

MATH 242, LECTURE 8

0.1. Practice with the definite integral and the Fundamental Theorem.

Example 1. Use both the Fundamental Theorem and a Riemann sum with 50 terms to evaluate $\int_1^3 \frac{x}{x+1}$.

1. FURTHER APPLICATIONS OF THE DEFINITE INTEGRAL

Many quantities can be well-approximated (to *first order*) by a Riemann sum $f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$. Any such quantity can be computed with a definite integral. The FTC can be applied to compute the integral, so the FTC is useful well beyond computing areas under or between curves.

The steps for setting up integrals to compute quantities in many different contexts are as follows:

- Name a variable through which the quantity can naturally be “broken up” (for the sake of concreteness, let us name that variable x).
- For small ranges of values for that variable, estimate the value of that quantity. Your answer should be of the form $f(x) \cdot \Delta x$, where Δx represents the change in the variable over that range and $f(x)$ is the function which makes the estimate work.
- Determine the range of values the variable must take on to account for the entire quantity (for the sake of concreteness, call those a and b).
- Evaluate $\int_a^b f(x)dx$ in order to compute the quantity !

We begin with examples coming from mathematics and physics, as we often like to do for first illustrations, since it is in these subjects which the calculus was developed. We will not emphasize these on homework and tests, but they are important because they beautifully illustrate the essence of modeling with the definite integral.

Example 2. Find the volume of the football obtained by rotating the curve $y = \frac{1}{2}(1 - x^2)$ between $x = -1$ and $x = 1$ around the x -axis in three-dimensional space. To be concrete, what is the volume of a foot-long football?

- *variable:* x we will “slice” the football.
- *begin by estimating the volume near $x = 0$ with $\Delta x = 0.1$ using (area of base) times (height).* namely $\pi(\frac{1}{2})^2 \times 0.1$. in general we get $\pi(\frac{1}{2}(1 - x^2))^2 \Delta x$.
- *x ranges from -1 to 1 .*
- *Evaluate $\int_{-1}^1 \pi(\frac{1}{2}(1 - x^2))^2 dx$.*

Using similar techniques one can compute the volume of spheres, cones, donuts, and satellite dishes.

These kinds of applied integration problems often comprise an entire course! In fact, the technique of chopping a problem into (tiny) pieces, analyzing the pieces, and then *integrating* the pieces into a whole is what gave the subject its name.

Example 3. A metal rod is made with an alloy of two metals, metal A which weighs 2 grams per cm of length and metal B which weighs 5 grams per cm of length. Suppose that one end of a 10cm rod is all metal B and the other end is a fifty-fifty mix, and the mix of metals changes linearly. How much does the rod weigh?

2. AREAS BETWEEN CURVES

We can use this same Riemann sum analysis to quickly evaluate areas between two curves.

To find the area between the graph of $f(x)$ and that of $g(x)$ between $x = a$ and $x = b$ we evaluate $\int_a^b |f(x) - g(x)| dx$.

Example 4. Find the area of the region between the functions $f(x) = x$ and $g(x) = x^2$ and the vertical lines at $x = -3$ and $x = -1$.

Sometimes there are no vertical lines supplied; one must figure out where the curves intersect.

Example 5. Find the total area of the football-shaped region between the graphs of $f(x) = x^2 - 2$ and $g(x) = 2 - x^2$.