MATH 241, LECTURE 20

1. Relative extrema

We see that if a function changes from increasing to decreasing at a stationary or singular point, that function obtains some kind of maximum (there is a similar statement of minima, which you should think about). We now formalize this.

Definition 1. *Let $P = (c, f(c))$ be a critical point of a function $f(x)$.*

- *P is a relative maximum if $f'(x) > 0$ for nearby $x < c$ and $f'(x) < 0$ for nearby $x > c$.*
- *P is a relative minimum if $f'(x) < 0$ for nearby $x < c$ and $f'(x) > 0$ for nearby $x > c$.*
- *P is not a relative extremum if $f'(x)$ has the same sign on both sides of c.*

If we are looking for largest or smallest values of a function, it often depends on where the function is defined. In particular, a relative maximum or minimum can also occur on an endpoint of the domain. For example, you were shortest at the moment you were born, so time $t = 0$ is where your height function has a minimum.

We now include endpoints of the domain whenever we refer to critical points (because it is critical to check for maxima and minima at those points!).

Definition 2. *If f is defined only over an interval from a to b, we say that f has a relative minimum at a if $f'(x)$ is positive for values of x close to a. We say that ... (you fill in the rest!).*

Example 3. *Classify the critical points (as relative maxima, relative minima, or neither) of $\frac{x^2-3}{e^x}$ over the interval from -4 to 4. Check the answer with the graph.*

Example 4. *Classify the critical points of the function $\ln(|x^2 - 2| + 1)$ over the interval from $-e$ to e.*

2. Optimization

Optimization, that is finding where a function takes on its largest or smallest values, is the most common application of calculus. The big theorem which we will use repeatedly is:

Theorem 5 (First Optimization Theorem)**.** *A continuous function defined everywhere on some interval I obtains its absolute maxima and minima either at the end(s) of I or at some critical number(s) in I.*

Let's do an example and then come back and formalize.

Example 6. *Maximize profit when the price at which q units can be sold is $p(q) = 25 - q$ and it costs \$10 to produce each unit.*

What we did was:

- Found all stationary points (by finding the derivative and setting it to zero).
- Found all singular points.
- Found the endpoints (these are often given).
- Compared values at all of these points - the largest is the max and the smallest is the min.

Example 7. *Find the maximum and minimum values of the function $\frac{x^2+3}{x+1}$ as x can take values from 0 to 5.*

3. Applied optimization problems

In applied problems one usually has to work in order to find a single function to maximize. One needs to keep track of relationships and constraints.

Example 8. *Suppose advertising costs \$1000 per unit (say for magazine adds), and product development costs \$20000 per unit. Suppose that the profits generated from x units of advertising and y units of product development are xy^2 thousands of dollars. If a company has \$10000 to spend on advertising and product development together, how should the money be allocated in order to maximize profits?*

Example 9. *Suppose the top of a can is made of a metal which costs 10 cents per square centimeter and the sides are made of a metal which costs 12 cents per square centimeter. What is the largest volume can which can be made from two dollars of material?*

We will formalize our techniques next time.