

RIGIDIFICATION OF QUASI-CATEGORIES

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ABSTRACT. We give a new construction for rigidifying a quasi-category into a simplicial category, and prove that it is weakly equivalent to the rigidification given by Lurie. Our construction comes from the use of necklaces, which are simplicial sets obtained by stringing simplices together. As an application of these methods, we use our model to reprove some basic facts from [L] about the rigidification process.

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1. INTRODUCTION

Quasi-categories are a certain generalization of categories, in which one has not only 1-morphisms but n -morphisms for every natural number n . They have been extensively studied by Cordier and Porter [CP], by Joyal [J1], [J2], and by Lurie [L]. If K is a quasi-category and x and y are two objects of K , then one may associate a “mapping space” $K(x, y)$ which is a simplicial set. There are many different constructions for these mapping spaces, but in [L] one particular model is given for which there are composition maps $K(y, z) \times K(x, y) \rightarrow K(x, z)$ giving rise to a simplicial category. This simplicial category is denoted $\mathfrak{C}(K)$, and it may be thought of as a *rigidification* of the quasi-category K . It is proven in [L] that the homotopy theories of quasi-categories and simplicial categories are equivalent via this functor.

In this paper we introduce some new models for the mapping spaces $K(x, y)$, which are particularly easy to describe and particularly easy to use—in fact they are just the nerves of ordinary categories (i.e., 1-categories). Like Lurie’s model, our models admit composition maps giving rise to a simplicial category; so we are giving a new method for rigidifying quasi-categories. We prove that our construction is homotopy equivalent (as a simplicial category) to Lurie’s $\mathfrak{C}(K)$. Moreover, because our mapping spaces are nerves of categories there are many standard tools available

for analyzing their homotopy types. We demonstrate the effectiveness of this by giving new proofs of some basic facts about the functor $\mathfrak{C}(-)$.

One payoff of this approach is that it is possible to give a streamlined proof of Lurie’s Quillen equivalence between the homotopy theory of quasi-categories and simplicial categories. This requires, however, a more detailed study of the model category structure on quasi-categories. We will take this up in a sequel [DS] and prove the Quillen equivalence there.

1.1. Mapping spaces via simplicial categories. Now we describe our results in more detail. A quasi-category is a simplicial set that has the right-lifting-property with respect to inner horn inclusions $\Lambda_i^n \rightarrow \Delta^n$, $0 < i < n$. It turns out that there is a unique model structure on $sSet$ where the cofibrations are the monomorphisms and the fibrant objects are the quasi-categories; this will be called the *Joyal model structure* and denoted $sSet_J$. The weak equivalences in $sSet_J$ will here be called *Joyal equivalences*. The existence of the Joyal model structure will not be needed in this paper, although it provides some useful context. The notions of quasi-categories and Joyal equivalences, however, will be used in several places. See Section 2.3 for additional background.

There is a functor, constructed in [L], which sends any simplicial set K to a corresponding simplicial category $\mathfrak{C}(K) \in sCat$. This is the left adjoint in a Quillen pair

$$\mathfrak{C}: sSet_J \rightleftarrows sCat: N,$$

where N is called the **coherent nerve**. The functor N can be described quite explicitly (see Section 2), but the functor \mathfrak{C} is in comparison a little mysterious. In [L] each $\mathfrak{C}(K)$ is defined as a certain colimit in the category $sCat$, but colimits in $sCat$ are notoriously difficult to understand.

Our main goal in this paper is to give a different model for the functor \mathfrak{C} . Define a **necklace** (which we picture as “unfastened”) to be a simplicial set of the form

$$\Delta^{n_0} \vee \Delta^{n_1} \vee \dots \vee \Delta^{n_k}$$

where each $n_i \geq 0$ and where in each wedge the final vertex of Δ^{n_i} has been glued to the initial vertex of $\Delta^{n_{i+1}}$.

The first and last vertex in any necklace T are denoted α_T and ω_T , respectively (or just α and ω if T is obvious from context). If S and T are two necklaces, then by $S \vee T$ we mean the necklace obtained in the evident way, by gluing the final vertex ω_S of S to the initial vertex α_T of T . Write Nec for the category whose objects are necklaces and where a morphism is a map of simplicial sets which preserves the initial and final vertices.

Let $S \in sSet$ and let $a, b \in S_0$. If T is a necklace, we use the notation

$$T \rightarrow S_{a,b}$$

to indicate a morphism of simplicial sets $T \rightarrow S$ which sends α_T to a and ω_T to b . Let $(Nec \downarrow S)_{a,b}$ denote the evident category whose objects are pairs $[T, T \rightarrow S_{a,b}]$ where T is a necklace. Note that for $a, b, c \in S$, there is a functor

$$(1.1) \quad (Nec \downarrow S)_{b,c} \times (Nec \downarrow S)_{a,b} \longrightarrow (Nec \downarrow S)_{a,c}$$

which sends the pair $[T_2, T_2 \rightarrow S_{b,c}] \times [T_1, T_1 \rightarrow S_{a,b}]$ to $[T_1 \vee T_2, T_1 \vee T_2 \rightarrow S_{a,c}]$.

Let $\mathfrak{C}^{nec}(S)$ be the function which assigns to any $a, b \in S_0$ the simplicial set $\mathfrak{C}^{nec}(S)(a, b) = N(Nec \downarrow S)_{a,b}$ (the classical nerve of the 1-category $(Nec \downarrow S)_{a,b}$).

The above pairings of categories induces pairings on the nerves, which makes $\mathfrak{C}^{nec}(S)$ into a simplicial category with object set S_0 .

Theorem 1.2. *There is a natural zig-zag of weak equivalences of simplicial categories between $\mathfrak{C}^{nec}(S)$ and $\mathfrak{C}(S)$, for all simplicial sets S .*

In the above result, the weak equivalences for simplicial categories are the so-called “DK-equivalences” used by Bergner in [B]. See Section 2 for this notion.

In this paper we also give an explicit description of the mapping spaces in the simplicial category $\mathfrak{C}(S)$. A rough statement is given below, but see Section 4 for more details.

Theorem 1.3. *Let S be a simplicial set and let $a, b \in S$. Then the mapping space $X = \mathfrak{C}(S)(a, b)$ is the simplicial set whose n -simplices are triples subject to a certain equivalence relations. The triples consist of a necklace T , a map $T \rightarrow S_{a,b}$, and a flag $\vec{T} = \{T^0 \subseteq \dots \subseteq T^n\}$ of vertices in T . For the equivalence relation, see Corollary 4.4. The face maps and degeneracy maps are obtained by removing or repeating elements T^i in the flag.*

The pairing

$$\mathfrak{C}(S)(b, c) \times \mathfrak{C}(S)(a, b) \longrightarrow \mathfrak{C}(S)(a, c)$$

sends the pair of n -simplices $([T \rightarrow S; \vec{T}^i], [U \rightarrow S; \vec{U}^i])$ to $[U \vee T \rightarrow S, \overline{U^i \cup T^i}]$.

Theorem 1.2 turns out to be very useful in the study of the functor \mathfrak{C} . There are many tools in classical homotopy theory for understanding the homotopy types of nerves of 1-categories, and via Theorem 1.2 these tools can be applied to understand mapping spaces in $\mathfrak{C}(S)$. We demonstrate this technique in Section 6 by proving, in a new way, the following two properties of \mathfrak{C} found in [L].

Theorem 1.4. *Let X and Y be simplicial sets.*

- (a) *The natural map $\mathfrak{C}(X \times Y) \rightarrow \mathfrak{C}(X) \times \mathfrak{C}(Y)$ is a weak equivalence of simplicial categories;*
- (b) *If $X \rightarrow Y$ is a Joyal equivalence then $\mathfrak{C}(X) \rightarrow \mathfrak{C}(Y)$ is a weak equivalence.*

1.5. Notation and Terminology. We will sometimes use $sSet_K$ to refer to the usual model structure on simplicial sets, which we’ll term the *Kan model structure*. The fibrations are the Kan fibrations, the weak equivalences (called Kan equivalences from now on) are the maps which induce homotopy equivalences on geometric realizations, and the cofibrations are the monomorphisms.

We will often be working with the category $sSet_{*,*} = (\partial\Delta^1 \downarrow sSet)$. Note that Nec is a full subcategory of $sSet_{*,*}$.

An object of $sSet_{*,*}$ is a simplicial set X with two distinguished points a and b . We sometimes (but not always) write $X_{a,b}$ for X , to remind us that things are taking place in $sSet_{*,*}$ instead of $sSet$.

If \mathfrak{C} is a (simplicial) category containing objects X and Y , we write $\mathfrak{C}(X, Y)$ for the (simplicial) set of morphisms from X to Y .

2. BACKGROUND ON QUASI-CATEGORIES

In this section we give the background on quasi-categories and simplicial categories needed in the rest of the paper.

2.1. Simplicial categories. A simplicial category is a category enriched over simplicial sets; it can also be thought of as a simplicial object of \mathcal{Cat} in which the categories in each level have the same object set. We use $s\mathcal{Cat}$ to denote the category of simplicial categories. A cofibrantly-generated model structure on $s\mathcal{Cat}$ was developed in [B]. A map of simplicial categories $F: \mathcal{C} \rightarrow \mathcal{D}$ is a weak equivalence (sometimes called a *DK-equivalence*) if

- (1) For all $a, b \in \text{ob } \mathcal{C}$, the map $\mathcal{C}(a, b) \rightarrow \mathcal{D}(Fa, Fb)$ is a Kan equivalence of simplicial sets;
- (2) The induced functor of ordinary categories $\pi_0 F: \pi_0 \mathcal{C} \rightarrow \pi_0 \mathcal{D}$ is surjective on isomorphism classes.

Likewise, the map F is a fibration if

- (1) For all $a, b \in \text{ob } \mathcal{C}$, the map $\mathcal{C}(a, b) \rightarrow \mathcal{D}(Fa, Fb)$ is a Kan fibration of simplicial sets;
- (2) For all $a \in \text{ob } \mathcal{C}$ and $b \in \text{ob } \mathcal{D}$, if $e: Fa \rightarrow b$ is a map in \mathcal{D} which becomes an isomorphism in $\pi_0 \mathcal{D}$, then there is an object $b' \in \mathcal{C}$ and a map $e': a \rightarrow b'$ such that $F(e') = e$ and e' becomes an isomorphism in $\pi_0 \mathcal{C}$.

The cofibrations are the maps which have the left lifting property with respect to the acyclic fibrations.

Remark 2.2. The second part of the fibration condition seems a little awkward at first. In this paper we will actually have no need to think about fibrations of simplicial categories, but have included the definition for completeness.

Bergner writes down sets of generating cofibrations and acyclic cofibrations in [B].

2.3. Quasi-categories and Joyal equivalences. As mentioned in the introduction, there is a unique model structure on $s\mathcal{Set}$ with the properties that

- (i) The cofibrations are the monomorphisms;
- (ii) The fibrant objects are the quasi-categories.

