MATH 241, LECTURE 16

0.1. **Partial substitution in applications of the chain rule.** When we are using the chain rule to get a numerical answer, as opposed to a formula for the derivative, we can save a small amount of work by not substituting fully to get a closed expression. The "intermediate form" is good enough to plug in and get numerical answers.

Example 1. *A shoe maker estimates that the profit for selling shoes as a function of price (accounting for the market equilibrium) is* $-5+p-\frac{p^2}{40}$ *per shoe. [Check that this makes sense.] Because of a price war, the shoe maker estimates that the price will be* $22-\sqrt{t}$ *dollars over the next* t *months. How fast will his profits changing over in* 4 *and* 9 *months?*

1. The derivative of exponential functions

Looking at the definition of the derivative of a^x, we have

$$\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}=\lim_{h\to 0}\frac{a^x[x^h-1]}{h}.$$

But notice that a^x has no dependence on h so we can pull it out of the limit and get $a^x\lim_{h\to 0}\frac{a^h-1}{h}$. This last limit is going to be some constant, independent of x. It turns out to be equal to $\ln(a)$!

Theorem 2. *The derivative of* $a^x = \ln(a)a^x$. *In particular* $\frac{d}{dx}e^x = e^x$.

It is remarkable that the derivative of e^x is itself. This is a special property of exponentiation and the number e.

When we compute a derivative, we should see how our computation fits with the graph of the function. In the case of a^x, we see that as this function gets larger it grows faster (and thus gets even larger and grows even faster...).

Knowing just the derivative of e^x allows us to compute the derivatives of functions made from e^x and polynomials using the chain, product and quotient rules.

Example 3. *Find the derivatives of*

- e^{x^2}
- 10^x
- $\frac{e^x}{x}$
- $A(t) = Pe^{rt}$

The fact that the derivative of e^{rx} is r times itself makes it useful in modeling populations, investments, temperatures...

2. The derivative of the natural logarithm function

Recall from the properties of logarithm functions that $\log_a(x) = \log_a e \times \ln x$. We differentiate $\ln x$, since all other logarithm functions differ from it by a constant.

Theorem 4. *The derivative of* $\ln(x)$ *is* $\frac{1}{x}$.

We will establish this fact once we learn about *implicit differertiation*.

Note that this fills in a spot which has been missing on the list of derivatives. In general the derivative of $\frac{1}{n+1}x^{n+1} = x^n$. But this does not work for $n = -1$. But in the function $\ln(x)$ has derivative x^{-1}.

Again, we should check the behavior of the derivative with the behavior of the graph.

And again, once we combine with other rules, we can now go wild taking derivatives of complicated functions.

Example 5. *Find the derivatives of the functions*

- $f(x) = x\ln(x)$
- $g(x) = \ln(x^{\frac{7}{2}} + 3)$
- $u(t) = \ln x^2$
- $(\frac{e^x \ln x}{x^2+1})^9$