`sect-1`

The Change of Variable Theorem

We wish to prove:

`thm-1.1` **Theorem 1.1.** *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. Let $\Phi : \mathcal{O} \to \mathbb{R}^n$ be C^1. Assume that Φ is $1-1$ and that $\det(\Phi'(x)) \neq 0$ for all $x \in \mathcal{O}$. Set $\mathcal{U} = \Phi(\mathcal{O})$; this is an open subset as well. Let f be integrable in the extended sense on $\mathcal{U}$. Then $\Phi^* f|\det \Phi'|$ is integrable in the extended sense on $\mathcal{O}$ and*

$$\int_{\mathcal{U}} f = \int_{\mathcal{O}} \Phi^* f |\det \Phi'| \,.$$

We begin our discussion with some technical remarks:

`lem-1.2` **Lemma 1.2.** *Let S be a Jordan measurable subset of $\mathbb{R}^n$, let R be a rectangle in $\mathbb{R}^n$, let P be a rectangle in $\mathbb{R}^{n-1}$, and let $\varepsilon > 0$ be given.*

1. *Let $f : P \to \mathbb{R}$ be integrable. Let $G_f := \{(x, f(x)) \in \mathbb{R}^n : x \in P\}$ be the graph of f. Then G_f has content 0.*
2. *There exists $\ell \in \mathbb{N}$, a finite collection $\mathcal{C}(\ell, R) = \{C_i\}$ of cubes all of whose sides have length ℓ^{-1}, and a subcollection $\mathcal{D}(\ell, R) = \{D_j\} \subset \mathcal{C}(\ell, R)$ so*
$$\begin{aligned}&\cup_j D_j \subset R \subset \cup_i C_i, \quad \text{and}\\ &\operatorname{Vol}(R) - \varepsilon \le \sum_j \mathrm{vol(D_j)} \le \operatorname{Vol}(R) \le \sum_i \operatorname{Vol}(C_i) \le \operatorname{Vol}(R) + \varepsilon \,.\end{aligned}$$
3. *There exists a finite collection of rectangles $\{R_i\}$ and a subcollection $\{\tilde{R}_j\}$ so*
$$\begin{aligned}&\cup_j \tilde{R}_j \subset S \subset \cup_i R_i,\\ &\operatorname{Vol}(S) - \varepsilon < \sum_j \operatorname{Vol}(\tilde{R}_j) \le \operatorname{Vol}(S) \le \sum_i \operatorname{Vol}(R_i) \le \operatorname{Vol}(S) + \varepsilon \,.\end{aligned}$$
4. *There exists a finite collection of cubes $\{C_i\}$ and a subcollection $\{\tilde{C}_j\}$ so*
$$\begin{aligned}&\cup_j \tilde{C}_j \subset S \subset \cup_i C_i,\\ &\operatorname{Vol}(S) - \varepsilon < \sum_j \operatorname{Vol}(\tilde{C}_j) \le \operatorname{Vol}(S) \le \sum_i \operatorname{Vol}(C_i) \le \operatorname{Vol}(S) + \varepsilon \,.\end{aligned}$$

Proof. Since f is assumed to be integrable, we may choose a partition $\mathcal{P} = \{P_i\}$ of P so that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$. If $P_i \in \mathcal{P}$, we shall set $m_i := \sup_{x \in R_i} f(x)$ and $M_i := \sup_{x \in R_i} f(x)$. Let $R_i := P_i \times [m_i, M_i]$ be rectangles in $\mathbb{R}^n$. Assertion (1) follows from the observations:

$$\begin{aligned}&\sum_i \operatorname{Vol}(R_i) = \sum_i \operatorname{Vol} P_i \cdot (M_i - m_i) = U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon,\\ &G_f \subset \cup_i R_i \,.\end{aligned}$$