It is easy to see that there is at most one such structure. To do this, let E^1 be the 0-coskeleton—see [AM], for instance—of the set $\{0, 1\}$ (note that the geometric realization of E^1 is essentially the standard model for S^∞). The map $E^1 \rightarrow *$ has the right lifting property with respect to all monomorphisms, and so it will be an acyclic fibration in this structure. Therefore $X \times E^1 \rightarrow X$ is also an acyclic fibration for any X , and hence $X \times E^1$ will be a cylinder object for X . Since every object is cofibrant, a map $A \rightarrow B$ will be a weak equivalence if and only if it induces bijections $[B, Z]_{E^1} \rightarrow [A, Z]_{E^1}$ for every quasi-category Z , where $[A, Z]_{E^1}$ means the coequalizer of $s\mathcal{Set}(A \times E^1, Z) \rightrightarrows s\mathcal{Set}(A, Z)$. Therefore the weak equivalences are determined by properties (i)–(ii), and since the cofibrations and weak equivalences are determined so are the fibrations.

Motivated by the above discussion, we define a map of simplicial sets $A \rightarrow B$ to be a **Joyal equivalence** if it induces bijections $[B, Z]_{E^1} \rightarrow [A, Z]_{E^1}$ for every quasi-category Z .

That there actually exists a model structure satisfying (i) and (ii) is not so clear, but it was established by Joyal (see [J1] or [J2], or [L] for another proof). For this reason, we will call it the **Joyal model structure** and denote it by $s\mathcal{Set}_J$. The weak equivalences are defined differently in both [J2] and [L], but of course turn out to be equivalent to the definition we have adopted here.

In the rest of the paper we will never use the Joyal model structure, only the notion of Joyal equivalence.

2.4. Background on \mathfrak{C} and N . Given a simplicial category S , one can construct a simplicial set called the *coherent nerve* of S [L, 1.1.5]. We will now describe this construction.

Recall the adjoint functors $F: \mathit{Grph} \rightleftarrows \mathit{Cat}: U$. Here Cat is the category of 1-categories, and Grph is the category of graphs: a graph consists of an object set and morphism sets, but no composition law. The functor U is a forgetful functor, and F is a free functor. Given any category \mathfrak{C} we may then consider the comonad resolution $(FU)_\bullet(\mathfrak{C})$ given by $[n] \mapsto (FU)^{n+1}(\mathfrak{C})$. This is a simplicial category.

There is a functor of simplicial categories $(FU)_\bullet(\mathfrak{C}) \rightarrow \mathfrak{C}$ (where the latter is considered a discrete simplicial category). This functor induces a weak equivalence on all mapping spaces, a fact which can be seen by applying U , at which point the comonad resolution picks up a contracting homotopy. Note that this means that the simplicial mapping spaces in $(FU)_\bullet(\mathfrak{C})$ are all homotopy discrete.

Recall that $[n]$ denotes the category $0 \rightarrow 1 \rightarrow \dots \rightarrow n$, where there is a unique map from i to j whenever $i \leq j$. We let $\mathfrak{C}(\Delta^n)$ denote the simplicial category $(FU)_\bullet([n])$. The mapping spaces in this simplicial category can be analyzed completely, and are as follows. For each i and j , let $P_{i,j}$ denote the poset of all subsets of $\{i, i+1, \dots, j\}$ containing i and j (ordered by inclusion). Note that the nerve of $P_{i,j}$ is isomorphic to the cube $(\Delta^1)^{j-i-1}$ if $j > i$, Δ^0 if $j = i$, and the emptyset if $j < i$. The nerves of the $P_{i,j}$'s naturally form the mapping spaces of a simplicial category with object set $\{0, 1, \dots, n\}$, using the pairings $P_{j,k} \times P_{i,j} \rightarrow P_{i,k}$ given by union of sets.

Lemma 2.5. *There is an isomorphism of simplicial categories $\mathfrak{C}(\Delta^n) \cong NP$.*

Remark 2.6. The proof of the above lemma is a bit of an aside from the main thrust of the paper, so it is given in Appendix A. In fact we could have *defined* $\mathfrak{C}(\Delta^n)$ to be NP , which is what Lurie does in [L], and avoided the lemma entirely; the construction $(FU)_\bullet([n])$ will never again be used in this paper. Nevertheless, the identification of NP with $(FU)_\bullet([n])$ seems informative to us.

For any simplicial category \mathcal{D} , the **coherent nerve** of \mathcal{D} is the simplicial set $N\mathcal{D}$ given by

$$[n] \mapsto s\mathit{Cat}(\mathfrak{C}(\Delta^n), \mathcal{D}).$$

It was proven by Lurie [L] that if every mapping space in \mathcal{D} is a Kan complex, then $N\mathcal{D}$ is a quasi-category; see also Lemma 6.5 below.

The functor N has a left adjoint denoted $\mathfrak{C}: s\mathit{Set} \rightarrow s\mathit{Cat}$. Any simplicial set K may be written as a colimit of simplices via the formula

$$K \cong \operatorname{colim}_{\Delta^n \rightarrow K} \Delta^n,$$

and consequently one has

$$(2.6) \quad \mathfrak{C}(K) \cong \operatorname{colim}_{\Delta^n \rightarrow K} \mathfrak{C}(\Delta^n)$$

where the colimit takes place in $s\mathit{Cat}$. This formula is a bit unwieldy, however, in the sense that it does not give much concrete information about the mapping spaces in $\mathfrak{C}(K)$. The point of the next three sections is to obtain such concrete information, via the use of necklaces.

3. NECKLACES

A necklace is a simplicial set obtained by stringing simplices together in succession. In this section we establish some basic facts about them, as well as facts about the more general category of ordered simplicial sets. When T is a necklace we are able to give a complete description of the mapping spaces in $\mathfrak{C}(T)$ as nerves of certain posets, generalizing what was said for $\mathfrak{C}(\Delta^n)$ in the last section. See Proposition 3.7.

As briefly discussed in the introduction, a **necklace** is defined to be a simplicial set of the form

$$\Delta^{n_0} \vee \Delta^{n_1} \vee \dots \vee \Delta^{n_k}$$

where each $n_i \geq 0$ and where in each wedge the final vertex of Δ^{n_i} has been glued to the initial vertex of $\Delta^{n_{i+1}}$. We say that the necklace is in **preferred form** if either $k = 0$ or each $n_i \geq 1$.

Let $T = \Delta^{n_0} \vee \Delta^{n_1} \vee \dots \vee \Delta^{n_k}$ be in preferred form. Each Δ^{n_i} is called a **bead** of the necklace. A **joint** of the necklace is either an initial or a final vertex in some bead. Thus, every necklace has at least one vertex, one bead, and one joint; Δ^0 is not a bead in any necklace except in the necklace Δ^0 itself.

Given a necklace T , write V_T and J_T for the sets of vertices and joints of T . Note that $V_T = T_0$ and $J_T \subseteq V_T$. Both V_T and J_T are totally ordered, by saying $a \leq b$ if there is a directed path in T from a to b . The initial and final vertices of T are denoted α_T and ω_T (and we sometimes drop the subscript); note that $\alpha_T, \omega_T \in J_T$.

Every necklace T comes with a particular map $\partial\Delta^1 \rightarrow T$ which sends 0 to the initial vertex of the necklace, and 1 to the final vertex. If S and T are two necklaces, then by $S \vee T$ we mean the necklace obtained in the evident way, by gluing the final vertex of S to the initial vertex of T . Let Nec denote the full subcategory of $sSet_{*,*} = (\partial\Delta^1 \downarrow sSet)$ whose objects are necklaces $\partial\Delta^1 \rightarrow T$. We sometimes talk about Nec as though it is a subcategory of $sSet$.

A simplex is a necklace with one bead. A **spine** is a necklace in which every bead is a Δ^1 . Every necklace T has an associated simplex and spine, which we now define. Let $\Delta[T]$ be the simplex whose vertex set is the same as the (ordered) vertex set of T . Likewise, let $\text{Spi}[T]$ be the longest spine inside of T . Note that there are inclusions $\text{Spi}[T] \hookrightarrow T \hookrightarrow \Delta[T]$. The assignment $T \rightarrow \Delta[T]$ is a functor, but $T \rightarrow \text{Spi}[T]$ is not (for instance, the unique map of necklaces $\Delta^1 \rightarrow \Delta^2$ does not induce a map on spines).

3.1. Ordered simplicial sets. If $T \rightarrow T'$ is a map of necklaces, then the image of T is also a necklace. To prove this, as well as for several other reasons scattered throughout the paper, it turns out to be very convenient to work in somewhat greater generality.

If X is a simplicial set, define a relation on its 0-simplices by saying that $x \preceq y$ if there exists a spine T and a map $T \rightarrow X$ sending $\alpha_T \mapsto x$ and $\omega_T \mapsto y$. In other words, $x \preceq y$ if there is a directed path from x to y inside of X . Note that this relation is clearly reflexive and transitive, but not necessarily antisymmetric: that is, if $x \preceq y$ and $y \preceq x$ it need not be true that $x = y$.

Definition 3.2. A simplicial set X is **ordered** if

- (i) The relation \preceq defined on X_0 is antisymmetric, and

- (ii) An simplex $x \in X_n$ is determined by its sequence of vertices $x(0) \preceq \cdots \preceq x(n)$; i.e. no two distinct n -simplices have identical vertex sequences.

Note the role of degenerate simplices in condition (ii). For example, notice that $\Delta^1/\partial\Delta^1$ is not an ordered simplicial set.

The following notion is also useful:

Definition 3.3. Let A and X be simplicial sets. A map $A \rightarrow X$ is called a **simple inclusion** if it has the right lifting property with respect to the canonical inclusions $\partial\Delta^1 \hookrightarrow T$ for all necklaces T . (Note that such a map really is an inclusion, because it has the lifting property for $\partial\Delta^1 \rightarrow \Delta^0$).

The notion of simple inclusion says that if there is a “path” (in the sense of a necklace) in X that starts and ends in A , then it must lie entirely in A . As an example, four out of the five inclusions $\Delta^1 \hookrightarrow \Delta^1 \times \Delta^1$ are simple inclusions.

Lemma 3.4. A simple inclusion $A \hookrightarrow X$ has the right lifting property with respect to the maps $\partial\Delta^k \hookrightarrow \Delta^k$ for all $k \geq 1$.

Proof. Suppose given a square

$$\begin{array}{ccc} \partial\Delta^k & \longrightarrow & A \\ \downarrow & & \downarrow \\ \Delta^k & \longrightarrow & X. \end{array}$$

By restricting the map $\partial\Delta^k \rightarrow A$ to $\partial\Delta^1 \hookrightarrow \partial\Delta^k$ (given by the initial and final vertices of $\partial\Delta^k$), we get a corresponding lifting square with $\partial\Delta^1 \hookrightarrow \Delta^k$. Since $A \rightarrow X$ is a simple inclusion, this new square has a lift $l: \Delta^k \rightarrow A$. It is not immediately clear that l restricted to $\partial\Delta^k$ equals our original map, but the two maps are equal after composing with $A \rightarrow X$; since $A \rightarrow X$ is a monomorphism, the two maps are themselves equal. \square

Lemma 3.5. Let X and Y denote ordered simplicial sets and let $f: X \rightarrow Y$ be a map.

