The Journal of Fourier Analysis and Applications

Volume 3, Number 5, 1997

Tight Frames of Multidimensional Wavelets

Marcin Bownik

ABSTRACT. In this paper we deal with multidimensional wavelets arising from a multiresolution analysis with an arbitrary dilation matrix A, namely we have scaling equations

$$\varphi^s(x) = \sum_{k\in\mathbb{Z}^n} h_k^s \sqrt{|\det A|}\varphi^1(Ax-k) \qquad \text{for } s = 1,\ldots,q\,,$$

where φ^1 is a scaling function for this multiresolution and $\varphi^2,\ldots,\varphi^q$ ($q = |\det A|$) are wavelets. Orthogonality conditions for $\varphi^1,\ldots,\varphi^q$ naturally impose constraints on the scaling coefficients $\{h_k^s\}_{k\in\mathbb{Z}^n}^{s=1,\ldots,q}$, which are then called the wavelet matrix. We show how to reconstruct functions satisfying the scaling equations above and show that $\varphi^2,\ldots,\varphi^q$ always constitute a tight frame with constant 1. Furthermore, we generalize the sufficient and necessary conditions of orthogonality given by Lawton and Cohen to the case of several dimensions and arbitrary dilation matrix A.

1. Preliminaries

In this section we fix some definitions and notations and we present theorems we use later. We use the following definition of the Fourier transform in $\mathbb{R}^n$.

$$\mathcal{F}f(x) = \hat{f}(x) = \int_{\mathbb{R}^n} f(y)e^{-2\pi i\langle x,y\rangle}dy\,. \tag{1.1}$$

This is well defined for integrable functions f. Nevertheless, $\mathcal{F}$ can be defined on $L^2(\mathbb{R}^n)$, and then $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is unitary (Plancherel theorem). Let $||\cdot||_2$ be the norm in $L^2(\mathbb{R}^n)$. Let us recall some useful properties of the Fourier transform. Let us denote by $\mathcal{T}_y$ the operator of translation by y, $\mathcal{T}_y f(x) = f(x-y)$ and by $\mathcal{U}_A$ the scaling operator by a non-degenerate matrix $A \in M_n(\mathbb{R})$, $\mathcal{U}_A f(x) = f(Ax)$. Then

$$\begin{aligned} \mathcal{F}\mathcal{T}_y f(x) &= e^{-2\pi i\langle x,y\rangle}\hat{f}(x)\,, && (1.2)\\ \mathcal{F}\mathcal{U}_A f(x) &= \frac{1}{|\det A|}\hat{f}\left(\left(A^T\right)^{-1}x\right)\,. && (1.3)\end{aligned}$$

Definition. A family of vectors $(v_j)_{j\in J}$ in Hilbert space $\mathcal{H}$ is called *a frame* if there are $A > 0$, $B < \infty$, such that for all $v \in \mathcal{H}$,

$$A||v||^2 \le \sum_{j\in J} |\langle v, v_j\rangle|^2 \le B||v||^2\,. \tag{1.4}$$

Math Subject Classifications. 42C15.
Keywords and Phrases. Tight frame, wavelets, multiresolution analysis.

ISSN 1069-5869

A and B are called frame constants. If $A = B$, then $(v_j)_{j\in J}$ is called *a tight frame.* $\square$

General information about frames can be found in [3]. For a tight frame with frame constant 1 we need only the simple fact that if $||v_j|| = 1$ for all $j \in J$, then $(v_j)_{j\in J}$ is an orthonormal basis of $\mathcal{H}$. To see this, note that linear combinations of v_j are dense in $\mathcal{H}$ and $||v_{j_0}||^2 = \sum_{j\in J} |\langle v_{j_0}, v_j\rangle|^2 = ||v_{j_0}||^4 + \sum_{j\in J, j\neq j_0} |\langle v_{j_0}, v_j\rangle|^2$; hence, $\langle v_{j_0}, v_j\rangle = 0$ for $j \neq j_0$.

Definition. For any integer $m > 0$ we introduce *the Sobolev space* (with exponent 2) by

$$W^m(\mathbb{R}^n) = \left\{f \in L^2(\mathbb{R}^n) : D^\alpha f \in L^2(\mathbb{R}^n) \text{ for } |\alpha| \le m\right\} . \tag{1.5}$$

with norm

$$||f||_{W^m} = \left(\sum_{|\alpha|\le m} ||D^\alpha f||_2^2\right)^{1/2} , \tag{1.6}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi index and by $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}$ is the distributional derivative. $W^m(\mathbb{R}^n)$ equipped with the norm (1.6) is Hilbert space. $\square$

We will use the Sobolev lemma and a simple lemma about Sobolev spaces (see[17]).

Lemma 1.

(Sobolev). If $f \in W^m(\mathbb{R}^n)$ and $m > n/2$, then (eventually after a change of values on a set of measure 0)

- *$f \in C^r(\mathbb{R}^n)$ for $r < m - n/2$,*
- *the derivatives $D^\alpha f$ for $|\alpha| \le r$ satisfy the inequality*

$$||D^\alpha f||_\infty \le c||f||_{W^m} , \tag{1.7}$$

with constant $c > 0$ independent of f.

Lemma 2.

Let $h \in W^m(\mathbb{R}^n)$ and $g \in C^\infty(\mathbb{R}^n)$ have compact support. Then the sequence $a_k = ||h\mathcal{T}_k g||_{W^m}$, $k \in \mathbb{Z}^n$, belongs to $l^2(\mathbb{Z}^n)$.

Proof.

$$\begin{aligned} |a_k|^2 = ||h\mathcal{T}_k g||_{W^m}^2 &= \sum_{|\alpha|\le m} ||D^\alpha(h\mathcal{T}_k g)||_2^2 \le C \sum_{|\alpha|+|\beta|\le m} ||D^\alpha h D^\beta(\mathcal{T}_k g)||_2^2 \\ &\le C \sup_{|\beta|\le m} ||D^\beta g||_\infty^2 \sum_{|\alpha|\le m} \int_{\operatorname{supp}\mathcal{T}_{-k}g} |D^\alpha h(x)|^2 dx . \end{aligned}$$

Because supp$g$ is bounded,

$$\sum_{k\in\mathbb{Z}^n} \int_{\operatorname{supp}\mathcal{T}_{-k}g} |D^\alpha h(x)|^2 dx \le C'' \int_{\mathbb{R}^n} |D^\alpha h(x)|^2 dx$$

which finishes the proof. $\square$

2. Introduction

Assume we have some matrix $A \in M_n(\mathbb{R})$ acting on a lattice Γ, ($\Gamma = P\mathbb{Z}^n$ for some non-degenerate matrix $P \in M_n(\mathbb{R})$), such that:

- A is a dilation matrix, i.e., all eigenvalues λ of A satisfy $|\lambda| > 1$.
- Γ is invariant for A, that is $A\Gamma \subset \Gamma$.

$P^{-1}AP$ is a matrix with integer entries hence $q = |\det A| = |\det P^{-1}AP|$ is an integer greater than 1.

Definition. By *a multiresolution analysis* associated with (A, Γ) we mean a sequence of closed subspaces $(V_i)_{i\in\mathbb{Z}} \subset L^2(\mathbb{R}^n)$, satisfying following conditions:

i. $V_i \subset V_{i+1}$ for $i \in \mathbb{Z}$.
ii. $\bigcup_{i\in\mathbb{Z}} V_i$ is dense in $L^2(\mathbb{R}^n)$.
iii. $\bigcap_{i\in\mathbb{Z}} V_i = \{0\}$.
iv. $f(x) \in V_i$ iff $f(A^{-i}x) \in V_0$.
v. There exists φ called *a scaling function* such that $\{\varphi(x-\gamma)\}_{\gamma\in\Gamma}$ is an orthonormal basis of V_0.

