Topics in Open Economy Macroeconomics

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Problem Set 4

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Note: The grade for will be on a $\checkmark^-$, $\checkmark$, $\checkmark^+$ scale.

Problem 1: Reputation

This problem intends to answer the following question: How much net uncollateralized lending can be supported by the threat of a capital markets embargo?

We assume that the only cost of default is a loss of reputation that brings immediate and permanent exclusion from capital markets. We focus on a small open economy that has stochastic output $Y_s = \overline{Y} + \epsilon_s$ for dates $s \geq t$. ϵ_s is a mean-zero i.i.d. shock that takes values $\epsilon_1, ..., \epsilon_N \in [\underline{\epsilon}, \overline{\epsilon}]$, and $\pi(\epsilon_i)$ is the probability that $\epsilon = \epsilon_i$.

On date t, the country's infinitely-lived representative agent maximizes

$$U_t = \mathrm{E}_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} u\left(C_s\right) \right\} \tag{1}$$

subject to the budget constraint

$$B_{s+1} = (1+r)B_s + \overline{Y} + \epsilon_s - C_s - P_s(\epsilon_s) \tag{2}$$

where u is concave; B denotes national holdings of non-contingent claims on foreigners, $B_t = 0$, and for every date s, insurance payments, P_s, satisfy

$$\sum_{i=1}^{\infty} \pi\left(\epsilon_i\right) P_s\left(\epsilon_i\right) = 0 \tag{3}$$

a) Assume that $\beta(1+r) = 1$. Show that with full insurance contracts satisfy $P_s(\epsilon) = \epsilon$. This will be the equilibrium contract if the country can precommit to meet its obligations with creditors. Show that this implies $C_s = \overline{Y} \forall s$ and $B = 0 \forall s$.

b) If the country cannot precommit to pay, is the threat of being cut off from world credit markets enough to support full insurance? We will answer this in the following steps.

Suppose that at date t a country contemplates default on the full insurance contract. Its short-run gain is

$$\text{Gain}_t = u\left(\overline{Y} + \epsilon_t\right) - u\left(\overline{Y}\right) \tag{4}$$

Explain the intuition for the latter expression.

c) The date t cost of default is given by

$$\text{Cost}_t = \sum_{s=t+1}^{\infty} \beta^{s-t} u\left(\overline{Y}\right) - \sum_{s=t+1}^{\infty} \beta^{s-t} \text{E}_t u\left(\overline{Y} + \epsilon_s\right) \tag{5}$$

What is the intuition for the latter?

d) Show that you can write the latter expression as

$$\text{Cost} = \frac{\beta}{1-\beta}\left[u\left(\overline{Y}\right) - \text{E}u\left(\overline{Y} + \epsilon\right)\right] \tag{6}$$

e) Is $u\left(\overline{Y}\right) > \text{E}u\left(\overline{Y} + \epsilon\right)$ or $u\left(\overline{Y}\right) < \text{E}u\left(\overline{Y} + \epsilon\right)$? Why? Given your response, what what can you say about the penalty for defaulting? Why? What happens with the cost of default as $\beta \rightarrow 1$? What is the the intuition behind this?

f) The full insurance contract is sustainable in all states of nature (and all dates) only if $\text{Gain}\left(\overline{\epsilon}\right) \leq \text{Cost}$. Why?

Clearly, this can also be expressed as

$$u\left(\overline{Y} + \overline{\epsilon}\right) - u\left(\overline{Y}\right) \leq \frac{\beta}{1-\beta}\left[u\left(\overline{Y}\right) - \text{E}u\left(\overline{Y} + \epsilon\right)\right] \tag{7}$$

g) Given the latter equation analyze the incentives for a country to default or to honor its debts.

h) What will happen if the same exercise has a finite horizon (say, T)? Explain.