- (1) The category of ordered simplicial sets is closed under taking finite limits.
- (2) Every necklace is an ordered simplicial set.
- (3) If $X' \subseteq X$ is a simplicial subset, then X' is also ordered.
- (4) The map f is completely determined by the map $f_0: X_0 \rightarrow Y_0$ on vertices.
- (5) If f_0 is injective then so is f .
- (6) The image of an n -simplex $x: \Delta^n \rightarrow X$ is of the form $\Delta^k \hookrightarrow X$ for some $k \leq n$.
- (7) If T is a necklace and $y: T \rightarrow X$ is a map, then its image is a necklace.
- (8) Suppose that $X \leftarrow A \rightarrow Y$ is a diagram of ordered simplicial sets, and both $A \rightarrow X$ and $A \rightarrow Y$ are simple inclusions. Then the pushout $B = X \amalg_A Y$ is an ordered simplicial set, and the inclusions $X \hookrightarrow B$ and $Y \hookrightarrow B$ are both simple.

Proof. For (1), the terminal object is a point with its unique ordering. Given a diagram of the form

$$X \longrightarrow Z \longleftarrow Y,$$

let $A = X \times_Z Y$. It is clear that if $(x, y) \preceq_A (x', y')$ then both $x \preceq_X x'$ and $y \preceq_Y y'$ hold, and so antisymmetry of \preceq_A follows from that of \preceq_X and \preceq_Y . Condition (ii) from Definition 3.2 is easy to check.

Parts (2)–(5) are easy, and left to the reader.

For (6), the sequence $x(0), \dots, x(n) \in X_0$ may have duplicates; let $d: \Delta^k \rightarrow \Delta^n$ denote any face such that $x \circ d$ contains all vertices $x(j)$ and has no duplicates. Note that $x \circ d$ is an injection by (5). A certain degeneracy of $x \circ d$ has the same vertex sequence as x . Since X is ordered, x is this degeneracy of $x \circ d$. Hence, $x \circ d: \Delta^k \hookrightarrow X$ is the image of x .

Claim (7) follows from (6).

For claim (8) we first show that the maps $X \hookrightarrow B$ and $Y \hookrightarrow B$ are simple inclusions. To see this, suppose that $u, v \in X$ are vertices, T is a necklace, and $f: T \rightarrow B_{u,v}$ is a map; we want to show that f factors through X . Note that any simplex $\Delta^k \rightarrow B$ either factors through X or through Y . Suppose that f does not factor through X . From the set of beads of T which do not factor through X , take any maximal subset T' in which all the beads are adjacent. Then we have a necklace $T' \subseteq T$ such that $f(T') \subseteq Y$. If there exists a bead in T prior to $\alpha_{T'}$, then it must map into X since T' was maximal; so $f(\alpha_{T'})$ lies in $X \cap Y = A$. Likewise, if there is no bead prior to $\alpha_{T'}$ then $f(\alpha_{T'}) = u$ and so again $f(\alpha_{T'})$ lies in $X \cap Y = A$. Similar remarks apply to show that $f(\omega_{T'})$ lies in A . At this point the fact that $A \hookrightarrow Y$ is a simple inclusion implies that $f(T') \subseteq A \subseteq X$, which is a contradiction. So in fact f factored through X .

We have shown that $X \hookrightarrow B$ (and dually $Y \hookrightarrow B$) is a simple inclusion. Now we show that B is ordered, so suppose $u, v \in B$ are such that $u \preceq v$ and $v \preceq u$. There are spines T and U and maps $T \rightarrow B_{u,v}, U \rightarrow B_{v,u}$. Consider the composite spine $T \vee U \rightarrow B_{u,u}$. If $u \in X$, then since $X \hookrightarrow B$ is a simple inclusion it follows that the image of $T \vee U$ maps entirely into X ; so $u \preceq_X v$ and $v \preceq_X u$, which means $u = v$ because X is ordered. The same argument works if $u \in Y$, so this verifies antisymmetry of \preceq_B .

To verify condition (ii) of Definition 3.2, suppose $p, q: \Delta^k \rightarrow B$ are k -simplices with the same sequence of vertices; we wish to show $p = q$. We know that p factors through X or Y , and so does q ; if both factor through Y , then the fact that Y is ordered implies that $p = q$ (similarly for X). So we may assume p factors through X and q factors through Y . By induction on k , the restrictions $p|_{\partial\Delta^k} = q|_{\partial\Delta^k}$ are equal, hence factor through A . By Lemma 3.4 applied to $A \hookrightarrow X$, the map p factors through A . Therefore it also factors through Y , and now we are done because q also factors through Y and Y is ordered. \square

3.6. Categorification of necklaces. Let T be a necklace. Our next goal is to give a complete description of the simplicial category $\mathfrak{C}(T)$. The object set of this category is precisely T_0 .

For vertices $a, b \in T_0$, let $V_T(a, b)$ denote the set of vertices in T between a and b , inclusive (with respect to the relation \preceq). Let $J_T(a, b)$ denote the union of $\{a, b\}$ with the set of joints between a and b . There is a unique subnecklace of T with joints $J_T(a, b)$ and vertices $V_T(a, b)$; let $\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_k$ denote its beads. There are canonical inclusions of each \tilde{B}_i to T . Hence, there is a natural map

$$\mathfrak{C}(\tilde{B}_k)(j_k, b) \times \mathfrak{C}(\tilde{B}_{k-1})(j_{k-1}, j_k) \times \cdots \times \mathfrak{C}(\tilde{B}_1)(j_1, j_2) \times \mathfrak{C}(\tilde{B}_0)(a, j_1) \rightarrow \mathfrak{C}(T)(a, b)$$

Proof. Follows at once from the previous lemma. \square

Remark 3.9. Given a necklace T , there is a heuristic way to understand faces (both codimension one and higher) in the cubes $\mathfrak{C}(T)(a, b)$ in terms of “paths” from a to b in T . To choose a face in $\mathfrak{C}(T)(a, b)$, one chooses three subsets $Y, N, M \subset V_T(a, b)$ which cover the set $V_T(a, b)$ and are mutually disjoint. The set Y is the set of vertices which we require our path to go through – it must contain $J_T(a, b)$; the set N is the set of vertices which we require our path to not go through; and the set M is the set of vertices for which we leave the question open. Such choices determine a unique face in $\mathfrak{C}(T)(a, b)$. The dimension of this face is precisely the number of vertices in M .

4. THE CATEGORIFICATION FUNCTOR

By this point, we fully understand $\mathfrak{C}(\Delta^n)$ as a simplicial category. Recall that $\mathfrak{C}: sSet \rightarrow sCat$ is defined for $S \in sSet$ by the formula

$$\mathfrak{C}(S) = \operatorname{colim}_{\Delta^n \rightarrow S} \mathfrak{C}(\Delta^n).$$

The trouble with this formula is that given a diagram $X: I \rightarrow sCat$ of simplicial categories, it is generally quite difficult to understand the mapping spaces in the colimit. In our case, however, something special happens because the simplicial categories $\mathfrak{C}(\Delta^n)$ are “directed” in a certain sense. It turns out by making use of necklaces one can write down a precise description of the mapping spaces for $\mathfrak{C}(S)$; this is the goal of the present section.

Fix a simplicial set S and elements $a, b \in S_0$. For any necklace T and map $T \rightarrow S_{a,b}$, there is an induced map $\mathfrak{C}(T)(\alpha, \omega) \rightarrow \mathfrak{C}(S)(a, b)$. Let $(Nec \downarrow S)_{a,b}$ denote the category whose objects are pairs $[T, T \rightarrow S_{a,b}]$ and whose morphisms are maps of necklaces $T \rightarrow T'$ giving commutative triangles over S . Then we obtain a map

$$(4.1) \quad \operatorname{colim}_{T \rightarrow S \in (Nec \downarrow S)_{a,b}} \left[\mathfrak{C}(T)(\alpha, \omega) \right] \longrightarrow \mathfrak{C}(S)(a, b).$$

Let us write $E_S(a, b)$ for the domain of this map. Note that there are composition maps

$$(4.2) \quad E_S(b, c) \times E_S(a, b) \longrightarrow E_S(a, c)$$

induced in the following way. Given $T \rightarrow S_{a,b}$ and $U \rightarrow S_{b,c}$ where T and U are necklaces, one obtains $T \vee U \rightarrow S_{a,c}$ in the evident manner. The composite

$$\begin{array}{ccc} \mathfrak{C}(U)(\alpha_U, \omega_U) \times \mathfrak{C}(T)(\alpha_T, \omega_T) & \longrightarrow & \mathfrak{C}(T \vee U)(\omega_T, \omega_U) \times \mathfrak{C}(T \vee U)(\alpha_T, \omega_T) \\ & & \downarrow \mu \\ & & \mathfrak{C}(T \vee U)(\alpha_T, \omega_U) \end{array}$$

induces the pairing of (4.2). One readily checks that E_S is a simplicial category with object set S_0 , and (4.1) yields a map of simplicial categories $E_S \rightarrow \mathfrak{C}(S)$. Moreover, the construction E_S is clearly functorial in S .

Here is our first result:

Proposition 4.3. *For every simplicial set S , the map $E_S \rightarrow \mathfrak{C}(S)$ is an isomorphism of simplicial categories.*

Proof. First note that if S is itself a necklace then the identity map $S \rightarrow S$ is a terminal object in $(\text{Nec} \downarrow S)_{a,b}$. It follows at once that $E_S(a, b) \rightarrow \mathfrak{C}(S)(a, b)$ is an isomorphism for all a and b .

Now let S be an arbitrary simplicial set, and choose vertices $a, b \in S_0$. We will show that $E_S(a, b) \rightarrow \mathfrak{C}(S)(a, b)$ is an isomorphism. Consider the commutative diagram of simplicial sets

$$\begin{array}{ccc} (\text{colim}_{\Delta^k \rightarrow S} E_{\Delta^k})(a, b) & \xrightarrow{t} & E_S(a, b) \\ \cong \downarrow & & \downarrow \\ (\text{colim}_{\Delta^k \rightarrow S} \mathfrak{C}(\Delta^k))(a, b) & \xlongequal{\quad} & \mathfrak{C}(S)(a, b). \end{array}$$

The bottom equality is the definition of \mathfrak{C} . The left-hand map is an isomorphism by our remarks in the first paragraph. It follows that the top map t is injective. To complete the proof it therefore suffices to show that t is surjective.

Choose an n -simplex $x \in E_S(a, b)_n$; it is represented by a necklace T , a map $f: T \rightarrow S_{a,b}$, and an element $\tilde{x} \in \mathfrak{C}(T)(\alpha, \omega)$. We have a commutative diagram

$$\begin{array}{ccc} (\text{colim}_{\Delta^k \rightarrow T} \mathfrak{C}(\Delta^k))(\alpha, \omega) & \longrightarrow & \mathfrak{C}(T)(\alpha, \omega) \\ \uparrow & & \uparrow \\ (\text{colim}_{\Delta^k \rightarrow T} E_{\Delta^k})(\alpha, \omega) & \longrightarrow & E_T(\alpha, \omega) \\ f \downarrow & & \downarrow E_f \\ (\text{colim}_{\Delta^k \rightarrow S} E_{\Delta^k})(a, b) & \xrightarrow{t} & E_S(a, b). \end{array}$$

The n -simplex in $E_T(\alpha, \omega)$ represented by $[T, \text{id}_T: T \rightarrow T; \tilde{x}]$ is sent to x under E_f . It suffices to show that the middle horizontal map is surjective, for then x will be in the image of t . But the top map is an isomorphism, and the vertical arrows in the top row are isomorphisms by the remarks from the first paragraph. Thus, we are done. \square

Corollary 4.4. *For any simplicial set S and elements $a, b \in S_0$, the simplicial set $\mathfrak{C}(S)(a, b)$ admits the following description. An n -simplex in $\mathfrak{C}(S)(a, b)$ consists of an equivalence class of triples $[T, T \rightarrow S, \vec{T}]$, where*

- T is a necklace;
- $T \rightarrow S$ is a map of simplicial sets which sends α_T to a and ω_T to b ; and
- \vec{T} is a flag of sets $T^0 \subseteq T^1 \subseteq \dots \subseteq T^n$ such that T^0 contains the joints of T and T^n is contained in the set of vertices of T .