□

Remark. Conditions (iv) and (v) can be expressed by saying that for each i $\{\varphi(A^i x - \gamma)\}_{\gamma\in\Gamma}$ is an orthonormal basis of V_i. Condition (i) then implies that *a scaling equation* is satisfied

$$\varphi(x) = \sum_{\gamma\in\Gamma} h_\gamma \sqrt{|\det A|}\varphi(Ax-\gamma)\,, \tag{2.1}$$

for some coefficients $(h_\gamma)_{\gamma\in\Gamma}$. Thus, the main ingredient of a multiresolution analysis is scaling function φ satisfying (2.1). □

If we have a multiresolution analysis with scaling function φ, then one can show there are numbers $\{h^s_\gamma\}_{\gamma\in\Gamma}^{s=2,\dots,q}$ so that the $q-1$ functions $\varphi^2, \dots; \varphi^q$, called *wavelets*, generated from φ by the formula

$$\varphi^s(x) = \sum_{\gamma\in\Gamma} h^s_\gamma \sqrt{|\det A|}\varphi(Ax-\gamma) \qquad \text{for } s = 2, \dots, q\,, \tag{2.2}$$

have the property that $\{\varphi^s(x-\gamma)\}_{\gamma\in\Gamma}^{s=2,\dots,q}$ is an orthonormal basis of $W_0 = V_1 \ominus V_0$ (the orthogonal compliment of V_0 in V_1), see [19]. Equations (2.1) and (2.2) can be expressed jointly by

$$\varphi^s(x) = \sum_{\gamma\in\Gamma} h^s_\gamma \sqrt{|\det A|}\varphi^1(Ax-\gamma) \qquad \text{for } s = 1, \dots, q\,, \tag{2.3}$$

where $\varphi^1 = \varphi$ is the scaling function and $h^1_\gamma = h_\gamma$, $\gamma \in \Gamma$. Therefore

$$\left\{|\det A|^{j/2}\varphi^s(A^j x - \gamma)\right\}_{\gamma\in\Gamma, j\in\mathbb{Z}}^{s=2,\dots,q} \tag{2.4}$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$.

In order to simplify many calculations, we will deal with a multiresolution analysis associated with $(A, \mathbb{Z}^n)$, where A is some dilation matrix with integer entries. One should stress that this is not an essential restriction. For any multiresolution analysis $(V_i)_{i\in\mathbb{Z}^n}$ associated with (A, Γ) with scaling function φ, we can consider another multiresolution analysis associated with $(P^{-1}AP, \mathbb{Z}^n)$, where P has the same meaning as above. Since A and $P^{-1}AP$ have the same characteristic polynomials $P^{-1}AP$ is also a dilation matrix. Consider the unitary operator U_P given by $U_P f(x) = \sqrt{|\det P|} f(Px)$. Because U_P preserves scalar product in $L^2(\mathbb{R}^n)$ $(U_P V_i)_{i\in\mathbb{Z}}$ is a multiresolution analysis with scaling function $U_P\varphi$. Scalings and translates of $U_P\varphi^2, \dots, U_P\varphi^q$ form an orthonormal basis. The operator U_P preserves other properties, such as tightness of the frame, smoothness, vanishing in infinity, compact support, etc. This is why we will deal only with dilation matrices A acting invariantly on $\Gamma = \mathbb{Z}^n$.

Self similar tilings of $\mathbb{R}^n$ arise naturally when one considers a multiresolution analysis for which the scaling function is the indicator function of some measurable set. This was first noticed in the paper [6]. Many other authors have worked on related subjects, see [8, 12] and [5]. The following fact which can be extracted from [9] is of great use.

Fact 1.

Let A be a dilation matrix and $\mathcal{D} = \{k_1, \dots, k_q\}$ be $q = |\det A|$ representatives of different cosets of $\mathbb{Z}^n / A\mathbb{Z}^n$ and $Q = Q(A, \mathcal{D}) = \{x \in \mathbb{R}^n : x = \sum_{i=1}^{\infty} A^{-i}\varepsilon_i, \quad \varepsilon_i \in \mathcal{D}\}$. If $f \in L^1_{loc}(\mathbb{R}^n)$ (locally integrable on $\mathbb{R}^n$) is $\mathbb{Z}^n$-periodic then

$$\int_Q f(x)\,dx = |Q| \int_{[0,1]^n} f(x)\,dx\,.$$

3. Solution of the Scaling Equation

Suppose we have some scaling coefficients $\{h_k^s\}_{k\in\mathbb{Z}^n}^{s=1,\dots,q}$ and using them we try to reconstruct the wavelets appearing in (2.3). Naturally, we should add some extra conditions on these coefficients. The orthonormality of translations of the scaling function and of the wavelets is the motivation for the following definition.

Definition. A sequence of vectors $(h^1, h^2, \dots, h^q) \in (l^1(\mathbb{Z}^n))^q$ is called *a wavelet matrix*, if

$$\sum_{k\in\mathbb{Z}^n} h^s_{k+Am} \overline{h^{s'}_{k+Am'}} = \delta_{s,s'}\delta_{m,m'} \tag{3.1}$$

for every $s, s' = 1, \dots, q$; $m, m' \in \mathbb{Z}^n$ and

$$\sum_{m\in\mathbb{Z}^n} h^1_m = \sqrt{|\det A|}\,. \tag{3.2}$$

The first vector is called the scaling vector, the others are called wavelet vectors. □

This definition in the case of one dimension appeared in [7], where the reader can find various examples of wavelet matrices. The simplest example of a wavelet matrix for the general dilation A is obtained by taking a unitary $q \times q$ matrix $U = (u_{ij})_{i=1,\dots,q}^{j=1,\dots,q}$ with a constant first row, that is $u_{1j} = 1/\sqrt{q}$, $j = 1, \dots, q$ and defining

$$h_k^s = \begin{cases} u_i^s & \text{if } k = k_i \text{ for some } i = 1, \dots, q\,, \\ 0 & \text{otherwise,} \end{cases}$$

where $\{k_1, \dots, k_q\}$ are representatives of different cosets of $\mathbb{Z}^n / A\mathbb{Z}^n$. Not much is known to the author about the existence, for a given dilation matrix, of wavelet matrices with coefficients of compact support or with strong decay at infinity.

Now we begin to study the existence of a scaling function satisfying (2.1) and hence wavelets given by (2.2).

For a given scaling vector $(h_k)_{k\in\mathbb{Z}^n}$ we define a function m by

$$m(x) = \frac{1}{\sqrt{|\det A|}} \sum_{k\in\mathbb{Z}^n} h_k e^{-2\pi i \langle k,x\rangle}\,. \tag{3.3}$$

Let us denote $B = A^T$ and let us choose any $l_1, \dots, l_q$ representatives of different cosets of $\mathbb{Z}^n / B\mathbb{Z}^n$, that is $\bigcup_{j=1}^q (l_j + B\mathbb{Z}^n) = \mathbb{Z}^n$.

Lemma 3.

$$\sum_{j=1}^{q} \frac{1}{|\det A|} e^{2\pi i \langle m, B^{-1} l_j \rangle} = \begin{cases} 0 & m \notin A\mathbb{Z}^n, \\ 1 & m \in A\mathbb{Z}^n . \end{cases}$$

The proof can be found in [5].

Lemma 4.

Let $(h_k)_{k\in\mathbb{Z}^n} \in l^1(\mathbb{Z}^n)$ and let $m(x)$ be a function given by (3.3). The condition

$$\sum_{k\in\mathbb{Z}^n} h_k \overline{h_{k+Am}} = \delta_{0,m} \qquad \text{for } m \in \mathbb{Z}^n \tag{3.4}$$

is equivalent to

$$\sum_{j=1}^{q} |m(B^{-1}(x + l_j))|^2 = 1 \qquad \text{for a.e. } x \in \mathbb{R}^n . \tag{3.5}$$

Proof.

$$\begin{aligned} \sum_{j=1}^{q} |m(x + B^{-1} l_j)|^2 &= \sum_{j=1}^{q} \frac{1}{|\det A|} \sum_{k,k'\in\mathbb{Z}^n} h_k \overline{h_{k'}} e^{-2\pi i \langle k-k', x+B^{-1} l_j \rangle} \\ &= \sum_{k,m\in\mathbb{Z}^n} h_k \overline{h_{k+m}} e^{2\pi i \langle m, x\rangle} \sum_{j=1}^{q} \frac{1}{|\det A|} e^{2\pi i \langle m, B^{-1} l_j \rangle} \\ &= \sum_{k,m\in\mathbb{Z}^n} h_k \overline{h_{k+Am}} e^{2\pi i \langle Am, x\rangle} \end{aligned}$$

In the last equation, we used Lemma 3. Since $\{e^{2\pi i \langle m, x\rangle}\}_{m\in\mathbb{Z}^n}$ is an orthonormal basis of $L^2([0,1]^n)$ (3.4) is equivalent to (3.5). □

If $(h_k)_{k\in\mathbb{Z}^n}$ is a scaling vector, then using (3.2) we can deduce that m is continuous, $|m(x)| \le 1$ and $m(0) = 1$. The next theorem tells about the existence of a scaling function φ.

Theorem 1.

(about the product) *Let m be given by (3.3), where h is a scaling vector. If the product $\prod_{i=1}^{\infty} m(B^{-i}x)$ converges pointwise to*

$$\hat{\varphi}(x) := \prod_{i=1}^{\infty} m(B^{-i}x) , \tag{3.6}$$

then $\hat{\varphi}$ belongs to $L^2(\mathbb{R}^n)$ and $||\hat{\varphi}||_2 \le 1$. Moreover, $\hat{\varphi}$ is the Fourier transform of the φ satisfying the scaling equation (2.1).

