Applied and Computational Harmonic Analysis **8**, 203–221 (2000)
doi:10.1006/acha.2000.0284, available online at http://www.idealibrary.com on IDEAL®

LETTER TO THE EDITOR

A Characterization of Affine Dual Frames in $L^2(\mathbb{R}^n)$

Marcin Bownik [1,2]

Communicated by Guido L. Weiss on February 23, 1999

Abstract—We give a characterization of all (quasi)affine frames in $L^2(\mathbb{R}^n)$ which have a (quasi)affine dual in terms of the two simple equations in the Fourier transform domain. In particular, if the dual frame is the same as the original system, i.e., it is a tight frame, we obtain the well-known characterization of wavelets. Although these equations have already been proven under some special conditions we show that these characterizations are valid without any decay assumptions on the generators of the affine system.

Key Words: affine frame; quasiaffine frame; dual frame; wavelet

1. INTRODUCTION

In this paper we try to unify several concepts that arise in the theory of wavelets. A classical *orthonormal* wavelet is a function ψ on the real line such that

$$\{\psi_{j,k} = 2^{j/2}\psi(2^j \cdot -k)\}_{j,k\in\mathbb{Z}}$$

forms an orthonormal basis of $L^2(\mathbb{R})$. The natural question is whether we can characterize such functions. It turns out that the necessary and sufficient condition is that $\|\psi\|_2 = 1$, and the following two equations are satisfied:

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}$$

$$\sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+s))} = 0 \qquad \text{for a.e. } \xi \in \mathbb{R}, s \in 2\mathbb{Z}+1.$$

There are several directions in which a notion of a wavelet can and has been extended, for example multiwavelets in $\mathbb{R}^n$ forming a tight frame or the ϕ and ψ-transforms of Frazier and Jawerth; see [9, 10].

In this paper we present a unified approach to these various means of analyzing and reconstructing functions, as well as the fact that translations need not always be performed before dilations. It is natural to consider what happens if we exchange this order in the definition of $\psi_{j,k}$. This will lead us to the consideration of quasiaffine systems. Our

[1] Department of Mathematics, Washington University, Campus Box 1146, St. Louis, Missouri 63130. E-mail: marbow@math.wustl.edu
[2] The author thanks Prof. Guido Weiss for the motivation and many fruitful discussions.

1063-5203/00 $35.00

unification of the characterization of all these concepts shows that two equations that are surprisingly not much different from the ones in the one-dimensional case apply to the situation in the more general settings. In fact we go beyond the cases just described by considering dilations that are not necessarily dyadic and translations by elements of certain general lattices. In order to describe these and also give proper references we need to establish some notation.

Assume we have a lattice Γ ($\Gamma = P\mathbb{Z}^n$ for some nondegenerate $n \times n$ matrix P) and a *dilation* matrix A preserving Γ, i.e., all eigenvalues λ of A satisfy $|\lambda| > 1$, and $A\Gamma \subset \Gamma$. Let Ψ be a finite family of functions $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$. The *affine system* (resp. *quasiaffine system*) generated by Ψ associated with (A, Γ) is the collection

$$X(\Psi) = \{\psi^l_{j,\gamma} : j \in \mathbb{Z},\ \gamma \in \Gamma,\ l = 1, \ldots, L\}$$
$$X^q(\Psi) = \{\tilde{\psi}^l_{j,\gamma} : j \in \mathbb{Z},\ \gamma \in \Gamma,\ l = 1, \ldots, L\},$$

where for $\psi \in L^2(\mathbb{R}^n)$ we use the convention

$$\psi_{j,\gamma}(x) = D_{A^j} T_\gamma \psi(x) = |\det A|^{j/2} \psi(A^j x - \gamma), \qquad j \in \mathbb{Z},\ \gamma \in \Gamma$$
$$\tilde{\psi}_{j,\gamma}(x) = \begin{cases} D_{A^j} T_\gamma \psi(x) = |\det A|^{j/2} \psi(A^j x - \gamma), & j \geq 0,\ \gamma \in \Gamma \\ |\det A|^{j/2} T_\gamma D_{A^j} \psi(x) = |\det A|^j \psi(A^j(x - \gamma)), & j < 0,\ \gamma \in \Gamma, \end{cases}$$

where $T_\gamma f(x) = f(x - \gamma)$ is a translation operator by the vector $\gamma \in \mathbb{R}^n$, and $D_A f(x) = \sqrt{|\det A|} f(Ax)$ is a dilation by the matrix A.

DEFINITION 1.1. $X \subset L^2(\mathbb{R}^n)$ is a *Bessel family* if there exists $B > 0$ so that

$$\sum_{\eta \in X} |\langle f, \eta \rangle|^2 \leq B \|f\|^2 \qquad \text{for } f \in L^2(\mathbb{R}^n). \tag{1.1}$$

If, in addition, there exist $0 < A \leq B$ so that

$$A\|f\|^2 \leq \sum_{\eta \in X} |\langle f, \eta \rangle|^2 \leq B\|f\|^2 \qquad \text{for } f \in L^2(\mathbb{R}^n), \tag{1.2}$$

X is a *frame* and it is *tight* if A, B can be chosen so that $A = B$. (Quasi)affine system $X(\Psi)$ (resp. $X^q(\Psi)$) is a *(quasi)affine frame* if (1.2) holds for $X = X(\Psi)$ ($X = X^q(\Psi)$).

DEFINITION 1.2. Let $\Psi = \{\psi^1, \ldots, \psi^L\}$, $\Phi = \{\phi^1, \ldots, \phi^L\} \subset L^2(\mathbb{R}^n)$ be two finite families of functions so that $X(\Psi)$, $X(\Phi)$ are Bessel families. Then Φ is called a *(quasi)affine dual* of Ψ, if (1.3) (resp. (1.4)) holds,

$$\langle f, g \rangle = \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{\gamma \in \Gamma} \langle f, \psi^l_{j,\gamma} \rangle \langle \phi^l_{j,\gamma}, g \rangle \qquad \text{for all } f, g \in L^2(\mathbb{R}^n) \tag{1.3}$$

$$\langle f, g \rangle = \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{\gamma \in \Gamma} \langle f, \tilde{\psi}^l_{j,\gamma} \rangle \langle \tilde{\phi}^l_{j,\gamma}, g \rangle \qquad \text{for all } f, g \in L^2(\mathbb{R}^n). \tag{1.4}$$

Note that by polarization identity for sesquilinear forms S

$$S(f, g) = \frac{1}{4} \sum_{k=0}^{3} S(f + i^k g, f + i^k g), \tag{1.5}$$

(1.3) (or (1.4)) holds if and only if it holds for all $f = g \in L^2(\mathbb{R}^n)$.

The concepts of affine and quasiaffine frames are closely related. This was observed by Ron and Shen in [17, 18] under some decay assumptions and proved by Chui *et al*. in full generality in [6].

THEOREM 1.3. *Suppose* $\Psi, \Phi \subset L^2(\mathbb{R}^n)$ *are finite sets with the same cardinality. Then*

(i) $X(\Psi)$ *is a Bessel family if and only if* $X^q(\Psi)$ *is a Bessel family. Furthermore, their exact upper bounds are equal.*
(ii) $X(\Psi)$ *is an affine frame if and only if* $X^q(\Psi)$ *is a quasiaffine frame. Furthermore, their lower and upper exact bounds are equal.*
(iii) Φ *is an affine dual of* Ψ *if and only if* Φ *is a quasiaffine dual of* Ψ.

Although the implication $\Leftarrow$ of (iii) in Theorem 1.3 is not stated and proved in [6] it does follow from the techniques developed in their paper.

Since $AP\mathbb{Z}^n \subset P\mathbb{Z}^n$, $P^{-1}AP$ is a matrix with integer entries and $q = |\det A| = |\det P^{-1}AP|$ is the order of the group $\Gamma/A\Gamma$; see [22]. Let Γ^* be the dual lattice; that is,

$$\Gamma^* = \{\gamma' \in \mathbb{R}^n : \forall \gamma \in \Gamma \ \langle \gamma, \gamma' \rangle \in \mathbb{Z}\} = (P^T)^{-1}\mathbb{Z}^n.$$

By taking the transpose of $P^{-1}AP$ we observe that $B = A^T$ is a dilation preserving the dual lattice: $B\Gamma^* \subset \Gamma^*$. Also let $\mathbb{S} = \Gamma^* \backslash B\Gamma^*$. We use the Fourier transform $\mathcal{F}$ given by

$$\mathcal{F}\psi(\xi) = \hat{\psi}(\xi) = \int_{\mathbb{R}^n} \psi(x) e^{-2\pi i \langle x, \xi \rangle}\, dx.$$

The main result of our paper is the characterization of affine dual frames in terms of two equations, (1.6) and (1.7), in the Fourier transform domain.