The equivalence relation is generated by considering $(T \rightarrow S; \vec{T})$ and $(U \rightarrow S; \vec{U})$ to be equivalent if there exists a map of necklaces $f: T \rightarrow U$ over S with $\vec{U} = f_*(\vec{T})$.

The i th face (resp. degeneracy) map omits (resp. repeats) the set T^i in the flag. That is, if $x = (T \rightarrow S; T^0 \subseteq \dots \subseteq T^n)$ represents an n -simplex of $\mathfrak{C}(S)(a, b)$ and $0 \leq i \leq n$, then

$$s_i(x) = (T \rightarrow S; T^0 \subseteq \dots \subseteq T^i \subseteq T^i \subseteq \dots \subseteq T^n)$$

and

$$d_i(x) = (T \rightarrow S; T^0 \subseteq \dots \subseteq T^{i-1} \subseteq T^{i+1} \subseteq \dots \subseteq T^n).$$

Proof. This is a straightforward interpretation of the colimit appearing in the definition of E_S from (4.1). Recall that every colimit can be written as a coequalizer

$$\operatorname{colim}_{T \rightarrow S \in (\mathcal{N}ec \downarrow S)_{a,b}} [\mathfrak{C}(T)(\alpha, \omega)] \cong \operatorname{coeq} \left[\prod_{T_1 \rightarrow T_2 \rightarrow S} \mathfrak{C}(T_1)(\alpha, \omega) \rightrightarrows \prod_{T \rightarrow S} \mathfrak{C}(T)(\alpha, \omega) \right],$$

and that elements of $\mathfrak{C}(T)$ are identified with flags of subsets of V_T , containing J_T , by Lemma 3.7. \square

Our next goal is to simplify the equivalence relation appearing in Corollary 4.4 somewhat. This analysis is somewhat cumbersome, but culminates in the important Proposition 4.10.

Let us begin by introducing some terminology. A **flagged necklace** is a pair $[T, \vec{T}]$ where T is a necklace and \vec{T} is a flag of subsets of V_T which all contain J_T . The **length of the flag** is the number of subset symbols, or one less than the number of subsets. A morphism of flagged necklaces $[T, \vec{T}] \rightarrow [U, \vec{U}]$ exists only if the flags have the same length, in which case it is a map of necklaces $f: T \rightarrow U$ such that $f(T^i) = U^i$ for all i . Finally, a flag $\vec{T} = (T^0 \subseteq \cdots \subseteq T^n)$ is called **flanked** if $T^0 = J_T$ and $T^n = V_T$. Note that if $[T, \vec{T}]$ and $[U, \vec{U}]$ are both flanked, then every morphism $[T, \vec{T}] \rightarrow [U, \vec{U}]$ is surjective (because its image will be a subnecklace of U having the same joints and vertices as U , hence it must be all of U).

Lemma 4.5. *Under the equivalence relation of Corollary 4.4, each of the triples $[T, T \rightarrow S, \vec{T}]$ is equivalent to one in which the flag is flanked. Moreover, two flanked triples are equivalent (in the sense of Corollary 4.4) if and only if they can be connected by a zig-zag of morphisms of flagged necklaces in which every triple of the zig-zag is flanked.*

Proof. Suppose given a flagged necklace $[T, T^0 \subseteq \cdots \subseteq T^n]$. There is a unique subnecklace $T' \hookrightarrow T$ whose set of joints is T^0 and whose vertex set is T^n . Then the pair $(T', T^0 \subseteq \cdots \subseteq T^n)$ is flanked. This assignment, which we call *flankification*, is actually functorial: a morphism of flagged necklaces $f: [T, \vec{T}] \rightarrow [U, \vec{U}]$ must map T' into U' and therefore gives a morphism $[T', \vec{T}] \rightarrow [U', \vec{U}]$.

Using the equivalence relation of Corollary 4.4, each triple $[T, T \rightarrow S, \vec{T}]$ will be equivalent to the flanked triple $[T', T' \rightarrow T \rightarrow S, \vec{T}]$ via the map $T' \rightarrow T$. If the flanked triple $[U, U \rightarrow S, \vec{U}]$ is equivalent to the flanked triple $[V, V \rightarrow S, \vec{V}]$ then there is a zig-zag of maps between triples which starts at the first and ends at the second, by Corollary 4.4. Applying the flankification functor gives a corresponding zig-zag in which every object is flanked. \square

Remark 4.6. By the previous lemma, we can alter our model for $\mathfrak{C}(S)(a, b)$ so that the n -simplices are equivalence classes of triples $[T, T \rightarrow S, \vec{T}]$ in which the flag is flanked, and the equivalence relation is given by maps (which are necessarily surjections) of flanked triples. Under this model the degeneracies and inner faces are given by the same description as before: repeating or omitting one of the subsets in the flag. The outer faces d_0 and d_n are now more complicated, however, because omitting the first or last subset in the flag may produce one which is no longer flanked; one must first remove the subset and then apply the flankification functor from Lemma 4.5. This model for $\mathfrak{C}(S)(a, b)$ was originally shown to us by Jacob Lurie; it will play only a very minor role in what follows.

Our next task will be to analyze surjections of flagged triples. Let T be a necklace and S a simplicial set. Say that a map $T \rightarrow S$ is **totally nondegenerate** if the image of each bead of T is a nondegenerate simplex of S . Note a totally nondegenerate map need not be an injection: for example, let $S = \Delta^1/\partial\Delta^1$ and consider the nondegenerate 1-simplex $\Delta^1 \rightarrow S$.

Recall that in a simplicial set S , if $z \in S$ is a degenerate simplex then there is a unique nondegenerate simplex z' and a unique degeneracy operator $s_\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ such that $z = s_\sigma(z')$; see [H, Lemma 15.8.4]. Using this, and the fact that degeneracy operators correspond to surjections of simplices, one finds that for any map $T \rightarrow S$ there is a necklace \bar{T} , a map $\bar{T} \rightarrow S$ which is totally nondegenerate, and a surjection of necklaces $T \rightarrow \bar{T}$ making the evident triangle commute; moreover, these three things are unique up to isomorphism.

Proposition 4.7. *Let S be a simplicial set and let $a, b \in S_0$.*

- (a) *Suppose that T and U are necklaces, $U \xrightarrow{u} S$ and $T \xrightarrow{t} S$ are two maps, and that t is totally nondegenerate. Then there is at most one surjection $f: U \rightarrow T$ such that $u = t \circ f$.*
- (b) *Suppose that one has a diagram*

$$\begin{array}{ccc} U & \xrightarrow{f} & T \\ g \downarrow & & \downarrow \\ V & \longrightarrow & S \end{array}$$

where T , U , and V are flagged necklaces, $T \rightarrow S$ is totally nondegenerate, and f and g are surjections. Then there exists a unique map of flagged necklaces $V \rightarrow T$ making the diagram commute.

Proof. We first make the observation that if $A \rightarrow B$ is a surjection of necklaces and $B \neq *$ then every bead of B is surjected on by a unique bead of A . Also, each bead of A is either collapsed onto a joint of B or else mapped surjectively onto a bead of B .

For (a), note that we may assume $T \neq *$ (or else the claim is trivial). Assume there are two distinct surjections $f, f': U \rightarrow T$ such that $tf = tf' = u$. Let B be the first bead of U on which f and f' disagree. Let j denote the initial vertex of U , and let C be the bead of T whose initial vertex is $f(j) = f'(j)$.

If u maps B to a point in S then B cannot surject onto the bead C (using that $T \rightarrow S$ is totally nondegenerate); so B must be collapsed to a point by both f and f' . Alternatively, if u does not map B to a point then B must surject onto the bead C via both f and f' ; this identifies the simplex $B \rightarrow U \rightarrow S$ with a degeneracy of the nondegenerate simplex $C \rightarrow S$. Then by uniqueness of degeneracies we have that f and f' must coincide on B , which is a contradiction.

Next we turn to part (b). Note that the map $V \rightarrow T$ will necessarily be surjective, so the uniqueness part is guaranteed by (a); we need only show existence.

Observe that if B is a bead in U which maps to a point in V then it maps to a point in T , by the reasoning above. It now follows that there exists a necklace U' , obtained by collapsing every bead of U that maps to a point in V , and a

commutative diagram

$$\begin{array}{ccc}
 U & \xrightarrow{f} & T \\
 \downarrow g & \searrow f' & \downarrow \\
 & U' & \\
 \downarrow g' & \swarrow & \downarrow \\
 V & \xrightarrow{\quad} & S
 \end{array}$$

Replacing U , f , and g by U' , f' , and g' , and dropping the primes, we can now assume that g induces a one-to-one correspondence between beads of U and beads of V . Let B_1, \dots, B_m denote the beads of U , and let C_1, \dots, C_m denote the beads of V .

Assume that we have constructed the lift $l: V \rightarrow T$ on the beads C_1, \dots, C_{i-1} . If the bead B_i is mapped by f to a point, then evidently we can define l to map C_i to this same point and the diagram will commute. Otherwise f maps B_i surjectively onto a certain bead D inside of T . We have the diagram

$$\begin{array}{ccc}
 B_i & \xrightarrow{f} & D \\
 \downarrow g & & \downarrow t \\
 C_i & \xrightarrow{v} & S
 \end{array}$$

where here f and g are surjections between simplices and therefore represent degeneracy operators s_f and s_g . We have that $s_f(t) = s_g(v)$. But the simplex t of S is nondegenerate by assumption, therefore by [H, Lemma 15.8.4] we must have $v = s_h(t)$ for some degeneracy operator s_h such that $s_f = s_g s_h$. The operator s_h corresponds to a surjection of simplices $C_i \rightarrow D$ making the above square commute, and we define l on C_i to coincide with this map. Continuing by induction, this produces the desired lift l . It is easy to see that l is a map of flagged necklaces, as $l(V^i) = l(g(U^i)) = f(U^i) = T^i$. \square

Corollary 4.8. *Let S be a simplicial set and $a, b \in S_0$. Under the equivalence relation from Corollary 4.4, every triple $[T, T \rightarrow S_{a,b}, \vec{T}]$ is equivalent to a unique triple $[U, U \rightarrow S_{a,b}, \vec{U}]$ which is both flanked and totally nondegenerate.*

Proof. Let $t = [T, T \rightarrow S_{a,b}, \vec{T}]$. Then t is clearly equivalent to at least one flanked, totally nondegenerate triple because we can replace t with $[T', T' \rightarrow S_{a,b}, \vec{T}']$ (flankification) and then with $[\vec{T}', \vec{T}' \rightarrow S_{a,b}, \vec{T}']$ (defined above Proposition 4.7).