We suppose $R = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$. Assume additionally that $\operatorname{Vol}(R) > 0$; the case $\operatorname{Vol}(R) = 0$ is handled similarly where the collection $\mathcal{D}(\ell, R)$ is empty. Let $\ell \in \mathbb{N}$ be any integer sufficiently large so

$$\ell^{-1} < \min_{1 \le k \le n} \frac{1}{b_k - a_k} \,.$$

For $1 \le k \le n$, let

$$\nu_k := \nu_k(\ell, R) = \operatorname{int}\left(\frac{\ell}{b_k - a_k}\right) \ge 1$$

be the number of intervals of width ℓ^{-1} one can fit inside the interval $[a_k, b_k]$. The collection $\mathcal{C}(\ell, R) = \{C_i\}$ consists of all cubes of the form

$$C = [a_1 + \frac{\mu_1}{\ell}, a_1 + \frac{\mu_1 + 1}{\ell}] \times [a_2 + \frac{\mu_2}{\ell}, a_2 + \frac{\mu_2 + 1}{\ell}] \times \ldots\}$$
$$\text{for} \quad 0 \le \mu \le \nu_1, 0 \le \mu_2 \le \nu_2, \ldots \,.$$

Similarly $\mathcal{D}(\ell,R)=\{D_j\}$ consists of all cubes of the form

$$D=[a_1+\frac{\mu_1}{\ell},a_1+\frac{\mu_1+1}{\ell}]\times[a_2+\frac{\mu_2}{\ell},a_2+\frac{\mu_2+1}{\ell}]\times...\}$$
$$\text{for}\quad 0\le\mu_1\le\nu_1(\ell)-1,0\le\mu_2\le\nu_2(\ell)-1,\dots.$$

It is then immediate from the construction that

$$\cup_j D_j\subset R\subset\cup_i C_i\,.$$

Because the volume of each cube is ℓ^{-n}, one has

$$\begin{aligned}\sum_j\mathrm{Vol}(D_j)&=\ell^{-n}\cdot\nu_1(\ell)\cdot\ldots\cdot\nu_n(\ell)\le\mathrm{Vol}(V)\\ \le&\quad\ell^{-n}(\nu_1(\ell)+1)\cdot\ldots\cdot(\nu_n(\ell)+1)=\sum_{C_i\in\mathcal{C}(\ell,R)}\mathrm{Vol}(C_i)\,.\end{aligned}$$

Since $\nu_k(\ell,R)\to\infty$ as $\ell\to\infty$, we have

$$\begin{aligned}&1=\lim_{\ell\to\infty}\frac{\nu_1(\ell)+1}{\nu_1(\ell)}\cdot\frac{\nu_2(\ell)+1}{\nu_2(\ell)}\cdot\ldots=\lim_{\ell\to\infty}\frac{\sum_i\mathrm{Vol}(C_i)}{\sum_j\mathrm{Vol}(D_j)}\quad\text{so}\\ &\lim_{\ell\to\infty}\sum_i\mathrm{Vol}(C_i)=\lim_{\ell\to\infty}\sum_j\mathrm{Vol}(D_j)=\mathrm{Vol}(R)\,.\end{aligned}$$

The desired estimate now follows if ℓ is sufficiently large.

Since C is Jordan measurable, C is a subset of some rectangle R and χ_C is integrable over R. Choose a partition $\mathcal{P}$ of R so $U(\chi_C,\mathcal{P})-L(\chi_C,\mathcal{P})<\varepsilon$. Let R_i be the collection of rectangles where $\max_{x\in R_i}\chi_C(x)=1$ and let $\tilde{R}_j$ be the sub collection of rectangles where $\min_{y\in R_i}\chi_C(y)=1$. The conclusions of Assertion (3) now follow immediately. We derive Assertion (4) from Assertion (3) by using Assertion (2) to approximate the rectangles R_i from outside and the rectangles $\tilde{R}_j$ from within by cubes of uniform side ℓ^{-1} for ℓ sufficiently large. □

Next, we establish:

`lem-1.3` **Lemma 1.3.** *Let R be a rectangle in $\mathbb{R}^n$. Let $\Psi:R\to\mathbb{R}^n$ be C^1. Let $||\Psi'||$ denote the operator norm of Ψ'. Assume $||\Psi'(x)||\le K$ for all $x\in R$.*