Problem 2: Optimal Financial Contract and Demand for Capital

In the paper by Bernanke, Gertler, and Gilchrist (1999), "The Financial Accelerator in a Quantitative Business Cycle Framework", (Handbook of Macroeconomics), we saw the existence of a monotonically increasing relationship between the capital/wealth ratio and the premium on external funds: $QK/N = \psi\left(R^K/R\right)$ with $\psi'(\bullet) > 0$. Now we will derive it formally.

Let profits per unit of capital equal ωR^K, where $\omega \in [0, \infty)$ is an idiosyncratic shock with $E(\omega) = 1$. We assume $F(x) = \Pr[\omega < x]$ is a continuous probability distribution with $F(0) = 0$. Denote by $f(\omega)$ the pdf of ω. Given an initial level of net worth N, and a price of capital Q, the entrepreneur borrows $QK - N$ to invest K units of capital in the project. The total return to the project is thus $\omega R^K QK$. Assume ω is unknown to both the entrepreneur and the lender prior to the investment decision. After the investment decision is made, the lender can only observe ω by paying the monitoring cost $\mu\omega R^K QK$, where $0 < \mu < 1$. Let the required return on lending equal R, with $R < R^K$.

The optimal contract specifies a cutoff value $\overline{\omega}$ such that if $\omega > \overline{\omega}$, the borrower pays the lender the fixed amount $\overline{\omega} R^K QK$ and keeps the equity $(\omega - \overline{\omega})R^K QK$. Alternatively, if $\omega < \overline{\omega}$, the borrower receives nothing, while the lender monitors the borrower and receives $(1-\mu)\omega R^K QK$ in residuals claims net of monitoring costs. In equilibrium, the lenders earns an expected return

$$\left[\omega \Pr(\omega \geq \overline{\omega}) + (1-\mu)E(\omega | \omega < \overline{\omega})\Pr(\omega < \overline{\omega})\right] R^K QK = R(QK - N) \tag{8}$$

a) Explain the intuition in the latter expression.

Given constant returns to scale, the cutoff $\overline{\omega}$ determines the division of expected gross profits $R^K QK$. Define $\Gamma(\overline{\omega})$ as the expected gross share of profits going to the lender

$$\Gamma(\overline{\omega}) \equiv \int_0^{\overline{\omega}} \omega f(\omega) d\omega + \overline{\omega} \int_{\overline{\omega}}^{\infty} f(\omega) d\omega \tag{9}$$

b) Show that

$$\Gamma'(\overline{\omega}) = 1 - F(\overline{\omega}) \tag{10}$$

and

$$\Gamma''(\overline{\omega}) = -f(\overline{\omega}) \tag{11}$$

c) What do these two conditions say?

d) Define the expected monitoring costs, $\mu G(\overline{\omega})$, as follows

$$\mu G(\overline{\omega}) \equiv \mu \int_0^{\overline{\omega}} \omega f(\omega) d\omega \tag{12}$$

Show that $\mu G'(\overline{\omega}) \equiv \mu \overline{\omega} f(\overline{\omega})$

e) Thus, the net share of profits going to the lender is $\Gamma(\overline{\omega}) - \mu G(\overline{\omega})$, and the share going to the entrepreneur is $1 - \Gamma(\overline{\omega})$, where by definition $\Gamma(\overline{\omega})$ satisfies $0 < \Gamma(\overline{\omega}) < 1$.

Show that this implies

$$\Gamma(\overline{\omega}) - \mu G(\overline{\omega}) > 0 \tag{13}$$

$$\lim_{\overline{\omega} \to 0} \Gamma(\overline{\omega}) - \mu G(\overline{\omega}) = 0 \tag{14}$$

$$\lim_{\overline{\omega} \to 0} \Gamma(\overline{\omega}) - \mu G(\overline{\omega}) = 1 - \mu \tag{15}$$

f) Thus, to preclude the firm to obtain – with probability one – unbounded profits under monitoring we assume that $R^K(1-\mu) < R$.