Now suppose that $[U, U \rightarrow S_{a,b}, \vec{U}]$ and $[V, V \rightarrow S_{a,b}, \vec{V}]$ are both flanked, totally nondegenerate, and equivalent in $\mathfrak{C}(S)(a, b)_n$. Then by Lemma 4.5 there is a zig-zag of maps between flanked necklaces (over S) connecting U to V :

$$\begin{array}{ccccccc}
 & W_1 & & W_2 & & \cdots & & W_k & & \\
 & \swarrow & & \swarrow & & \swarrow & & \swarrow & & \\
 U = U_1 & & U_2 & & U_3 & & \cdots & & U_k & & U_{k+1} = V
 \end{array}$$

Using Proposition 4.7, we inductively construct surjections of flanked necklaces $U_i \rightarrow U$ over S . This produces a surjection $V \rightarrow U$ over S . Similarly, we obtain a surjection $U \rightarrow V$ over S . By Proposition 4.7(a) these maps must be inverses of each other; that is, they are isomorphisms. \square

Remark 4.9. Again, as in Remark 4.6 the above corollary shows that we can describe $\mathfrak{C}(S)(a, b)$ as the simplicial set whose n -simplices are triples $[T, T \rightarrow S, \vec{T}]$ which are both flanked and totally nondegenerate. The degeneracies and inner faces are again easy to describe—they are repetition or omission of a set in the flag—but for the outer faces one must first omit a set and then modify the triple appropriately. The usefulness of this description is limited because of these complications with the outer faces, but it does make a brief appearance in Corollary 4.13 below.

The following result is the culmination of our work in this section, and will turn out to be a key step in the proof of our main theorems. Fix a simplicial set S and vertices $a, b \in S_0$, and let F_n denote the category of flagged triples over $S_{a,b}$ that have length n . That is, the objects of F_n are triples $[T, T \rightarrow S_{a,b}, T^0 \subseteq \dots \subseteq T^n]$ and morphisms are maps of necklaces $f: T \rightarrow T'$ over S such that $f(T^i) = (T')^i$ for all i .

Proposition 4.10. *For each $n \geq 0$, the nerve of F_n is homotopy discrete in $sSet_K$.*

Proof. Recall from Lemma 4.5 that there is a functor $\phi: F_n \rightarrow F_n$ which sends any triple to its ‘flankification’. There is a natural transformation from ϕ to the identity, and the image of ϕ is the subcategory $F'_n \subseteq F_n$ of flanked triples. It will therefore suffice to prove that (the nerve of) F'_n is homotopy discrete.

Recall from Corollary 4.8 that every component of F'_n contains a unique triple t which is both flanked and totally nondegenerate. Moreover, following the proof of that corollary one sees that every triple in the same component as t admits a unique map to t —that is to say, t is a final object for its component. Therefore its component is contractible. This completes the proof. \square

4.11. The functor \mathfrak{C} applied to ordered simplicial sets.

Note that even if a simplicial set S is small—say, in the sense that it has finitely many nondegenerate simplices—the space $\mathfrak{C}(S)(a, b)$ may be quite large. This is due to the fact that there are infinitely many necklaces mapping to S (if S is nonempty). For certain simplicial sets S , however, it is possible to restrict to necklaces which lie *inside of* S ; this cuts down the possibilities. The following results and subsequent example demonstrate this. Recall the definition of ordered simplicial sets from Definition 3.2.

Lemma 4.12. *Let D be an ordered simplicial set and let $a, b \in D_0$. Then every n -simplex in $\mathfrak{C}(D)(a, b)$ is represented by a unique triple $[T, T \rightarrow D, \vec{T}]$ in which T is a necklace, \vec{T} is a flanked flag of length n , and the map $T \rightarrow D$ is injective.*

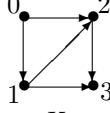
Proof. By Corollary 4.8, every n -simplex in $\mathfrak{C}(D)(a, b)$ is represented by a unique triple $[T, T \rightarrow D, \vec{T}]$ which is both flanked and totally non-degenerate. It suffices to show that if D is ordered, then any totally non-degenerate map $T \rightarrow D$ is injective. This follows from Lemma 3.5(6). \square

Corollary 4.13. *Let D be an ordered simplicial set, and $a, b \in D_0$. Let $M_D(a, b)$ denote the simplicial set for which $M_D(a, b)_n$ is the set of triples $[T, T \xrightarrow{f} D_{a,b}, \vec{T}]$, where f is injective and \vec{T} is a flanked flag of length n ; face and boundary maps are as in Remark 4.6. Then there is a natural isomorphism*

$$\mathfrak{C}(D)(a, b) \xrightarrow{\cong} M_D(a, b).$$

Proof. This follows immediately from Lemma 4.12. □

Example 4.14. Consider the simplicial set $S = \Delta^2 \amalg_{\Delta^1} \Delta^2$ depicted as follows:



We will describe the mapping space $X = \mathfrak{C}(S)(0, 3)$ by giving its non-degenerate simplices and face maps. Note that S is ordered.

By Lemma 4.12, it suffices to consider flanked necklaces that inject into S . There are only five such necklaces that have endpoints 0 and 3. These are $T = \Delta^1 \vee \Delta^1$, which maps to S in two different ways f, g ; and $U = \Delta^1 \vee \Delta^1 \vee \Delta^1$, $V = \Delta^1 \vee \Delta^2$, and $W = \Delta^2 \vee \Delta^1$, each of which maps uniquely into $S_{0,3}$. The image of T_0 under f is $\{0, 1, 3\}$ and under g is $\{0, 2, 3\}$. The images of U_0, V_0 , and W_0 are all $\{0, 1, 2, 3\}$.

We find that X_0 consists of three elements $[T; \{0, 1, 3\}]$, $[T; \{0, 2, 3\}]$ and $[U; \{0, 1, 2, 3\}]$. There are two nondegenerate 1-simplices, $[V; \{0, 1, 3\} \subset \{0, 1, 2, 3\}]$ and $[W; \{0, 2, 3\} \subset \{0, 1, 2, 3\}]$. These connect the three 0-simplices in the obvious way, resulting in two 1-simplices with a common final vertex. There are no higher non-degenerate simplices. Thus $\mathfrak{C}(S)(0, 3)$ looks like

$$\bullet \longleftarrow \bullet \longrightarrow \bullet.$$

5. HOMOTOPICAL MODELS FOR CATEGORIFICATION

In the last section we gave a very explicit description of the mapping spaces $\mathfrak{C}(S)(a, b)$, for arbitrary simplicial sets S and $a, b \in S_0$. While this description was explicit, in some ways it is not very useful from a homotopical standpoint—in practice it is hard to use this description to identify the homotopy type of $\mathfrak{C}(S)(a, b)$.

In this section we will discuss a functor $\mathfrak{C}^{nec}: sSet \rightarrow sCat$ that has a simpler description than \mathfrak{C} and which is more homotopical. We prove that for any simplicial set S there is a natural zigzag of weak equivalences between $\mathfrak{C}(S)$ and $\mathfrak{C}^{nec}(S)$. Variants of this construction are also introduced, leading to a collection of functors $sSet \rightarrow sCat$ all of which are weakly equivalent to \mathfrak{C} .

Let $S \in sSet$. A choice of $a, b \in S_0$ will be regarded as a map $\partial\Delta^1 \rightarrow S$. Let $(Nec \downarrow S)_{a,b}$ be the overcategory for the inclusion functor $Nec \hookrightarrow (\partial\Delta^1 \downarrow S)$. Finally, define

$$\mathfrak{C}^{nec}(S)(a, b) = N(Nec \downarrow S)_{a,b}.$$

Then $\mathfrak{C}(S)$ is a simplicial category in an evident way; see (1.1).

Remark 5.1. Both the functor \mathfrak{C} and the functor \mathfrak{C}^{nec} have distinct advantages and disadvantages. The main advantage to \mathfrak{C} is that it is left adjoint to the coherent nerve functor N (in fact it is a left Quillen functor $sSet_J \rightarrow sCat$); as such, it preserves colimits. However, as mentioned above, the functor \mathfrak{C} can be difficult to use in practice because the mapping spaces have an awkward description.

It is at this point that our functor \mathfrak{C}^{nec} becomes useful, because the mapping spaces are given as nerves of 1-categories. Many tools are available for determining when a morphism between nerves is a Kan equivalence. This will be an important point in [DS], where we show the \mathfrak{C} functor gives a Quillen equivalence between $sSet_J$ and $sCat$. See also Section 6 below.

Our main theorem is that there is a simple zigzag of weak equivalences between $\mathfrak{C}(S)$ and $\mathfrak{C}^{nec}(S)$; that is, there is a functor $\mathfrak{C}^{hoc}: sSet \rightarrow sCat$ and natural weak equivalences $\mathfrak{C} \leftarrow \mathfrak{C}^{hoc} \rightarrow \mathfrak{C}^{nec}$. We begin by describing the functor \mathfrak{C}^{hoc} .

Fix a simplicial set S . Define $\mathfrak{C}^{hoc}(S)$ to have object set S_0 , and for every $a, b \in S_0$

$$\mathfrak{C}^{hoc}(S)(a, b) = \operatorname{hocolim}_{T \in (\mathcal{Nec} \downarrow S)_{a,b}} \mathfrak{C}(T)(\alpha, \omega).$$

Note the similarities to Proposition 4.3, where it was shown that $\mathfrak{C}(S)(a, b)$ has a similar description in which the hocolim is replaced by the colim. In our definition of $\mathfrak{C}^{hoc}(S)(a, b)$ we mean to use a particular model for the homotopy colimit, namely the diagonal of the bisimplicial set whose (k, l) -simplices are pairs

$$(F: [k] \rightarrow (\mathcal{Nec} \downarrow S)_{a,b}; x \in \mathfrak{C}(F(0))(\alpha, \omega)_l),$$

where $F(0)$ denotes the necklace obtained by applying F to $0 \in [k]$ and then applying the forgetful functor $(\mathcal{Nec} \downarrow S)_{a,b} \rightarrow \mathcal{Nec}$. The composition law for \mathfrak{C}^{hoc} is defined just as for the E_S construction from Section 4.

We proceed to establish natural transformations $\mathfrak{C}^{hoc} \rightarrow \mathfrak{C}^{nec}$ and $\mathfrak{C}^{hoc} \rightarrow \mathfrak{C}$. Note that $\mathfrak{C}^{nec}(S)(a, b)$ is the homotopy colimit of the constant functor $\{*\}: (\mathcal{Nec} \downarrow S)_{a,b} \rightarrow sSet$ which sends everything to a point. The map $\mathfrak{C}^{hoc}(S)(a, b) \rightarrow \mathfrak{C}^{nec}(S)(a, b)$ is the map of homotopy colimits induced by the evident map of diagrams. Since the spaces $\mathfrak{C}(T)(\alpha, \omega)$ are all contractible simplicial sets (see Corollary 3.8), the induced map $\mathfrak{C}^{hoc}(S)(a, b) \rightarrow \mathfrak{C}^{nec}(S)(a, b)$ is a Kan equivalence. We thus obtain a natural weak equivalence of simplicial categories $\mathfrak{C}^{hoc}(S) \rightarrow \mathfrak{C}^{nec}(S)$.