THEOREM 1.4. *Suppose two affine systems* $X(\Psi)$, $X(\Phi)$ *form Bessel families, where* $\Psi = \{\psi^1, \ldots, \psi^L\}$, $\Phi = \{\phi^1, \ldots, \phi^L\} \subset L^2(\mathbb{R}^n)$. *Then* Φ *is a (quasi)affine dual of* Ψ *if and only if*

$$\sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \hat{\psi}^l(B^j \xi) \overline{\hat{\phi}^l(B^j \xi)} = |\det P| \qquad \textit{a.e. } \xi \in \mathbb{R}^n \tag{1.6}$$

$$t_s(\xi) \equiv \sum_{l=1}^{L} \sum_{j=0}^{\infty} \hat{\phi}^l(B^j \xi) \overline{\hat{\psi}^l(B^j(\xi + s))} = 0 \qquad \textit{a.e. } \xi \in \mathbb{R}^n \textit{ for } s \in \mathbb{S}. \tag{1.7}$$

This result was obtained by Frazier *et al*. [8] for dyadic dilations $A = 2Id$, even though they did not use the language of affine dual frames. Ron and Shen [18] and, independently, Han [13] have obtained the above characterization under some decay assumptions of the Fourier transform of the generators Ψ and Φ. Finally, in the case when $\Phi = \Psi$ the above characterization was established by Calogero in [4]. The proof that we present has elements similar to all these cited papers. We think our approach is more direct and also avoids unnecessary assumptions like decay at infinity. In fact, one of the purposes of this work is to show that the decay assumptions can be eliminated.

Without loss of generality and in order to simplify the proofs we will deal with (quasi)affine systems associated with dilation matrices preserving the standard lattice $\mathbb{Z}^n$.

Indeed, for any (quasi)affine system $X(\Psi)$ ($X^q(\Psi)$) associated with (A, Γ) we consider the unitary operator D_P given by $D_P f(x) = \sqrt{|\det P|} f(Px)$. $X(D_P\Psi)$ (or $X^q(D_P\Psi)$) as a (quasi)affine system associated with $(\tilde{A}, \mathbb{Z}^n)$ (where $\tilde{A} = P^{-1}AP$ is a dilation matrix with integer entries) is equivalent to $X(\Psi)$ (or $X^q(\Psi)$); we see this from

$$X(D_P\Psi) = D_P X(\Psi), \qquad X^q(D_P\Psi) = D_P X^q(\Psi),$$

which follow from the following identities:

$$D_{\tilde{A}^j} T_k D_P = D_P[D_{A^j} T_{Pk}], \qquad T_k D_{\tilde{A}^j} D_P = D_P[T_{Pk} D_{A^j}], \qquad j \in \mathbb{Z},\ k \in \mathbb{Z}^n.$$

Since the unitary operator D_P preserves the scalar product in $L^2(\mathbb{R}^n)$ it also preserves properties like being a Bessel family, a frame, duality of affine systems, etc.

Moreover, Eqs. (1.6) and (1.7) are also invariant under this transformation. Let $\tilde{B} = \tilde{A}^T = P^T B (P^T)^{-1}$; since $\mathcal{F} D_P \psi(\xi) = |\det P|^{-1/2} \mathcal{F}\psi((P^T)^{-1}\xi)$, we have

$$\begin{aligned} \mathcal{F} D_P \psi(\tilde{B}^j \tilde{\xi}) &= |\det P|^{-1/2} \mathcal{F}\psi(B^j \xi) \\ \mathcal{F} D_P \psi(\tilde{B}^j(\tilde{\xi} + \tilde{s})) &= |\det P|^{-1/2} \mathcal{F}\psi(B^j(\xi + s)), \end{aligned} \tag{1.8}$$

where $\tilde{\xi} \in \mathbb{R}^n$, $\tilde{s} \in \mathbb{Z}^n \backslash \tilde{B}\mathbb{Z}^n$, and $\xi = (P^T)^{-1}\tilde{\xi}$, $s = (P^T)^{-1}\tilde{s} \in \Gamma^* \backslash B\Gamma^*$. Formulae (1.8) guarantee that (1.6) and (1.7) hold for the affine systems $X(\Psi)$, $X(\Phi)$ associated with (A, Γ) if and only if they hold for $X(D_P\Psi)$, $X(D_P\Phi)$ associated with $(\tilde{A}, \mathbb{Z}^n)$.

Theorem 1.4 and other results in this paper could be written in slightly greater generality involving subspaces of $L^2(\mathbb{R}^n)$ of the form

$$FL^2(S) = \{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \hat{f} \subset S\},$$

where $S \subset \mathbb{R}^n$ satisfies $BS = S$; see [13]. This would inevitably lead to even more complicated notation, the presence of which is not justified by the only natural example known to the author, i.e., $FL^2(0, \infty) = \mathbb{H}^2(\mathbb{R})$.

2. PREPARATORY FACTS ABOUT LATTICES AND FRAMES

For the rest of the paper we will assume we have a dilation A with integer entries. Since A (and, therefore, $B = A^T$) is a dilation there exist constants $\lambda > 1$ and $c > 0$ such that

$$|B^j \xi| > c\lambda^j |\xi|, \qquad |B^{-j}\xi| < 1/c\lambda^{-j}|\xi| \qquad \text{for } j > 0. \tag{2.1}$$

Throughout this paper we will follow the convention that the support of the function f is $\operatorname{supp} f = \{x \in \mathbb{R}^n : f(x) \neq 0\}$ and $I_n = (-1/2, 1/2)^n$.

LEMMA 2.1. *Let C be any nonsingular matrix in $\mathbb{R}^n$; then*

$$\lim_{M\to\infty} \frac{\#((-M, M)^n \cap C\mathbb{Z}^n)}{(2M)^n} \to \frac{1}{|\det C|} \qquad \text{as } M \to \infty.$$

Proof. Let $\delta = \operatorname{diam}(CI_n)$. Define $Z_0 = \{m \in \mathbb{Z}^n : C(I_n + m) \subset (-M, M)^n\}$, $Z_1 = \{m \in \mathbb{Z}^n : C(I_n + m) \cap (-M, M)^n \neq \emptyset\}$. Clearly $(-M + \delta, M - \delta)^n \subset \bigcup_{m\in Z_0} C(I_n + m)$ and $\bigcup_{m\in Z_1} C(I_n + m) \subset (-M - \delta, M + \delta)^n$ modulo sets of measure zero. Hence

$$|(-M+\delta, M-\delta)^n| \le (\det C)\#Z_0 \le (\det C)\#((-M,M)^n \cap C\mathbb{Z}^n)$$
$$\le (\det C)\#Z_1 \le |(-M-\delta, M+\delta)^n|.$$

Dividing this inequality by $|(-M,M)^n| = (2M)^n$ and taking the limit as $M \to \infty$ we obtain the desired conclusion. ■

LEMMA 2.2.

$$\#((2I_n) \cap B^j\mathbb{Z}^n) \le 2^n q^{-j} \qquad \textit{for } j < 0 \textit{ and } q = |\det B|. \tag{2.2}$$

Proof. Note first that $(2I_n + m) \cap B^j\mathbb{Z}^n = (2I_n) \cap B^j\mathbb{Z}^n$ since $j < 0$. For any $k \in \mathbb{Z},\ k \ge 0$ let $Z = \{m = (m_1, \dots, m_n) \in \mathbb{Z}^n : |m_i| \le k,\ i = 1, \dots, n\}$.

$$\# \bigcup_{m\in Z} (2I_n + 2m) \cap B^j\mathbb{Z}^n = \#Z \cdot \#((2I_n) \cap B^j\mathbb{Z}^n) = (2k+1)^n \#((2I_n) \cap B^j\mathbb{Z}^n).$$

Since $\bigcup_{m\in Z}(2I_n + 2m) \subset (-2k-1, 2k+1)^n$ then, by Lemma 2.1,

$$\frac{\#\bigcup_{m\in Z}(I_n + 2m) \cap B^j\mathbb{Z}^n}{(4k+2)^n} = \frac{(2k+1)^n\#((2I_n) \cap B^j\mathbb{Z}^n)}{(4k+2)^n}$$
$$= 2^{-n}\#((2I_n) \cap B^j\mathbb{Z}^n) \le q^{-j}. \ ■$$

LEMMA 2.3. *Supppose* $0 < a < b < \infty$. *Then for any* $\xi \in \mathbb{R}^n$

$$\#\{j \in \mathbb{Z} : a < |B^j\xi| < b\} \le M, \tag{2.3}$$

where $M = M(b/a)$ *depends monotonically only on* b/a.