Proof. Let $\mathcal{L} = \{l_1, \dots, l_q\}$ denote the set of representatives of the distinct q cosets of $\mathbb{Z}^n / B\mathbb{Z}^n$ and

$$Q = \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{\infty} B^{-i} \varepsilon_i, \quad \varepsilon_i \in \mathcal{L} \right\} .$$

We can use the fact 1 for the set Q.

Let us define the sequence $\{f_k\}_{k\ge 0}$ by

$$\begin{aligned} f_0(x) &= \chi_{[-1/2,1/2]^n}(x) , \\ f_k(x) &= \prod_{i=1}^{k} m(B^{-i}x) \chi_{[-1/2,1/2]^n}(B^{-k}x) \quad \text{for } k > 0 . \end{aligned}$$

Since B is a dilation matrix f_k converges pointwise to $\hat{\varphi}$.

We compute the norms of f_k in $L^2(\mathbb{R}^n)$.

$$\int_{\mathbb{R}^n} |f_k(x)|^2 dx = \int_{\mathbb{R}^n} \prod_{i=1}^{k} |m(B^{-i}x)|^2 \chi_{[-1/2,1/2]^n}(B^{-k}x)\, dx$$

$$= |\det B|^k \int_{[-1/2,1/2]^n} \prod_{i=1}^{k} |m(B^{k-i}x)|^2 dx = |\det B|^k \int_{[0,1]^n} \prod_{i=1}^{k} |m(B^{k-i}x)|^2\, dx$$

$$= \frac{|\det B|^k}{|Q|} \int_Q \prod_{i=1}^{k} |m(B^{k-i}x)|^2 dx \qquad \text{(by fact 1)}$$

$$= \frac{|\det B|^{k-1}}{|Q|} \int_{BQ} \prod_{i=1}^{k} |m(B^{k-i-1}x)|^2 dx$$

$$= \frac{|\det B|^{k-1}}{|Q|} \int_{\bigcup_{j=1}^{q}(l_j+Q)} \prod_{i=1}^{k} |m(B^{k-i-1}x)|^2 dx \text{ (by the self similarity of } Q)$$

$$= \frac{|\det B|^{k-1}}{|Q|} \int_Q \prod_{i=1}^{k-1} |m(B^{k-i-1}x)|^2 \left[\sum_{j=1}^{q} |m(B^{-1}(x+l_j))|^2 \right] dx$$

(By (3.5) the sum in the bracket $= 1$.)

$$= |\det B|^{k-1} \int_{[-1/2,1/2]^n} \prod_{i=1}^{k-1} |m(B^{k-i-1}x)|^2 dx = \int_{\mathbb{R}^n} |f_{k-1}(x)|^2 dx\,.$$

Therefore, we have shown that for every $k > 0$

$$||f_k||_2 = ||f_{k-1}||_2 = \ldots = ||f_0||_2 = 1\,.$$

Hence, by Fatou's lemma

$$\int_{\mathbb{R}^n} |\hat{\varphi}(x)|^2 dx \le \liminf_{k\to\infty} \int_{\mathbb{R}^n} |f_k(x)|^2 dx = 1\,.$$

and we have $||\hat{\varphi}||_2 \le 1$.

It is easy to see that $\hat{\varphi}$ satisfies

$$\hat{\varphi}(x) = m(B^{-1}x)\hat{\varphi}(B^{-1}x)\,, \tag{3.7}$$

Using (1.2) and (1.3) and the fact that $h \in l^1(\mathbb{Z}^n)$ is a scaling vector, we can see that (3.7) is an equivalent form of the scaling equation (2.1). □

In order to apply Theorem 1, we need to ensure convergence of the product in (3.6). Therefore, we add some conditions on m in the next definition.

Definition. A function m given by (3.3) is *A-regular*, if the product $\prod_{i=1}^{\infty} m(B^{-i}x)$ is convergent uniformly on compact sets in $\mathbb{R}^n$, where $B = A^T$. □

Remark. In our considerations, the matrix A does not change so we will simply say that m is regular. For regular m Theorem 1 is valid and the function $\hat{\varphi}$ is continuous and $\hat{\varphi}(0) = 1$. Therefore, for any regular scaling vector there exists a non-vanishing scaling function φ satisfying (2.1) and unique up to a constant factor. □

For example, to assure regularity of m, it is sufficient to assume there exists some $\varepsilon > 0$ such that

$$\sum_{k\in\mathbb{Z}^n} |h_k||k|^\varepsilon < \infty .$$

Indeed, we can assume $\varepsilon \leq 1$ and for any $R > 0$ and every $|x| \leq R$

$$\begin{aligned}
\sum_{j=1}^{\infty} |m(B^{-j}x) - 1| &= \sum_{j=1}^{\infty} \left| \sum_{k\in\mathbb{Z}^n} \frac{h_k}{\sqrt{|\det A|}} e^{-2\pi i\langle k, B^{-j}x\rangle} - 1 \right| \\
&\leq 2\sum_{j=1}^{\infty} \frac{1}{\sqrt{|\det A|}} \sum_{k\in\mathbb{Z}^n} |h_k||\sin(\pi\langle k, B^{-j}x\rangle)| \\
&\leq 2\sum_{k\in\mathbb{Z}^n} |h_k| \sum_{j=1}^{\infty} C|k|^\varepsilon |B^{-j}x|^\varepsilon \leq 2\sum_{k\in\mathbb{Z}^n} |h_k|C'|k|^\varepsilon \sum_{j=1}^{\infty} \lambda^{-j\varepsilon} R^\varepsilon \\
&\leq C'' \sum_{k\in\mathbb{Z}^n} |h_k||k|^\varepsilon < \infty .
\end{aligned}$$

We used the elementary inequality $|\sin x| \leq C|x|^\varepsilon$ for $x \in \mathbb{R}$, $0 < \varepsilon \leq 1$ and since B is dilation matrix there exist constants $\lambda > 1$ and $C' > 0$ such that

$$|B^{-j}x| \leq C'\lambda^{-j}|x| \qquad \text{for } x \in \mathbb{R}^n,\ j > 0 . \tag{3.8}$$

The next result extends Theorem 1.

Theorem 2.

Let h, m, and $\hat{\varphi}$ be as in Theorem 1. If $||\varphi||_2 = 1$, then the translates of φ are orthonormal, i.e., $\langle\varphi, \mathcal{T}_k\varphi\rangle = \delta_{k,0}$ for $k \in \mathbb{Z}^n$.

Proof. Denote by $S : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ the bounded operator given by

$$Sf(x) = \sum_{k\in\mathbb{Z}^n} h_k\sqrt{|\det A|}f(Ax - k) . \tag{3.9}$$

Using (3.4) it is easy to check that if the translates of f are orthonormal, i.e., $\langle f, \mathcal{T}_k f\rangle = \delta_{k,0}$ for $k \in \mathbb{Z}^n$, then the translates of Sf are too, $\langle Sf, \mathcal{T}_k Sf\rangle = \delta_{k,0}$. Denoting $\hat{S} = \mathcal{F}S\mathcal{F}^{-1}$ and using (1.2) and (1.3) we get

$$\hat{S}g(x) = m(B^{-1}x)g(B^{-1}x) . \tag{3.10}$$

Let $f_0 = \chi_{[-1/2,1/2]^n}$, $f_k = S^k f_0$, $k > 0$. Then

$$\hat{f}_k(x) = \hat{S}^k \hat{f}_0(x) = \prod_{i=1}^{k} m(B^{-i}x)\hat{f}_0(B^{-k}x) .$$

The functions $\hat{f}_k$ converge pointwise to $\hat{\varphi}$ as $k \to \infty$ because $\hat{f}_0$ is continuous in a neighborhood of 0 and $\hat{f}_0(0) = 1$. Since the $\hat{f}_k$ are uniformly bounded by 1, the $\hat{f}_k$ converge weakly to $\hat{\varphi}$. Weak convergness of $\hat{f}_k$ to $\hat{\varphi}$ and $||\hat{f}_k||_2 = ||\hat{\varphi}||_2 = 1$ imply $\hat{f}_k$ converge in norm to $\hat{\varphi}$. Hence, the f_k converge in norm to φ so translates of φ are orthonormal. □

It is worth noting that using the ideas in the previous proof one can simplify the proof of Theorem 1, avoiding calculations.

Theorem 3.

Assume that the $\mathbb{Z}^n$-periodic function m is of class C^N for some $N = 1, 2, \dots, \infty$. Then, $\hat{\varphi}$ given by (3.6) is also of class C^N. Moreover $\hat{\varphi} \in W^N(\mathbb{R}^n)$.

Proof.

