MATH 242, LECTURE 14

0.1. **The dot product.** Recall vectors, which are rows or columns of numbers, and the dot product.

Definition 1. *The dot product of a row vector* $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ *and a column vector* $\begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$ *is the sum* $a_1b_1 + a_2b_2 + \cdots a_nb_n$.

Note that the dot product of two vectors is not another vector but a number. Dot products can supply us with alternate notation for linear equations.

Example 2.

$$\begin{bmatrix} 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 5$$

is an alternate notation for the equation $3x + 2y = 5$.

Any collection of related data (that is any series of numbers) can (and rightfully should) be collected in a vector. The dot product is then useful in describing linear relationships in that data.

Example 3. *Let* V *denote the number of vampires in Sunnydale and* S *be the number of vampire slayers. We may collect these data in the vector* $\begin{bmatrix} V \\ S \end{bmatrix}$. *Let* W *be the number of vampires next year, which is 1.2 times the number of vampires this year minus 26 times the number of slayers. Then we have*

$$W = \begin{bmatrix} 1.2 & -26 \end{bmatrix} \begin{bmatrix} V \\ S \end{bmatrix}.$$

1. Matrix multiplication

The multiplication of matrices often throws people for a loop since it seems unnatural. It builds on the dot product of vectors, which is a helpful starting point.

Definition 4. *The product* $M \cdot N$ *of two matrices is the matrix whose entry in the* i*th row and* j*th column is the dot product of the* i*th row of* M *and the* j*th column of* N, *when this is defined.*

Example 5. *(Try to) perform the following matrix multiplications:*

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ \frac{1}{2} & -2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$$

So, the matrices M and N can only be multiplied if the number of columns of M equals the number of rows of N. So in the second example, while the two matrices given can be multiplied, they could not be if their order is reversed. Reversing order of matrices is a problem even when the multiplication is still defined. Compare our first example with the product $\begin{bmatrix} -1 & 0 \\ \frac{1}{2} & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Matrix multiplication is not commutative! In general $M \cdot N$ *is not equal to* $N \cdot M$, *even when both are defined!*

You should be asking at this point: why should I multiply matrices like this? Why can't I say $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 21 & 32 \end{bmatrix}$? At least that would be commutative...

The answer is that you could, but you would not find it as useful as the matrix multiplication defined using the dot product. Experience with many kinds of problems ranging from geometry and computer graphics to iterative processes and political science has shown that the matrix multiplication just defined is a remarkably expedient.

Example 6. *The system of equations*

$$\begin{aligned} 2x + 3y &= 5 \\ x - 2y &= 3 \end{aligned}$$

can be written in matrix-vector notation as

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Because matrix multiplication *is* associative, some applications can be developed cleanly.

Example 7. *Suppose that the number of vampires and slayers in Sunnydale changes from one year to the next according to*

$$\begin{bmatrix} V_{new} \\ S_{new} \end{bmatrix} = \begin{bmatrix} 1.2 & -26 \\ \frac{1}{100} & 1 \end{bmatrix} \begin{bmatrix} V_o \\ S_o \end{bmatrix}.$$

(Interpret this equation.) Then after ten year you would expect

$$\begin{bmatrix} 1.2 & -26 \\ \frac{1}{100} & 1 \end{bmatrix} \begin{bmatrix} 1.2 & -26 \\ \frac{1}{100} & 1 \end{bmatrix} \cdots \begin{bmatrix} 1.2 & -26 \\ \frac{1}{100} & 1 \end{bmatrix} \begin{bmatrix} V_o \\ S_o \end{bmatrix}.$$

vampires and slayers to be on the prowl. Because matrix multiplication is associative, this can be written and computed more succinctly as

$$\begin{bmatrix} 1.2 & -26 \\ \frac{1}{100} & 1 \end{bmatrix}^{10} \begin{bmatrix} V_o \\ S_o \end{bmatrix}.$$

This example is a taste of population modeling using matrices. For a more complete story, take Math 342.

2. Solving systems of equations using inverses of matrices

Matrices are collections of numbers which behave in some ways just like numbers themselves. We can add, subtract and multiply them, and there is a zero matrix. The similarities only go so far, though - only square matrices can both be added and multiplied, the multiplication is complicated and it depends on the order in which the matrices appear!

