Exercises

11.1. Let $a_n = 3 + 2(-1)^n$ for $n \in \mathbb{N}$.

(a) List the first eight terms of the sequence (a_n).
(b) Give a subsequence that is constant [takes a single value]. Specify the selection function σ.

11.2. Consider the sequences defined as follows:

$$a_n = (-1)^n, \qquad b_n = \frac{1}{n}, \qquad c_n = n^2, \qquad d_n = \frac{6n+4}{7n-3}.$$

(a) For each sequence, give an example of a monotone subsequence.
(b) For each sequence, give its set of subsequential limits.
(c) For each sequence, give its lim sup and lim inf.
(d) Which of the sequences converges? diverges to $+\infty$? diverges to $-\infty$?
(e) Which of the sequences is bounded?

11.3. Repeat Exercise 11.2 for the sequences:

$$s_n = \cos(n\pi/3), \qquad t_n = \frac{3}{(4n+1)}, \qquad u_n = \left(-\frac{1}{2}\right)^n, \qquad v_n = (-1)^n + \frac{1}{n}.$$

11.4. Repeat Exercise 11.2 for the sequences:

$$w_n = (-2)^n, \qquad x_n = 5^{(-1)^n}, \qquad y_n = 1 + (-1)^n, \qquad z_n = n\cos(n\pi/4).$$

11.5. Let (q_n) be an enumeration of all the rational numbers in the interval $(0, 1]$.

(a) Give the set of subsequential limits for (q_n).
(b) Give the values of $\limsup q_n$ and $\liminf q_n$.

11.6. Show that every subsequence of a subsequence of a given sequence is itself a subsequence of the given sequence. *Hint*: Define subsequences as in (3) of Definition 11.1.

12.1. Let (s_n) and (t_n) be sequences and suppose that there exists N_0 such that $s_n \leqslant t_n$ for all $n > N_0$. Show that $\liminf s_n \leqslant \liminf t_n$ and $\limsup s_n \leqslant \limsup t_n$. *Hint*: Use Definition 10.6 and Exercise 9.9(c).

12.2. Prove that $\limsup |s_n| = 0$ if and only if $\lim s_n = 0$.

12.3. Let (s_n) and (t_n) be the following sequences that repeat in cycles of four:

$$(s_n) = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, \ldots)$$

$$(t_n) = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, \cdots)$$

Find

(a) $\liminf s_n + \liminf t_n$,
(b) $\liminf(s_n + t_n)$,
(c) $\liminf s_n + \limsup t_n$,
(d) $\limsup(s_n + t_n)$,
(e) $\limsup s_n + \limsup t_n$,
(f) $\liminf(s_n t_n)$,
(g) $\limsup(s_n t_n)$

12.4. Show that $\limsup(s_n + t_n) \leqslant \limsup s_n + \limsup t_n$ for bounded sequences (s_n) and (t_n). *Hint*: First show

$$\sup\{s_n + t_n : n > N\} \leqslant \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$$

Then apply Exercise 9.9(c).

12.5. Use Exercises 11.8(a) and 12.4 to prove

$$\liminf(s_n + t_n) \geqslant \liminf s_n + \liminf t_n$$

for bounded sequences (s_n) and (t_n).

12.9. (a) Prove that if $\lim s_n = +\infty$ and $\liminf t_n > 0$, then $\lim s_n t_n = +\infty$.
(b) Prove that if $\limsup s_n = +\infty$ and $\liminf t_n > 0$, then $\limsup s_n t_n = +\infty$.
(c) Observe that Exercise 12.7 is the special case of (b) where $t_n = k$ for all $n \in \mathbb{N}$.

12.10. Prove that (s_n) is bounded if and only if $\limsup |s_n| < +\infty$.

12.11. Prove the first inequality in Theorem 12.2.

12.12. Let (s_n) be a sequence of nonnegative numbers and for each n define $\sigma_n = (s_1 + s_2 + \cdots + s_n)/n$.

(a) Show that

$$\liminf s_n \leqslant \liminf \sigma_n \leqslant \limsup \sigma_n \leqslant \limsup s_n.$$

Hint: For the last inequality, show first that $M > N$ implies

$$\sup\{\sigma_n : n > M\} \leqslant (s_1 + s_2 + \cdots + s_N)/M + \sup\{s_n : n > N\}.$$

(b) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.

12.13. Let (s_n) be a bounded sequence in $\mathbb{R}$. Let A be the set of $a \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n < a\}$ is finite, i.e., all but finitely many s_n are $\geqslant a$. Let B be the set of $b \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n > b\}$ is finite. Prove that $\sup A = \liminf s_n$ and $\inf B = \limsup s_n$.

12.14. Calculate $\lim(n!)^{1/n}$.