MATH 242, LECTURE 16

0.1. **Multiple linear inequalities.** Many familiar shapes can be described algebraically using multiple linear inequalities.

Definition 1. *The solution set of a system of linear inequalities is the set of all points which satisfy all of the inequalities.*

Example 2. *Graph the solution set for the system of inequalities:*

$$\begin{cases} x+y \geq 1 \\ x+y \leq 3 \\ 2x-y < 2 \end{cases}$$

Some basic terminology:

Definition 3. *The* boundary lines *for a system of inequalities are the lines defining by replacing inequalities by equalities. The* boundary points *are points on the boundary lines which lie next to the solution set. The* corner points *of a system are those boundary points which lie on more than one of the boundary lines.*

Example 4. *Graph the solution set, and identify the corner points for the systems:*

$$A : \begin{cases} x-y \geq -1 \\ x+y \leq 2 \\ y \geq -1 \end{cases} \qquad B : \begin{cases} x-y \geq 2 \\ x+y \geq 5 \\ x, y \geq 0 \end{cases}$$

One more piece of terminology:

Definition 5. *A solution set is bounded if it is contained in some (larger) rectangle. If there is no such rectangle, then it is unbounded.*

Informally, an unbounded region has points which go to infinity in some direction.

1. Linear programming

Linear programming is the subject of optimizing linear functions over constraint regions. In a single variable, the theory is a very simple case of problems we studied in Math 241.

Example 6. *Find the maximum and minimum of the function* $f(x) = 2x + 3$ *over the interval from* -1 *to* 3.

Our results in this example are consistent with what we saw in optimization of single-variable functions using calculus. The main theorem was that maxima and minima occurred either at critical points (where the derivative of the function in question is zero or does not exist) or at the endpoints of the interval. Since a linear function has a constant, usually non-zero derivative, the maxima and minima must occur at the endpoints of the interval.

Remarkably, the same basic principal applies in many variables!!

Theorem 7 (Main theorem of linear programming)**.** *A linear function obtains its maxima and minima, if they exist, at one or more of the corners of the constraint region, as well as the edges (faces, in higher dimensions) between corners with the same maximum/minimum value. If the constraint region is bounded, maxima and minima both exist.*

We will see how this main theorem works in an example, formalize the process of applying the theorem, and develop intuition as to why the theorem holds.

Example 8. *Maximize the function* $F(x, y) = x - y$ *subject to the constraints* $\begin{cases} x \geq 1 \\ y \geq 0 \\ 2x + 3y \leq 6 \end{cases}$.

Before we go further, a little terminology is helpful.

Definition 9. *In a linear programming problem, the linear function to be maximized is often called the objective function, and the region over which the optimization occurs is often called the feasible region.*

In general, we follow these steps to solve a linear programming problem (in two variables):

- (For word problems) Identify the objective function and feasible region.
- Sketch the feasible region. (This is in some sense optional, and isn't possible in more than two or three variables, but very helpful when we first see these problems).
- Find the corner points of the feasible region.
- Evaluate the objective function at the corner points.
- If the constraint region is bounded, take the biggest and smallest values at corner points; those are the maxima and minima, and they are achieved at those corner points as well as edges which connect optimal corner points with the same value.
- If the constraint region is unbounded, one must understand values at boundary points which "go to infinity". If these values are greater than the max of corner values or less than the min of corner values, then the max or min does not exist.

Example 10. *Maximize* $F(x, y) = x + 2y$ *subject to*

$$\begin{cases} x + y \leq 5 \\ 2x + y \leq 8 \\ x \geq 0; y \geq 0. \end{cases}$$

Maximize $3x - 2y$ *subject to the same constraints.*

Linear programming, and our steps for finding maxima and minima, are applicable in a number of kinds of applied optimization problems.

Example 11. *Farmer Lynn raises chickens and goats. She wants to raise no more than 16 animals, including no more than 10 chickens. She spends \$5 to raise a chicken and \$15 to raise a goat. She has \$180 available to spend. Each chicken generates \$6 in profit and each goat \$20. How many of each animal should she raise in order to maximize profits?*