MATH 242, LECTURE 3

0.1. **A first application of integration.**

Example 1. *Basic physics dictates that, neglecting air speed, the downward acceleration of an object dropped due to gravity is constant,* 32 *feet per second per second. Find a formula for the distance travelled by an object dropped (not thrown, so it starts with no speed) over* t *seconds. How long would it take an object to fall 5000 feet (for example dropping it over the edge of the Grand Canyon)?*

This example marks the beginning of the subject of *differential equations*, which is the study of how to solve equations involving derivatives. All of the laws of basic, classical physics are written in terms of differential equations.

1. Techniques of integration

So far, our method for finding anti-derivatives has been to guess, check (by taking the derivative of our guess) and modify until the answer is correct. It would be better to have a more systematic approach. Unfortunately, while differentiation requires only a few rules to take the derivatives of most familiar functions, anti-differentiation is a much more complicated game, involving many more rules (so many that at MIT they hold an "Integration Bee"). Moreover, there are many functions, such as e^{x^2}, which have no simple anti-derivative.

The first way to be systematic about integration is to find analogues of the first rules we used for differentiation. For example, if $F(x)$ is an anti-derivative for $f(x)$ - so that $\frac{d}{dx}F(x) = f(x)$, and similarly $\frac{d}{dx}G(x) = g(x)$, then $\frac{d}{dx}[F(x) + G(x)] = f(x) + g(x)$. So that $F(x) + G(x)$ is an anti-derivative for $f(x) + g(x)$. This is the first of the rules listed below. All of these rules may be checked by differentiating.

1.1. **First integration rules.**

- $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$.
- For any constant number c, $\int cf(x)dx = c \int f(x)dx$.
- $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$ if $n \neq -1$.
- $\int e^{kx}dx = \frac{1}{k}e^{kx} + C$.
- $\int \frac{1}{x}dx = \ln|x| + C$.

Looking at the first two rules, do you think there will be an easy rule involving $\int f(x) \cdot g(x)dx$?

Example 2. *Evaluate the following indefinite integrals:*

- $\int (2x^3 - 5x + 3)dx$.
- $\int e^{3x}dx$.
- $\int [\frac{5}{x} + \frac{3}{x^2}]dx$.

1.2. **Integration by substitution.** While we might not have thought about it, the chain rule was by far the rule we used most frequently in finding derivatives. In the world of integration, the cousin of the chain rule is the technique of "u-substitution" for integration. We start with a couple examples which we attempt before formalizing the process.

Example 3. *Evaluate* $\int (2x - 3)^{10} 20dx$.

Example 4. *Evaluate* $\int (x^2 - 1)^5 (10x)dx$ *and* $\int x(x^2 - 1)^5 dx$

We make this systematic through steps similar to the steps we took when applying the chain rule. To evaluate $\int f(x)dx$,

- First we identify a function traditionally called $u(x)$ (hence the name "u-substitution") which we use as a building block to make the integrand. In particular, we want to have some $g(u(x))$ appearing naturally as part of the integrand.
- Then, we take the derivative of $u(x)$, which is part of the chain rule.
- We substitute the variable u in for $u(x)$ (this is easy) and then - here's the rub! - try to match what's remaining with du (which is $\frac{du}{dx}dx$). In this step we often use the trick of multiplying by some number k inside the integral and then $\frac{1}{k}$ outside the integral.
- If the substitution meshes, we have successfully translated the integral $\int f(x)dx = \int g(u)du$.
- If we can integrate $\int g(u)du = G(u) + C$ then we can substitute $u(x)$ for u to find the original integral.

The only way to learn this technique is through plenty of practice. (It's a bit of an art).

Example 5. *Evaluate the following integrals:*

- $\int (x^4 + 2x - 1)^{-5}(2x^3 + 1)dx$
- $\int e^{x^3-1}x^2dx$
- $\int \frac{1}{x^2-4}xdx$
- $\int \frac{1}{x\ln x}dx$
- $\int \frac{x^2+2}{x+1}dx$
- $\int x^2\sqrt{x^3+5}dx$
- $\int \frac{e^x}{e^x+1}dx$
- $\int e^{-x^2}dx$