MATH 242, LECTURE 7

We continue our study of Riemann sums, today emphasizing the connection to areas.

Definition 1. *If f is a function over some interval $[a, b]$ then the Riemann sum of f with n terms is defined as*

$$RS_n f|_a^b = \Delta x[f(x_0) + f(x_1) + \cdots + f(x_n)],$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Example 2. *Compute the Riemann sum for the function $f(x) = x^2 - 5$ over the interval from 0 to 4 with $n = 8$.*

Riemann sums are easy to do with the help of a computer.

Example 3. *Use Excel to compute the Riemann sum with everything as in the previous example but with $n = 20$.*

1. Areas and Riemann sums

We have seen that Riemann sums arise natural when one is trying to compute a total displacement from velocity data, total cost from marginal cost data, etc. They first arose arose historically in computing areas under curves, a first step to computing areas of arbitrary shapes.

Example 4. *Use rectangles to approximate the area of the trapezoid which is under the graph of $y = 100 - 32x$, above the x-axis and between the lines $x = 3$ and $x = 4$. Compare this question with the first example from the previous lecture. Use geometry to get an exact answer.*

As we see in this example, the areas of individual rectangles used to approximate area are exactly the terms which occur in a Riemann sum. This relationship is so natural, we won't muddy the waters by trying to formalize it. We'll just give one more example.

Example 5. *Write down and (use computer to) evaluate a Riemann sum with twenty terms which approximates the area under the curve $y = \frac{x}{x^2+2}$ between $x = 1$ and $x = 3$.*

2. The Fundamental Theorem

We return to the the Fundamental Theorem of Calculus.

The way mathematicians usually organize these ideas is by defining $\int_a^b f(x)dx$ to be the limit as $N \to \infty$ of $RS_N|_a^b f(x)$.

Theorem 6 (Fundamental Theorem)**.** *$\int_a^b f(x)dx = F(b) - F(a)$ where F is any anti-derivative for f.*

This theorem now makes some intuitive sense, in the case where $f(x)$ is measuring the marginal change in some quantity $F(x)$. Adding up all of those changes yields the total change $F(b) - F(a)$. Indeed, we can almost (ignoring issues about limits) see a mathematically complete proof by writing in the definition of the derivative of $F(x_i)$ everywhere $f(x_i)$ appears in the Riemann sums defining $\int_a^b f(x)dx$.

Example 7. *Use the Fundamental Theorem to find exact answers for each of the questions we have considered so far in this lecture.*