Let $h(\overline{\omega}) \equiv f(\overline{\omega}) / (1 - F(\overline{\omega}))$ be the hazard rate. Assume that $\overline{\omega} h(\overline{\omega})$ is increasing in $\overline{\omega}$[1]. Show that differentiating $\Gamma(\overline{\omega}) - \mu G(\overline{\omega})$, there exists an $\overline{\omega}^*$ such that

$$\Gamma'(\overline{\omega}) - \mu G'(\overline{\omega}) = (1 - F(\overline{\omega}))(1 - \mu \overline{\omega} h(\overline{\omega})) \lesseqgtr 0 \text{ for } \overline{\omega} \gtreqless \overline{\omega}^* \tag{16}$$

This implies that the net payoff reaches a global maximum at $\overline{\omega}^*$.

g) The optimal contracting problem with non-stochastic monitoring results from

$$\max_{K, \overline{\omega}} (1 - \Gamma(\overline{\omega})) R^K Q K \tag{17}$$

subject to

$$[\Gamma(\overline{\omega}) - \mu G(\overline{\omega})] R^K Q K = R(QK - N) \tag{18}$$

[1] Any monotonically increasing transformation of the normal distribution satisfies this condition.

To simplify the analysis, define the premium on external fund $s \equiv R^K/R$, and given constant returns to scale, normalizing by wealth, use $k = QK/N$, the capital/wealth ratio, as the choice variable. Let the Lagrange multiplier associated with the constraint that lenders earn their required rate of return in expectation be λ. Show that the first order conditions are

$$\Gamma'(\overline{\omega}) - \lambda \left[\Gamma'(\overline{\omega}) - \mu G'(\overline{\omega})\right] = 0 \tag{19}$$

$$\left[(1 - \Gamma(\overline{\omega})) + \lambda \left(\Gamma(\overline{\omega}) - \mu G(\overline{\omega})\right)\right] s - \lambda = 0 \tag{20}$$

$$\left[\Gamma(\overline{\omega}) - \mu G(\overline{\omega})\right] sk - (k-1) = 0 \tag{21}$$

h) Since $\Gamma(\overline{\omega}) - \mu G(\overline{\omega})$ is increasing on $(0, \overline{\omega}^*)$ and decreasing on $(\overline{\omega}^*, \infty)$, the lender would never choose $\overline{\omega} > \overline{\omega}^*$. Consider the case $0 < \overline{\omega} < \overline{\omega}^*$, which implies an interior solution. A sufficient condition for an interior solution is

$$s < \frac{1}{\Gamma(\overline{\omega}^*) - \mu G(\overline{\omega}^*)} \equiv s^* \tag{22}$$

From the FOC's, we can write

$$\lambda(\overline{\omega}) = \frac{\Gamma'(\overline{\omega})}{\Gamma'(\overline{\omega}) - \mu G'(\overline{\omega})} \tag{23}$$

Taking derivatives, show that

$$\lambda'(\overline{\omega}) = \frac{\mu \left[\Gamma'(\overline{\omega}) G''(\overline{\omega}) - \Gamma''(\overline{\omega}) G'(\overline{\omega})\right]}{\left[\Gamma'(\overline{\omega}) - \mu G'(\overline{\omega})\right]^2} \tag{24}$$

for $\overline{\omega} \in (0, \overline{\omega}^*)$. Taking limits, show

$$\lim_{\overline{\omega} \to 0} \lambda(\overline{\omega}) = 1 \tag{25}$$

$$\lim_{\overline{\omega} \to \overline{\omega}^*} \lambda(\overline{\omega}) = +\infty \tag{26}$$

i) Define

$$\rho(\overline{\omega}) \equiv \frac{\lambda(\overline{\omega})}{1-\Gamma(\overline{\omega})+\lambda\left[\Gamma(\overline{\omega})-\mu G(\overline{\omega})\right]} \tag{27}$$

Thus, show that $\overline{\omega}$ satisfies

$$s = \rho(\overline{\omega}) \tag{28}$$

This means that $\rho(\overline{\omega})$ is the wedge between the expected rate of return on capital and the safe return demanded by lenders.

