EXERCISES

1.1. Prove $1^2+2^2+\cdots+n^2=n(n+1)(2n+1)/6$ for all natural numbers n.

1.2. Prove $3+11+\cdots+(8n-5)=4n^2-n$ for all natural numbers n.

1.3. Prove $1^3+2^3+\cdots+n^3=(1+2+\cdots+n)^2$ for all natural numbers n.

1.12. For $n\in\mathbb{N}$, let $n!$ [read "n factorial"] denote the product $1\cdot2\cdot3\cdots n$. Also let $0!=1$ and define

$$\binom{n}{k}=\frac{n!}{k!\,(n-k)!}\qquad\text{for}\quad k=0,1,\ldots,n.$$

2.1. Show that $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{24}$, and $\sqrt{31}$ are not rational numbers.

2.2. Show that $2^{1/3}$, $5^{1/7}$, and $(13)^{1/4}$ do not represent rational numbers.

2.3. Show that $(2+\sqrt{2})^{1/2}$ does not represent a rational number.

2.4. Show that $(5-\sqrt{3})^{1/3}$ does not represent a rational number.

EXERCISES

3.1. (a) Which of the properties A1–A4, M1–M4, DL, O1–O5 fail for $\mathbb{N}$?
(b) Which of these properties fail for $\mathbb{Z}$?

3.2. (a) The commutative law A2 was used in the proof of (ii) in Theorem 3.1. Where?
(b) The commutative law A2 was also used in the proof of (iii) in Theorem 3.1. Where?

3.3. Prove (iv) and (v) of Theorem 3.1.

3.1 Theorem. *The following are consequences of the field properties:*

(i) $a+c=b+c$ *implies* $a=b$;
(ii) $a\cdot0=0$ *for all* a;
(iii) $(-a)b=-ab$ *for all* a,b;
(iv) $(-a)(-b)=ab$ *for all* a,b;
(v) $ac=bc$ *and* $c\neq0$ *imply* $a=b$;
for $a,b,c\in\mathbb{R}$.

3.6. (a) Prove that $|a+b+c|\leqslant|a|+|b|+|c|$ for all a, b, $c\in\mathbb{R}$. *Hint*: Apply the triangle inequality twice. Do *not* consider eight cases.
(b) Use induction to prove

$$|a_1+a_2+\cdots+a_n|\leqslant|a_1|+|a_2|+\cdots+|a_n|$$

for n numbers $a_1,a_2,\ldots,a_n$.

3.7. (a) Show that $|b|<a$ if and only if $-a<b<a$.
(b) Show that $|a-b|<c$ if and only if $b-c<a<b+c$.
(c) Show that $|a-b|\leqslant c$ if and only if $b-c\leqslant a\leqslant b+c$.

3.8 Let $a,b\in\mathbb{R}$. Show that if $a\leqslant b_1$ for every $b_1>b$, then $a\leqslant b$.