MATH 242, LECTURE 18

1. Introduction to multivariable functions

The linear functions we have been studying are our first examples of multivariable functions, which are the topic of study for the last third of the term.

A multivariable function describes a dependence of one quantity on two or more others. In some linear programming examples we saw, expenditures depended on the number of two different kinds of items purchased; temperature depends on where you are, which is described by longitude and latitude; production levels depend on capital and labor.

As we saw, the graphs of two-variable functions are surfaces which sit in three-dimensional space. We will not focus on graphing functions on our own, but it is helpful to look at graphs the computer generates as we go along.

Example 1. *Look at the graphs of the following functions:*

- $f(x, y) = \frac{1}{2}(x^2 + y^2)$
- $f(x, y) = \frac{1}{2}(x^2 - y^2)$
- $f(x, y) = e^{x-y}$.

We can ask many of the same kinds of questions as we did for single-variable functions, but with some additional complications. Sometimes we see what are maxima and minima, but in order to talk about whether the function is increasing or decreasing, we have to pick a direction. It is helpful to draw on intuition from hiking, thinking of the graph as a landscape.

1.1. Level curves.

We saw with linear functions that given a two-variable function $f(x, y)$ it is helpful to look at the sets of points where $f(x, y)$ has a fixed value like 0, 1, 35, ... As we mentioned previously, such collections are depicted in topographical maps as contour lines, and on weather maps as isotherms (lines of constant temperature).

Definition 2. *A* level curve *for a function $f(x, y)$ is the collection of all points (x, y) such that $f(x, y) = c$ for a given c.*

Example 3. *What are some level curves of the functions $f(x, y) = x^2 + y^2$, $f(x, y) = x^2 - y^2$, and e^{x+y}?*

In some specific applications, level curves have specialized names (e.g. contour lines for the height function on a topographical map). In economics, the *utility function* U of two (or more variables) measures "satisfaction" - also called *utility* - a consumer derives from having different amounts of goods. Level curves in this setting are called *indifference curves*, because consumers are supposed to be indifferent to choices which lead to the same level of satisfaction.

2. Partial derivatives

By fixing one of the variables in a two-variable function, we get a one-variable function. For example, if we take the function $x^2 + y^2$ and set $x = 2$ we get the function $4 + y^2$. This resulting one-variable can then have its derivative taken; that's the idea behind partial derivatives. Before seeing the general technique we do one example ad hoc.

Example 4. *Find the derivative of the function $f(x, y) = xy^2$ when $x = 3, 5, -1, 100, \sqrt{2}, \pi$. Notice that this derivative is always equal to $2xy$.*

Definition 5. *If $f(x,y)$ is a function of two variables, its partial derivative with respect to x, denoted either $\frac{\partial f}{\partial x}$ or $f_x(x,y)$, is the function obtained by treating y as a constant and differentiating with respect to x. Similarly, the partial derivative with respect to y, denoted $\frac{\partial f}{\partial y}$ or f_y, is obtained by treating x as a constant and differentiating with respect to y.*

You will need to take partial derivatives for the quiz this week.

Example 6. *Find the partial derivatives with respect to x and y of*

- $\frac{2y}{x}$.
- $2x^2\sqrt{3y+1}$.
- $\ln(xy)$.
- $e^{-(x^2+y^2)}$.