MATH 242, LECTURE 11

1. Values of revenue streams

Often in business and economics, money is paid (back) in installments, as with student loans, mortgages and retirement accounts. Standard application of the Fundamental Theorem allows us to extract a total amount paid given the rate.

Theorem 1. *If money is paid at a rate of $R(t)$ dollars per unit of time, the total amount paid between time a and b is $\int_a^b R(t)dt$.*

Example 2 (Stupid example)**.** *What does this theorem say when $R(t)$ is constant (say \$100 per month between the second and sixth months of some year)?*

But as any pensioner or lottery winner will tell you, getting a constant amount of money means that your purchasing power will go down over time. How can we determine the current or future value of installment payments?

Example 3. *\$100 deposited per month in an account earning 5% interest has a present value of* $100 + 100e^{-0.05\frac{1}{12}} + \cdots + 100e^{-0.05\frac{11}{12}}$ *dollars.*

What if, instead of deposited as \$100 per month, \$1200 was deposited as a continuous income stream, evenly over the year? Then, over a short fraction of the year $\Delta t = \frac{1}{N}$th of a year, we would deposit $1200\Delta t$ dollars. Its present value would be $(1200\Delta t)e^{-0.5t}$, where t is a time at which this money is deposited. The sum of all of these pieces would be

$$1200\Delta te^{-0.5t_0} + 1200\Delta te^{-0.5t_1} + \cdots .$$

We have gone through the steps of setting up a Riemann sum, leading to an integral; by passing to a limit and using the definite integral, we see that the present value will be $\int_0^1 1200e^{0.5t}$. (Note how much this income stream is worth - making payments more frequently can make a big difference!)

Theorem 4. *The present value of a revenue stream $R(t)$ between times a and b is $\int_a^b R(t)e^{r(a-t)}dt$.*

The future value of a revenue stream $R(t)$ between times a and b is $\int_a^b R(t)e^{r(b-t)}dt$.

Let's justify this again by analyzing the present and future value of the money deposited over a short period of time.

Then we're ready for an example. Note that the term e^{ra} or e^{rb} which occurs can be "pulled out" of the integral.

Example 5. *What is the present value of a million dollars paid constantly over twenty years, assuming a prevailing inflation rate of four percent per year?*

But what happens if the revenue function isn't constant, but is even a linear function? We run into integrals like that of te^{rt}. These are done in section 7.1, but we will just use a general formula.

Theorem 6.

$$\int t^n e^{rt} dt =$$

$$\frac{1}{r} t^n e^{rt} - \frac{n}{r^2} t^{n-1} e^{rt} + \frac{n(n-1)}{r^3} t^{n-2} e^{rt} - \cdots \pm \frac{n(n-1)(n-2)\cdots 1}{r^{n+1}} e^{rt} + C.$$

We are mainly concerned in the cases of $n = 0$ or 1, in which case we can verify this by taking the derivative.

We can use this for more complicated revenue stream problems.

Example 7. *If you invest $R(t) = 2000 + 400t$ dollars per year in a retirement account earning nine percent per year, how much will you have after forty years? How much of that is principal and how much is interest?*

The other class of revenue streams for which exact answers are within are reach are those described by exponential functions.

Example 8. *The revenues of Startup.com is modeled by $10e^{1.2t} - 20$ millions of dollars. Adjusting for inflation of 4%, estimate the present value of its revenues over its first five years.*