MATH 242, LECTURE 21

1. Finding maxima and minima using partial derivatives

From our main theorem of last time, we can extract the following procedure to find maxima, minima and saddle points of a two-variable function $f(x, y)$.

- Calculate the partial derivatives f_x and f_y.
- Set the partial derivatives equal to zero and solve for x and y to find critical points.
- Given a critical point, compute the value of $D = f_{xx}f_{yy} - (f_{xy})^2$ at that critical point. (This quantity is called H in the book.)
- If $D > 0$ and $f_{xx} < 0$ the critical point is a relative maximum.
- If $D > 0$ and $f_{yy} > 0$ the critical point is a relative minimum.
- If $D < 0$ the critical point is a saddle point.
- If none of these hold (for example $D = 0$), further analysis (which we will not develop) is needed to understand the critical point.

Example 1. *Find and classify all critical points of the functions* $\ln(x^2+y^2+1)$ *and* $f(x, y) = x^3 - y^3 + 6xy$.

For applications we should add to the beginning of our list our standard procedures of identifying relevant variables for a problem and determining the function to be optimized.

Example 2. *A computer company is introducing two new systems marketed to larger businesses. They estimate that if the systems are priced at* x *and* y *hundred dollars, respectively, then* $40 - 8x + 5y$ *customers will buy the first system and and* $50 + 9x - 7y$ *will buy the second. If the costs of manufacturing the systems are $1000 and $3000 respectively, how should the company price these items?*

In the next example we see that optimization methods of calculus can lead to solutions of geometric problems, which historically has been one of the main applications of the calculus. To this day, using calculus is central in mathematical areas such as finding surfaces with least area (soap bubbles) and areas in physics, biology and chemistry such as finding positions of molecules which minimize energy.

Example 3. *Four dormitories on campus are located at points with coordinates* $(-5, 0)$, $(1, 7)$, $(9, 0)$ *and* $(0, -8)$ *on a campus map. Where on that map should a cafeteria be placed to minimize the total of distances (squared) from the dorms to the cafeteria?*

One more, just for kicks (and to remind ourselves of the fun we had in single-variable calculus, plugging in for one variable to end up with an appropriate number of variables).

Example 4. *Suppose it costs $8 per square foot for material on the bottom of a large rectangular fish tank and $10 per square foot for the sides. What dimensions should the fish tank be to maximize volume with a total cost of $1000?*