MATH 241, LECTURE 18

1. Implicit differentiation

Here are the basic steps to use implicit differentiation to find the derivative of y with respect to x if the two are related by an equation but not a function.

- Differentiate the equation, treating y as a function of x even though it is not expressed as one. Remember to use the chain rule to differentiate y, since we are treating it as a function of x!
- The resulting expression has $\frac{dy}{dx}$ in it, since we have used the chain rule on y. Now do algebra to solve for it.
- Now that we know a formula for $\frac{dy}{dx}$, we can plug in values for x *and* y to find a numerical value for the derivative.

Example 1. *Find $\frac{dy}{dx}$ when $e^{xy+4x-3} = 5x^2$. (Do this twice, the second time by taking the natural logarithm of both sides).*

Example 2. *Two manufacturers of widgets are in direct competition. Because of the many variables in pricing and publicity, the number of widgets they sell does not add up to a constant, but $6x^2 + xy + 5y^2 = 120,000$, where x is the number of widgets sold per day by the first company and y by the second. If both companies are currently selling 100 widgets, what would be the effect on the first company if the second is increase sales by ten widgets per day.*

Example 3. *The Cobb-Douglas model of production uses functions of the form $P = x^a y^{1-a}$ to model production level P, where x represents an amount of capital and y represents an amount labor. Why does this make sense? Suppose widget production level P is modeled by $x^{\frac{1}{4}} y^{\frac{3}{4}}$, where P is measured in thousands of units, x in millions of dollars, and y in hundreds of workers. If currently x is 16 and y is 81, what would $\frac{dP}{dt}$ be if there is currently a move to capitalize further at a rate of \$2 million per quarter and increase workers at a rate of 300 per quarter. What is the expected increase in production per quarter?*

Example 4. *When the price of gadgets is p dollars each, the manufacturer is willing to supply x hundred units where $x^2 - 2\frac{x}{5+p} - p^2 = 20$. How fast is the supply changing when the price is \$5 and increasing at the rate of 20 cents per week?*

1.1. Understanding the derivative of the logarithm function.

Now that we have implicit differentiation, we can use the fact that $\frac{d}{dx}e^x = e^x$ to show that

Theorem 5. *The derivative of $\ln(x)$ with respect to x is $\frac{1}{x}$.*

Proof: Start with the fact that $e^{\ln x} = x$, and differentiate both sides. Remember that we don't really know yet what the derivative of $\ln x$ is. To take the derivative of the left side, we must use the chain rule; the derivative of x is straightforward:

$$\frac{d}{dx} e^{\ln x} = e^{\ln x} \frac{d}{dx} \ln x = x \frac{d}{dx} \ln x,$$

while $\frac{d}{dx} x = 1$. So we have $x \frac{d}{dx} \ln x = 1$, so $\frac{d}{dx} \ln x = \frac{1}{x}$.

2. Critical points

Our main application of derivatives will be using them to find where a function is biggest or smallest. This process is called optimization/maximization/minimization. The usefulness is clear - we'd all like to maximize our income and minimize our expenses (if only this were as easy in our personal lives as we will find doing it through equations).

Back when we were determining where a function increased or decreased, we saw that the first step was finding places where $f'(x) = 0$. These were places where, for example, a function's value stopped growing and started falling (which means that they "peaked"). This condition is useful in many contexts, so we give it a name.

Definition 6. *A number x in the domain of a function $f(x)$ is a stationary number if $f'(x) = 0$ and is a singular number if the derivative at x does not exist. If x is a stationary number for $f(x)$, the point $(x, f(x))$ is a stationary point on the graph of $f(x)$. (And similarly for singular points)...*

The derivative does not exist when the limit in the algebraic definition of the derivative does not exist.

We sometimes refer to both stationary points and singular points together as critical points.

Example 7. *Find all the stationary points for the function $f(x) = x^3 - \frac{9}{2}x^2 + 2x - 5$. Find all the singular points of the function $g(x) = |5 + 4x - x^2|$. Graph these functions and say what you see at critical points.*