Sketch of the proof. Assume that m is of class C^N. Calculate the partial derivatives

$$\frac{\partial}{\partial x_i}\hat{\varphi}(x) = \sum_{k=1}^{\infty}\left[\sum_{s=1}^{n}(B^{-k})_{i,s}\frac{\partial}{\partial x_s}m(B^{-k}x)\right]\prod_{\substack{j=1\\ j\neq k}}^{\infty} m(B^{-j}x)$$

and the series is convergent because $|m(x)| \le 1$, the derivatives $\frac{\partial}{\partial x_s}m$ are bounded and (3.8) is true. Analogously we compute derivatives of higher order.

To show $\hat{\varphi} \in W^N(\mathbb{R}^n)$ we take the sequence $\{f_k\}_{k\ge 0}$ defined in the proof of Theorem 1 which converges pointwise to $\hat{\varphi}$.

$$\frac{\partial}{\partial x_i}f_k = \sum_{l=1}^{k}\left[\sum_{s=1}^{n}(B^{-l})_{i,s}\frac{\partial}{\partial x_s}m(B^{-l}x)\right]\prod_{\substack{j=1\\ j\neq l}}^{k} m(B^{-j}x)\chi_{[-1/2,1/2]^n}(B^{-k}x)\,.$$

Using calculations similar to those in the proof of Theorem 1 we get

$$\int_{\mathbb{R}^n}\prod_{\substack{j=1\\ j\neq l}}^{k}|m(B^{-j}x)|^2\chi_{[-1/2,1/2]^n}(B^{-k}x)\,dx = |\det B|\,,$$

for $l = 1, \dots, k$, $k > 0$. Therefore, by (3.8)

$$\left\|\frac{\partial}{\partial x_i}f_k\right\|_2 \le |\det B| n \sup_{s=1,\dots,n}\left\|\frac{\partial}{\partial x_s}m\right\|_\infty \sum_{l=1}^{k} C\lambda^{-l}\,.$$

Using Fatou's lemma, we get $\|\frac{\partial}{\partial x_i}\hat{\varphi}\|_2 < \infty$. Using similar methods we can estimate derivatives of higher order. □

Corollary 1.

Assume the $\mathbb{Z}^n$-periodic function m is of class C^N for $N > n/2$ and $\hat{\varphi}$ is given by (3.6). Then the series $\sum_{k\in\mathbb{Z}^n}|\hat{\varphi}(x+k)|^2$ is convergent uniformly on compact sets and its limit is a continuous function.

Proof. Let g be of class C^∞ with compact support and $g(x) = 1$ for $x \in B(0,r)$, $r > 0$. For any $\varepsilon > 0$ Lemma 2 guarantees there exists $m > 0$ such that

$$\sum_{|k|>m}\|\hat{\varphi}\mathcal{T}_k g\|^2_{W^N} < \varepsilon\,.$$

Since $N > n/2$, we may use the Sobolev lemma

$$\sum_{|k|>m}|\hat{\varphi}(x+k)|^2 < C\varepsilon, \qquad \text{for } x \in B(0,r)\,,$$

which assures uniform convergence on $B(0,r)$. Since $r > 0$ was arbitrary, this finishes the proof. □

4. Tight Frames of Wavelets

We start this section with some facts associated with the definition of the wavelet matrix. If $(h^1, \dots, h^q)$ is the wavelet matrix and $m = m_1$ is regular, then by Theorem 1 $\varphi = \varphi^1$ satisfies (2.1) and we can define φ^s, $s = 2, \dots, q$ by (2.2). The main result of this section says that dilates and translates of φ^s for $s = 2, \dots, q$ form a tight frame. This generalizes results in [14] (a proof can also be found in [3], Proposition 6.2.3), where dimension $n = 1$, dilation $A = [2]$ and $h_k^2 = (-1)^k \overline{h_{-k+1}}$.

For any matrix wavelet $(h_1, \dots, h_q)$ we denote by $m_1, \dots, m_q$ the functions given by

$$m_s(x) = \frac{1}{\sqrt{|\det A|}} \sum_{k \in \mathbb{Z}^n} h_k^s e^{-2\pi i \langle k, x \rangle} . \tag{4.1}$$

The next fact is parallel to Lemma 4.

Lemma 5.

Condition (3.1) is equivalent to

$$\sum_{i=1}^{q} m_s(B^{-1}(x + l_i)) \overline{m_{s'}(B^{-1}(x + l_i))} = \delta_{s,s'} , \qquad \textit{for } s, s' = 1, \dots, q \tag{4.2}$$

and for a.e. $x \in \mathbb{R}^n$.

Proof.

$$\begin{aligned}
&\sum_{i=1}^{q} m_s(x + B^{-1} l_i) \quad \overline{m_{s'}(x + B^{-1} l_i)} \\
&\qquad = \sum_{i=1}^{q} \frac{1}{|\det A|} \sum_{k, k' \in \mathbb{Z}^n} h_k^s \overline{h_{k'}^{s'}} e^{-2\pi i \langle k - k', x + B^{-1} l_i \rangle} \\
&\qquad = \sum_{k, m \in \mathbb{Z}^n} h_k^s \overline{h_{k+m}^{s'}} e^{2\pi i \langle m, x \rangle} \sum_{i=1}^{q} \frac{1}{|\det A|} e^{2\pi i \langle m, B^{-1} l_i \rangle} \\
&\qquad = \sum_{k, m \in \mathbb{Z}^n} h_k^s \overline{h_{k+Am}^{s'}} e^{2\pi i \langle Am, x \rangle}
\end{aligned}$$

the last equality being a consequence of Lemma 3. $\{e^{2\pi i \langle m, x \rangle}\}_{m \in \mathbb{Z}^n}$ is an orthonormal basis of $L^2([0, 1]^n)$, thus

$$\sum_{i=1}^{q} m_s(x + B^{-1} l_i) \overline{m_{s'}(x + B^{-1} l_i)} = \delta_{s,s'} \qquad \text{for a.e. } x \in [0, 1]^n$$

iff

$$\sum_{k, m \in \mathbb{Z}^n} h_k^s \overline{h_{k+Am}^{s'}} = \delta_{m,0} \delta_{s,s'}$$

which turns out to be (3.1). □

Another way of stating (3.1) is given by the next lemma.

Lemma 6.

Let $(h_1, \dots, h_q) \in (l^1(\mathbb{Z}^n))^q$. *Equality holds in (3.1) iff*

$$\sum_{k \in \mathbb{Z}^n} \sum_{s=1}^{q} h_{m-Ak}^s \overline{h_{m'-Ak}^s} = \delta_{m,m'} \qquad \textit{for } m, m' \in \mathbb{Z}^n . \tag{4.3}$$

Proof. Consider

$$\begin{pmatrix} \sum_{k\in d_1+A\mathbb{Z}^n} h_k^1 e^{-2\pi i\langle k,x\rangle} & \dots & \sum_{k\in d_1+A\mathbb{Z}^n} h_k^q e^{-2\pi i\langle k,x\rangle} \\ \vdots & & \vdots \\ \sum_{k\in d_q+A\mathbb{Z}^n} h_k^1 e^{-2\pi i\langle k,x\rangle} & \dots & \sum_{k\in d_q+A\mathbb{Z}^n} h_k^q e^{-2\pi i\langle k,x\rangle} \end{pmatrix}$$

where $d_1, \dots, d_q$ are representatives of different cosets of $\mathbb{Z}^n/A\mathbb{Z}^n$. This matrix is unitary for a.e. x iff equality holds in (3.1). To show this, it suffices to compute the scalar product of two columns s and s'.

$$\begin{aligned}
&\sum_{j=1}^{q}\Big(\sum_{k\in d_j+A\mathbb{Z}^n} h_k^s e^{-2\pi i\langle k,x\rangle}\Big)\Big(\overline{\sum_{k\in d_j+A\mathbb{Z}^n} h_k^{s'} e^{-2\pi i\langle k,x\rangle}}\Big)\\
&=\sum_{j=1}^{q}\sum_{k,k'\in d_j+A\mathbb{Z}^n} h_k^s\overline{h_{k'}^{s'}}e^{-2\pi i\langle k-k',x\rangle}\\
&=\sum_{j=1}^{q}\sum_{k,k'\in\mathbb{Z}^n} h_{d_j+Ak}^s\overline{h_{d_j+Ak'}^{s'}}e^{-2\pi i\langle d_j+Ak-d_j-Ak',x\rangle}\\
&=\sum_{j=1}^{q}\sum_{k,k'\in\mathbb{Z}^n} h_{d_j+Ak}^s\overline{h_{d_j+Ak+Ak'}^{s'}}e^{2\pi i\langle Ak',x\rangle}\\
&=\sum_{k'\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n} h_k^s\overline{h_{k+Ak'}^{s'}}e^{2\pi i\langle Ak',x\rangle}=\sum_{k'\in\mathbb{Z}^n}\delta_{k',0}\delta_{s,s'}e^{2\pi i\langle Ak',x\rangle}=\delta_{s,s'}
\end{aligned}$$