Proof. For any $\xi \ne 0$ let $j_0 \in \mathbb{Z}$ be the smallest integer such that $|B^{j_0}\xi| > a$. Then by (2.1) $|B^{j_0+k}\xi| > c\lambda^k|B^{j_0}\xi| > c\lambda^k a$ for $k \ge 0$, where λ, c are the same as in (2.1). Let $k_0 > 0$ be the smallest integer such that $c\lambda^{k_0}a > b$, i.e., $k_0 = \lceil \log_\lambda(b/(ac))\rceil$. Then we have

$$\{j \in \mathbb{Z} : a < |B^j\xi| < b\} \subset \{j_0, \dots, j_0 + k_0 - 1\},$$

and $M = k_0$ works. Therefore $M\colon \mathbb{R}_+ \to \mathbb{N}$ defined by $M(x) = \lceil \log_\lambda(x/c)\rceil$ performs the job. ■

LEMMA 2.4. *Suppose* $a > 0$, $g \in L^\infty(\mathbb{R}^n)$, $\operatorname{supp} g \subset \{\xi \in \mathbb{R}^n : |\xi| > a\}$, *and* $\operatorname{supp} g \subset B^{j_0}I_n + \xi_0$ *for some* $\xi_0 \in \mathbb{R}^n$ *and* $j_0 \in \mathbb{Z}$ *then*

$$\sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n\setminus\{0\}} q^i |g(B^j\xi)g(B^j(\xi+m))| \le 2^n q^{j_0} M((a+\delta)/a)\|g\|_\infty^2 \mathbf{1}_\Upsilon(\xi) \qquad \textit{a.e. } \xi \in \mathbb{R}^n, \tag{2.4}$$

where $\delta = \operatorname{diam}(B^{j_0}I_n)$ *and* $\Upsilon = \Upsilon(\xi_0, j_0) = \bigcup_{j<j_0} B^{-j}(B^{j_0}I_n + \xi_0)$.

Proof. For simplicity assume $\|g\|_\infty = 1$. If $|g(B^j\xi)g(B^j(\xi+m))| \ne 0$ for some $\xi \in \mathbb{R}^n$ then $B^{-j}(B^{j_0}I_n + \xi_0) \cap (B^{-j}(B^{j_0}I_n + \xi_0) - m) \ne \emptyset \Leftrightarrow (I_n + B^{-j_0}\xi_0) \cap (I_n + B^{-j_0}\xi_0 - B^{j-j_0}m) \ne \emptyset \Leftrightarrow I_n \cap (I_n - B^{j-j_0}m) \ne \emptyset \Leftrightarrow B^{j-j_0}m \in 2I_n$. Since $B^{j-j_0}\mathbb{Z}^n \cap$

$2I_n = \{0\}$ for all $j - j_0 \geq 0$, only the terms with $j < j_0$ may contribute to the sum. Moreover, by Lemma 2.2

$$\#\{m \in \mathbb{Z}\colon\ |g(B^j\xi)g(B^j(\xi+m))| \neq 0\} \leq 2^n q^{j_0-j} \tag{2.5}$$

for any $j < j_0$ and $\xi \in \mathbb{R}^n$.

Clearly we can find $a' \geq a$ so that $\operatorname{supp} g \subset \{\xi \in \mathbb{R}^n : a' < |\xi| < a' + \delta\}$, and for any $\xi \neq 0$ denote $Z = \{j \in \mathbb{Z} : a' < |B^j\xi| < a' + \delta\}$. Then by (2.3) and (2.5)

$$\begin{aligned}&\sum_{j<j_0}\sum_{m\in\mathbb{Z}^n\setminus\{0\}} q^j |g(B^j\xi)g(B^j(\xi+m))|\\ &\quad\leq \sum_{j\in Z} q^j 2^n q^{j_0-j} = 2^n q^{j_0}\#Z \leq 2^n q^{j_0} M((a'+\delta)/a') \leq 2^n q^{j_0} M((a+\delta)/a).\end{aligned}$$

Since only terms with $j < j_0$ contribute to the sum, $g(B^j\xi) \neq 0$ implies $\xi \in B^{-j}(B^{j_0}I_n + \xi_0)$ and only for $\xi \in \Upsilon = \bigcup_{j<j_0} B^{-j}(B^{j_0}I_n + \xi_0)$ the sum is nonzero. ■

For the sake of completness we will prove the following simple lemma.

LEMMA 2.5. *Suppose $F, G \in L^2(\mathbb{R}^n)$, and* $\operatorname{supp} F$, $\operatorname{supp} G$ *are bounded. Then*

$$\sum_{k\in\mathbb{Z}^n} \hat{F}(k)\overline{\hat{G}(k)} = \int_{\mathbb{R}^n}\left(\sum_{m\in\mathbb{Z}^n} F(\xi+m)\right)\overline{G(\xi)}\,d\xi.$$

Proof. Consider $\mathbb{Z}^n$ periodization of F and G:

$$\tilde{F}(\xi) = \sum_{m\in\mathbb{Z}^n} F(\xi+m), \qquad \tilde{G}(\xi) = \sum_{m\in\mathbb{Z}^n} G(\xi+m).$$

Clearly $\tilde{F}$, $\tilde{G}$ belong to $L^2(I_n)$, because only a finite number of terms contributes to the above sum. Since

$$\hat{\tilde{F}}(k) = \int_{I_n} \tilde{F}(\xi)e^{-2\pi i\langle k,\xi\rangle}\,d\xi = \hat{F}(k), \qquad \text{for } k \in \mathbb{Z}^n,$$

hence by Plancherel formula

$$\int_{\mathbb{R}^n} \tilde{F}(\xi)\overline{G(\xi)}\,d\xi = \int_{I_n} \tilde{F}(\xi)\overline{\tilde{G}(\xi)}\,d\xi = \sum_{k\in\mathbb{Z}^n} \hat{F}(k)\overline{\hat{G}(k)}. \ ■$$

To investigate duality of frames we need two lemmas (see [8]).

LEMMA 2.6. *Let $\{e_i\}_{i\in\mathbb{N}}$ be a seguence of vectors in Hilbert space $\mathcal{H}$. If $\sum_{i=1}^{\infty} e_i$ converges unconditionally in $\mathcal{H}$, then*

$$\sum_{i=1}^{\infty} \|e_i\|^2 < \infty.$$

LEMMA 2.7. *Suppose $\{e_i\}_{i\in\mathbb{N}}$, $\{f_i\}_{i\in\mathbb{N}}$ are Bessel sequences in $\mathcal{H}$, i.e., there is $C > 0$*

$$\sum_{i\in\mathbb{N}} |\langle h, e_i\rangle|^2 \leq C\|h\|^2 \qquad \sum_{i\in\mathbb{N}} |\langle h, f_i\rangle|^2 \leq C\|h\|^2 \qquad \textit{for all } h \in \mathcal{H}. \tag{2.6}$$

Then the following are equivalent:

$$\|h\|^2 = \sum_{i\in\mathbb{N}} \langle h, e_i\rangle\langle f_i, h\rangle \qquad \textit{for all } h \in \mathcal{D},\ \textit{where } \mathcal{D} \textit{ is dense in } \mathcal{H} \tag{2.7}$$

$$\|h\|^2 = \sum_{i\in\mathbb{N}} \langle h, e_i\rangle\langle f_i, h\rangle \qquad \textit{for all } h \in \mathcal{H} \tag{2.8}$$

$$h = \sum_{i\in\mathbb{N}} \langle h, e_i\rangle f_i = \sum_{i\in\mathbb{N}} \langle h, f_i\rangle e_i \qquad \textit{unconditionally in } \mathcal{H} \textit{ for all } h \in \mathcal{H}. \tag{2.9}$$

3. GENERALIZED DUALITY OF AFFINE SYSTEMS

In this section we prove a general result about some kind of weak duality between two affine systems $X(\Psi)$, $X(\Phi)$ without even assuming that these systems are Bessel families. This is a generalization of the result by Frazier *et al*. originally proved for dilations $A = 2Id$; see Theorem 3 in [8]. First we start with the lemma which provides necessary condition for family $X(\{\psi\})$ to be a Bessel family. We will make extensive use of

$$\mathcal{D} = \{f \in L^2(\mathbb{R}^n) : \hat{f} \in L^\infty(\mathbb{R}^n), \operatorname{supp} \hat{f} \subset K \text{ for some compact } K \subset \mathbb{R}^n\backslash\{0\}\},$$

which is a dense subspace of $L^2(\mathbb{R}^n)$.

