MATH 242, LECTURE 20

1. The gradient

Because we have talked about vectors, we can talk about the construction which assembles partial derivatives into what might be called a "total derivative", namely the gradient vector. The book does not cover this topic, and it will not appear on homework, quizzes and the main part of the last exam, but still might appear somehow (wink wink).

Definition 1. *The gradient of a function* $f(x,y)$ *is the vector* $\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$.

Example 2. *Find the gradient of the function* $f(x,y) = x^2 - y^2$ *and its value at the point* $(1,2)$.

Using the dot product, we can use the gradient to find the rate of change of a function in any direction.

Theorem 3. *The rate of change of a two-variable function at a point in a given direction is the dot product of the gradient vector at that point with a* unit vector *in that direction.*

Example 4. *Find the rate of change of the function* $x^2 + y^2$ *at the point* $(1,1)$ *in the direction of the unit vectors* $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ *and* $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$.

2. Higher-order partial derivatives

As in the case of a single variable, we are free to take a derivative of a derivative. The notation works as follows:

Definition 5. *The partial derivative with respect to* x *of the partial with respect to* x *is* f_{xx} *or* $\frac{\partial^2 f}{\partial x^2}$.
The partial derivative with respect to y *of the partial with respect to* x *is* f_{xy} *or* $\frac{\partial^2 f}{\partial y \partial x}$.
The partial derivative with respect to x *of the partial with respect to* y *is* f_{yx} *or* $\frac{\partial^2 f}{\partial x \partial y}$.
The partial derivative with respect to y *of the partial with respect to* y *is* f_{yy} *or* $\frac{\partial^2 f}{\partial y^2}$.

Example 6. *Find second-order partial derivatives of* $f(x,y) = x^2y^3 + e^{xy}$ *and* $f(x,y) = \ln(x+y)$

3. Finding relative maxima and minima of multivariable functions

In one variable, we found that maxima and minima can occur either at endpoints of a constraint interval or at critical points, which were mainly where the derivative of the function vanished. In the second third of the class, we learned about finding maxima and minima of linear functions, which all occur on the boundary of the constraint region (aka linear programming). Now we will focus on multivariable critical points.

Definition 7. *A critical point of a multivariable function is a point at which all partial derivatives vanish.*

We will formalize what can happen at a critical point after looking at some basic examples.

Example 8. *Find the critical point(s), and look at the graphs of the functions near those points, of the following:*

- $f(x, y) = x^2 + y^2$
- $f(x, y) = -(x-2)^2 - (y+1)^2$
- $f(x, y) = x^2 - y^2$

So at a critical point, in every direction we go from increasing to decreasing, or decreasing to increasing. At (what looks like) a local maximum, a function goes from increasing to decreasing in every direction. At a local minimum, the change is from decreasing to increasing in every direction. But in some cases, called saddle points, the function goes from decreasing to increasing in some directions and increasing to decreasing in others.

One way to test what kind of behavior is occurring is to take values of many points near the critical point. If, for example, they are all less than the value at the critical point, then the critical point is probably a local maximum. But one cannot be absolutely sure with this method. There is an intricate series of computations which gives certainty in this determination.

Theorem 9. *Let $f(x, y)$ be a function in two variables and let (a, b) be a critical point, so that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let D be the quantity $f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$. If $D < 0$ then (a, b) is a saddle point. If $D > 0$ and $f_{xx}(a, b) < 0$ then (a, b) is a relative maximum. If $D > 0$ and $f_{xx}(a, b) > 0$ then (a, b) is a relative minimum.*

We will summarize this theorem as a technique in the next lecture. For now, we give one illustration.

Example 10. *Find and classify all critical points of the function $f(x, y) = x^3 - y^3 + 6xy$.*