If we compute the scalar products of two rows r and r', we arrive at

$$\begin{aligned}
&\sum_{s=1}^{q}\Big(\sum_{k\in d_r+A\mathbb{Z}^n} h_k^s e^{-2\pi i\langle k,x\rangle}\Big)\Big(\overline{\sum_{k\in d_{r'}+A\mathbb{Z}^n} h_k^{s} e^{-2\pi i\langle k,x\rangle}}\Big)\\
&=\sum_{s=1}^{q}\sum_{k,k'\in\mathbb{Z}^n} h_{d_r+Ak}^s\overline{h_{d_{r'}+Ak'}^{s}}e^{-2\pi i\langle d_r+Ak-d_{r'}-Ak',x\rangle}\\
&=\sum_{k,k'\in\mathbb{Z}^n}\sum_{s=1}^{q} h_{d_r+Ak}^s\overline{h_{d_{r'}+Ak+Ak'}^{s}}e^{-2\pi i\langle d_r-d_{r'}-Ak',x\rangle}\\
&=\sum_{k'\in\mathbb{Z}^n}\Big(\sum_{k\in\mathbb{Z}^n}\sum_{s=1}^{q} h_{d_r+Ak}^s\overline{h_{d_{r'}+Ak+Ak'}^{s}}\Big)e^{-2\pi i\langle d_r-d_{r'}-Ak',x\rangle}
\end{aligned}$$

Since the matrix is unitary, the last expression in brackets is equal to $\delta_{d_r,d_{r'}+Ak'}$ which turns out to be (4.3). $\square$

Theorem 4.

(about tight frames). *Suppose $(h^1, \dots, h^q)$ is a wavelet matrix and m_1 is regular. Then the family of wavelets $\{\varphi_{j,k}^s\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}^{s=2,\dots,q}$, where $\varphi_{j,k}^s(x) = |\det A|^{j/2}\varphi^s(A^j x - k)$, forms a tight frame with constant 1 in $L^2(\mathbb{R}^n)$, i.e.,*

$$\sum_{\substack{s=2,\dots,q,\\ j\in\mathbb{Z},k\in\mathbb{Z}^n}} |\langle f, \varphi_{j,k}^s\rangle|^2 = ||f||^2 \qquad \text{for } f \in L^2(\mathbb{R}^n)\,. \tag{4.4}$$

Moreover, if $||\varphi|| = 1$, then this family forms an orthonormal basis.

Proof. Let f be any function of class C^∞ with compact support. Since these functions are dense in $L^2(\mathbb{R}^n)$, it is sufficient to show (4.4) for any such f.

1. First we show that for every j series $\sum_{k\in\mathbb{Z}^n} |\langle f, \varphi_{j,k}\rangle|^2$ is convergent (using the notation $\varphi_{j,k} = \varphi^1_{j,k}$).

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^n} |\langle f, \varphi_{j,k}\rangle|^2 &\leq q^j \sum_k \left(\int_{\mathbb{R}^n} |f(x)||\varphi(A^j x - k)|\,dx\right)^2 \\
&\leq q^j \sum_k ||f||_2^2 \int_{\mathrm{supp} f} |\varphi(A^j x - k)|^2 dx \\
&= q^j ||f||_2^2 q^{-j} \sum_k \int_{A^j(\mathrm{supp} f)+k} |\varphi(x)|^2 dx\,.
\end{aligned}$$

Since $\mathrm{supp} f$ is compact, there exists an integer $K > 0$ such that $A^j(\mathrm{supp} f) \subset (-K, K]^n$. Continuing the calculations

$$\sum_{k\in\mathbb{Z}^n} \int_{A^j(\mathrm{supp} f)+k} |\varphi(x)|^2 dx = \sum_{l\in\mathbb{Z}^n} \sum_{r\in(-K,K]^n,\ r\in\mathbb{Z}^n} \int_{A^j(\mathrm{supp} f)+2Kl+r} |\varphi(x)|^2 dx$$

and since $k \in \mathbb{Z}^n$ is uniquely represented as $k = 2Kl + r$ where $l \in \mathbb{Z}^n$ and the remainder $r \in (-K, K]^n \cap \mathbb{Z}^n$ we can proceed

$$\sum_{r\in(-K,K]^n\cap\mathbb{Z}^n} \sum_{l\in\mathbb{Z}^n} \int_{A^j(\mathrm{supp} f)+r+2Kl} |\varphi(x)|^2 \leq \sum_{r\in(-K,K]^n\cap\mathbb{Z}^n} \int_{\mathbb{R}^n} |\varphi(x)|^2 dx\,,$$

because $((-K, K]^n + 2Kl) \cap ((-K, K]^n + 2Kl') = \emptyset$ for $l \neq l'$. This gives us an estimate of the sum of series.

2. Now we compute $\sum_{s=1}^q \sum_{k\in\mathbb{Z}^n} |\langle f, \varphi^s_{0,k}\rangle|^2$. Since $\varphi^s_{0,0} = \sum_{m\in\mathbb{Z}^n} h^s_m \varphi_{1,m}$, then

$$\begin{aligned}
\varphi^s_{0,k}(x) &= \varphi^s_{0,0}(x-k) = \sum_{m\in\mathbb{Z}^n} h^s_m \varphi_{1,m}(x-k) \\
&= \sum_{m\in\mathbb{Z}^n} h^s_m \varphi_{1,m+Ak}(x) = \sum_{m\in\mathbb{Z}^n} h^s_{m-Ak} \varphi_{1,m}(x)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{s=1}^q \sum_{k\in\mathbb{Z}^n} |\langle f, \varphi^s_{0,k}\rangle|^2 &= \sum_{k\in\mathbb{Z}^n} \sum_{s=1}^q |\langle f, \sum_{m\in\mathbb{Z}^n} h^s_{m-Ak}\varphi_{1,m}\rangle|^2 \\
&= \sum_{k\in\mathbb{Z}^n} \sum_{m,m'\in\mathbb{Z}^n} \left(\sum_{s=1}^q h^s_{m-Ak}\overline{h^s_{m'-Ak}}\right) \langle \varphi_{1,m}, f\rangle \langle f, \varphi_{1,m'}\rangle \\
&= \sum_{m\in\mathbb{Z}^n} \langle \varphi_{1,m}, f\rangle \langle f, \varphi_{1,m}\rangle = \sum_{m\in\mathbb{Z}^n} |\langle \varphi_{1,m}, f\rangle|^2\,.
\end{aligned}$$

The third equality is a consequence of Lemma 6.

3. The operator $U_A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by $U_A f(x) = \sqrt{|\det A|} f(Ax)$ is unitary. Using this and Item 2 we can transform our last equality to get

$$\sum_{k\in\mathbb{Z}^n} \sum_{s=1}^q |\langle f, \varphi^s_{i,k}\rangle|^2 = \sum_{k\in\mathbb{Z}^n} |\langle f, \varphi_{i+1,k}\rangle|^2 \qquad \text{for } i \in \mathbb{Z}\,.$$

Using this repeatedly for $i \in \{-I, -I+1, \dots, I\}$, $I > 0$, $I \in \mathbb{Z}$ we obtain

$$\sum_{i=-I}^{I} \sum_{k \in \mathbb{Z}^n} \sum_{s=2}^{q} |\langle f, \varphi_{I,k}^s \rangle|^2 = \sum_{k \in \mathbb{Z}^n} |\langle f, \varphi_{I+1,k} \rangle|^2 - \sum_{k \in \mathbb{Z}^n} |\langle f, \varphi_{-I,k} \rangle|^2 .$$

4. To finish the proof we need to show

$$\lim_{I \to \infty} \sum_{k \in \mathbb{Z}^n} |\langle f, \varphi_{-I,k} \rangle|^2 = 0 , \tag{4.5}$$

$$\lim_{I \to \infty} \sum_{k \in \mathbb{Z}^n} |\langle f, \varphi_{I,k} \rangle|^2 = ||f||^2 \tag{4.6}$$

Let us first prove (4.5). Using the computations in Item 1 we have the estimate

$$\sum_{k \in \mathbb{Z}^n} |\langle f, \varphi_{-I,k} \rangle|^2 \le ||f||_\infty^2 |\mathrm{supp} f| \sum_k \int_{A^{-I}(\mathrm{supp} f)+k} |\varphi(x)|^2 dx .$$

