MATH 242, LECTURE 24

1. Using Lagrange multipliers to solve constrained optimization problems

The method of Lagrange multipliers is one of the most commonly used optimization techniques. For example, most equilibria in basic (graduate-level) economics are found using this method.

In the last lecture, we saw the theoretical idea behind the method of Lagrange multipliers, namely that an optima for a function constrained to some curve will happen at a points where the level set for the function is tangent to the curve. This observation led to the Lagrange equations. In this lecture we use the Lagrange equations to solve both abstract and applied constrained optimization problems.

We summarize the method as a step-by-step process:

- Identify the objective function $f(x,y)$, and if necessary rewrite the constraint curve as the set of all (x,y) such that $g(x,y)=k$ for some k. (The constraint is often given in some other form).
- Find the partial derivatives of both f and g.
- Introduce another variable λ (!) and solve the following system of equations:
  (1) $f_x(x,y)=\lambda g_x(x,y)$.
  (2) $f_y(x,y)=\lambda g_y(x,y)$.
  (3) $g(x,y)=k$.
  The (simultaneous) solutions of these equations will be critical points of the constrained equation.

It is remarkable that introducing the variable λ allows one to find these critical points - without it we would not find them all.

Example 1. *Find the minimum value of the function* x^2+xy+y^2 *subject to the constraint* $\sqrt{x^2+y}=5$.

The first step can sometimes be the most difficult in applied problems.

Example 2. *A production team has been budgeted \$60 million for the development and promotion of a new product line. Market experience predicts that of x million dollars is spent on development and y million on promotion, then $40x^{3/2}y$ units will be sold in the first year. How much money should be alloted to development and how much to promotion?*

In economics, Lagrange multipliers are often used to maximize *utility functions*. In the idealized world of theoretical economics, a utility function gives a numerical measure of "satisfaction" one experiences when one has given amounts of various goods. Intuitively, there is some utility function for beer and nachos whose maximum involves a combination of both - if you have nachos and no beer you feel kind of thirsty; beer and no nachos and you're inebriated and hungry.

Example 3. *Suppose beer costs \$4 for a pint and nachos \$6 per platter. Suppose a group of customers has \$120 to spend and they derive utility from x pints and y platters according to a* Cobb-Douglas utility function $10x^{0.6}y^{0.4}$. *How much beer and nachos should this group of customers order to maximize their utility?*

Example 4. *Find the optimal dimensions for a fish tank if it is to hold fifty thousand cubic centimeters of water, is supposed to have a total surface area of five thousand square centimeters, and costs one dollar per square centimeter for the base and fifty cents per square centimeter for the sides.*