MATH 241, LECTURE 12

1. Limits and their use for derivatives

Last time we focussed on intuitive methods for finding limits, and introduced a few algebraic methods. Let's begin by practicing our algebraic limit techniques.

Example 1. *Find the following limits:*

- $\lim_{h\to 0} \frac{h^3+h}{3h^2+2h}$.
- $\lim_{h\to 0} \frac{h^3+1}{3h^2+2h}$.
- $\lim_{x\to 3} 2^x$.
- $\lim_{x\to \infty} \frac{3x}{x^2+4}$.
- $\lim_{x\to \infty} \frac{5x^2+1}{x-3x^2}$.

For another example, we revisit the famous computation of the derivative of $f(x) = x^n$.

Theorem 2. *For any number n, the derivative of x^n is nx^{n-1}.*

We can now justify this theorem carefully. The key is to understand a general formula for $(x+h)^n$, which starts out $x^n + n \cdot x^{n-1} + \cdots$.

2. Marginal analysis

As we have mentioned, the derivative is commonly used in theoretical business management and economics. One primary application goes under the name of marginal analysis. Marginal refers to the idea of trying to understand the costs, benefits, etc of "one more."

The main idea mathematically is that since the derivative is $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ the derivative is approximately equal to the difference quotient when $h = 1$, if $f(x)$ is much larger than one (for example, if $f(x)$ is national debt).

In other words, sometimes we have the following approximate equality

$$f'(x) \simeq \frac{f(x+1) - f(x)}{1} = f(x+1) - f(x).$$

This is equivalent to the expression which we would get from approximation by increments, namely

$$f(x+1) \simeq f(x) + f'(x) \cdot 1.$$

This leads us to the following definitions.

Definition 3.
- *The marginal cost function $C'(x)$ is the derivative of the cost function. It approximates the cost of producing one more item after x items have already been produced.*
- *The marginal revenue function $R'(x)$ is the derivative of the revenue function. It approximates the revenue generated by producing one more item after x have already been produced.*
- *The marginal profit function $P'(x)$ is the derivative of the profit function. It approximates ...*

The relation between these functions is...

It may take some getting used to thinking of these quantities as functions of the number of goods produced, rather than say functions of time. But as we'll see in doing problems, these functions better allow you to make decisions about production.

Example 4. *If it costs approximately* $1000 - \frac{500}{x}$ *dollars to produce* x *CD's, what is the marginal cost of producing the* $20th$ *CD?*

Example 5. *Suppose the value of the marginal profit function for producing the* $1000th$ *jar of peanut butter was* -0.25*. Should you produce more or fewer jars of peanut butter? In general what does the sign of the marginal profit function tell you?*

Example 6. *It costs* $\frac{n^2}{4} + 50n + 100$ *cents to produce* n *rubber duckies. Moreover, all* n *duckies will be sold when the price is* $(200 - \frac{n}{2})$ *cents. Find the following:*

- *The marginal cost for producing the* nth *duckie.*
- *The marginal profit made by producing and selling the* nth *duckie.*
- *The values of marginal cost and profit when* $n = 10$*.*
- *The actual cost and profit for producing the tenth duckie.*

Moral of the story - the marginal cost, revenue and profit functions are only *approximations* to the amounts needed/earned for producing one more item. But they are good approximations and (because of the rules we learn for calculus) are easier to work with than the true values.