LEMMA 3.1. *Suppose $\psi \in L^2(\mathbb{R}^n)$, $f \in \mathcal{D}$, and $J \in \mathbb{Z}$. Then*

$$\sum_{j\le J}\sum_{k\in\mathbb{Z}^n} |\langle f, \psi_{j,k}\rangle|^2 < \infty.$$

Moreover,

$$\sum_{j\in\mathbb{Z}} |\psi(B^j\xi)|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) \Leftrightarrow \sum_{j\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n} |\langle f, \psi_{j,k}\rangle|^2 < \infty \qquad \textit{for all } f \in \mathcal{D}. \tag{3.1}$$

Proof. Note that

$$\hat{\psi}_{j,k}(\xi) = q^{-j/2}\psi(B^{-j}\xi)e^{-2\pi i\langle k, B^{-j}\xi\rangle}, \qquad q = |\det B|;$$

therefore,

$$\begin{aligned}\langle f, \psi_{j,k}\rangle = \langle \hat{f}, \hat{\psi}_{j,k}\rangle &= q^{-j/2}\int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{\psi}(B^{-j}\xi)}e^{2\pi i\langle k, B^{-j}\xi\rangle}\,d\xi \\ &= q^{-j/2}\int_{\mathbb{R}^n} \hat{f}(B^j\xi)\overline{\hat{\psi}(\xi)}e^{2\pi i\langle k,\xi\rangle}q^j\,d\xi = q^{j/2}\int_{\mathbb{R}^n} \hat{f}(B^j\xi)\overline{\hat{\psi}(\xi)}e^{2\pi i\langle k,\xi\rangle}\,d\xi.\end{aligned} \tag{3.2}$$

By (3.2) we can write the series as

$$I = \sum_{j\le J}\sum_{k\in\mathbb{Z}^n} |\langle f, \psi_{j,k}\rangle|^2 = \sum_{j\le J}\sum_{k\in\mathbb{Z}^n} q^j \left|\int_{\mathbb{R}^n} \hat{f}(B^j\xi)\overline{\hat{\psi}(\xi)}e^{2\pi i\langle k,\xi\rangle}\,d\xi\right|^2. \tag{3.3}$$

For fixed $j \in \mathbb{Z}$ let $F(\xi) \equiv \hat{f}(B^j\xi)\overline{\hat{\psi}(\xi)}$; then, using Lemma 2.5 applied when $F = G$, we have

$$\sum_{k\in\mathbb{Z}^n}\left|\int_{\mathbb{R}^n}\hat f(B^j\xi)\overline{\hat\psi(\xi)}e^{2\pi i\langle k,\xi\rangle}\,d\xi\right|^2$$
$$=\int_{\mathbb{R}^n}\overline{\hat f(B^j\xi)}\hat\psi(\xi)\left[\sum_{m\in\mathbb{Z}^n}\hat f(B^j(\xi+m))\overline{\hat\psi(\xi+m)}\right]d\xi;$$

hence,

$$I=\sum_{j\le J}q^j\int_{\mathbb{R}^n}|\hat f(B^j\xi)|^2|\hat\psi(\xi)|^2\,d\xi$$
$$+\sum_{j\le J}q^j\int_{\mathbb{R}^n}\overline{\hat f(B^j\xi)}\hat\psi(\xi)\left[\sum_{m\in\mathbb{Z}^n\setminus\{0\}}\hat f(B^j(\xi+m))\overline{\hat\psi(\xi+m)}\right]d\xi. \tag{3.4}$$

Since

$$2|\hat\psi(\xi)\hat\psi(\xi+m)|\le|\hat\psi(\xi)|^2+|\hat\psi(\xi+m)|^2,$$

the second sum is absolutely convergent in $L^1(\mathbb{R}^n)$ and, thus, absolutely summable for a.e. ξ even if we extend the sumation over all $j\in\mathbb{Z}$; i.e.,

$$\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n\setminus\{0\}}q^j|\hat f(B^j\xi)\hat\psi(\xi)||\hat f(B^j(\xi+m))\hat\psi(\xi+m)|\,d\xi$$
$$\le\frac12\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n\setminus\{0\}}(q^j|\hat f(B^j\xi)\hat f(B^j(\xi+m))|$$
$$+q^j|\hat f(B^j(\xi-m))\hat f(B^j\xi)|)|\hat\psi(\xi)|^2\,d\xi$$
$$=\int_{\mathbb{R}^n}|\hat\psi(\xi)|^2\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n\setminus\{0\}}q^j|\hat f(B^j\xi)\hat f(B^j(\xi+m))|\,d\xi$$
$$\le C\sum_{l=1}^{L}\int_{\mathbb{R}^n}|\hat\psi(\xi)|^2\,d\xi<\infty, \tag{3.5}$$

where C is the constant appearing in Lemma 2.4 depending on the size and the location of supp $\hat f$.

The first sum apearing in (3.4) can be estimated crudely by

$$\sum_{j\le J}q^j\int_{\mathbb{R}^n}|\hat f(B^j\xi)|^2|\hat\psi(\xi)|^2\,d\xi\le\|\hat f\|_\infty^2\sum_{j\le J}q^j\int_{\mathbb{R}^n}|\hat\psi(\xi)|^2\,d\xi$$
$$=\frac{q^{J+1}}{q-1}\|\hat f\|_\infty^2\|\psi\|^2. \tag{3.6}$$

To show the second part of the lemma, note that we have

$$\sum_{j\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}|\langle f,\psi_{j,k}\rangle|^2=\sum_{j\in\mathbb{Z}}q^j\int_{\mathbb{R}^n}|\hat f(B^j\xi)|^2|\hat\psi(\xi)|^2\,d\xi$$
$$+\sum_{j\in\mathbb{Z}}q^j\int_{\mathbb{R}^n}\overline{\hat f(B^j\xi)}\hat\psi(\xi)\left[\sum_{m\in\mathbb{Z}^n\setminus\{0\}}\hat f(B^j(\xi+m))\overline{\hat\psi(\xi+m)}\right]d\xi,$$

where the second expression in this decomposition is always finite by (3.5).

The implication "⇒" follows from

$$\sum_{j\in\mathbb{Z}^n} q^j \int_{\mathbb{R}^n} |\hat{f}(B^j\xi)|^2|\hat{\psi}(\xi)|^2\,d\xi = \sum_{j\in\mathbb{Z}^n}\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2|\hat{\psi}(B^{-j}\xi)|^2\,d\xi$$

$$\leq \|\hat{f}\|_\infty^2 \int_{\operatorname{supp}\hat{f}} \sum_{j\in\mathbb{Z}^n} |\hat{\psi}(B^{-j}\xi)|^2\,d\xi < \infty,$$

whereas the converse "$\Leftarrow$" is the consequence of applying the above to $\hat{f} = \mathbf{1}_K$ for compact $K \subset \mathbb{R}^n\backslash\{0\}$, since we have equality (instead of inequality) in the above formula. ■

THEOREM 3.2. *Suppose* $\Psi = \{\psi^1,\ldots,\psi^L\}$, $\Phi = \{\phi^1,\ldots,\phi^L\} \subset L^2(\mathbb{R}^n)$. *Then*

$$\|f\|^2 = \lim_{J\to\infty}\sum_{l=1}^{L}\sum_{j\leq J}\sum_{k\in\mathbb{Z}^n}\langle f,\psi^l_{j,k}\rangle\langle\phi^l_{j,k}, f\rangle \qquad \textit{for all } f\in\mathcal{D} \tag{3.7}$$

iff

$$\lim_{J\to\infty}\sum_{l=1}^{L}\sum_{j\geq -J}\hat{\psi}^l(B^j\xi)\overline{\hat{\phi}^l(B^j\xi)} = 1 \qquad \textit{weakly in } L^1(K),\ \forall \textit{compact } K\subset\mathbb{R}^n\backslash\{0\} \tag{3.8}$$

$$t_s(\xi) \equiv \sum_{l=1}^{L}\sum_{j=0}^{\infty}\hat{\phi}^l(B^j\xi)\overline{\hat{\psi}^l(B^j(\xi+s))} = 0 \qquad \textit{for a.e. } \xi\in\mathbb{R}^n \textit{ for } s\in\mathbb{S} = \mathbb{Z}^n\backslash B\mathbb{Z}^n. \tag{3.9}$$