For any diagram in a model category there is a canonical natural transformation from the homotopy colimit to the colimit of that diagram. Hence there is a morphism

$$\mathfrak{C}^{hoc}(S)(a, b) \rightarrow \operatorname{colim}_{T \in (\mathcal{Nec} \downarrow S)_{a,b}} \mathfrak{C}(T)(\alpha, \omega) \cong \mathfrak{C}(S)(a, b).$$

(For the isomorphism we are using Proposition 4.3.) As this is natural in $a, b \in S_0$ and natural in S , we have a natural transformation $\mathfrak{C}^{hoc} \rightarrow \mathfrak{C}$.

Theorem 5.2. *For every simplicial set S , the maps $\mathfrak{C}(S) \leftarrow \mathfrak{C}^{hoc}(S) \rightarrow \mathfrak{C}^{nec}(S)$ defined above are weak equivalences of simplicial categories.*

Proof. We have already established that the natural transformation $\mathfrak{C}^{hoc} \rightarrow \mathfrak{C}^{nec}$ is an objectwise equivalence, so it suffices to show that for each simplicial set S and objects $a, b \in S_0$ the natural map $\mathfrak{C}^{hoc}(S)(a, b) \rightarrow \mathfrak{C}(S)(a, b)$ is also a Kan equivalence.

Recall that $\mathfrak{C}^{hoc}(S)(a, b)$ is the diagonal of a bisimplicial set whose l th ‘horizontal’ row is the nerve NF_l of the category of flagged necklaces mapping to S , where the flags have length l . Also recall from Corollary 4.4 that $\mathfrak{C}(S)(a, b)$ is the simplicial set which in level l is $\pi_0(NF_l)$. But Proposition 4.10 says that $NF_l \rightarrow \pi_0(NF_l)$ is a Kan equivalence, for every l . It follows that $\mathfrak{C}^{hoc}(S)(a, b) \rightarrow \mathfrak{C}(S)(a, b)$ is also a Kan equivalence. \square

5.3. Other models for categorification. One can imagine variations of our basic construction in which one replaces necklaces with other convenient simplicial sets— which we might term “gadgets,” for lack of a better word. We will see in Section 6, for instance, that using *products* of necklaces leads to a nice theorem about the categorification of a product. In [DS] several key arguments will hinge on a clever

choice of what gadgets to use. In the material below we give some basic requirements of the “gadgets” which will ensure they give a model equivalent to that of necklaces.

Suppose \mathcal{P} is a subcategory of $sSet_{*,*} = (\partial\Delta^1 \downarrow sSet)$ containing the terminal object. For any simplicial set S and vertices $a, b \in S_0$, let $(\mathcal{P} \downarrow S)_{a,b}$ denote the overcategory whose objects are pairs $[P, P \rightarrow S]$, where $P \in \mathcal{P}$ and the map $P \rightarrow S$ sends $\alpha_P \mapsto a$ and $\omega_P \mapsto b$. Define

$$\mathfrak{C}^{\mathcal{P}}(S)(a, b) = N(\mathcal{P} \downarrow S)_{a,b}.$$

The object $\mathfrak{C}^{\mathcal{P}}$ is simply an assignment which takes a simplicial set S with two distinguished vertices and produces a “ \mathcal{P} -mapping space.” However, if \mathcal{P} is closed under the wedge operation (i.e. for any $P_1, P_2 \in \mathcal{P}$ one has $P_1 \vee P_2 \in \mathcal{P}$), then $\mathfrak{C}^{\mathcal{P}}$ may be given the structure of a functor $sSet \rightarrow sCat$ in the evident way.

Definition 5.4. *We call a subcategory $\mathcal{G} \subseteq sSet_{*,*}$ a **category of gadgets** if it satisfies the following properties:*

- (1) \mathcal{G} contains the category Nec ,
- (2) For every object $X \in \mathcal{G}$ and every necklace T , all maps $T \rightarrow X$ are contained in \mathcal{G} , and
- (3) For any $X \in \mathcal{G}$, the simplicial set $\mathfrak{C}(X)(\alpha, \omega)$ is contractible.

The category \mathcal{G} is said to be **closed under wedges** if it is also true that

- (4) For any $X, Y \in \mathcal{G}$, the wedge $X \vee Y$ also belongs to \mathcal{G} .

The above definition can be generalized somewhat by allowing $Nec \rightarrow \mathcal{G}$ to be an arbitrary functor over a natural transformation in $sSet$; we do not need this generality in the present paper.

Proposition 5.5. *Let \mathcal{G} be a category of gadgets. Then for any simplicial set S and any $a, b \in S_0$, the natural map*

$$\mathfrak{C}^{nec}(S)(a, b) \longrightarrow \mathfrak{C}^{\mathcal{G}}(S)(a, b)$$

(induced by the inclusion $Nec \hookrightarrow \mathcal{G}$) is a Kan equivalence. If \mathcal{G} is closed under wedges then the map of simplicial categories $\mathfrak{C}^{nec}(S) \rightarrow \mathfrak{C}^{\mathcal{G}}(S)$ is a weak equivalence.

Proof. Let $j: (Nec \downarrow S)_{a,b} \rightarrow (\mathcal{G} \downarrow S)_{a,b}$ be the functor induced by the inclusion map $Nec \hookrightarrow \mathcal{G}$. The map in the statement of the proposition is just the nerve of j . To verify that it is a Kan equivalence, it is enough by Quillen’s Theorem A [Q] to verify that all the overcategories of j are contractible. So fix an object $[X, X \rightarrow S]$ in $(\mathcal{G} \downarrow S)_{a,b}$. The overcategory $(j \downarrow [X, X \rightarrow S])$ is precisely the category $(Nec \downarrow X)_{\alpha, \omega}$, the nerve of which is $\mathfrak{C}^{nec}(X)(\alpha, \omega)$. By Theorem 5.2 and our assumptions on \mathcal{G} , this is contractible.

The second statement of the result is a direct consequence of the first. \square

6. PROPERTIES OF CATEGORIFICATION

In this section we establish two main properties of the categorification functor \mathfrak{C} . First, we prove that there is a natural weak equivalence $\mathfrak{C}(X \times Y) \simeq \mathfrak{C}(X) \times \mathfrak{C}(Y)$. Second, we prove that whenever $S \rightarrow S'$ is a Joyal equivalence it follows that $\mathfrak{C}(S) \rightarrow \mathfrak{C}(S')$ is a weak equivalence in $sCat$. These properties are also proven in [L], but the proofs we give here are of a different nature and make central use of the \mathfrak{C}^{nec} functor.

If T_1, \dots, T_n are necklaces then they are, in particular, ordered simplicial sets in the sense of Definition 3.2. So $T_1 \times \dots \times T_n$ is also ordered, by Lemma 3.5. Let \mathcal{G} be the full subcategory of $sSet_{*,*} = (\partial\Delta^1 \downarrow sSet)$ whose objects are products of necklaces with a map $f: \partial\Delta^1 \rightarrow T_1 \times \dots \times T_n$ that has $f(0) \preceq f(1)$.

Proposition 6.1. *The category \mathcal{G} is a category of gadgets in the sense of Definition 5.4.*

For the proof of this one needs to verify that $\mathfrak{C}(T_1 \times \dots \times T_n)(\alpha, \omega) \simeq *$. This is not difficult, but is a bit of a distraction; we prove it later as Proposition A.4.

Proposition 6.2. *For any simplicial sets X and Y , both $\mathfrak{C}(X \times Y)$ and $\mathfrak{C}(X) \times \mathfrak{C}(Y)$ are simplicial categories with object set $X_0 \times Y_0$. For any $a_0, b_0 \in X$ and $a_1, b_1 \in Y$, the natural map*

$$\mathfrak{C}(X \times Y)(a_0 a_1, b_0 b_1) \rightarrow \mathfrak{C}(X)(a_0, b_0) \times \mathfrak{C}(Y)(a_1, b_1)$$

induced by $\mathfrak{C}(X \times Y) \rightarrow \mathfrak{C}(X)$ and $\mathfrak{C}(X \times Y) \rightarrow \mathfrak{C}(Y)$ is a Kan equivalence. Consequently, the map of simplicial categories

$$\mathfrak{C}(X \times Y) \rightarrow \mathfrak{C}(X) \times \mathfrak{C}(Y)$$

is a weak equivalence in $sCat$.

Proof. Let \mathcal{G} denote the above category of gadgets, in which the objects are products of necklaces. By Theorem 5.2 and Proposition 5.5 it suffices to prove the result for $\mathfrak{C}^{\mathcal{G}}$ in place of \mathfrak{C} .

Consider the functors

$$(\mathcal{G} \downarrow X \times Y)_{a_0 a_1, b_0 b_1} \begin{array}{c} \xrightarrow{\phi} \\ \xleftarrow{\theta} \end{array} (\mathcal{G} \downarrow X)_{a_0, b_0} \times (\mathcal{G} \downarrow Y)_{a_1, b_1}$$

given by

$$\phi: [G, G \rightarrow X \times Y] \mapsto ([G, G \rightarrow X \times Y \rightarrow X], [G, G \rightarrow X \times Y \rightarrow Y])$$

and

$$\theta: ([G, G \rightarrow X], [H, H \rightarrow Y]) \mapsto [G \times H, G \times H \rightarrow X \times Y].$$

Note that we are using that the subcategory \mathcal{G} is closed under finite products.

It is very easy to see that there is a natural transformation $\text{id} \rightarrow \theta\phi$, obtained by using diagonal maps, and a natural transformation $\phi\theta \rightarrow \text{id}$, obtained by using projections. As a consequence, the maps θ and ϕ induce inverse homotopy equivalences on the nerves. This completes the proof. \square

Let $E: \text{Set} \rightarrow sSet$ denote the 0-coskeleton functor (see [AM]). For any simplicial set X and set S , we have $sSet(X, ES) = \text{Set}(X_0, S)$. In particular, if $n \in \mathbb{N}$ we denote $E^n = E\{0, 1, \dots, n\}$.

Lemma 6.3. *For any $n \geq 0$, the simplicial category $\mathfrak{C}(E^n)$ is contractible in $sCat$ — that is to say, all the mapping spaces in $\mathfrak{C}(E^n)$ are contractible.*

Proof. By Theorem 5.2 it is sufficient to prove that the mapping space $\mathfrak{C}^{nec}(E^n)(i, j)$ is contractible, for every $i, j \in \{0, 1, \dots, n\}$. This mapping space is the nerve of the overcategory $(Nec \downarrow E^n)_{i, j}$.

Observe that if T is a necklace then any map $T \rightarrow E^n$ extends uniquely over $\Delta[T]$. This is because maps into E^n are determined by what they do on the 0-skeleton, and $T \hookrightarrow \Delta[T]$ is an isomorphism on 0-skeleta.

Consider two functors

$$f, g: (\mathcal{Nec} \downarrow E^n)_{i,j} \rightarrow (\mathcal{Nec} \downarrow E^n)_{i,j}$$

given by

$$f: [T, T \xrightarrow{x} E^n] \mapsto [\Delta[T], \Delta[T] \xrightarrow{\bar{x}} E^n] \text{ and } g: [T, T \xrightarrow{x} E^n] \mapsto [\Delta^1, \Delta^1 \xrightarrow{z} E^n].$$

Here \bar{x} is the unique extension of x to $\Delta[T]$, and z is the unique 1-simplex of E^n connecting i to j . Observe that g is a constant functor.

