MATH 242, LECTURE 5

1. The definite integral and the fundamental theorem of calculus

Today we give an overview of sections 6.3, 6.4 and 6.5 together. We will go over these ideas again more separately.

Areas naturally correspond to aggregate amounts. If the bars on a bar graph of profit are of unit width, then the area of the bar graph represents total profit. We will see that for a continuous(ly modelled) revenue stream, the area under the marginal revenue curve represents total revenue.

The Fundamental Theorem of Calculus will tell us that anti-derivative computes area under a curve.

1.1. Approximating areas by rectangles.

While we do not know how to find areas of regions with curved boundaries, we do know how to find areas of regions with straight boundaries, especially rectangles. We approximate the area of a curved region by trying to fill it (or cover it, with some "spillage") with rectangles. This method for finding curved areas, such as that inside a circle, was known to some Greek mathematicians but was lost until the Renaissance and the (re?)invention of integral calculus.

Example 1. *Approximate the area under the curve $f(x) = x^2$ between $x = 0$ and $x = 1$ using ten rectangles.*

We can see that ten rectangles gives a pretty good fit. The area of each rectangle is computed as base times height. Each rectangle has base of $\frac{1}{10} = 0.1$ and the ith rectangle has height equal to $f(0.1 \cdot i)$. Thus, the total area is

$$\begin{aligned}
&0.1(0.1)^2 + 0.1(0.2)^2 + 0.1(0.3)^2 + \cdots + 0.1(1.0)^2 \\
&\qquad = 0.1\left[(0.1)^2 + (0.2)^2 + \cdots + (0.9)^2 + (1.0)^2\right] \\
&\qquad = 0.1\left[0.01 + 0.04 + \cdots + 0.21 + 1.00\right] \\
&\qquad = 0.1(3.85) = 0.385
\end{aligned}$$

We will see soon that the exact answer is $\frac{1}{3}$, so this approximation is not bad (but not spectacularly good either).

The general technique is as follows: we find the area under $f(x)$ between $x = a$ and $x = b$ by approximating it by N rectangles, each of width $\frac{b-a}{N}$. We let $\Delta x = \frac{b-a}{N}$ to save writing, and also let $x_i = a + \Delta x \cdot i$, the place of the dividing line between the ith rectangle and the $i+1$st. Then the area of these N rectangles is

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_N)\Delta x = \Delta x\left[f(x_1) + f(x_2) + \cdots + f(x_N)\right] \tag{1}$$

If we let N get larger and larger, we get a better approximation to the area, leading to the following.

Theorem 2. *If $f(x)$ is continuous, the area under the graph of f between $x = a$ and $x = b$ is*

$$\lim_{N\to\infty}\left(\frac{b-a}{N}\right)\left[f(x_1) + \cdots + f(x_N)\right]$$

.

1.2. **The (First) Fundamental Theorem.** As we know well, computing limits can be a difficult process, and in this case the function whose limit we take is complicated. One of the best theorems in all of mathematics gives us an easy way to find this area using the anti-derivative.

Theorem 3 (First Fundamental Theorem)**.** *Let $F(x)$ be an anti-derivative of $f(x)$. Then the area under $f(x)$ between $x = a$ and $x = b$ is equal to $F(b) - F(a)$.*

Example 4.

- *Find the area under $f(x) = x^2$ between $x = 0$ and $x = 1$ and between $x = 2$ and $x = 5$.*
- *Find the area under $f(x) = e^x$ between $x = 0$ and $x = 3$.*
- *Find the area under $f(x) = \frac{1}{x}$ between $x = 1$ and $x = 2$ and between 1 and 4 and between 1 and 10 and in general between 1 and a where $a > 1$ is any number.*

Question: why would using any anti-derivative give the same answer when using the Fundamental Theorem?

We will revisit the process of defining the definite integral as a sum and the Fundamental Theorem in our next lectures.