MATH 241, LECTURE 7

1. Average rate of change

A fundamental philosophical truth is that everything changes. In physics, the change in position is known as velocity or speed. In economics, the change in price is known as inflation. In business, the change in costs is sometimes known as trend. In mathematics, the change in values of a function is known as the derivative. But to understand the derivative, which will measure "instantaneous" change, you need to to first be comfortable with "average" change over some intervals.

1.1. **Calculating average speed.** If you travel 200 miles in four hours, what is your average speed? What about 75 miles in one and a half hours?

$$\text{Average speed} = \frac{\text{distance travelled}}{\text{total time}}$$

Questions to ponder: Do these calculations necessarily mean you went 50mph for four hours?

Next, what if, instead of giving you total distance travelled, you need to calculate from a position function which describes where you are?

Example 1. *A ball which is dropped from the top of the Tower of Pisa has travelled down $16t^2$ feet after t seconds. What is its average speed over the first three seconds? over the first five seconds? between the second and fifth second?*

1.2. **Calculating slopes of secant lines to a curve.** Next we look at what at first appears to be unrelated to dropping a ball.

Definition 2. *A secant line goes through two points on the graph of the function. In symbols, it is a line through $(a, f(a))$ and $(b, f(b))$ for some a and b.*

Example 3. *Find the secant lines to the graph of $f(x) = 16x^2$ through the points with: $a = 0, b = 3$, $a = 0, b = 5$, $a = 2, b = 5$.*

What do you notice about this and the previous problem?

Example 4. *Some functions will be sketched on the board; find the slopes of secant lines as indicated.*

1.3. **Formula for average rate of change.**

Definition 5. *In general, the average rate of change of some function $f(x)$ as x varies between values a and b is*

$$\frac{f(b) - f(a)}{b - a}.$$

This can be computed in any way that f is presented, through a formula, through a graph, or in a table.

Example 6. *Find the average rate of change for \$1000 invested at a rate of five percent over four years. (Note: this is* not *the interest rate).*

Example 7. *Analyze different measured and predicted rates of change for world population according to:* `http://www.unfpa.org/6billion/pages/worldpopgrowth.htm`

2. The derivative

The derivative is the central topic of study in this class. It measures the instantaneous rate of change of a function at all times.

Before formalizing it, which is difficult, we will try to understand it in examples parallel to those we have done for average rate of change.

2.1. Instantaneous speed.

Question to ponder: how does your speedometer calculate how fast you are going at one moment?

(What does "how fast you are going at one moment" even mean?)

Example 8. *How fast was the ball falling two seconds after it was dropped from the Tower of Pisa? Five second after?*

2.2. Slope of tangent lines.

Definition 9. *A tangent line to a curve is a line which intersects the curve at some point, but does not cross the curve.*

Informally, a tangent line "kisses" the curve.

Question 10. *How could we calculate the equation of the line which is tangent to $f(x) = 16x^2$ at $x = 2$? at $x = 5$?*

2.3. Formula for instantaneous rate of change.

We would see similarities in trying to compute average and instantaneous differences regardless of what our functions are measuring. The following definitions work in all such cases. First, for average rate of change, we re-write $a = x$ and $b = x + h$, so that the difference quotient becomes $\frac{f(x+h)-f(x)}{h}$. So for example if $a = 3$ and $b = 4$, we would instead think of $x = 3$ and $h = 1$. The reason we change from a and b to x and h is to be consistent with the derivative.

Definition 11. *The derivative of $f(x)$ with respect to x is the function*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We will talk about lim, which means the "limit", later. For now, we will adopt a working notion that we may compute this for smaller and smaller h; if all of those computations seem to approach a single number, for now we will call that the limit.

Example 12. *What is the derivative of $f(x) = 16x^2$? What is the value of the derivative when $x = 2$? What does this mean for the velocity of a ball which is dropped from the Tower of Pisa? or the equation of the line tangent to $f(x)$ at $x = 2$?*

Example 13. *Suppose price index, measuring the aggregate price for a large cross-section of household goods measured in thousands of dollars, has values $1 - 0.5\sqrt{x} + 0.15x$ over two years. What is the rate of price increase over two years? What is the "instantaneous rate of inflation" at six months, one year, and two years?*