Compute the derivative of $\rho(\overline{\omega})$ to obtain

$$\rho'(\overline{\omega}) = \rho(\overline{\omega})\frac{\lambda'(\overline{\omega})}{\lambda(\overline{\omega})}\left[\frac{1-\Gamma(\overline{\omega})}{1-\Gamma(\overline{\omega})+\lambda\left[\Gamma(\overline{\omega})-\mu G(\overline{\omega})\right]}\right] > 0 \text{ for } \overline{\omega} \in (0, \overline{\omega}^*) \tag{29}$$

$$\lim_{\overline{\omega}\to 0} \rho(\overline{\omega}) = 1 \tag{30}$$

$$\lim_{\overline{\omega}\to\overline{\omega}^*} \rho(\overline{\omega}) = \frac{1}{\left[\Gamma(\overline{\omega}^*)-\mu G(\overline{\omega}^*)\right]} \equiv s^* < \frac{1}{1-\mu} \tag{31}$$

Thus, for $s < s^*$, these conditions guarantee a one-to-one mapping between the optimal cutoff $\overline{\omega}$ and the premium on external funds.

j) Notice that inverting (28), we may express $\overline{\omega} = \overline{\omega}'(s)$ where $\overline{\omega}'(s) > 0$ for $s \in (1, s^*)$. This establishes a monotonically increasing relationship between default probabilities and the premium on external funds.

Define

$$\Psi(\overline{\omega}) \equiv 1 + \frac{\lambda[\Gamma(\overline{\omega})-\mu G(\overline{\omega})]}{1-\Gamma(\overline{\omega})} \tag{32}$$

Then, given a cutoff $\overline{\omega} \in (0, \overline{\omega}^*)$, show that FOC imply a unique capital/wealth ratio (and hence leverage) ratio

$$k = \Psi(\overline{\omega}) \tag{33}$$

Now compute the derivatives and limits:

$$\Psi'(\overline{\omega}) = \frac{\lambda'(\overline{\omega})}{\lambda(\overline{\omega})}\left[\Psi(\overline{\omega}) - 1\right] + \frac{\Gamma'(\overline{\omega})}{1-\Gamma(\overline{\omega})}\Psi(\overline{\omega}) \text{ for } \overline{\omega} \in (0, \overline{\omega}^*) \tag{34}$$

$$\lim_{\overline{\omega}\to 0} \Psi(\overline{\omega}) = 1 \tag{35}$$

$$\lim_{\overline{\omega}\to \overline{\omega}^*} \Psi(\overline{\omega}) = +\infty \tag{36}$$

Explain why from (28) and (33), you may express the capital/wealth ratio as an increasing function of the premium on external funds $k = \psi(s)$, with $\psi' > 0$ for $s \in (1, s^*)$.

k) Let's now introduce the possibility of aggregate risk. Let profits per unit of capital expenditures now equal $\tilde{u}\omega R^K$, where ω represents the idiosyncratic shock, and $\tilde{u}$ the aggregate shock to the profit rate, and $E[\omega] = E[\tilde{u}] = 1$. Since entrepreneurs are risk neutral, assume that they bear all the risk associated with the contract.

Thus, in this case, the optimal contracting problem may be written as

$$\max_{k,\overline{\omega}} E\left\{\left[1-\Gamma(\overline{\omega})\right]\tilde{u}sk + \lambda\left[\left(\Gamma(\overline{\omega}) - \mu G(\overline{\omega})\right)\tilde{u}sk - (k-1)\right]\right\} \tag{37}$$

where λ is the *ex-post* value (after the realization of the aggregate shock $\tilde{u}$) of the Lagrange multiplier on the constraint that lenders earn their required return, and $E\{\bullet\}$ refers to the expectation taken over the distribution of the aggregate shock $\tilde{u}$.

