MATH 242, LECTURE 13

1. Outcomes of linear systems and larger systems of equations

When we talked about linear systems in two variables, it seemed like they always had a single solution. This isn't always the case.

Example 1. *Solve the system of equations*

$$\begin{aligned} x + 2y &= 4 \\ 3x + 6y &= 8. \end{aligned}$$

What happens if in the last equation 8 *is replaced by* 9*?*

These results can be explained by geometry, as we illustrate with this example. The set of points which satisfies one linear equation in two variables is a line. And we saw last time that an intersection point corresponds to a solution. If there are no intersection points or many that corresponds to no solutions or many.

We can use similar techniques to understand larger systems of equations.

Example 2. *Analyze the system of equations*

$$\begin{aligned} x + 2y &= 3 \\ 2x - y &= 5 \\ 3x + y &= 8 \end{aligned}$$

and the system one obtains by replacing 8 *in the last equation by* 9.

2. The basics of matrices

Matrices are a useful piece of notation whenever arrays of numbers occur in a problem, especially linear problems.

Definition 3. *A matrix is a collection of numbers arranged (indexed) in a rectangular array.*

Matrices are usually represented within brackets, like $\begin{bmatrix} 1 & 5 \\ -2 & \frac{1}{2}. \end{bmatrix}$ To specify a number we specify its *row* and *column*. For example, in this matrix the number 5 is in the first row, second column or in the 1,2 position. We call a matrix an n by m matrix if it has n rows and m columns. The example above is a two by two matrix.

We may use matrices to represent and efficiently solve systems of equations by putting both the coefficients of the system and the values of the equations in a matrix and mimicking our usual procedure for solving them.

Example 4. *Translate into matrix notation and solve the system*

$$\begin{aligned} x + 2y &= 2 \\ x + 3y &= 5. \end{aligned}$$

Matrix notation has uses well beyond solving systems.

2.1. **Arithmetic of matrices.** Before doing involved applications of matrices, we develop ways to manipulate them and combine them algebraically. (Note: though our ultimate topics of interest lie in section 1.5 of *Linear Systems* packet, we need topics from sections 1.2-1.4 to develop them).

Matrices add by adding their entries.

Example 5.
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 6 & -1 \end{bmatrix}.$$

Important note: you can only add matrices of the same size!

Matrices also subtract in the way you would expect.

Example 6.
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 2 & -1 \\ 3 & -5 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -3 & 0 & 4 \\ 1 & 10 & 5\frac{1}{2} \end{bmatrix}.$$

Addition and subtraction of matrices obey the same rules as for numbers.

The zero matrix has zero for all of its entries. Matrices do not change when the zero matrix is add to or subtracted from them.

We will see next time that the most natural way to multiply matrices is not what you would expect.

Some manipulations of matrices are not just translation from those on numbers.

Definition 7. *The transpose of an n by m matrix M is the m by n matrix called M^T whose i,jth entry is the j,ith entry of M.*

Example 8. *The transpose of* $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ *is* $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

3. Vectors

Matrices with only one column or row are called (column or row) *vectors*. Vectors naturally sit on a line, plane, three-space, etc.

Example 9. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and* $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$ *are vectors, which we can represent by points in the plane or in space.*

Because they are special cases of matrices, they can be added and subtracted. Vectors also have a fundamental way to multiply each other.

Definition 10. *The dot product of a row vector* $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ *and a column vector* $\begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$ *is the sum* $a_1b_1 + a_2b_2 + \cdots a_nb_n$.

Note that the dot product of two vectors is not another vector but a number. The dot product arises in many contexts, especially geometry.

Theorem 11. *The dot product of v and w is zero if and only if the line between O and v is perpendicular to the line between O and w, where O is the origin - the point $(0,\ldots,0)$.*

Example 12. *Verify this theorem when* $v = \begin{bmatrix} 1 & 2 \end{bmatrix}$ *and* $w = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$.