1. *If $x,y\in R$, then $|\Psi x-\Psi y|\le K|x-y|$.*
2. *If C is a cube of side c, then $\Psi(C)$ is contained in a cube of side $cK\sqrt{n}$.*
3. *If S is a subset of R which has content 0, then $\Psi(S)$ has content 0.*

Proof. Since rectangles are convex, we may express:

$$\Psi(x)-\Psi(y)=\int_{t=0}^1\partial_t\Psi(x+t(y-x))dt=\int_{t=0}^1\{\Psi'(x+t(y-x))dt\}(x-y)\,.$$

The first estimate now follows. Next, let C be a cube of side c. Since we have that $\mathrm{diam}(C)=c\sqrt{n}$, we may conclude that C is contained in a ball of radius $c\sqrt{n}/2$. The first estimate now yields $\Psi(C)$ is contained in a ball of radius $cK\sqrt{n}/2$ which is in turn contained in a cube of side $cK\sqrt{n}$ which establishes Assertion (2). Finally, suppose that S is a subset of R which has content 0. Let $\varepsilon>0$ be given. Choose ϱ so that $K^nn^{n/2}\varrho<\varepsilon$. A judicious application of Lemma 1.2 (2) shows that we can find a finite number of cubes C_i of sides s_i so that $S\subset\cup_iC_i$ and so that $\sum_is_i^n<\varrho$. But then $\Psi(S)\subset\cup_i\Psi(C_i)$. Since the images $\Psi(C_i)$ are contained in cubes $\tilde{C}_i$ of sides $Ks_i\sqrt{n}$, we have

$$\sum_i\mathrm{Vol}(\tilde{C}_i)\le K^nn^{n/2}\varrho<\varepsilon\,.$$

The Lemma is now established. □

The following is a useful remark.

`lem-1.4` **Lemma 1.4.** *Let* $\Phi : R \to \mathbb{R}^n$ *be* C^1. *Assume* $\det \Phi'(x) \neq 0$ *for all* $x \in R$. *If* S *is a Jordan measurable subset of* R, *then* $\Phi(S)$ *is Jordan measurable.*

Proof. Since $\bar{S}$ is compact, $\phi(\bar{S})$ is compact and hence closed. Thus the closure of $\Phi(S)$ is contained in $\phi(\bar{S})$. Conversly, suppose y belongs to $\phi(\bar{S})$. Then $y = \phi(x)$ where $x \in \bar{S}$. Thus we can find $x_n \in S$ with $x_n \to x$. Consequently $\phi(x_n) \to \phi x$ and y is in the closure of $\phi(S)$. We decompose $\bar{S} = bd(S) \cup \operatorname{int}(S)$. Since $\det(\Phi') \neq 0$, $\Phi(\operatorname{int}(S))$ is an open set. Thus we may decompose $\overline{\Phi S} = \Phi(bd)(S) \cup \Phi(\operatorname{int}(S))$ and clearly $\Phi(\operatorname{int}(S)) \subset \operatorname{int}(\Phi(S))$. It now follows that $bd(\Phi(S)) \subset \Phi(bd(S))$ and hence by Lemma 1.3 (3), $bd(\Phi(S))$ has content 0. It now follows that $\Phi(S)$ is Jordan measurable. ☐

The linear case is central.

`lem-1.5` **Lemma 1.5.** *Let* $T : \mathbb{R}^m \to \mathbb{R}^m$ *be a linear map with* $\det(T) \neq 0$. *If* S *is Jordan measurable, then* $\operatorname{Vol}(\Phi(S)) = |\det(T)| \operatorname{Vol}(S)$.