Since A is a dilation matrix, there exists I_0 such that $A^{-I}(\mathrm{supp} f) \subset (-1/2, 1/2)^n$ for $I > I_0$. If $I > I_0$, the last expression becomes

$$||f||_\infty^2 |\mathrm{supp} f| \int_{\mathbb{R}^n} |\varphi(x)|^2 \chi_I(x) dx ,$$

where $\chi_I(x) = \sum_{k \in \mathbb{Z}^n} \chi_{A^{-I}(\mathrm{supp} f)+k}(x)$. For every $\varepsilon > 0$, $A^{-I}(\mathrm{supp} f) \subset B(0, \varepsilon)$ for sufficient big I; thus, χ_I converges pointwise to 0 when $I \to \infty$. By Lebesgue's dominated convergence theorem we obtain

$$\lim_{I \to \infty} \int_{\mathbb{R}^n} |\varphi(x)|^2 \chi_I(x) dx = 0 .$$

Now we compute (4.6).

$$\begin{aligned}
\sum_{k \in \mathbb{Z}^n} |\langle f, \varphi_{I,k} \rangle|^2 &= \sum_{k \in \mathbb{Z}^n} \left| \int_{\mathbb{R}^n} \hat{f}(x) \overline{\hat{\varphi}_{I,k}(x)} dx \right|^2 \\
&= \sum_{k \in \mathbb{Z}^n} \left| \frac{1}{|\det A|^I} \int_{\mathbb{R}^n} \hat{f}(x) e^{-2\pi i \langle k, B^{-I} x \rangle} \overline{\hat{\varphi}(B^{-I} x)} dx \right|^2 \\
&= \sum_{k \in \mathbb{Z}^n} \left| \int_{B^I [0,1]^n} \frac{1}{|\det A|^I} e^{-2\pi i \langle k, B^{-I} x \rangle} \left[\sum_{l \in \mathbb{Z}^n} \hat{f}(x + B^I l) \overline{\hat{\varphi}(B^{-I} x + l)} \right] dx \right|^2 \\
&= \int_{B^I [0,1]^n} \left| \sum_{l \in \mathbb{Z}^n} \hat{f}(x + B^I l) \overline{\hat{\varphi}(B^{-I} x + l)} \right|^2 dx
\end{aligned}$$

(since $\{|\det A|^{-I} e^{-2\pi i \langle k, B^{-I} x \rangle}\}_{k \in \mathbb{Z}^n}$ is orthonormal basis in $L^2(B^I[0, 1]))$

$$\begin{aligned}
&= \int_{B^I [0,1]^n} \sum_{l, l' \in \mathbb{Z}^n} \hat{f}(x + B^I l) \overline{\hat{\varphi}(B^{-I} x + l) \hat{f}(x + B^I l')} \hat{\varphi}(B^{-I} x + l') \, dx \\
&= \sum_{l' \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \hat{f}(x) \overline{\hat{f}(x + B^I l') \hat{\varphi}(B^{-I} x)} \hat{\varphi}(B^{-I} x + l') \, dx \\
&= \int_{\mathbb{R}^n} |\hat{f}(x)|^2 |\hat{\varphi}(B^{-I} x)|^2 dx + R(f) ,
\end{aligned}$$

where $R(f)$ represents the terms summed with $l' \in \mathbb{Z}^n \setminus \{0\}$. The dominated convergence theorem yields

$$\lim_{I\to\infty} \int_{\mathbb{R}^n} |\hat{f}(x)|^2 |\hat{\varphi}(B^{-I}x)|^2 dx = ||\hat{f}||^2 = ||f||^2 ,$$

since $\hat{\varphi}(0) = 1$, $\hat{\varphi}$ is continuous in a neighborhood of 0 and $\hat{\varphi}(B^{-I}x)$ tends to 1 as $I \to \infty$.

Now it suffices to estimate $R(f)$.

$$\begin{aligned}
|R(f)| &\le \sum_{l\in\mathbb{Z}^n\setminus\{0\}} \left| \int_{\mathbb{R}^n} \hat{f}(x)\overline{\hat{f}(x+B^I l)\hat{\varphi}(B^{-I}x)}\hat{\varphi}(B^{-I}x+l)\,dx \right| \\
&\le \sum_{l\in\mathbb{Z}^n\setminus\{0\}} ||\hat{\varphi}||_\infty^2 \int_{\mathbb{R}^n} |\hat{f}(x)||\hat{f}(x+B^I l)|dx \\
&\le \sum_{l\in\mathbb{Z}^n\setminus\{0\}} \int_{\mathbb{R}^n} |\hat{f}(x)||\hat{f}(x+B^I l)|dx
\end{aligned}$$

because $||\hat{\varphi}||_\infty \le 1$. Since f is of class C^∞, there exists $C > 0$ such that $|\hat{f}(x)| \le C(1+|x|)^{-4n}$. Continuing the estimates

$$\begin{aligned}
&\le \sum_{l\in\mathbb{Z}^n\setminus\{0\}} \int_{\mathbb{R}^n} |\hat{f}(x)||\hat{f}(x+B^I l)|dx \\
&\le \sum_{l\in\mathbb{Z}^n\setminus\{0\}} C^2 \int_{\mathbb{R}^n} (1+|x|)^{-4n}(1+|x+B^I l|)^{-4n}dx \\
&\le C^2 \sum_{l\in\mathbb{Z}^n\setminus\{0\}} \int_{\mathbb{R}^n} (1+|x|)^{-2n}((1+|x|)(1+|x+B^I l|))^{-2n}dx \\
&\le C^2 \int_{\mathbb{R}^n} (1+|x|)^{-2n}dx \sum_{l\in\mathbb{Z}^n\setminus\{0\}} \frac{4^n}{|B^I l|^{2n}} .
\end{aligned}$$

We used the elementary inequality

$$\sup_{x\in\mathbb{R}} \frac{1}{(1+|x|)(1+|x+y|)} \le \frac{2}{|y|} ,$$

for $y \ne 0$. $\sum_{l\in\mathbb{Z}^n\setminus\{0\}} |l|^{-2n} \le C'$, for some $C' > 0$. Since B is a dilation, there exist $\lambda > 1$ and $C'' > 0$ such that

$$|B^I x| \ge C''\lambda^I |x| \qquad \text{for } x \in \mathbb{R}^n,\ I > 0 .$$

Therefore,

$$|R(f)| \le C^2 \int_{\mathbb{R}^n} (1+|x|)^{-2n}dx \frac{4^n C'}{(C'')^{2n}\lambda^{2nI}} \to 0$$

when $I \to \infty$. This shows the tightness of the frame.

5. If $||\varphi|| = 1$, then by Theorem 2 the translates of φ are orthonormal; hence, $||\varphi^2|| = \dots = ||\varphi^q|| = 1$ and the family $\{\varphi^s_{j,k}\}^{s=2,\dots,q}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ by the remark in Section 1. □

5. Orthonormality of Wavelets

In this section we give necessary and sufficient conditions for orthonormality of the translates of φ given by (3.6), where m is given by (3.3). When these criteria are satisfied, the tight frame of wavelets becomes orthonormal basis. The first condition is found in [1] in the case where $A = 2Id$ and it generalizes in a straightforward way to arbitrary dilation matrices A.

Theorem 5.

(Cohen). *Suppose m is regular and $\hat{\varphi}$ is given by (3.6).*
Suppose there exists a compact set $K \subset \mathbb{R}^n$ such that

- *K contains neighborhood of zero,*
- *$|K \cap (l + K)| = \delta_{l,0}$ for $l \in \mathbb{Z}^n$,*
- *$m(B^{-i}x) \neq 0$ for $x \in K$, $i > 0$.*

Then $\langle \varphi, T_l \varphi \rangle = \delta_{l,0}$ for $l \in \mathbb{Z}^n$.
Moreover, if m is of class C^N, where $N > n/2$, then the converse is true.