Before we start the proof let us see that statements (3.7)–(3.9) are meaningful by showing that all three series are absolutely convergent. Since

$$2|\langle f,\psi^l_{j,k}\rangle\langle\phi^l_{j,k}, f\rangle| \leq |\langle f,\psi^l_{j,k}\rangle|^2 + |\langle\phi^l_{j,k}, f\rangle|^2$$

the series in (3.7) is summable by Lemma 3.1. Moreover, by the polarization identity (1.5), condition (3.7) is equivalent to

$$\langle f, g\rangle = \lim_{J\to\infty}\sum_{l=1}^{L}\sum_{j\leq J}\sum_{k\in\mathbb{Z}^n}\langle f,\psi^l_{j,k}\rangle\langle\phi^l_{j,k}, g\rangle \qquad \text{for all } f, g\in\mathcal{D}. \tag{3.10}$$

Note that for any $\psi\in L^2(\mathbb{R}^n)$, and $s\in\mathbb{R}^n$,

$$\int_{\mathbb{R}^n}\sum_{j\geq -J}|\hat{\psi}(B^j(\xi+s))|^2\,d\xi = \int_{\mathbb{R}^n}\sum_{-j\leq J} q^{-j}|\hat{\psi}(\xi+B^j s)|^2\,d\xi$$

$$= \frac{q^{J+1}}{q-1}\int_{\mathbb{R}^n}|\hat{\psi}(\xi)|^2\,d\xi < \infty;$$

hence,

$$\sum_{j\geq -J}|\hat{\psi}(B^j(\xi+s))|^2 < \infty \qquad \text{for a.e. } \xi. \tag{3.11}$$

Using the above when $s = 0$ yields

$$2\sum_{l=1}^{L}\sum_{j\geq -J}|\hat{\psi}^l(B^j\xi)\overline{\hat{\phi}^l(B^j\xi)}| \leq \sum_{l=1}^{L}\sum_{j\geq -J}|\hat{\psi}^l(B^j\xi)|^2 + |\hat{\phi}^l(B^j\xi)|^2 < \infty \qquad \text{for a.e. } \xi.$$

And, similarly, (3.11) applied when $J = 0$ implies

$$2\sum_{l=1}^{L}\sum_{j=0}^{\infty}|\hat{\phi}^l(B^j\xi)\overline{\hat{\psi}^l(B^j(\xi+s))}| \le \sum_{l=1}^{L}\sum_{j=0}^{\infty}|\hat{\phi}^l(B^j\xi)|^2 + |\hat{\psi}^l(B^j(\xi+s))|^2 < \infty \qquad \text{for a.e. } \xi.$$

Proof (3.8) *and* (3.9) $\Rightarrow$ (3.7). Suppose $f, g \in \mathcal{D}$. By (3.2)

$$\langle f, \psi^l_{j,k}\rangle\langle\phi^l_{j,k}, g\rangle = q^j\int_{\mathbb{R}^n}\hat{f}(B^j\xi)\overline{\hat{\psi}^l(\xi)}e^{2\pi i\langle k,\xi\rangle}\,d\xi\int_{\mathbb{R}^n}\overline{\hat{g}(B^j\xi)}\hat{\phi}^l(\xi)e^{-2\pi i\langle k,\xi\rangle}\,d\xi.$$

For fixed $l = 1, \ldots, L$, and $j \in \mathbb{Z}$, let $F(\xi) \equiv \hat{f}(B^j\xi)\overline{\hat{\psi}^l(\xi)}$, $G(\xi) \equiv \hat{g}(B^j\xi)\overline{\hat{\phi}^l(\xi)}$; then, using the above and Lemma 2.5, we have

$$\sum_{k\in\mathbb{Z}^n}\langle f, \psi^l_{j,k}\rangle\langle\phi^l_{j,k}, g\rangle = \int_{\mathbb{R}^n}\left[\sum_{m\in\mathbb{Z}^n}\hat{f}(B^j(\xi+m))\overline{\hat{\psi}^l(\xi+m)}\right]\overline{\hat{g}(B^j\xi)}\hat{\phi}^l(\xi)\,d\xi. \tag{3.12}$$

Hence,

$$I = I(J) = \sum_{l=1}^{L}\sum_{j\le J}\sum_{k\in\mathbb{Z}^n}\langle f, \psi^l_{j,k}\rangle\langle\phi^l_{j,k}, g\rangle = I_0 + I_1, \tag{3.13}$$

where

$$\begin{cases} I_0 = I_0(J) = \displaystyle\sum_{l=1}^{L}\sum_{j\le J}q^j\int_{\mathbb{R}^n}\hat{f}(B^j\xi)\overline{\hat{g}(B^j\xi)}\overline{\hat{\psi}^l(\xi)}\hat{\phi}^l(\xi)\,d\xi \\ I_1 = I_1(J) = \displaystyle\sum_{l=1}^{L}\sum_{j\le J}q^j\int_{\mathbb{R}^n}\overline{\hat{g}(B^j\xi)}\hat{\phi}^l(\xi)\left[\sum_{m\in\mathbb{Z}^n\setminus\{0\}}\hat{f}(B^j(\xi+m))\overline{\hat{\psi}^l(\xi+m)}\right]d\xi \end{cases}$$

by splitting the sum (3.12) into terms corresponding to $m = 0$ and $m \neq 0$. We can interchange the summation and integration in I_0 and I_1 since for $h \in \mathcal{D}$, defined by $\hat{h} = \max(|\hat{f}|, |\hat{g}|)$, we have

$$\begin{aligned} &\sum_{l=1}^{L}\sum_{j\le J}q^j\int_{\mathbb{R}^n}|\hat{h}(B^j\xi)|^2|\hat{\psi}^l(\xi)\hat{\phi}^l(\xi)|\,d\xi < \infty \\ &\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}q^j\int_{\mathbb{R}^n}|\hat{h}(B^j\xi)\hat{\phi}^l(\xi)|\left[\sum_{m\in\mathbb{Z}^n\setminus\{0\}}|\hat{h}(B^j(\xi+m))\hat{\psi}^l(\xi+m)|\right]d\xi < \infty. \end{aligned} \tag{3.14}$$

Indeed, using $2|\hat{\psi}^l(\xi)\hat{\phi}^l(\xi)| \le |\hat{\psi}^l(\xi)|^2 + |\hat{\phi}^l(\xi)|^2$ and, similarly, $2|\hat{\phi}^l(\xi)\hat{\psi}^l(\xi+m)| \le |\hat{\phi}^l(\xi)|^2 + |\hat{\psi}^l(\xi+m)|^2$ the estimate (3.14) follows from estimates (3.5) and (3.6).

Therefore, we can manipulate the sums

$$I_1 = \sum_{l=1}^{L}\sum_{j\le J}q^j\int_{\mathbb{R}^n}\overline{\hat{g}(B^j\xi)}\hat{\phi}^l(\xi)\left[\sum_{m\in\mathbb{Z}^n\setminus\{0\}}\hat{f}(B^j(\xi+m))\overline{\hat{\psi}^l(\xi+m)}\right]d\xi$$

$$= \sum_{l=1}^{L} \sum_{j \le J} \int_{\mathbb{R}^n} \overline{\hat{g}(\xi)} \hat{\phi}^l(B^{-j}\xi) \Bigg[\sum_{m \in \mathbb{Z}^n \setminus \{0\}} \hat{f}(\xi + B^j m)) \overline{\hat{\psi}^l(B^{-j}\xi + m)} \Bigg] d\xi$$

$$= \sum_{l=1}^{L} \sum_{j \le J} \int_{\mathbb{R}^n} \overline{\hat{g}(\xi)} \hat{\phi}^l(B^{-j}\xi) \sum_{r \ge 0} \sum_{s \in \mathbb{S}} \hat{f}(\xi + B^j B^r s)) \overline{\hat{\psi}^l(B^{-j}\xi + B^r s)}\, d\xi$$

$$= \int_{\mathbb{R}^n} \overline{\hat{g}(\xi)} \sum_{l=1}^{L} \sum_{s \in \mathbb{S}} \sum_{r \ge 0} \sum_{j \le J} \hat{\psi}^l(B^r(B^{-r-j}\xi)) \hat{f}(\xi + B^{j+r}s) \overline{\hat{\psi}^l(B^r(B^{-r-j}\xi + s))}\, d\xi$$

$$= \int_{\mathbb{R}^n} \overline{\hat{g}(\xi)} \sum_{l=1}^{L} \sum_{s \in \mathbb{S}} \sum_{r \ge 0} \sum_{p \le J+r} \hat{\phi}^l(B^r(B^{-p}\xi)) \overline{\hat{\psi}^l(B^r(B^{-p}\xi + s))} \hat{f}(\xi + B^p s)\, d\xi$$

