MATH 242: CALCULUS FOR BUSINESS AND THE SOCIAL SCIENCES, II

1. Outline of the course

We will first continue with one of the greatest mathematical inventions of all time, the calculus, now in its *integral* (as opposed to differential) version, based on the *fundamental theorems of calculus.*

We will next lay the algebraic foundation for solving problems involving many variables and in particular development of multivariable calculus. Recalling how important lines are in calculus, it is no surprise that we will study *linear algebra* in many variables.

Finally, we will develop optimization techniques, as were the main focus at the end of last term, in the setting of multiple variables.

2. Why are we studying these topics?

The study of mathematics places stringent demands on one's ability to reason clearly and completely – this is why the university has a general requirement in mathematics.

The integral calculus is the second, crucial, half of one of the greatest intellectual stories of all time, namely the calculus. The fundamental theorems, which relate integral to differential calculus, are two of the most important, often used theorems in all of mathematics.

Most often in real-world modeling it is not practical or useful to isolate one or two variables. Starting to work on many-variable problems adds a key tool to our mathematical modeling toolbox.

Multivariable optimization through calculus is ubiquitous in economics and other social sciences. Graduate programs in economics often give crash courses in this material to their incoming students.

3. Practicalities

- Course materials, including a complete syllabus, and announcements are at: http://noether.uoregon.edu/∼dps/242/
- Lectures will be projected in outline with details worked out in writing. The outlines will be available online before class.
- Homework is due on Fridays at the beginning of class. All assignments are posted on the syllabus.
- Sections are on Wednesdays or Thursdays. There will often be quizzes (as posted on syllabus).
- There will be three fifty-minute exams, each covering one third of the material. They will be non-cumulative, as a significant fraction of the material is non-cumulative. Accordingly, there will be no final exam.

4. First (half) lecture: Discrete calculus, a chance to both review and to look forward

The functions with which we worked in differential calculus depended on a *continuous* variable - one which could take on any value (such as $\frac{1}{2}$ or $\sqrt{2}$ or π or e). This variable was often measuring time. For some problems it is more natural to have a discrete variable (for example, when measuring financials every quarter as opposed to continuously).

Definition 1. *A discrete function is one whose domain consists of integers (but whose range consists of any kind of number)*

Any of the familiar examples of functions (polynomials, exponentials) can be *restricted* to be discrete functions. We often use n to name the variable instead of x, so that the squaring function is named $f(n) = n^2$.

4.1. **Difference functions.**

Definition 2. *Given a discrete function $f(n)$ define its difference function $Df(n)$ by $Df(n) = f(n) - f(n-1)$.*

Example 3.
- *If $f(n) = n^2$, then $Df(n) = n^2 - (n-1)^2 = n^2 - (n^2 - 2n + 1) = n^2 - n^2 + 2n - 1 = 2n - 1$.*
- *If $f(n) = n^3$, then $Df(n) = n^3 - (n-1)^3 = n^3 - n^3 + 3n^2 - 3n + 1 = 3n^2 - 3n + 1$.*
- *If $f(n) = 2^n$, then $Df(n) = 2^n - 2^{n-1} = 2^{n-1}(2-1) = 2^{n-1}$.*

Notice the similarities with the derivative!

One can use the difference function much as we used the derivative. In particular, a function achieves a relative maximum or minimum where its difference function changes sign.

4.2. **Aggregate functions.** Aggregate functions measure total amounts (for example, total sales when given monthly sales).

Definition 4. *Given a discrete function $f(n)$ define the aggregate function $Af(n)$ to be $f(1) + f(2) + \cdots + f(n)$.*

Example 5.
- *If $f(n) = n$, $Af(n) = \frac{n(n+1)}{2}$ (as Gauss figured out when he was eight years old!)*
- *If $f(n) = n^2$, $Af(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$.*
- *If $f(n) = 2^{n-1}$, $Af(n) = 2^n - 1$.*

Notice that the derivative of $Af(n)$ looks related to $f(n)$ itself.

4.3. **The fundamental theorems of discrete calculus.** If one were given a total sales function, the way one would find the sales just for the fifth month would be to subtract the total for the first four months from the total for the first five months. This simple observation leads to the first fundamental theorem of discrete calculus.

Theorem 6 (First fundamental theorem)**.** *If $f(n)$ is a discrete function, then $D(Af(n)) = f(n)$.*

Proof: We compute

$$\begin{aligned} D(Af(n)) &= Af(n) - Af(n-1) \\ &= \{f(1) + \cdots + f(n)\} - \{f(1) + \cdots + f(n-1)\} \\ &= f(n). \end{aligned}$$

Along similar lines, we have the following.

Theorem 7 (Second fundamental theorem)**.** *If $f(n)$ is a discrete function then $A(Df(n)) = f(n) - f(0)$.*

Proof:

$$\begin{aligned} A(Df(n)) &= Df(1) + Df(2) + \cdots + Df(n) \\ &= (f(1) - f(0)) + (f(2) - f(1)) + \cdots \\ &\quad + (f(n) - f(n-1)) \\ &= -f(0) + f(1) - f(1) + f(2) - f(2) + \cdots \\ &\quad + f(n-1) - f(n-1) + f(n) \\ &= f(n) - f(0) \end{aligned}$$

For discrete functions, these theorems are the result of simple arithmetic. Their continuous analogues are much more subtle. The difference function is the discrete analogue of the derivative, and the aggregate function is the discrete analogue of something we will call the (definite) integral. Both of the former have to do with differences and change. Both of the latter have to do with total and accumulation. Their interrelationship through fundamental theorems will be of both conceptual and computational use.