Proof. Suppose first R is a rectangle. We may use Gaussian elimination to convert T to the identity transformation. This implies that we can write T as a product of elementary row operations, i.e. linear maps which either (1) permute the elements in the basis, (2) rescale one of the coordinates, or (3) are a shear transformation. Verifying $\operatorname{Vol}(TR) = |\det(T)| \operatorname{Vol}(R)$ is straightforward for linear transformations of this form; Fubini's Theorem plays a central role. This establishes the Lemma if S is a rectangle. More generally, given $\varepsilon > 0$, we apply Lemma 1.2 (3) to find a collection $\mathcal{C}$ of rectangles R_i and a sub-collection $\mathcal{D}$ of rectangles $\tilde{R}_j$ so that

`eqn-1.a` (1.a) $$\cup_j \tilde{R}_j \subset S \subset \cup_i R_i,$$

`eqn-1.b` (1.b) $$\operatorname{Vol}(S) - \varepsilon \leq \sum_j \operatorname{vol}(\tilde{R}_j) \leq \operatorname{Vol}(S) \leq \sum_i \operatorname{Vol}(R_i) \leq \operatorname{Vol}(S) + \varepsilon\,.$$

From Equation (1.a), we have the estimate

$$\sum_j \chi_{\operatorname{int}(T\tilde{R}_j)} \leq \chi_{TS} \leq \sum_i \chi_{TR_i}\,.$$

By Lemma 1.2 (1), $bd(\tilde{R}_j)$ has content 0. Thus $T(bd(\tilde{R}_j))$ has content 0 by Lemma 1.3. Since $T\tilde{R}_j - \operatorname{int}(\widetilde{TR}_j) = T(bd\tilde{R}_j)$ has content 0, $\operatorname{Vol}(T\tilde{R}_j) = \operatorname{Vol}(\operatorname{int}(T\tilde{R}_j))$. Thus from Assertion (1), we may estimate:

$$\begin{aligned} \sum_j \operatorname{Vol}(\operatorname{int}(T\tilde{R}_j)) &= \sum_j \operatorname{Vol}(T\tilde{R}_j) = \sum_j \det(T) \operatorname{Vol}(\tilde{R}_j) \\ &\leq \operatorname{Vol}(TS) \leq \sum_i \operatorname{Vol}(TR_i) = |\det(T)| \sum_i \operatorname{Vol}(R_i)\,. \end{aligned}$$

Multiplying Equation (1.b) by $|\det(T)|$ then yields the estimate

$$\begin{aligned} |\det(T)|\{\operatorname{Vol}(S) - \varepsilon\} &\leq |\det(T)| \sum_j \operatorname{Vol}(\tilde{R}_j) \leq |\det(T)| \operatorname{Vol}(S) \\ &\leq |\det(T)| \sum_i \operatorname{Vol}(R_i) \leq |\det(T)|\{\operatorname{Vol}(S) + \varepsilon\}\,. \end{aligned}$$

We combine these estimates to see

$$|\det(T)|\{\operatorname{Vol}(S) - \varepsilon\} \leq \operatorname{Vol}(TS) \leq |\det(T)|\{\operatorname{Vol}(S) + \varepsilon\}\,.$$

The Lemma now follows by letting $\varepsilon \to 0$. ☐

We now establish a fundamental estimate in the general setting.

`lem-1.6` **Lemma 1.6.** *Let* R *be a rectangle in* $\mathbb{R}^n$. *Let* $\Phi : R \to \mathbb{R}^n$ *be* C^1. *Assume* $\det \Phi'(x) \neq 0$ *for all* $x \in R$. *Let* S *be a Jordan measurable subset of* R.

(1) *Suppose* $||\Phi'(x) - \operatorname{Id}|| < \varepsilon$ *for all* $x \in R$. *Then* $\operatorname{Vol}(\Phi(S)) \leq (1 + \sqrt{n}\varepsilon)^n \operatorname{Vol}(S)$.

(2) *Suppose given an invertible linear tranformation* T_0 *of* $\mathbb{R}^n$ *so for all* $x \in R$, $||\Phi'(x) - T_0|| < \varepsilon$. *Then* $\operatorname{Vol}(\Phi(S)) \le |\det(T_0)|(1 + ||T_0^{-1}||\sqrt{n}\varepsilon)^n \operatorname{Vol}(S)$.