$$= \int_{\mathbb{R}^n} \overline{\hat{g}(\xi)} \sum_{l=1}^{L} \sum_{s \in \mathbb{S}} \sum_{r \ge 0} \sum_{p \in \mathbb{Z}} \hat{\phi}^l(B^r(B^{-p}\xi)) \overline{\hat{\psi}^l(B^r(B^{-p}\xi + s))} \hat{f}(\xi + B^p s)\, d\xi,$$

for J sufficiently large so that $\hat{g}(\xi)\hat{f}(\xi + B^p s) = 0$ for all $p \ge J$, $s \in \mathbb{S}$, i.e., $(\operatorname{supp} \hat{f} - \operatorname{sup} \hat{g}) \cap B^p\mathbb{S} = \emptyset$ for all $p \ge J$. If $b = \sup\{|\xi| : \xi \in (\operatorname{supp} \hat{f} - \operatorname{supp} \hat{g})\}$; thus, by (2.1) any $J \ge \lceil \log_\lambda(b/c) \rceil$ works. Therefore, we have for any $f, g \in \mathcal{D}$ and sufficiently large J,

$$I(J) = I_0(J) + I_1(J), \qquad \text{where} \begin{cases} I_0(J) = \displaystyle\sum_{l=1}^{L} \sum_{j \ge -J} \int_{\mathbb{R}^n} \hat{f}(\xi) \overline{\hat{g}(\xi) \hat{\psi}^l(B^j\xi)} \hat{\phi}^l(B^j\xi)\, d\xi \\ I_1(J) = \displaystyle\int_{\mathbb{R}^n} \overline{\hat{g}(\xi)} \sum_{p \in \mathbb{Z}} \sum_{s \in \mathbb{S}} \hat{f}(\xi + B^p s) t_s(B^{-p}\xi)\, d\xi. \end{cases} \tag{3.15}$$

Note that the formula for I_0 follows by a simple change of variables, and I_1 does not depend on J. Equation (3.15), combined with assumptions (3.8) and (3.9), immediately implies $\lim_{J\to\infty} I(J) = \lim_{J\to\infty} I_0(J) + I_1(J) = \langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$. ■

Proof (3.7) ⇒ (3.9). Fix $s_0 \in \mathbb{S}$ and $d > 0$ and define

$$\Omega(d) = \{\xi \in \mathbb{R}^n : |\xi| > d, |\xi + s_0| > d\}.$$

For any $\xi_0 \in \Omega(d)$ and $j \ge 0$ define

$$\hat{f}_j(\xi) = |B^{-j} I_n|^{-1/2} \arg t_{s_0}(\xi) \mathbf{1}_{B^{-j} I_n + \xi_0}(\xi)$$
$$\hat{g}_j(\xi) = |B^{-j} I_n|^{-1/2} \mathbf{1}_{B^{-j} I_n + \xi_0 + s_0}(\xi),$$

where, for the purposes of the proof, we define, for $z \in \mathbb{C}$,

$$\arg z = \begin{cases} z/|z| & z \ne 0 \\ 1 & z = 0. \end{cases}$$

By separating the term corresponding to $p = 0$ and $s = s_0$ in formula (3.15) for $I_1(J)$ $f = f_j$, $g = g_j$, from the rest, which we denote by $R(j)$, we have

$$I_1(J) = \frac{1}{|B^{-j} I_n|} \int_{B^{-j} I_n + \xi_0} |t_{s_0}(\xi)|\, d\xi + \int_{\mathbb{R}^n} \overline{\hat{g}_j(\xi)} \sum_{\substack{p \in \mathbb{Z}, s \in \mathbb{S} \\ (p,s) \ne (0,s_0)}} \hat{f}_j(\xi + B^p s) t_s(B^{-p}\xi)\, d\xi. \tag{3.16}$$

Indeed, if $|\hat{g}_j(\xi)\hat{f}_j(\xi + B^p s)| \neq 0$ for some $\xi \in \mathbb{R}^n$ then $(B^{-j}I_n + \xi_0) \cap (B^{-j}I_n + \xi_0 + s_0 - B^p s) \neq \emptyset$ so $B^{-j}(2I_n) \cap (s_0 - B^p\mathbb{S}) \neq \emptyset$ which means $2I_n \cap (B^j s_0 - B^{p+j}\mathbb{S}) \neq \emptyset$. If $p + j \geq 0$ then $B^j s_0 - B^{p+j}\mathbb{S} \subset \mathbb{Z}^n$, and since $2I_n \cap \mathbb{Z}^n = \{0\}$, $s_0 \notin B^p\mathbb{S}$ for $p \neq 0$, the only nonzero term happens for $p = 0$ and $s = s_0$. Therefore, the other nonzero terms can contribute only if $p + j < 0$, so we can restrict the sum in (3.16) to $p < -j$.

Using the estimate

$$2|t_s(\xi)| \leq \sum_{l=1}^{L} \sum_{m\geq 0} |\hat{\phi}^l(B^m\xi)|^2 + |\hat{\psi}^l(B^m(\xi + s))|^2 \leq T(\xi) + T(\xi + s),$$

where $T(\xi) \equiv \sum_{l=1}^{L} \sum_{m\geq 0} |\hat{\phi}^l(B^m\xi)|^2 + |\hat{\psi}^l(B^m\xi)|^2 \in L^1$, we have

$$\begin{aligned}
|R(j)| &\leq \frac{1}{2}\int_{\mathbb{R}^n} \sum_{p<-j} \sum_{s\in\mathbb{S}} q^p |\hat{g}_j(B^p\xi)||\hat{f}_j(B^p(\xi + s))|T(\xi)\, d\xi \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n} \sum_{p<-j} \sum_{s\in\mathbb{S}} q^p |\hat{g}_j(B^p\xi)||\hat{f}_j(B^p(\xi + s))|T(\xi + s)\, d\xi \\
&= \frac{1}{2}\int_{\mathbb{R}^n} \sum_{p<-j} \sum_{s\in\mathbb{S}} q^p |\hat{g}_j(B^p\xi)||\hat{f}_j(B^p(\xi + s))|T(\xi)\, d\xi \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n} \sum_{p<-j} \sum_{s\in\mathbb{S}} q^p |\hat{g}_j(B^p(\xi - s))||\hat{f}_j(B^p\xi)|T(\xi)\, d\xi.
\end{aligned} \tag{3.17}$$

Using $|\hat{f}_j(\xi)| = |\hat{g}_j(\xi - s_0)|$ we have

$$\begin{aligned}
&\sum_{p<-j} \sum_{s\in\mathbb{S}} q^p |\hat{g}_j(B^p\xi)||\hat{f}_j(B^p(\xi + s))| \\
&\quad = \sum_{p<-j} \sum_{s\in\mathbb{S}} q^p |\hat{g}_j(B^p\xi)||\hat{g}_j(B^p(\xi + s) - s_0)| \\
&\quad = \sum_{p<-j} \sum_{s\in\mathbb{S}} q^p |\hat{g}_j(B^p\xi)||\hat{g}_j(B^p(\xi + s - B^{-p}s_0)| \\
&\quad \leq \sum_{p<-j} \sum_{m\in\mathbb{Z}^n\setminus\{0\}} q^p |\hat{g}_j(B^p\xi)||\hat{g}_j(B^p(\xi + m)| \\
&\quad \leq 2^n q^j M((a + \delta)/a)\|\hat{g}_j\|_\infty^2 \mathbf{1}_\Upsilon(\xi) = 2^n M((a + \delta)/a)\mathbf{1}_\Upsilon(\xi),
\end{aligned} \tag{3.18}$$

by Lemma 2.4, assuming $a > 0$, where $a = a(j) = \inf\{|\xi| : \xi \in B^{-j}I_n + \xi_0\}$, $\delta = \delta(j) = \operatorname{diam}(B^{-j}I_n)$, $\Upsilon = \Upsilon(j) = \bigcup_{p<-j} B^{-p}(B^{-j}I_n + \xi_0)$. Similarly,

$$\begin{aligned}
&\sum_{p<-j}\sum_{s\in\mathbb{S}} q^p|\hat{g}_j(B^p(\xi-s))||\hat{f}_j(B^p(\xi)| \\
&\quad\le \sum_{p<-j}\sum_{m\in\mathbb{Z}^n\setminus\{0\}} q^p|\hat{f}_j(B^p\xi)||\hat{f}_j(B^p(\xi+m)| \\
&\quad\le 2^n q^j M((b+\delta)/b)\|\hat{f}_j\|_\infty^2 \mathbf{1}_{\Upsilon'}(\xi) = 2^n M((b+\delta)/b)\mathbf{1}_{\Upsilon'}(\xi) \qquad (3.19)
\end{aligned}$$

by Lemma 2.4, assuming $b>0$, where $b=b(j)=\inf\{|\xi|:\xi\in B^{-j}I_n+\xi_0+s_0\}$, $\Upsilon'=\Upsilon'(j)=\bigcup_{p<-j}B^{-p}(B^{-j}I_n+\xi_0+s_0)$.

