MATH 241, LECTURE 8

1. The derivative

1.1. Slopes of tangent lines.

Definition 1. *A tangent line to a curve is a line which intersects the curve at some point, but does not cross the curve.*

Informally, a tangent line "kisses" the curve. The derivative at x which measures instantaneous change at x, is the slope of the tangent line at x.

We already know one point on the tangent line, namely the point on the curve which it is kissing. What we can do to find the slope of the tangent line is to look at slopes of secant lines for points close to the point of tangency.

Example 2. *Estimate the slope of the line which is tangent to $f(x) = 16x^2$ at $x = 2$? at $x = 5$?*

Example 3. *Calculate the equation of the line tangent to $f(x) = \frac{1}{x}$ when $x = 1$.*

1.2. The sign of the derivative.

Knowledge of the derivative sheds a lot of light on the original function. One of the most basic facts about the derivative, since it measures the rate of change of a function, is based on simply seeing whether the derivative is greater than or less than zero.

Theorem 4. *A function is increasing at x if its derivative there is positive. A function is decreasing at x if the derivative at x is negative.*

Example 5. *Find where the function $f(x) = 2x - x^2$ is increasing or decreasing. Check the answer against the graph of the function.*

The next example might require some audience participation.

Example 6. *The derivative game: given some graphs of derivative functions, sketch possible graphs for the original functions.*

1.3. Formula for the derivative - the algebraic viewpoint.

We would see similarities in trying to compute average and instantaneous differences regardless of what our functions are measuring. The following definitions work in all such cases, given an algebraic way to compute the derivative. First, for average rate of change, we re-write $a = x$ and $b = x + h$, so that the difference quotient becomes $\frac{f(x+h)-f(x)}{h}$. So for example if $a = 3$ and $b = 4$, we would instead think of $x = 3$ and $h = 1$. The reason we change from a and b to x and h is to be consistent with the derivative.

Definition 7. *The derivative of $f(x)$ with respect to x is the function*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We will talk about lim, which means the "limit", later. For now, we will adopt a working notion that we may compute this for smaller and smaller h; if all of those computations seem to approach a single number, for now we will call that the limit. We will often leave h in the equation "until the very end", at which point we "let it become very small - so small it is negligible".

Example 8. *What is the derivative of $f(x) = 16x^2$? What is the value of the derivative when $x = 2$? What does this mean for the velocity of a ball which is dropped from the Tower of Pisa? or the equation of the line tangent to $f(x)$ at $x = 2$?*

Example 9. *Suppose price index, measuring the aggregate price for a large cross-section of household goods measured in thousands of dollars, has values $2 - \frac{1}{1+x}$ over two years. What is the rate of price increase over two years? What is the "instantaneous rate of inflation" at six months, one year, and two years?*