Proof. Given $\delta > 0$, we can find a collection of rectangles $\{R_i\}$ so $S \subset \cup_i R_i$ and $\sum_i \operatorname{Vol}(R_i) < \operatorname{Vol}(S) + \delta$. By Lemma 1.2 (4) we may find a collection of cubes C_j so $S \subset \cup_i R_i \subset \cup_j C_j$ and $\sum_j \operatorname{Vol}(C_j) \le \operatorname{Vol}(S) + 2\delta$. Applying Lemma 1.3 (1) to the function $\tilde{\Psi} := \Phi - \mathrm{id}$ with $K = \varepsilon$ shows

$$||\Phi(x) - \Phi(y) - (x - y)|| \le \varepsilon||x - y||\,.$$

If C_j is a cube of side s_j and if $x, y \in C_j$, then for $1 \le i \le n$, we may estimate

$$\begin{aligned}&||\Phi(x)_i - \Phi(y)_i - (x_i - y_i)|| \le ||\Phi(x) - \Phi(y) - (x - y)|| \le \varepsilon||x - y|| \le \varepsilon s_j\sqrt{n},\\ &||\Phi(x)_i - \Phi(y)_i|| \le s_j(1 + \sqrt{n}\varepsilon)\,.\end{aligned}$$

Consequently $\Phi(C_j)$ is contained a cube of side $s_j(1 + \sqrt{n}\varepsilon)$. This shows

$$\begin{aligned}\operatorname{Vol}(\Phi(S)) &\le \sum_j \operatorname{Vol}(\Phi(C_j)) \le \sum_j c_j^n(1 + \sqrt{n}\varepsilon)^n\\ &\le (1 + \sqrt{n}\varepsilon)^n \sum_j \operatorname{Vol}(C_j) \le (1 + \sqrt{n}\varepsilon)^n(\operatorname{Vol}(S) + \delta)\,.\end{aligned}$$

This estimate for all $\delta > 0$ then yields the desired estimate in Assertion (1).

We apply Assertion (1) to establish Assertion (2). We replace Φ by $= T_0^{-1}\Phi$ and ε by $||T_0^{-1}||\varepsilon$ and apply Assertion (1) to conclude

$$\operatorname{Vol}(T_0^{-1}\Phi(S)) \le (1 + \sqrt{n}||T_0^{-1}||\varepsilon)^n \operatorname{Vol}(S)\,.$$

Applying T_0 to both sides and using Lemma 1.5 then yields Assertion (2). □

We continue our investigations:

`lem-1.7` **Lemma 1.7.** *Let* R *be a rectangle in* $\mathbb{R}^n$ *and let* $\Phi : R \to \mathbb{R}^n$ *be* C^1. *Assume* Φ *is* $1 - 1$ *and that* $\det \Phi'(x) \ne 0$ *for all* $x \in R$. *Note that* $\Phi(R)$ *is Jordan measurable. Let* $f : \Phi(R) \to \mathbb{R}$ *be integrable.*

(1) $\Phi^* f : R \to \mathbb{R}$ *is integrable.*
(2) *If* $f \ge 0$, *then* $\int_{\Phi(R)} f \le \int_R |\det(\Phi')|\Phi^* f$.
(3) *Let* S *be a Jordan measurable subset of* R *such that* $\Phi(S)$ *is a Jordan measurable subset of some rectangle contained in* $\Phi(R)$. *If* $f \ge 0$, *then* $\int_{\Phi(S)} f = \int_S |\det(\Phi')|\Phi^* f$.
(4) *If* $f \ge 0$, *then* $\int_{\Phi(R)} f = \int_R |\det(\Phi')|\Phi^* f$.
(5) $\int_{\Phi(R)} f = \int_R |\det(\Phi')|\Phi^* f$ *for any integrable* f.

