# MATH 242, LECTURE 19

## 1. Partial derivatives

**Definition 1.** *If $f(x, y)$ is a function of two variables, its partial derivative with respect to $x$, denoted either $\frac{\partial f}{\partial x}$ or $f_x(x, y)$, is the function obtained by treating $y$ as a constant and differentiating with respect to $x$. Similarly, the partial derivative with respect to $y$, denoted $\frac{\partial f}{\partial y}$ or $f_y$, is obtained by treating $x$ as a constant and differentiating with respect to $y$.*

So, the game is to treat one variable as a constant while differentiating with respect to the other variable.

**Example 2.** *Find the partial derivatives of the following functions:*

- $f(x, y) = x^3y^2 - 7xy^3 + \sqrt{5}x^2 + 5.$
- $f(x, y) = x^2e^{\sqrt{xy}}$

### 1.1. Geometric and practical interpretations.

Recall that one of the first and most important interpretations of the derivative was that it was the slope of the tangent line to a curve. There is a similar first interpretation of the partial derivative. As we illustrate on the overhead, the partial derivative with respect to $x$ is the slope, within the plane where $y$ is fixed at some $c$ and $x$ and $z$ are allowed to vary, of the *tangent line to the graph of the function*.

Thus, practically speaking, if you were walking along some surface which you could then think of as the graph of a two-variable function, then the partial derivatives tell you about how steep your climb or descent will be if you walk parallel to the $x$ or $y$ axes (due north-south or east-west). You might wonder how you could find out about the steepness of the climb or descent if you travel northeast - that's the topic of the *gradient* of a function.

### 1.2. Higher-order partial derivatives.

As in the case of a single variable, we are free to take a derivative of a derivative. The notation works as follows:

**Definition 3.** *The partial derivative with respect to $x$ of the partial with respect to $x$ is $f_{xx}$ or $\frac{\partial^2 f}{\partial x^2}$.*
*The partial derivative with respect to $y$ of the partial with respect to $x$ is $f_{xy}$ or $\frac{\partial^2 f}{\partial y \partial x}$.*
*The partial derivative with respect to $x$ of the partial with respect to $y$ is $f_{yx}$ or $\frac{\partial^2 f}{\partial x \partial y}$.*
*The partial derivative with respect to $y$ of the partial with respect to $y$ is $f_{yy}$ or $\frac{\partial^2 f}{\partial y^2}$.*

**Example 4.** *Find all four second-order partial derivatives of $f(x, y) = x^2y^3 + e^{xy}$ and $f(x, y) = \ln(x + y)$*

### 1.3. The gradient.

Because we have talked about vectors, we can talk about the construction which assembles partial derivatives into what might be called a "total derivative", namely the gradient vector. The book does not cover this topic, and it will not appear on homework, quizzes and the main part of the last exam, but....

**Definition 5.** *The gradient of a function $f(x, y)$ is the vector $\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$.*

**Example 6.** *Find the gradient of the function $f(x, y) = x^2 - y^2$ and its value at the point $(1, 2)$.*

Using the dot product, we can use the gradient to find the rate of change of a function in any direction.

**Theorem 7.** *The rate of change of a two-variable function at a point in a given direction is the dot product of the gradient vector at that point with a* unit vector *in that direction.*

**Example 8.** *Find the rate of change of the function* $x^2 + y^2$ *at the point* $(1,1)$ *in the direction of the unit vectors* $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ *and* $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.