MATH 241, LECTURE 13

Next we continue to learn rules for differentiation. These are basic skills we must memorize in order to apply the derivative in different settings.

1. The product rule

We have already seen that the derivative of the sum of two functions is the sum of their derivatives. So what should the derivative of a product of two functions be? The product of their derivatives, right? WRONG.

Example 1. *If $f(x) = x$ and $g(x) = x$ then $\frac{d}{dx}f(x) = 1 = \frac{d}{dx}g(x)$. So the product of the derivatives is $1 \cdot 1 = 1$. But the derivative of the product is the derivative of $x \cdot x = x^2$, which is $2x$, a function very different from the function 1.*

So what is the rule for the product of the derivative?

Theorem 2.

$$\frac{d}{dx}\left(f(x)g(x)\right) = \frac{d}{dx}[f(x)] \cdot g(x) + f(x) \cdot \frac{d}{dx}[g(x)].$$

Example 3.

- *Apply the theorem with $f(x) = x$, $g(x) = x$.*
- *Apply the theorem with $f(x) = x^2$, $g(x) = (x - 5)$, and check the result by multiplying the two and expanding and then taking the derivative. Find all places where the tangent line to this curve is horizontal.*
- *Find the tangent line to the curve $y = (2\sqrt{x} - 1)(x + 4)$ at $x = 1$.*

Where did the product rule come from? It helps to think geometrically, with $f(x)$ and $g(x)$ measuring lengths of a rectangle, and their product $f(x) \cdot g(x)$ measuring area.

Or, if you like to think in terms of your marginal analysis, suppose that you currently sell 100 items at \$20 each per day, but demand has been good so you plan to make 105 items and sell them at \$22 each. What are the changes in production level, price, and profit, and how are they related?

The at first surprising nature of the product rule has confused many bright people in many settings, including Leibniz who was one of the inventors of calculus.

1.1. Establishing the product rule.

We give a careful justification of the product rule, The ingredients needed to establish that the product rule is valid are all at our fingertips: the definition of derivative, rules about limits of sums and products, and some (clever) algebraic manipulations. All of the steps, except on inspired piece of algebra, are basically common sense.

2. The quotient rule

The quotient rule is even more difficult to remember, since it is further from being what you would expect.

Theorem 4.

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{\frac{d}{dx}[f(x)]g(x) - f(x)\frac{d}{dx}[g(x)]}{(g(x))^2}$$

The seven dwarves had a much simpler way to remember the quotient rule.

Example 5. *Find the derivative of $\frac{\sqrt{x}+5}{3x^2-5}$.*

Example 6. *Find the tangent line to the curve* $y = \frac{x-1}{x+1}$ *when* $x = 1$*. Sketch the curve and the tangent line.*

Example 7. *There are* $35 - t$ *cases of a flu after* t *months in a town of* $5000 + t^2$ *people. What is the percentage of people with flu? What is the derivative of this percentage?*

Note that there is a big difference between the derivative of the percentage and what you get by "turning the derivative into a percentage" by taking $\frac{f'(x)}{f(x)}$. This latter quantity is called the percent rate of change.