Proof. Let $\mathfrak{o}(f, x)$ denote the oscillation. Set

$$\begin{aligned}S_\ell &= \{y \in \Phi(R) : \mathfrak{o}(f, y) \ge \tfrac{1}{\ell}\},\\ S &= \cup_{\ell\in\mathbb{N}} S_\ell = \{y \in \Phi(R) : \mathfrak{o}(f, y) > 0\}\,.\end{aligned}$$

Since f is integrable, S has measure 0. Since $\Phi(R)$ is compact and S_ℓ is closed, S_ℓ is compact. Since $S_\ell \subset S$, S_ℓ has measure 0 and hence content 0. Since Φ is a local diffeomorphism, applying Lemma 1.3 (3) to Φ^{-1} yields $\Phi^{-1}(S_\ell)$ has content 0 and hence $\Phi^{-1}(S) = \cup_{\ell\in\mathbb{N}}\Phi^{-1}(S_\ell)$ has measure 0. Since

$$\Phi^{-1}(S) = \{x \in R : \mathfrak{o}(\Phi^* f, x) > 0\}$$

has measure 0, $\Phi^* f$ is integrable. Assertion (1) now follows.

Let $\varepsilon > 0$ be given. Choose $\delta > 0$ so that $|x_1 - x_2| < \delta$ implies

$$||\Phi'(x_1) - \Phi'(x_2)|| < \varepsilon\,.$$

Also choose $\delta > 0$ so that if $\mathcal{P}$ is any partition of R of mesh less than δ, then

`eqn-1.c`

$$\begin{aligned} &U(|\det \Phi'|\Phi^* f, \mathcal{P}) - L(|\det \Phi'|\Phi^* f, \mathcal{P}) < \varepsilon, \\ &U(|\det \Phi'|, \mathcal{P}) - L(|\det \Phi'|, \mathcal{P}) < 1\,. \end{aligned} \tag{1.c}$$

Choose K so $||\Phi'(x)^{-1}|| \le K$ and $||\Phi'(x)|| \le K$ for all $x \in R$.

Let $\mathcal{P} = \{R_i\}$ be a partition of R of mesh less than δ. Then

$$\int_{\Phi(R)} f \le \sum_i \sup_{y \in \Phi(R_i)} f(y) \cdot \operatorname{Vol}(\Phi(R_i))\,.$$

For each i, choose $x_i \in R_i$ so $\sup_{y\in\Phi(R_i)} f(y) \le \Phi^* f(x_i) + \varepsilon$. Since $\operatorname{diam}(R_i) < \delta$, we may apply Lemma 1.6 to conclude

$$\begin{aligned} &\sup_{y \in \Phi(R_i)} f(y) \cdot \operatorname{Vol}(\Phi(R_i)) \\ \le\;& (\Phi^* f(x_i) + \varepsilon) \cdot \operatorname{Vol}(\Phi(R_i)) \\ \le\;& (\Phi^* f(x_i) + \varepsilon) \cdot \det(\Phi'(x_i))(1 + K\varepsilon\sqrt{n})^n \operatorname{Vol}(R_i) \\ \le\;& \sup_{x \in R_i} (\Phi^* f(x) + \varepsilon)|\det(\Phi'(x))(1 + K\varepsilon\sqrt{n})^n \operatorname{Vol}(R_i)\,. \end{aligned}$$

Summing and applying Equation (1.c) then yields

$$\begin{aligned} \int_{\Phi(R)} f \;&\le\; (1 + K\varepsilon\sqrt{n})^n U\{|\det(\Phi'(x)|(\Phi^* f + \varepsilon)\} \\ &\le\; (1 + K\varepsilon\sqrt{n})^n (1 + \varepsilon) \int_R |\det(\Phi')|(\Phi^* f + \varepsilon)\,. \end{aligned}$$

Taking the limit as $\varepsilon \to 0$ establishes (2).

We use (2) to see

$$\int_{\Phi(S)} f \le \int_S |\det(\Phi')|\Phi^* f\,.$$

Applying this observation to the function $|\det(\Phi')|\Phi^* f$ and the local diffeomorphism Φ^{-1} establishes the reverse inequality; it was for this reason we assumed that $\Phi(S)$ to be contained in a rectangle within the domain of Φ^{-1}; this establishes (3). Assertion (4) follows from a partition of unity argument from Assertion (3).

Since f is integrable, it is bounded by some constant K. Apply Assertion (4) of Lemma 1.7 to $f + K$ and to K and then use the additivity of the integral. ☐

Using a partition of unity then yields the change of variable formula in full generality as stated in Theorem 1.1.

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