Proof. Define a sequence of functions $\{g_k\}_{k \geq 0}$ by

$$\begin{aligned} g_0(x) &= \chi_K(x)\,, \\ g_k(x) &= \prod_{i=1}^{k} m(B^{-i}x)\chi_K(B^{-k}x) \qquad \text{for } k > 0\,. \end{aligned}$$

Since B is a dilation and K contains a neighborhood of zero, g_k tends pointwise to $\hat{\varphi}$ when $k \to \infty$. Because m is continuous there exists $c > 0$ such that $|m(B^{-i}x)| > c$ for $x \in K$, $i > 0$. Since the product $\prod_{i=1}^{\infty} m(B^{-i}x)$ converges uniformly on K, there exists some N such that $\prod_{i=N}^{\infty} m(B^{-i}x) > c'$ for $x \in K$, where $c' > 0$ is some constant independent of x. Therefore,

$$|\hat{\varphi}(x)| = \prod_{i=1}^{N-1} |m(B^{-i}x)| \prod_{i=N}^{\infty} |m(B^{-i}x)| \geq c^{N-1}c' = c'' \qquad \text{for } x \in K\,,$$

which can be written as

$$\chi_K(x) \leq \frac{1}{c''}|\hat{\varphi}(x)| \qquad \text{for } x \in \mathbb{R}^n\,.$$

Using the last inequality we can estimate g_k from above

$$|g_k(x)| \leq \prod_{i=1}^{k} |m(B^{-1}x)| \frac{1}{c''}|\hat{\varphi}(B^{-k}x)| = \frac{1}{c''}|\hat{\varphi}(x)|\,.$$

We compute

$$\begin{aligned} &\int_{\mathbb{R}^n} |\hat{g}_k(x)|^2 e^{-2\pi i\langle l,x\rangle} = \int_{\mathbb{R}^n} \prod_{i=1}^{k} |m(B^{-i}x)|^2 e^{-2\pi i\langle l,x\rangle} \chi_K(B^{-k}x)dx \\ &= |\det B|^k \int_K \prod_{i=1}^{k} |m(B^{k-i}x)|^2 e^{-2\pi i\langle l,B^k x\rangle} dx \\ &= |\det B|^k \int_{[0,1]^n} \prod_{i=1}^{k} |m(B^{k-i}x)|^2 e^{-2\pi i\langle l,B^k x\rangle} dx \\ &\text{(by } \mathbb{Z}^n\text{-periodicity of the integrand)} \\ &= \frac{|\det B|^k}{|Q|} \int_Q \prod_{i=1}^{k} |m(B^{k-i}x)|^2 e^{-2\pi i\langle l,B^k x\rangle} dx \qquad \text{(by fact 1)} \end{aligned}$$

$$= \frac{|\det B|^{k-1}}{|Q|} \int_{BQ} \prod_{i=1}^{k} |m(B^{k-i-1}x)|^2 e^{-2\pi i \langle l, B^{k-1}x\rangle} dx$$

$$= \frac{|\det B|^{k-1}}{|Q|} \int_{Q} \prod_{i=1}^{k-1} |m(B^{k-i-1}x)|^2 e^{-2\pi i \langle l, B^{k-1}x\rangle} \left[\sum_{j=1}^{q} |m(B^{-1}(x+l_j))|^2 \right] dx$$

(since $BQ = \bigcup_{j=1}^{q}(l_j + Q)$)

$$= |\det B|^{k-1} \int_{[0,1]^n} \prod_{i=1}^{k} |m(B^{k-i-1}x)|^2 e^{-2\pi i \langle l, B^{k-1}x\rangle} dx$$

$$= |\det B|^{k-1} \int_{K} \prod_{i=1}^{k} |m(B^{k-i-1}x)|^2 e^{-2\pi i \langle l, B^{k-1}x\rangle} dx = \int_{\mathbb{R}^n} |\hat{g}_{k-1}(x)|^2 e^{-2\pi i \langle l, x\rangle} .$$

Therefore, by induction we have for every $k > 0$,

$$\int_{\mathbb{R}^n} |\hat{g}_k(x)|^2 e^{-2\pi i \langle l, x\rangle} = \int_{\mathbb{R}^n} |\hat{g}_0(x)|^2 e^{-2\pi i \langle l, x\rangle} = \delta_{l,0} .$$

Lebesgue's dominated convergence theorem gives orthonormality

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |\hat{g}_k(x)|^2 e^{-2\pi i \langle l, x\rangle} = \int_{\mathbb{R}^n} |\hat{\varphi}(x)|^2 e^{-2\pi i \langle l, x\rangle} .$$

Now we show the implication in the opposite direction.

$$\begin{aligned} \langle \varphi, T_l \varphi \rangle &= \langle \mathcal{F}\varphi, \mathcal{F}T_l\varphi \rangle \\ &= \int_{\mathbb{R}^n} |\hat{\varphi}(x)|^2 e^{2\pi i \langle l, x\rangle} dx = \int_{[0,1]^n} \left[\sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(x+k)|^2 \right] e^{2\pi i \langle l, x\rangle} dx \qquad (5.1) \end{aligned}$$

and using the fact that $\{e^{2\pi i \langle l, x\rangle}\}_{l\in\mathbb{Z}^n}$ is an orthonormal basis in $L^2([0,1]^n)$ we obtain that the orthonormality condition is equivalent to

$$\sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(x+k)|^2 = 1 \qquad \text{for a.e. } x \in \mathbb{R}^n$$

In fact, the series $\sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(x+k)|^2$ converges uniformly on compact sets and equality holds for every $x \in \mathbb{R}^n$ by virtue of Corollary 1. Thus, there exists $l > 0$ such that

$$\sum_{k\in\mathbb{Z}^n, |k|<l} |\hat{\varphi}(x+k)|^2 > \frac{1}{2} \qquad \text{for } x \in [-1/2, 1/2]^n .$$

The last sum is finite; hence, there exists $c > 0$ such that for any $x \in [-1/2, 1/2]^n$ we can find a cube U_x with center in x and a translation $k_x \in \mathbb{Z}^n$, $|k_x| < l$ such that

$$|\hat{\varphi}(y + k_x)| > c \qquad \text{for } y \in U_x .$$

We can find finite subcover $\{U_{x_i}\}_{i=1,\dots,m}$ with $x_1 = 0$ of the covering $\{U_x\}_{x\in[-1/2,1/2]^n}$ of $[-1/2, 1/2]^n$. Now we define by induction sets $K_1, \dots, K_m$.

$$\begin{aligned} K_1 &= \overline{U_{x_1}} \cap [-1/2, 1/2]^n \\ K_{i+1} &= \overline{U_{x_{i+1}} \setminus \cup_{j=1}^{i} K_j} \cap [-1/2, 1/2]^n \end{aligned}$$

Since $\bigcup_{i=1}^{m} K_i = [-1/2, 1/2]^n$ and $|K_i \cap K_j| = 0$ for $i \neq j$, $K = \bigcup_{i=1}^{m}(K_i + k_{x_i})$ satisfies $|K \cap (l + K)| = \delta_{l,0}$ for $l \in \mathbb{Z}^n$ and K contains a neighborhood of zero. Moreover, $|\hat{\varphi}(x)| > c$ for $x \in K$. By the definition of $\hat{\varphi}$ every element of the product $|m(B^{-i}x)| > c$ for $i > 0$. Therefore, K has all the required properties and the proof is done. □

Another necessary and sufficient condition is due to Lawton which originally appeared in the context of one dimension. For any fixed scaling vector h we define an operator $\hat{C} : L^2([0, 1]^n) \to L^2([0, 1]^n)$ by the formula

$$\hat{C}f(x) = \sum_{j=1}^{q} |m(B^{-1}(x + l_j))|^2 f(B^{-1}(x + l_j)), \tag{5.2}$$

where $L^2([0, 1]^n)$ is the space of $\mathbb{Z}^n$-periodic functions and $l_1, \ldots, l_q$ are representatives of different cosets of $\mathbb{Z}^n / B\mathbb{Z}^n$. The choice of representatives does not affect definition of $\hat{C}$.

The following lemma justifies the notation of $\hat{C}$.

Lemma 7.

Operator $\hat{C}$ is unitary equivalent (by the Fourier transform) to $C : l^2(\mathbb{Z}^n) \to l^2(\mathbb{Z}^n)$ which is represented by the matrix

$$C = (c_{p,r})_{p,r \in \mathbb{Z}^n}, \quad c_{p,r} = \sum_{k \in \mathbb{Z}^n} h_k \overline{h_{k+Ar-p}}. \tag{5.3}$$

Proof. Suppose $f(x) = e^{2\pi i \langle p,x \rangle}$ and compute Cf

$$\begin{aligned} \hat{C}f(x) &= \sum_{j=1}^{q} |m(B^{-1}(x + l_j))|^2 e^{2\pi i \langle p, B^{-1}(x+l_j) \rangle} \\ &= \frac{1}{|\det A|} \sum_{k,m \in \mathbb{Z}^n} \sum_{j=1}^{q} h_k \overline{h_m} e^{-2\pi i \langle k-m, B^{-1}(x+l_j) \rangle} e^{2\pi i \langle p, B^{-1}(x+l_j) \rangle} \\ &\sum_{k,m \in \mathbb{Z}^n} h_k \overline{h_m} e^{2\pi i \langle m-k+p, B^{-1}x \rangle} \left(\frac{1}{|\det A|} \sum_{j=1}^{q} e^{2\pi i \langle m-k+p, B^{-1} l_j \rangle} \right) \end{aligned}$$

By Lemma 3 the expression in brackets is equal to 1 when $m - k + p \in A\mathbb{Z}^n$ or 0 when $m - k + p \notin A\mathbb{Z}^n$. Hence, we can assume that $m - k + p = Ar$ for some $r \in \mathbb{Z}^n$. Therefore, we can write the last expression as

$$\sum_{r \in \mathbb{Z}^n} e^{2\pi i \langle r,x \rangle} \left(\sum_{k \in \mathbb{Z}^n} h_k \overline{h_{k+Ar-p}} \right),$$

which shows the unitary equivalence of C and $\hat{C}$. □

Theorem 6.