For any $\varepsilon>0$, there exists $r>0$, so that $\int_{|\xi|>r}T(\xi)\,d\xi<\varepsilon$. By (2.1) we can find $j_0>0$ so that $\delta(j)<d/2$, and consequently $a(j)>d/2$, $b(j)>d/2$ for $j>j_0$. Furthermore, by (2.1) we can choose (a possibly larger) j_0 so that

$$\inf\{|\xi|:\xi\in\Upsilon(j)\} = \inf\left\{|\xi|:\xi\in\bigcup_{p>j}B^p(B^{-j}I_n+\xi_0)\right\} > c\lambda^j d/2 > r$$

$$\inf\{|\xi|:\xi\in\Upsilon'(j)\} = \inf\left\{|\xi|:\xi\in\bigcup_{p>j}B^p(B^{-j}I_n+\xi_0+s_0)\right\} > c\lambda^j d/2 > r$$

for all $j>j_0$.

Hence, by placing (3.18) and (3.19) into (3.17) we have

$$\begin{aligned}
|R(j)| &\le 2^{n-1}M(2)\int_{\Upsilon(j)}T(\xi)\,d\xi + 2^{n-1}\int_{\Upsilon'(j)}T(\xi)\,d\xi \\
&\le 2^n M(2)\int_{|\xi|>r}T(\xi)\,d\xi < 2^n M(2)\varepsilon \qquad (3.20)
\end{aligned}$$

for $j>j_0$ independent of the choice of $\xi_0\in\Omega(d)$. Since the supports of $\hat{f}_j$ and $\hat{g}_j$ are disjoint $I_0(J)=0$; moreover, (3.7) (and thus (3.10)) implies $0=\langle f_j,g_j\rangle=\lim_{J\to\infty}I(J)=\lim_{J\to\infty}I_1(J)=I_1$.

Since $\varepsilon>0$ was arbitrary, by (3.16) and (3.20)

$$\lim_{j\to\infty}\sup_{\xi_0\in\Omega(d)}\frac{1}{|B^{-j}I_n|}\int_{B^{-j}I_n+\xi_0}|t_{s_0}(\xi)|\,d\xi = 0. \qquad (3.21)$$

Consider any ball $B(r)$ with radius $r>0$ such that $B(r)\subset\Omega(2d)$. Let $Z=\{B^{-j}m: B^{-j}(I_n+m)\cap B(r)\neq\emptyset, m\in\mathbb{Z}^n\}$. If j is sufficiently large then $\operatorname{diam}(B^{-j}I_n)<\min(d,r)$, so

$$\tilde{Z}=\bigcup_{\xi_0\in Z}(B^{-j}I_n+\xi_0)\subset\Omega(d)\cap B(2r).$$

Hence,

$$\begin{aligned}
\int_{B(r)}|t_{s_0}(\xi)|\,d\xi &\le \int_{\tilde{Z}}|t_{s_0}(\xi)|\,d\xi \le \sum_{\xi_0\in Z}\int_{B^{-j}I_n+\xi_0}|t_{s_0}(\xi)|\,d\xi \\
&\le \sum_{\xi_0\in Z}|B^{-j}I_n+\xi_0|\varepsilon = |\tilde{Z}|\varepsilon = 2^n|B(r)|\varepsilon
\end{aligned}$$

for sufficiently large $j=j(\varepsilon)$ by (3.21). Since $\varepsilon>0$ is arbitrary $\int_{B(r)}|t_{s_0}(\xi)|\,d\xi=0$ for any ball $B(r)\subset\Omega(2d)$. Therefore, $\int_{\Omega(2d)}|t_{s_0}(\xi)|\,d\xi=0$ and since $d>0$ is arbitrary $\int_{\mathbb{R}^n}|t_{s_0}(\xi)|\,d\xi=0$ which implies $t_{s_0}(\xi)=0$ for a.e. $\xi\in\mathbb{R}^n$, $s_0\in\mathbb{S}$. ■

Proof (3.7) $\Rightarrow$ (3.8). Equation (3.8) follows easily from (3.9) and (3.15) since any function $h \in L^\infty(K)$ can be represented as $h = \overline{\hat{f}\hat{g}}$ for some $f, g \in \mathcal{D}$. ■

EXAMPLE. We present an example for which we cannot replace the limit in (3.7) by the sum over all $j \in \mathbb{Z}$ simply because the series diverges absolutely. For simplicity let us work in $\mathbb{R}$, the dilation $A = 2$ (multiplication by 2). Let $A_j = (2^{-j-1}, 2^{-j})$ for $j \in \mathbb{Z}$. Define $\psi, \phi \in L^2(\mathbb{R})$ by

$$\hat{\psi}(\xi) = \mathbf{1}_{A_1}(|\xi|) + \sum_{l=1}^{\infty} \mathbf{1}_{A_{2l}}(|\xi|), \qquad \hat{\phi}(\xi) = \mathbf{1}_{A_1}(|\xi|) + \sum_{l=1}^{\infty} \mathbf{1}_{A_{2l+1}}(|\xi|).$$

Since $\hat{\psi}(\xi)\hat{\phi}(\xi) = \mathbf{1}_{A_1}(|\xi|)$ we have

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(2^j\xi)\hat{\phi}(2^j\xi) = \sum_{j\in\mathbb{Z}} \mathbf{1}_{A_1}(2^j|\xi|) = \sum_{j\in\mathbb{Z}} \mathbf{1}_{2^{-j}A_1}(|\xi|) = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}.$$

Since $\operatorname{supp}\hat{\psi}, \operatorname{supp}\hat{\phi} \subset (-1/2, 1/2)$

$$\operatorname{supp}\hat{\psi}(2^j\cdot) \subset (-2^{-j-1}, 2^{-j-1}), \qquad \operatorname{supp}\hat{\phi}(2^j(\cdot + s)) \subset (-2^{-j-1}, 2^{-j-1}) - s;$$

hence, for $j \geq 0$, $s \in \mathbb{S} = \mathbb{Z}\backslash(2\mathbb{Z})$ the supports of the above functions are disjoint and $t_s(\xi) \equiv 0$. Therefore, by Theorem 3.2 we have

$$\|f\|^2 = \lim_{J\to\infty} \sum_{j\leq J} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle\langle\phi_{j,k}, f\rangle \qquad \text{for all } f \in \mathcal{D}.$$

Take $f \in \mathcal{D}$ given by $\hat{f} = \mathbf{1}_{(1,4)}$. By a simple calculation

$$|\langle f, \psi_{j,k}\rangle| \geq 2^{-j/2-1}, \qquad |\langle f, \phi_{j,k}\rangle| \geq 2^{-j/2-1} \qquad \text{for } j \geq 5,\ |k| \leq 2^{j-5}.$$

Therefore the above sum over all $j \in \mathbb{Z}$ is not absolutely convergent. This is not surprising because, in the light of Lemma 3.1, we cannot expect, in general, anything better if $\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = \infty$, $\sum_{j\in\mathbb{Z}} |\hat{\phi}(2^j\xi)|^2 = \infty$.

The next corollary is a positive step in this direction.

