

## MATH 242, LECTURE 12

### 0.1. Finishing a revenue stream example.

**Example 1.** If you invest  $R(t) = 2000 + 400t$  dollars per year in a retirement account earning nine percent per year, how much will you have after forty years? How much of that is principal and how much is interest?

**Answer:** The future value is  $\int_0^{40} (2000 + 400t)e^{0.09(40-t)} dt$ . Using the fact that  $\int e^{rt} = \frac{e^{rt}}{r}$  and  $\int te^{rt} = \frac{te^{rt}}{r} - \frac{e^{rt}}{r^2}$ , we can evaluate this integral (or we can just use a numerical integrator). The value is:

$$e^{3.6} \left[ \frac{2000e^{-0.09t}}{-0.09} \right]_0^{40} + e^{3.6} \left[ \frac{te^{-0.09t}}{-0.09} - \frac{e^{-0.09t}}{-0.09^2} \right]_0^{40}.$$

The answer is \$2,371,230, which in current dollars assuming a 4% rate of inflation is \$478,743 - enough to live on for a good number of years. The amount of principal deposited is  $\int_0^{40} (2000 + 400t) dt = 400,000$ , which leaves roughly two million of interest - the lion's share. Note that even the first \$2000 deposited becomes \$73,200.

## 1. SYSTEMS OF LINEAR EQUATIONS

In our studies of differential and integral calculus, and in much of the mathematics you studied through high-school, there was essentially only one variable which was allowed to vary freely. Many problems, both theoretical and applied, are not well-addressed by having only one quantity which can vary freely. But instead of moving directly to multivariable calculus, we are going to first cover more basic topics in algebra and geometry.

When you first learned algebra, the equations you focussed on most were *linear*, equations like  $3x+2=8$ , which you then learned to solve them by *isolating the variable*. We will similarly choose to look at *linear algebra* first, and find that similar strategies are useful.

**Definition 2.** A linear equation in two variables  $x$  and  $y$  is one of the form  $ax + by = c$ , where  $a$ ,  $b$ , and  $c$  are given numbers.

Linear equations in two variables should be familiar from high-school, Math 111, and Math 241. The set of all  $x$  and  $y$  which satisfy the equation form a line.

**Example 3.** Plot all points  $(x, y)$  which satisfy the linear equation  $-2x + y = 1$ .

**Definition 4.** A system of linear equations in two variables is simply a collection of such linear equations. A solution to a system of linear equations is a pair of numbers which satisfies all equations in the system.

**Example 5.** The collection

$$\begin{aligned} -2x + y &= 1 \\ x + y &= 4 \end{aligned}$$

is a system of linear equations in two variables.

The pair of numbers  $(x, y) = (1, 3)$  is a solution, as can be verified by substitution.

Solutions to a system of linear equations are places where the lines of points which satisfy those equations intersect, as can be illustrated with the system in the previous example. This observation leads to the first of three methods we develop for solving these systems.

**1.1. Methods for solving linear systems.** There are three basic methods we discuss, illustrating each one with the system of Example ??.

**Graphical Method:** This method is more helpful in getting approximate solutions and in verifying that answers obtained by more precise means are reasonable. We simply graph the solutions of all of the linear equations involved and estimate their point of intersection.

**Substitution:** We use one equation solve for one variable as a function of the other, and then substitute this function in for that variable in the other equation(s). In a system of two variables and two equations, the resulting equation will have only one variable. This method is very effective in solving small systems.

**Addition and subtraction of equations:** This is perhaps the most interesting method. We use the fact that one can always multiply, add, subtract and divide (but not by 0) equalities and end up with equalities. So we add and subtract multiples of equations so that the resulting equations are simpler.

**Example 6.** *Solve the following system of linear equations using the three methods:*

$$\begin{aligned} 2x + y &= 1 \\ -x + 2y &= 4 \end{aligned}$$

There are many problems, both toy (as occur on IQ tests) and real-world in which systems of linear equations naturally occur.

**Example 7.** *Last year I was three times as old as my sister and in four years I will be twice her age. How old are we?*

**Example 8.** *Almonds cost \$3 per pound and Cashews \$8. These two types of nuts make Nuttyfun.<sup>TM</sup> How many of each should be used to produce 100 pounds at \$5 per pound? 100 pounds at \$6 per pound? 120 pounds at \$4 per pound?*