It is easy to see that there are natural transformations $\text{id} \rightarrow f \leftarrow g$. The functor g factors through the terminal category $\{*\}$, so after taking nerves the identity map is null homotopic. Hence $(\mathcal{Nec} \downarrow E^n)_{i,j}$ is contractible. \square

For completeness (and because it is short) we include the following lemma, established in [L, Proof of 2.2.5.1]:

Lemma 6.4. *The functor $\mathfrak{C}: s\text{Set} \rightarrow s\text{Cat}$ takes monomorphism to cofibrations.*

Proof. Every cofibration in $s\text{Set}$ is obtained by compositions and cobase changes from boundary inclusions of simplices. It therefore suffices to show that for each $n \geq 0$ the map $f: \mathfrak{C}(\partial\Delta^n) \rightarrow \mathfrak{C}(\Delta^n)$ is a cofibration in $s\text{Cat}$. Let $0 \leq i, j \leq n$. If $i > 0$ or $j < n$ then $(\mathcal{Nec} \downarrow \Delta^n)_{i,j} \cong (\mathcal{Nec} \downarrow \partial\Delta^n)_{i,j}$, whereby

$$f(i, j): \mathfrak{C}(\partial\Delta^n)(i, j) \rightarrow \mathfrak{C}(\Delta^n)(i, j)$$

is an isomorphism by Proposition 4.3. For the remaining case $i = 0, j = n$, the map $f(0, n)$ is the inclusion of the boundary of a cube $b: \partial((\Delta^1)^{n-1}) \rightarrow (\Delta^1)^{n-1}$.

Let $U: s\text{Set} \rightarrow s\text{Cat}$ denote the functor which sends a simplicial set S to the unique simplicial category $U(S)$ with two objects x, y and morphisms $\text{Hom}(x, x) = \text{Hom}(y, y) = \{*\}$, $\text{Hom}(y, x) = \emptyset$, and $\text{Hom}(x, y) = S$. In view of the generating cofibrations for $s\text{Cat}$ (see [B]), it is easy to show that U preserves cofibrations. Hence $U(b)$ is a cofibration. Notice that f is the pushout of $U(b)$ along the obvious map $U[\partial((\Delta^1)^{n-1})] \rightarrow \mathfrak{C}(\partial\Delta^n)$ sending $x \mapsto 0$ and $y \mapsto n$. Thus, f is a cofibration. \square

Recall that a simplicial category is fibrant in $s\text{Cat}$ if all its mapping spaces are Kan fibrant.

Lemma 6.5. *If \mathcal{D} is a fibrant simplicial category then $N\mathcal{D}$ is a quasi-category.*

Proof. By adjointness it suffices to show that each $\mathfrak{C}(j^{n,k})$ is an acyclic cofibration in $s\text{Cat}$, where $j^{n,k}: \Lambda_k^n \hookrightarrow \Delta^n$ is an inner horn inclusion ($0 < k < n$). It is a cofibration by Lemma 6.4, so we must only verify that it is a weak equivalence. Just as in the proof of (6.4) above, $\mathfrak{C}(\Lambda_k^n)(i, j) \rightarrow \mathfrak{C}(\Delta^n)(i, j)$ is an isomorphism unless $i = 0$ and $j = n$. It only remains to show that $\mathfrak{C}(\Lambda_k^n)(0, n) \rightarrow \mathfrak{C}(\Delta^n)(0, n)$ is a Kan equivalence. An analysis as in Example 4.14 identifies $\mathfrak{C}(\Lambda_k^n)(0, n)$ with the result of removing one face from the boundary of $(\Delta^1)^{n-1}$, which clearly has the same homotopy type as the cube $(\Delta^1)^{n-1}$. \square

Proposition 6.6. *If $S \rightarrow S'$ is a map of simplicial sets which is a Joyal equivalence then $\mathfrak{C}(S) \rightarrow \mathfrak{C}(S')$ is a weak equivalence of simplicial categories.*

Proof. For any simplicial set X , the map $\mathfrak{C}(X \times E^n) \rightarrow \mathfrak{C}(X)$ induced by projection is a weak equivalence in $s\mathcal{C}at$. This follows by combining Proposition 6.2 with Lemma 6.3:

$$\mathfrak{C}(X \times E^n) \xrightarrow{\sim} \mathfrak{C}(X) \times \mathfrak{C}(E^n) \xrightarrow{\sim} \mathfrak{C}(X).$$

Since $X \amalg X \hookrightarrow X \times E^1$ is a cofibration in $sSet$, $\mathfrak{C}(X) \amalg \mathfrak{C}(X) = \mathfrak{C}(X \amalg X) \rightarrow \mathfrak{C}(X \times E^1)$ is a cofibration in $s\mathcal{C}at$, by Lemma 6.4. It follows that $\mathfrak{C}(X \times E^1)$ is a cylinder object for $\mathfrak{C}(X)$ in $s\mathcal{C}at$. So if \mathcal{D} is a fibrant simplicial category we may compute homotopy classes of maps $[\mathfrak{C}(X), \mathcal{D}]$ as the coequalizer

$$\operatorname{coeq}\left(s\mathcal{C}at(\mathfrak{C}(X \times E^1), \mathcal{D}) \rightrightarrows s\mathcal{C}at(\mathfrak{C}(X), \mathcal{D})\right).$$

But using the adjunction, this is isomorphic to

$$\operatorname{coeq}\left(sSet(X \times E^1, N\mathcal{D}) \rightrightarrows sSet(X, N\mathcal{D})\right).$$

The above coequalizer is $[X, N\mathcal{D}]_{E^1}$, and we have identified

$$(6.7) \quad [\mathfrak{C}(X), \mathcal{D}] \cong [X, N\mathcal{D}]_{E^1}.$$

Now let $S \rightarrow S'$ be a Joyal equivalence. Then $\mathfrak{C}(S) \rightarrow \mathfrak{C}(S')$ is a map between cofibrant objects of $s\mathcal{C}at$. To prove that it is a weak equivalence in $s\mathcal{C}at$ it is sufficient to prove that the induced map on homotopy classes

$$[\mathfrak{C}(S'), \mathcal{D}] \rightarrow [\mathfrak{C}(S), \mathcal{D}]$$

is a bijection, for every fibrant object $\mathcal{D} \in s\mathcal{C}at$. Since $N\mathcal{D}$ is a quasi-category by Lemma 6.5 and $S \rightarrow S'$ is a Joyal equivalence, we have that $[S', N\mathcal{D}]_{E^1} \rightarrow [S, N\mathcal{D}]_{E^1}$ is a bijection; the result then follows by (6.7). \square

Remark 6.8. In fact it turns out that a map of simplicial sets $S \rightarrow S'$ is a Joyal equivalence *if and only if* $\mathfrak{C}(S) \rightarrow \mathfrak{C}(S')$ is a weak equivalence of simplicial categories. This was proven in [L], and will be reproven in [DS] using an extension of the methods from the present paper.

APPENDIX A. LEFTOVER PROOFS

In this section we give two proofs which were postponed in the body of the paper.

A.1. Products of necklaces. Our first goal is to prove Proposition 6.1. Let T_1, \dots, T_n be necklaces, and consider the product $X = T_1 \times \dots \times T_n$. The main thing we need to prove is that whenever $a \preceq_X b$ in X the mapping space $\mathfrak{C}(X)(a, b) \simeq *$ is contractible.

Definition A.2. An ordered simplicial set (X, \preceq) is called **strongly ordered** if, for all $a \preceq b$ in X , the mapping space $\mathfrak{C}(X)(a, b)$ is contractible.

Note that in any ordered simplicial set X with $a, b \in X_0$, we have $a \preceq b$ if and only if $\mathfrak{C}(X)(a, b) \neq \emptyset$. Thus if X is strongly ordered then its structure as a simplicial category, up to weak equivalence, is completely determined by the ordering on its vertices. We also point out that every necklace $T \in Nec$ is strongly ordered by Corollary 3.8.

Lemma A.3. *Suppose given a diagram*

$$X \xleftarrow{f} A \xrightarrow{g} Y$$

where X, Y , and A are strongly ordered simplicial sets and both f and g are simple inclusions. Let $B = X \amalg_A Y$ and assume the following conditions hold:

- (1) A has finitely many vertices;
- (2) Given any $x \in X$, the set $A_{x \preceq} = \{a \in A \mid x \preceq_B a\}$ has an initial element (an element which is smaller than every other element).
- (3) For any $y \in Y$ and $a \in A$, if $y \preceq_Y a$ then $y \in A$.

Then B is strongly ordered.

Proof. By Lemma 3.5(8), B is an ordered simplicial set and the maps $X \hookrightarrow B$ and $Y \hookrightarrow B$ are simple inclusions. We must show that for $u, v \in B_0$ with $u \preceq v$, the mapping space $\mathfrak{C}(B)(u, v)$ is contractible. Suppose that u and v are both in X ; then since $X \hookrightarrow B$ is simple, any necklace $T \rightarrow B_{u, v}$ must factor through X . It follows that $\mathfrak{C}(B)(u, v) = \mathfrak{C}(X)(u, v)$, which is contractible since X is strongly ordered. The case $u, v \in Y$ is analogous. We claim we cannot have $u \in Y \setminus A$ and $v \in X \setminus A$. For if this is so and if $T \rightarrow B$ is a spine connecting u to v , then there is a last vertex j of T that maps into Y . The 1-simplex leaving that vertex then cannot belong entirely to Y , hence it belongs entirely to X . So j is in both X and Y , and hence it is in A . Then we have $u \preceq j$ and $j \in A$, which by assumption (3) implies $u \in A$, a contradiction.

The only remaining case to analyze is when $u \in X$ and $v \in Y \setminus A$. Consider the poset A_0 of vertices of A , under the relation \preceq . Let P denote the collection of linearly ordered subsets S of A_0 having the property that $u \preceq a \preceq v$ for all $a \in S$. That is, each element of P is a chain $u \preceq a_1 \preceq \cdots \preceq a_n \preceq v$ where each $a_i \in A$. We regard P as a category, where the maps are inclusions. Also let P_0 denote the subcategory of P consisting of all subsets except \emptyset .

Define a functor $D: P^{op} \rightarrow \mathfrak{Cat}$ by sending $S \in P$ to

$$\{[T, T \hookrightarrow B_{u, v}] \mid S \subseteq J_T\},$$

the full subcategory of $(\mathit{Nec} \downarrow B)_{u, v}$ spanned by objects $T \xrightarrow{m} B_{u, v}$ for which m is an injection and $S \subseteq J_T$. Let us adopt the notation

$$M_S(u, v) = \text{colim}_{T \in D(S)} \mathfrak{C}(T)(\alpha, \omega).$$

Note that there is a natural map

$$M_{\emptyset}(u, v) \longrightarrow \text{colim}_{T \in (\mathit{Nec} \downarrow S)_{u, v}} \mathfrak{C}(T)(\alpha, \omega) \cong \mathfrak{C}(B)(u, v).$$

The first map is not *a priori* an isomorphism because in the definition of $D(\emptyset)$ we require that the map $T \rightarrow B$ be an injection. However, using Lemma 4.12 (or Corollary 4.13) it follows at once that the map actually is an isomorphism.

