EXERCISES

25.1. Derive 25.1(b) from 25.1(a). *Hint*: Apply (a) twice, once to g and $|g|$ and once to $-|g|$ and g.

25.2. Let $f_n(x)=x^n/n$. Show that (f_n) is uniformly convergent on $[-1,1]$ and specify the limit function.

25.3. Let $f_n(x)=(n+\cos x)/(2n+\sin^2 x)$ for all real numbers x.

(a) Show that (f_n) converges uniformly on $\mathbb{R}$. *Hint*: First decide what the limit function is; then show (f_n) converges uniformly to it.

(b) Calculate $\lim_{n\to\infty}\int_2^7 f_n(x)\,dx$. *Hint*: Don't integrate f_n.

25.4. Let (f_n) be a sequence of functions on a set $S\subseteq\mathbb{R}$ and suppose that $f_n\to f$ uniformly on S. Prove that (f_n) is uniformly Cauchy on S. *Hint*: Use the proof of Lemma 10.9 as a model, but be careful.

25.5. Let (f_n) be a sequence of bounded functions on a set S and suppose that $f_n\to f$ uniformly on S. Prove that f is a bounded function on S.

25.6. (a) Show that if $\sum|a_k|<\infty$, then $\sum a_k x^k$ converges uniformly on $[-1,1]$ to a continuous function.

(b) Does $\sum_{n=1}^{\infty}(1/n^2)x^n$ represent a continuous function on $[-1,1]$?

25.7. Show that $\sum_{n=1}^{\infty}(1/n^2)\cos nx$ converges uniformly on $\mathbb{R}$ to a continuous function.

25.8. Show that $\sum_{n=1}^{\infty}x^n/(n^2 2^n)$ has radius of convergence 2 and that the series converges uniformly to a continuous function on $[-2,2]$.

25.9. (a) Let $0<a<1$. Show that the series $\sum_{n=0}^{\infty}x^n$ converges uniformly on $[-a,a]$ to $1/(1-x)$.

(b) Does the series $\sum_{n=0}^{\infty}x^n$ converge uniformly on $(-1,1)$ to $1/(1-x)$? Explain.

25.10. (a) Show that $\sum x^n/(1+x^n)$ converges for $x\in[0,1)$.

(b) Show that the series converges uniformly on $[0,a]$ for each a, $0<a<1$.

(c) Does the series converge uniformly on $[0,1)$? Explain.

25.11. (a) Sketch the functions g_0, g_1, g_2 and g_3 in Example 3.

(b) Prove that the function f in Example 3 is continuous.

25.12. Suppose that $\sum_{k=1}^{\infty}g_k$ is a series of continuous functions g_k on $[a,b]$ that converges uniformly to g on $[a,b]$. Prove that

$$\int_a^b g(x)\,dx=\sum_{k=1}^{\infty}\int_a^b g_k(x)\,dx.$$