We wish to establish that with the addition of aggregate risk, the capital/wealth ratio k is still an increasing function of the *ex-ante* premium on external funds. Define $\Upsilon(\overline{\omega}) \equiv 1 - \Gamma(\overline{\omega}) + \lambda\left[\Gamma(\overline{\omega}) - \mu G(\overline{\omega})\right]$. Show that now the first order conditions are

$$\Gamma'(\overline{\omega}) - \lambda\left[\Gamma'(\overline{\omega}) - \mu G'(\overline{\omega})\right] = 0 \tag{38}$$

$$E\left\{\Upsilon(\overline{\omega})\tilde{u}s - \lambda(\overline{\omega})\right\} = 0 \tag{39}$$

$$\left[\Gamma(\overline{\omega}) - \mu G(\overline{\omega})\right]\tilde{u}s - (k-1) = 0 \tag{40}$$

l) Note that here the function $\lambda(\overline{\omega})$ is similar to the one without aggregate risk. Now the constraint that lenders earn their required rate of return defines an implicit function of the cutoff $\overline{\omega} = \overline{\omega}(\tilde{u}, s, k)$. Show that

$$\frac{\partial\overline{\omega}}{\partial s} = \frac{-\left[\Gamma(\overline{\omega}) - \mu G(\overline{\omega})\right]}{\left[\Gamma'(\overline{\omega}) - \mu G'(\overline{\omega})\right]s} < 0 \tag{41}$$

$$\frac{\partial \overline{\omega}}{\partial k} = \frac{1}{\left[\Gamma'(\overline{\omega}) - \mu G'(\overline{\omega})\right] \tilde{u} s} > 0 \tag{42}$$

m) To obtain a relationship of the form $k = \psi(s)$, $\psi'(s) > 0$, totally differentiate the FOC with respect to capital, to obtain

$$E\left\{\tilde{u}\Upsilon(\overline{\omega})ds + \tilde{u}s\Upsilon'(\overline{\omega})\left(\frac{\partial \overline{\omega}}{\partial s}ds + \frac{\partial \overline{\omega}}{\partial k}dk\right) - \lambda'(\overline{\omega})\left(\frac{\partial \overline{\omega}}{\partial s}ds + \frac{\partial \overline{\omega}}{\partial k}dk\right)\right\} = 0 \tag{43}$$

Show that this is the same as

$$\frac{dk}{ds} = \frac{E\left\{\left[\tilde{u}s\Upsilon'(\overline{\omega}) - \lambda'(\overline{\omega})\right] \frac{\partial \overline{\omega}}{\partial s} + \tilde{u}\Upsilon(\overline{\omega})\right\}}{E\left\{\left[\lambda'(\overline{\omega}) - \tilde{u}s\Upsilon'(\overline{\omega})\right] \frac{\partial \overline{\omega}}{\partial k}\right\}} \tag{44}$$

n) Show that

$$\Upsilon'(\overline{\omega}) = \lambda'(\overline{\omega})\left[\Gamma(\overline{\omega}) - \mu G(\overline{\omega})\right] \tag{45}$$

Then obtain

$$\lambda'(\overline{\omega}) - \Upsilon'(\overline{\omega})\tilde{u}s = \lambda'(\overline{\omega})\left\{1 - \left[\Gamma(\overline{\omega}) - \mu G(\overline{\omega})\right]\tilde{u}s\right\} = \frac{\lambda'(\overline{\omega})}{k} \tag{46}$$

Thus,

$$\frac{dk}{ds} = \frac{E\left\{\tilde{u}s\Upsilon(\overline{\omega}) - \lambda'(\overline{\omega})\frac{\partial \overline{\omega}}{\partial s}\right\}}{E\left\{\lambda'(\overline{\omega})\frac{\partial \overline{\omega}}{\partial k}\right\}} \tag{47}$$

o) What are the signs of $\partial\overline{\omega}/\partial s$, $\partial\overline{\omega}/\partial k$, and $\lambda'(\overline{\omega})$? Given these, what is the sign of dk/ds?