MATH 242, LECTURE 4

1. Integration by substitution

While we might not have thought about it, the chain rule was by far the rule we used most frequently in finding derivatives. In the world of integration, the cousin of the chain rule is the technique of "u-substitution" for integration. We make steps similar to the steps we took when applying the chain rule. To evaluate $\int f(x)dx$,

- First we identify a function traditionally called $u(x)$ (hence the name "u-substitution") which we use as a building block to make the integrand. In particular, we want to have some $g(u(x))$ appearing naturally as part of the integrand.
- Then, we take the derivative of $u(x)$, which is part of the chain rule.
- We substitute the variable u in for $u(x)$ (this is easy) and then - here's the rub! - try to match what's remaining with du (which is $\frac{du}{dx}dx$). In this step we often use the trick of multiplying by some number k inside the integral and then $\frac{1}{k}$ outside the integral.
- If the substitution meshes, we have successfully translated the integral $\int f(x)dx = \int g(u)du$.
- If we can integrate $\int g(u)du = G(u) + C$ then we can substitute $u(x)$ for u to find the original integral.

The only way to learn this technique is through plenty of practice. (It's a bit of an art).

Example 1. *Evaluate the following integrals:*

- $\int (x^4 + 2x - 1)^{-5}(2x^3 + 1)dx$
- $\int e^{x^3-1}x^2dx$
- $\int \frac{1}{x^2-4}xdx$
- $\int \frac{1}{x\ln x}dx$
- $\int \frac{x^2+2}{x+1}dx$
- $\int \frac{e^x}{e^x+1}dx$
- $\int e^{-x^2}dx$

1.1. **Tips and more practice.** A few tips to keep in mind while doing integration problems:

- It is crucial to substitute in du as well as u, and not have any expressions involving x left when the substitution is made.
- Most u substitutions will not result in a manageable integral; it takes some practice to know which substitutions have the greatest chance of success. (In fact, it can be helpful to see some "dead-end" substitutions.)
- In fact, not all indefinite integrals are computable explicitly.

Example 2. *Evaluate the following integrals:*

- $\int x^2\sqrt{x^3+5}dx$
- $\int \frac{e^x}{(e^x+5)^3}dx$
- $\int \frac{x}{x+1}dx$