(Lawton). *Suppose a scaling vector h has a finite number of non-zero elements, m is given by (3.3) and $\hat{\varphi}$ by (3.6).*

The following conditions are equivalent:

1. $\langle \varphi, T_l \varphi \rangle = \delta_{l,0}$ *for* $l \in \mathbb{Z}^n$.
2. *There is no non-constant trigonometric polynomial* $\psi(x) = \sum_{l \in \mathbb{Z}^n} z_l e^{2\pi i \langle l,x \rangle}$ *satisfying* $\hat{C}\psi = \psi$.

Proof. Let us consider $\psi(x) = \sum_{k \in \mathbb{Z}^n} |\hat{\varphi}(x + k)|^2$. By Corollary 1, ψ is continuous. Moreover, ψ is a trigonometric polynomial

$$\psi(x) = \sum_{l \in \mathbb{Z}^n} z_l e^{2\pi i \langle l,x \rangle}, \qquad \text{where } z_l = \langle \varphi, T_{-l} \varphi \rangle,$$

because φ has bounded support and (5.1) holds. In order to check $\hat{C}\psi = \psi$, it suffices to show $Cz = z$ where $z = (z_l)_{l\in\mathbb{Z}^n}$ by Lemma 7.

$$\begin{aligned}(Cz)_r &= \sum_{k,p\in\mathbb{Z}^n} h_k \overline{h_{k+Ar-p}} \langle \varphi, \mathcal{T}_{-p}\varphi\rangle = \sum_{k\in\mathbb{Z}^n} h_k \left\langle \varphi, \sum_{p\in\mathbb{Z}^n} h_{k+Ar-p}\mathcal{T}_{-p}\varphi\right\rangle \\ &= \sum_{k\in\mathbb{Z}^n} h_k \left\langle \mathcal{T}_{k+Ar}\varphi, \sum_{p\in\mathbb{Z}^n} h_{k+Ar-p}\mathcal{T}_{k+Ar-p}\varphi\right\rangle \\ &= \left\langle \mathcal{T}_{Ar}(\sum_{k\in\mathbb{Z}^n} h_k\mathcal{T}_k\varphi), \sum_{p\in\mathbb{Z}^n} h_p\mathcal{T}_p\varphi\right\rangle \\ &= \langle \mathcal{T}_r\varphi, \varphi\rangle = \langle \varphi, \mathcal{T}_{-r}\varphi\rangle\end{aligned}$$

where the fifth equality is true by (2.1). Therefore, we have showed that if Item 1 is not true, then ψ is a non-constant trigonometric polynomial satisfying $\hat{C}\psi = \psi$.

Conversely, suppose that there exists such a polynomial ψ. Since $\overline{\psi}$ is also an eigenvector of $\hat{C}$, hence without loss of generality we can assume ψ is real valued and positive (by adding some constant).

Let us define m_1 by

$$m_1(x) = m(x)\sqrt{\psi(x)/\psi(Bx)}\,.$$

The function m_1 is given by (3.3) by some scaling vector. This can be seen easily by Lemma 4 and the calculations below:

$$\sum_{j=1}^{q} |m_1(B^{-1}(x+l_j))|^2 = \sum_{j=1}^{q} |m(B^{-1}(x+l_i))|^2 \psi(B^{-1}(x+l_j))/\psi(x) = \psi(x)/\psi(x) = 1\,.$$

Hence, we can define φ_1 by

$$\begin{aligned}\hat{\varphi}_1(x) &= \prod_{i=1}^{\infty} m_1(B^{-i}x) = \prod_{i=1}^{\infty} m(B^{-i}x)\sqrt{\psi(B^{-i}x)/\psi(B^{-i+1}x)} \\ &= \hat{\varphi}(x)\sqrt{\psi(0)/\psi(x)}\,.\end{aligned}$$

Since $m(x)$ and $m_1(x)$ vanish on the same set, m satisfies the Cohen condition iff m_1 does. Suppose Item 1 holds, hence

$$1 = \sum_{k\in\mathbb{Z}^n} |\hat{\varphi}_1(x+k)|^2 = \sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(x+k)|^2 \frac{\psi(0)}{\psi(x+k)} = \frac{\psi(0)}{\psi(x)}\,.$$

Therefore, ψ is constant–contradiction. □

Remark. The last theorem can be presented with more general assumptions about the regularity of m. Nevertheless, the most interesting case from a practical point of view is when m is a polynomial. This theorem gives us a method of constructively checking of orthonormality by computing the spectrum of C. Those calculations can be done on computers. This was already known in the case of self similar tilings of $\mathbb{R}^n$ in the work of [5]. For more criteria of orthogonality of compactly supported scaling functions in $\mathbb{R}^n$, see [13]. □

Acknowledgments

This work is taken from my Master's Thesis at the University of Warsaw. I would like to thank my thesis advisor, Prof. P. Wojtaszczyk.

References

[1] Cohen, A. (1990). Ondelettes, analyses multirésolutions et filtres en quadrature. *Ann. Inst. H. Poincairé - Analyse non linéaire.* **7**, 439–459.

[2] Cohen, A. and Daubechies, I. (1993). Non-separable bidimensional wavelet bases, *Revista Mat. Iberoamericana,* 51–137.

[3] Daubechies, I. (1992). Ten Lectures on Wavelets. *CBMS-NSF Reg. Conf. Ser. Appl. Math., SIAM,* Philadelphia, PA.

[4] Gröchenig, K. (1994). Orthogonality Criteria for Compactly Supported Scaling Functions. *Appl. Computational Harmonic Anal.,* **1**, 242–245.

[5] Gröchenig, K. and Haas, A. (1994). Self-Similar Lattice Tilings. *J. Fourier Analysis and Appl.,* **1**, 131–170.

[6] Gröchenig, K. and Madych, W.R. (1992). Multiresolution Analysis, Haar Bases, and Self-Similar Tilings of $\mathbb{R}^n$. *IEEE Trans. Inform. Theory.* **38**, 556-568.

[7] Heller, P.N., Resnikoff, K.L. and Wells, Jr., R.O. (1992). *Wavelet Matrices and the Representation of Discrete Functions.* Wavelets: A Tutorial in Theory and Applications. Chui, C.K., Ed., 15–50.

[8] Lagarias, J.C. and Wang, Y. (1996). Self-Affine Tiles in $\mathbb{R}^n$. *Adv. Math.* **121**, 21–49.

[9] Lagarias, J.C. and Wang, Y. (1996). Integral Self-Affine Tiles in $\mathbb{R}^n$. I. Standard and nonstandard digit sets. *J. London Math. Soc.* **54**, 161–179.

[10] Lagarias, J.C. and Wang, Y. Integral Self-Affine Tiles in $\mathbb{R}^n$. II. Lattice tilings. preprint.

[11] Lagarias, J.C. and Wang, Y. (1996). Haar Bases for $L^2(\mathbb{R}^n)$ and Algebraic Number Theory. *J. Number Theory.* **57**, 181–197.

[12] Lagarias, J.C. and Wang, Y. (1995). Haar-Type Orthonormal Wavelet Bases in $\mathbb{R}^2$. *J. Fourier Analysis and Appl.* **2**, 1–14.

[13] Lagarias, J.C. and Wang, Y. Orthogonality Criteria for Compactly Supported Scaling Functions in $\mathbb{R}^n$. preprint.

[14] Lawton, W.M. (1990). Tight frames of compactly supported affine wavelets. *J. Math. Phys.* **31**, 1898–1901.

[15] Lawton, W.M. (1991). Necessary and sufficient conditions for constructing orthonormal wavelet bases. *J. Math. Phys.* **32**, 57–61.

[16] Madych, W.R. (1992). *Some Elementary Properties of Multiresolution Analyses of* $L^2(\mathbb{R}^n)$. Wavelets: A Tutorial in Theory and Applications. Chui, C.K., Ed., 259–294.

[17] Meyer, Y. (1992). Wavelets and operators. *Cambridge studies in advanced mathematics.* 37.

[18] Strichartz, R.S. (1993). Wavelets and self-affine tilings. *Constructive Approx.* **9**, 327–346.

[19] Wojtaszczyk, P. (1996). A Mathematical Introduction to Wavelets. *London Mathematical Society Student Texts.* **37**.

Received May 20, 1996

Department of Mathematics, Washington University, St. Louis, MO 63130
e-mail: marbow@math.wustl.edu