MATH 241, LECTURE 4

1. Quadratic functions

After linear functions, the next simplest kinds of functions are quadratic functions. We will review basic properties of these functions now, so they are familiar when we apply calculus to them later.

Theorem 1. *A parabola intersects the y-axis at the point $(0, c)$. A parabola intersects the x-axis at points given by the quadratic equation. A parabola reaches its maximum or minimum when $x = \frac{-b}{2a}$.*

We can easily verify the first two statements. The third statement can also be verified through the algebra of "completing the square", but will be very easy to see with a little calculus.

Example 2. *Graph the function $f(x) = -\frac{1}{2}x^2 + 3x - 4$, indicating the places where the graph crosses the axes, as well as the minimum or maximum.*

Example 3. *Suppose the total profit made by manufacturing x thousands of units of Freshies doggie biscuits is given by $P(x) = -0.4x^2 + 80x - 20$ (measured in tens of thousands of dollars).*

Explain why this is a reasonably-shaped profit function. For example, $P(0) = -20$; does that make sense? Find the optimum number of Freshies to produce.

2. Exponential functions

Our next goal is to increase our toolbox of functions beyond polynomials, etc. Our first new functions will be the exponential functions. These functions arise in population modeling, finance, the study of heat transfer and rates of reaction in physics, chemistry and biology.

Exponential functions arise by taking one number to the power of another, just as polynomial functions like x^3 do. For exponential functions, the variable occurs as the power, not as the base.

It is an interesting story to see how the exponential function is defined for different kinds of numbers; its careful development required many of the fundamental ideas in calculus.

We develop the function 4^x, to be concrete and to be clear that for exponential functions, the variable appears in the power.

Definition 4.
- $4^0 = 1$.
- *If x is a positive integer, the value of 4^x is $4 \times 4 \times \cdots \times 4$, where 4 is multiplied by itself x times. For example $4^3 = 4 \times 4 \times 4 = 64$.*
- *The two laws of exponents state that $4^x 4^y = 4^{x+y}$ and $(4^x)^y = 4^{xy}$. For example $4^2 \times 4^3 = 16 \times 64 = 1024 = 4^5 = 4^{2+3}$ and $(4^2)^2 = 16^2 = 256 = 4^4 = 4^{2\times 2}$. These laws of exponents determine what 4^x must be for other kinds of x.*
- *If x is a positive integer, $4^{-x} = \frac{1}{4^x}$. For example $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$. This rule must hold since $4^x \times 4^{-x} = 4^{x+(-x)} = 4^0 = 1$ by the law of exponents, so that 4^x and 4^{-x} are reciprocals.*
- *If x is a positive integer $4^{\frac{1}{x}} = \sqrt[x]{4}$. For example, $4^{\frac{1}{2}} = \sqrt{4} = 2$. This rule must hold because $(4^{\frac{1}{x}})^x = 4^{\frac{1}{x}x} = 4^1 = 4$. (Interesting question: why must there be a $\sqrt[3]{4}$?)*
- *If x is a fraction $x = \frac{p}{q}$ then $4^x = (4^{\frac{1}{q}})^p$, as must be the case immediately from the laws of exponents. For example $4^{\frac{-3}{2}} = (4^{\frac{1}{2}})^{-3} = \sqrt{4}^{-3} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$.*

- *Finally, note that any real number x is approximated arbitrarily closely by rational numbers $\frac{p}{q}$. If x is an arbitrary real number we define 4^x as the* limit *of $4^{\frac{p}{q}}$ as the fraction $\frac{p}{q}$ approaches x . For example 4^π is the limit of the sequence $4^3, 4^{\frac{31}{10}}, 4^{\frac{314}{100}}, \ldots$. We will develop the notion of limit more fully later.*

Fortunately, our calculators can compute any of these values for us.

Important fact: if $4^x = 4^y$ then $x = y$. In other words, the function 4^x doesn't repeat any values (in contrast with most other functions – for example, $x^2 + 2$ has the same value when $x = 2$ or $x = -2$.)

The graph of an exponential function always starts or ends close to the x-axis (why?) and then gets far away from the axis very quickly.

Example 5. *Sketch the graphs of the exponential functions $f(x) = -3 \cdot 2^x$ and $y = (\frac{2}{3})^x$.*

3. Compound interest and the exponential base e

- When making a loan to a good friend, you charge no interest. (If you loan P, you get back P).
- A personal loan between strangers often has a one-time interest charge. If P dollars is loaned, $P + rP$ dollars is paid back, where r is the interest rate. For example, \$100 loaned with a one-time 7% rate is repaid as $100 + 0.07 \times 100 = 107$ dollars. It will be helpful to rewrite $P + rP$ as $P(1 + r)$.
- Some loans/investments accrue interest yearly. In these cases, one is paying/earning interest on top of interest. For example, after two years of 7% interest, \$100 becomes $[100 \times (1.07)] \times (1.07) = 114.49$ dollars. (in this case, the 49 cents is the interest on top of interest). In general, if for a principal of P, the value after n years will be $P(1 + r)^n$. Note that interest on top of interest becomes much more significant as n gets larger.
- In order to be more sensitive to when transactions are made, it makes sense to compound interest more frequently. To compound things monthly, we would charge an interest of $\frac{r}{12}$ each month, resulting in a total return of $P(1 + \frac{r}{12})^n$, where n is the number of months. Note that the total for $n = 12$, which is a year, will be greater than if an interest of r percent is changed once. To compound interest every day would result in a total of $P(1 + \frac{r}{365})^n$, after n days, for $P(1 + \frac{r}{365})^{365}$ after a year.
- We could even compoud every hour, every minute, etc. The general formula is that after a year one has $P(1 + \frac{r}{N})^N$, if one compounds N times. For a fraction q of a year (such as $\frac{1}{2}$ for six monts), the formula is $P(1 + \frac{r}{N})^{qN}$. For example, if one compounds 5% yearly interest every day, then after 6 months \$100 becomes $100(1 + \frac{0.05}{365})^{182} = \$102.53$
- Theoretically, we can compound *continuously* by taking the limit as N goes to ∞ in the formula above. For example, suppose we invest \$1 with yearly interest of 100%, so that $r = 1$. How much would there be if the interest is compounded continuously?

N	1	2	10	100	1000	100000
$(1 + \frac{1}{N})^N$	2	2.25	2.594	2.705	2.71815	2.71827

The answer to this question is the magic number e, whose value is approximately 2.71828.

- In general, if P dollars are invested at an annual rate of $r \times 100$ percent, then the balance $B(t)$ after t years is Pe^{rt} dollars.

The number e is called the natural base for exponentiation, since it occurs raised to a power in problems from almost every quantitative subject.

Example 6. *What are you paying out to the credit card company if you have a* \$1000 *balance at a* 19% *APR compounded continuously, which you finally pay after two years?*

Example 7. *Find the value of* \$325,000 *after* 1 *year with* 7% *interest compounded yearly, monthly, daily and continuously. (Ans: 347750; 348494.28; 348562.82; 348565.16)*