Our next step will be to learn how to "divide" matrices, which will be essential in solving matrix equations. Remember that for numbers to solve the equation $3x + 2 = 8$ we first need to subtract to get $3x = 6$ and then we divide both sides by 3 - the key step! - to get $x = 2$.

2.1. **Identity matrices.** Division by x is just multiplication by $\frac{1}{x}$, so first we must understand what "1" is for matrices. Your first guess might be the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. But how would we know this is right? The key fact about 1 is that $1 \cdot y = y$ for any y. We can check whether $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ satisfies this key fact: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 4 & 6 \end{bmatrix}$. So it doesn't! Fortunately, there is a matrix which satisfies this key property.

Definition 8. *The n by n identity matrix is the square matrix with 1's for the $1,1,\ 2,2,\dots,n,n$ entries, and 0's everywhere else. For example, the 3 by 3 identity matrix is* $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$. *Identity matrices are often denoted by the letter I (which then does not specify the size of the matrix).*

Theorem 9. *For any identity matrix I and square matrix M of the same size, $IM = MI = M$.*

Example 10. *We verify this theorem for the 2×2 identity matrix.*

We have already seen the identity matrix when we first looked at solving systems of equations using matrix notation. In fact, linear equations expressed in terms of the identity matrix couldn't be any easier!

Example 11. *Translate the matrix equation* $\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix}$.

2.2. **Inverses of matrices.**

Definition 12. *The inverse of a square matrix M is one which we call M^{-1} (as opposed to $\frac{1}{M}$) where $MM^{-1} = I = M^{-1}M$.*

Theorem 13. *The inverse of the 2×2 matrix* $\begin{bmatrix}a&b\\c&d\end{bmatrix}$ *is* $\begin{bmatrix}\frac{d}{ad-bc}&\frac{-b}{ad-bc}\\\frac{-c}{ad-bc}&\frac{a}{ad-bc}\end{bmatrix}$.

Note that the number $ad - bc$ which appears everywhere in this formula has its own special name; it is the *determinant* of the matrix. If it is zero, then an inverse does not exist.

Example 14. *Verify this theorem for the matrices* $\begin{bmatrix}1&2\\3&4\end{bmatrix}$ *and* $\begin{bmatrix}5&7\\2&3\end{bmatrix}$.

There are formulae for inverting larger matrices, but they are complicated! Fortunately, we do not have to invert matrices by hand. Many graphing calculators, including the TI-83, have a function to invert a matrix. See the manual, linked to from the class page, to learn how.

2.3. **Using inverses to solve linear systems.** Let M be a square matrix, X be a column vector of variables and C be a column vector of constants (of compatible sizes). To solve the equation $MX = C$ we first find M^{-1} (if it exists) and then multiplying both sides by M^{-1} we get $M^{-1}MX = M^{-1}C$, or $IX = M^{-1}C$ or $X = M^{-1}C$.

So the steps to solve $MX = C$ are:

(1) Find M^{-1}.
(2) Compute the product $M^{-1}C$.
(3) (For thoroughness) check that your answer works.

Example 15. *Use matrix algebra to solve the system of equations $5x + 7y = 3$, $2x + 3y = -1$.*

One great advantage to matrix methods is in solving related systems.

Example 16. *Solve the matrix equation* $\begin{bmatrix}3&2\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$. *What if we replace* $\begin{bmatrix}5\\6\end{bmatrix}$ *by* $\begin{bmatrix}1\\2\end{bmatrix}$? *By* $\begin{bmatrix}-1\\\pi\end{bmatrix}$?

With these techniques (and calculators in hand) we can move on to swiftly solving problems with more variables.

Example 17. *The average yield on A-bonds is 6%, on B-bonds is 7% and on C-bonds is 10%. Because of a hedging scheme, you must invest twice as much money in A bonds as C bonds. Find the amounts to invest for the following desired outcomes:*

- *\$25K invested with an annual return of \$1.8K.*
- *\$30K invested with an annual return of \$2.2K.*
- *\$40K invested with an annual return of \$2.9 K.*