We claim that for each S in P_0 the ‘‘latching’’ map

$$L_S: \text{colim}_{S' \supset S} M_{S'}(u, v) \rightarrow M_S(u, v)$$

is an injection, where the colimit is over sets $S' \in P$ which strictly contain S . To see this, suppose that one has a triple $[T, T \hookrightarrow B_{u, v}, t \in \mathfrak{C}(T)(\alpha, \omega)_n]$ giving an n -simplex of $M_{S'}(u, v)$ and another triple $[T', T' \hookrightarrow B_{u, v}, t' \in \mathfrak{C}(T')(\alpha, \omega)_n]$ giving an n -simplex of $M_S(u, v)$. If these become identical in $M_S(u, v)$ then it must be

that they have the same flankification $\bar{T} = \bar{U}$ and $t = t'$. Note that every joint of T is a joint of \bar{T} , so the joints of \bar{T} include both S' and S'' . Because the joints of any necklace are linearly ordered, it follows that $S' \cup S''$ is linearly ordered. Since $T \rightarrow \bar{T}$ is an injection, we may consider the triple $[\bar{T}, \bar{T} \hookrightarrow B_{u,v}, t]$ as an n -simplex in $M_{S' \cup S''}(u, v)$, which maps to the two original triples in the colimit; this proves injectivity.

We claim that the latching map $L_\emptyset: \operatorname{colim}_{S \in P_0^{op}} M_S(u, v) \rightarrow M_\emptyset(u, v)$ is an isomorphism. Injectivity was established above. For surjectivity, one needs to prove that if T is a necklace and $T \hookrightarrow B_{u,v}$ is an inclusion, then T must contain at least one vertex of A as a joint. To see this, recall that every simplex of B either lies entirely in X or entirely in Y . Since $v \notin X$, there is a last joint j_1 of T which maps into X . If C denotes the bead whose initial vertex is j_1 , then the image of C can not lie entirely in X ; so it lies entirely in Y , which means that j_1 belongs to both X and Y —hence it belongs to A .

From here the argument proceeds as follows. We will show:

- (i) The natural map $\operatorname{hocolim}_{S \in P_0^{op}} M_S(u, v) \rightarrow \operatorname{colim}_{S \in P_0^{op}} M_S(u, v)$ is a Kan equivalence;
- (ii) Each $M_S(u, v)$ is contractible, hence the above homotopy colimit is Kan equivalent to the nerve of P_0^{op} ;
- (iii) The nerve of P_0 (and hence also P_0^{op}) is contractible.

This will prove that $M_\emptyset(u, v) = \mathfrak{C}(B)(u, v)$ is contractible, as desired.

For (i) we refer to [D, Section 13] and use the fact that P_0^{op} has the structure of a directed Reedy category. Indeed, we can assign a degree function to P that sends a set $S \subseteq A_0$ to the nonnegative integer $|A_0 - S|$; all non-identity morphisms in P_0^{op} strictly increase this degree. By [D, Proposition 13.3] (but with $\mathcal{J}op$ replaced by $sSet$) it is enough to show that all the latching maps L_S are cofibrations, and this has already been established above.

For claim (iii), write θ for the initial vertex of $A_{u \prec}$. Define a functor $F: P_0 \rightarrow P_0$ by $F(S) = S \cup \{\theta\}$; note that $S \cup \{\theta\}$ will be linearly ordered, so this makes sense. Clearly there is a natural transformation from the identity functor to F , and also from the constant $\{\theta\}$ functor to F . It readily follows that the identity map on NP_0 is homotopic to a constant map, hence NP_0 is contractible.

Finally, for (ii) fix some $S \in P_0$ and let $u = a_0 \prec a_1 \prec \dots \prec a_n \prec a_{n+1} = v$ denote the complete set of elements of $S \cup \{u, v\}$. A necklace $T \hookrightarrow B_{u,v}$ whose joints include the elements of S can be split along the joints, and thus uniquely written as the wedge of necklaces $T_i \hookrightarrow B_{a_i, a_{i+1}}$, one for each $0 \leq i \leq n$. Under this identification, one has

$$\mathfrak{C}(T)(\alpha, \omega) \cong \mathfrak{C}(T_0)(\alpha_0, \omega_0) \times \dots \times \mathfrak{C}(T_n)(\alpha_n, \omega_n).$$

Thus $D(p)$ is isomorphic to the category

$$(\operatorname{Nec} \downarrow^m X)_{u, a_1} \times (\operatorname{Nec} \downarrow^m A)_{a_1, a_2} \times \dots \times (\operatorname{Nec} \downarrow^m A)_{a_{n-1}, a_n} \times (\operatorname{Nec} \downarrow^m Y)_{a_n, v},$$

where $(\operatorname{Nec} \downarrow^m X)_{s,t}$ denotes the category whose objects are $[T, T \rightarrow X_{s,t}]$ where the map $T \rightarrow X$ is a monomorphism.

Now, it is a general fact about colimits taken in the category of (simplicial) sets, that if M_i is a category and $F_i: M_i \rightarrow sSet$ is a functor, for each $i \in \{1, \dots, n\}$,

then there is an isomorphism of simplicial sets

$$(A.2.2) \quad \operatorname{colim}_{M_1 \times \cdots \times M_n} (F_1 \times \cdots \times F_n) \xrightarrow{\cong} \left(\operatorname{colim}_{M_1} F_1 \right) \times \cdots \times \left(\operatorname{colim}_{M_n} F_n \right).$$

Applying this in our case, we find that

$$M_S(u, v) \cong \mathfrak{C}(X)(u, a_1) \times \mathfrak{C}(A)(a_1, a_2) \times \cdots \times \mathfrak{C}(A)(a_{n-1}, a_n) \times \mathfrak{C}(Y)(a_n, v).$$

Note that this is always contractible, since X , A , and Y are strongly ordered. This proves (ii) and completes the argument. \square

Proposition A.4. *Let T_1, \dots, T_m be necklaces. Then their product $T_1 \times \cdots \times T_m$ is a strongly ordered simplicial set.*

Proof. We begin with the case $P = \Delta^{n_1} \times \cdots \times \Delta^{n_m}$, where each necklace is a simplex, and show that P is strongly ordered. It is ordered by Lemma 3.5, so choose vertices $a, b \in P_0$ with $a \preceq b$. If T is a necklace, any map $T \rightarrow \Delta^j$ extends uniquely to a map $\Delta[T] \rightarrow \Delta^j$. It follows that any map $T \rightarrow P_{a,b}$ extends uniquely to $\Delta[T] \rightarrow P_{a,b}$. Consider the two functors

$$f, g: (\mathcal{Nec} \downarrow P)_{a,b} \rightarrow (\mathcal{Nec} \downarrow P)_{a,b}$$

where f sends $[T, T \rightarrow P]$ to $[\Delta[T], \Delta[T] \rightarrow P]$ and g is the constant functor sending everything to $[\Delta^1, x: \Delta^1 \rightarrow P]$ where x is the unique edge of P connecting a and b . Then clearly there are natural transformations $\operatorname{id} \rightarrow f$ and $g \rightarrow f$, showing that the three maps id , f , and g induce homotopic maps on the nerves. So the identity induces the null map, hence $\mathfrak{C}^{\mathcal{Nec}}(P)(a, b) = N((\mathcal{Nec} \downarrow P)_{a,b})$ is contractible. The result for P now follows by Theorem 5.2.

For the general case, assume by induction that we know the result for all products of necklaces in which at most $k-1$ of them are not equal to beads. The case $k=1$ was handled by the previous paragraph. Consider a product

$$Y = T_1 \times \cdots \times T_k \times D$$

where each T_i is a necklace and D is a product of beads. Write $T_k = B_1 \vee B_2 \vee \cdots \vee B_r$ where each B_i is a bead, and let

$$P_j = (T_1 \times \cdots \times T_{k-1}) \times (B_1 \vee \cdots \vee B_j) \times D.$$

We know by induction that P_1 is strongly ordered, and we will prove by a second induction that the same is true for each P_j . So assume that P_j is strongly ordered for some $1 \leq j < r$.

Let us denote $A = (T_1 \times \cdots \times T_{k-1}) \times \Delta^0 \times D$ and

$$Q = (T_1 \times \cdots \times T_{k-1}) \times B_{j+1} \times D.$$

Then we have $P_{j+1} = P_j \amalg_A Q$, and we know that P_j , A , and Q are strongly ordered. Note that the maps $A \rightarrow P_{j+1}$ and $A \rightarrow Q$ are simple inclusions: they are the products of $\Delta^0 \rightarrow B_j$ (resp. $\Delta^0 \rightarrow B_{j+1}$) with identity maps, and any inclusion $\Delta^0 \rightarrow \Delta^m$ is clearly simple. It is easy to check that hypothesis (1)–(3) of Lemma A.3 are satisfied, and so this finishes the proof. \square

Proof of Proposition 6.1. This follows immediately from Proposition A.4. \square

A.5. **The category $\mathfrak{C}(\Delta^n)$.** Our final goal is to give the proof of Lemma 2.5. Recall that this says there is an isomorphism

$$\mathfrak{C}(\Delta^n)(i, j) \rightarrow N(P_{i,j})$$

for $n \in \mathbb{N}$ and $0 \leq i, j \leq n$, where $P_{i,j}$ is the poset of subsets of $\{i, i+1, \dots, j\}$ containing i and j .

Proof of Lemma 2.5. The result is obvious when $n = 0$, so we assume $n > 0$. Let $X = (FU)_\bullet([n])(i, j)$ and $Y = P_{i,j}$. For each $\ell \in \mathbb{N}$, we will provide an isomorphism $X_\ell \cong Y_\ell$, and these will be compatible with face and degeneracy maps.

One understands $X_0 = FU([n])(i, j)$ as the set of free compositions of sequences of morphisms in $[n]$ which start at i and end at j . By keeping track of the set of objects involved in this chain, we identify X_0 with the set of subsets of $\{i, i+1, \dots, j\}$ which contain i and j . This gives an isomorphism $X_0 \rightarrow Y_0$.

Similarly, for $\ell > 0$ one has that X_ℓ is the set of free compositions of sequences of morphisms in $X_{\ell-1}$. It is readily seen that X_ℓ (even when $\ell = 0$) is in one-to-one correspondence with the set of ways to “parenthesize” the sequence i, \dots, j in such a way that every element is contained in $(\ell + 1)$ -many parentheses (and no closed parenthesis directly follows an open parenthesis). Given such a parenthesized sequence, one can rank the parentheses by “interiority” (so that interior parentheses have higher rank). The face and degeneracy maps on X are given by deleting or repeating all the parentheses of a fixed rank.

Under this description, a vertex in an ℓ -simplex of X is given by choosing a rank and then ignoring all parentheses except those of that rank. Then by taking only the last elements before a close-parenthesis, we get a subset of $\{i+1, \dots, j\}$ containing j ; by unioning with $\{i\}$, we get a well-defined element of Y_0 . Given two ranks, the subset of $\{i+1, \dots, j\}$ corresponding to the higher rank will contain the subset corresponding to the lower rank. One also sees immediately that an ℓ -simplex in X is determined by its set of vertices, and so we can identify X_ℓ with the set of sequences $S_0 \subseteq S_1 \subseteq \dots \subseteq S_\ell \subseteq \{i, i+1, \dots, j\}$ containing i and j . This is precisely the set of ℓ -simplices of Y , so we have our isomorphism. It is clearly compatible with face and degeneracy maps. \square

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