph, or in a table.

Example 10. *Analyze different measured and predicted rates of change for world population according to:* `http://www.unfpa.org/6billion/pages/worldpopgrowth.htm`