COROLLARY 3.3. *Suppose* $\Psi = \{\psi^1, \ldots, \psi^L\}$, $\Phi = \{\phi^1, \ldots, \phi^L\} \subset L^2(\mathbb{R}^n)$ *satisfy*

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}^l(B^j\xi)|^2,\ \sum_{j\in\mathbb{Z}} |\hat{\phi}^l(B^j\xi)|^2 \in L^1_{\text{loc}}(\mathbb{R}^n\backslash\{0\}) \qquad \textit{for } l = 1, \ldots, L. \tag{3.22}$$

Then

$$\|f\|^2 = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \langle f, \psi^l_{j,k}\rangle\langle\phi^l_{j,k}, f\rangle \qquad \textit{for all } f \in \mathcal{D} \tag{3.23}$$

if and only if

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \hat{\psi}^l(B^j\xi)\overline{\hat{\phi}^l(B^j\xi)} = 1 \qquad \textit{a.e. } \xi \in \mathbb{R}^n \tag{3.24}$$

$$t_s(\xi) \equiv \sum_{l=1}^{L}\sum_{j=0}^{\infty} \hat{\phi}^l(B^j\xi)\overline{\hat{\psi}^l(B^j(\xi+s))} = 0 \qquad \textit{a.e. } \xi \in \mathbb{R}^n \textit{ for } s \in \mathbb{S} = \mathbb{Z}^n \backslash B\mathbb{Z}^n. \quad (3.25)$$

Proof. By Lemma 3.1 and (3.22) the series in (3.23) is absolutely convergent. Also by (3.22) the series in (3.24) converges absolutely in $L^1_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$ and, therefore, is absolutely convergent for a.e. ξ. Therefore, under the hypothesis (3.22), (3.8) $\Leftrightarrow$ (3.23) and (3.9) $\Leftrightarrow$ (3.24). Hence, the corollary follows from Theorem 3.2. ■

4. CHARACTERIZATION OF (QUASI)AFFINE DUAL SYSTEMS

In this section we prove the characterization announced in Theorem 1.4.

THEOREM 4.1. *Suppose* $\Psi = \{\psi^1, \dots, \psi^L\}$, $\Phi = \{\phi^1, \dots, \phi^L\} \subset L^2(\mathbb{R}^n)$. *Then the following are equivalent:*

(i) Φ *is an affine dual of* Ψ.
(ii) Φ *is a quasiaffine dual of* Ψ.
(iii) *The series in* (4.1) *converges unconditionally in* $L^2(\mathbb{R}^n)$*:*

$$f = \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}\langle f, \psi^l_{j,k}\rangle\phi^l_{j,k} = \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}\langle f, \phi^l_{j,k}\rangle\psi^l_{j,k} \qquad \textit{for all } f \in L^2(\mathbb{R}^n). \quad (4.1)$$

(iv) *The series in* (4.2) *converges unconditionally in* $L^2(\mathbb{R}^n)$*:*

$$f = \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}\langle f, \tilde{\psi}^l_{j,k}\rangle\tilde{\phi}^l_{j,k} = \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}\langle f, \tilde{\phi}^l_{j,k}\rangle\tilde{\psi}^l_{j,k} \qquad \textit{for all } f \in L^2(\mathbb{R}^n). \quad (4.2)$$

(v) $X(\Psi)$, $X(\Phi)$ *are Bessel families, and* (3.24) *and* (3.25) *hold.*

Proof. First note that if for some $l = 1, \dots, L$, ϕ^l or ψ^l is the zero function then each of the properties (i)–(v) holds for Ψ, Φ if and only if the corresponding property holds for $\Psi\backslash\{\psi^l\}$, $\Phi\backslash\{\phi^l\}$. So without loss of generality we can assume that all functions in Ψ and Φ are nonzero.

(i) $\Rightarrow$ (ii) was already proved in [6]. To show (ii) $\Rightarrow$ (i) take any $f \in L^2(\mathbb{R}^n)$ with compact support. By Lemma 4 in [6]

$$\sum_{j<0}\sum_{k\in\mathbb{Z}^n}|\langle D^N f, \tilde{\psi}^l_{j,k}\rangle\langle\tilde{\phi}^l_{j,k}, D^N f\rangle| \le \frac{1}{2}\sum_{j<0}\sum_{k\in\mathbb{Z}^n}|\langle D^N f, \tilde{\psi}^l_{j,k}\rangle|^2 + |\langle\tilde{\phi}^l_{j,k}, D^N f\rangle|^2 \to 0 \quad \text{as } N \to \infty,$$

for $l = 1, \dots, L$, where $Df(x) = |\det A|^{1/2} f(Ax)$. Hence by (1.4)

$$I(N) = \sum_{l=1}^{L}\sum_{j\ge 0}\sum_{k\in\mathbb{Z}^n}\langle D^N f, \tilde{\psi}^l_{j,k}\rangle\langle\tilde{\phi}^l_{j,k}, D^N f\rangle \to \|D^N f\|^2 = \|f\|^2 \qquad \text{as } N \to \infty.$$

But, on the other hand,

$$I(N) = \sum_{l=1}^{L}\sum_{j\ge 0}\sum_{k\in\mathbb{Z}^n}\langle D^N f, \psi^l_{j,k}\rangle\langle\phi^l_{j,k}, D^N f\rangle = \sum_{l=1}^{L}\sum_{j\ge -N}\sum_{k\in\mathbb{Z}^n}\langle D^N f, \psi^l_{j,k}\rangle\langle\phi^l_{j,k}, D^N f\rangle,$$

which yields (1.3) for $f \in L^2(\mathbb{R}^n)$ with compact support. By invoking Lemma 2.7 we obtain (i).

Assume either (iii) or (iv). Equations (4.1) or (4.2) and Lemma 2.6 imply

$$\infty > \sum_{l=1}^{L} \sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} |\langle f, \psi_{j,k}^l \rangle|^2 \|\phi_{j,k}^l\|^2 \geq \inf_{l=1,\dots,L} \|\phi^l\|^2 \sum_{l=1}^{L} \sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} |\langle f, \psi_{j,k}^l \rangle|^2.$$

Therefore the mapping $T\colon L^2(\mathbb{R}^n) \to l^2(\mathbb{N} \times \mathbb{Z}^n \times \{1, \dots, L\})$

$$Tf = \{\langle f, \psi_{j,k}^l \rangle\}_{j \in \mathbb{N}, k \in \mathbb{Z}^n}^{l=1,\dots,L} \qquad \text{for } f \in L^2(\mathbb{R}^n)$$

is well defined. It is clear that the graph of T is closed and therefore T is bounded; i.e., there is $C > 0$ so that

$$\sum_{l=1}^{L} \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}^n} |\langle f, \psi_{j,k}^l \rangle|^2 \leq C\|f\|^2 \qquad \text{for } f \in L^2(\mathbb{R}^n).$$

This implies that $X(\Psi)$ is the Bessel family. Indeed, for any $l = 1, \dots, L$, and $N > 0$

$$\begin{aligned}\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |\langle f, \psi_{j,k}^l \rangle|^2 &= \lim_{N \to \infty} \sum_{j \geq -N} \sum_{k \in \mathbb{Z}^n} |\langle f, \psi_{j,k}^l \rangle|^2 \\ &= \lim_{N \to \infty} \sum_{j \geq 0} \sum_{k \in \mathbb{Z}^n} |\langle D^N f, \psi_{j,k}^l \rangle|^2 \leq C\|f\|^2.\end{aligned}$$

By interchanging the roles of ϕ^l and ψ^l we obtain $X(\Phi)$ is a Bessel family; hence by Theorem 1.3 $X^q(\Psi)$, $X^q(\Phi)$ are Bessel. This shows (iii) $\Leftrightarrow$ (i) and (iv) $\Leftrightarrow$ (ii) by virtue of Lemma 2.7.

Finally assume (i). Since $X(\Psi)$, $X(\Phi)$ are Bessel, the condition (3.22) is satisfied by Lemma 3.1; hence we can apply Corollary 3.3 to conclude (v). Conversely, again by Corollary 3.3 (v) implies (3.23) and, by Lemma 2.7, (i) follows. ∎

In the special case when $\Phi = \Psi = \{\psi^1, \dots, \psi^L\}$ we obtain the characterization of wavelets proved by Calogero in [4]. This result in dimension 1 and for dilation $A = 2$ was first proved by [11] and independently by [21]; see also [14].

THEOREM 4.2. *Suppose $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$, then*

$$\|f\|^2 = \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |\langle f, \psi_{j,k}^l \rangle|^2 \qquad \textit{for all } f \in L^2(\mathbb{R}^n) \tag{4.3}$$

iff

$$\sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}^l(B^j \xi)|^2 = 1 \qquad \textit{a.e. } \xi \in \mathbb{R}^n \tag{4.4}$$

$$t_s(\xi) \equiv \sum_{l=1}^{L} \sum_{j=0}^{\infty} \hat{\psi}^l(B^j \xi)\overline{\hat{\psi}^l(B^j(\xi + s))} = 0 \qquad \textit{a.e. } \xi \in \mathbb{R}^n \textit{ for } s \in \mathbb{S} = \mathbb{Z}^n \setminus B\mathbb{Z}^n. \tag{4.5}$$

In particular, $X(\Psi)$ *is an orthonormal basis of* $L^2(\mathbb{R}^n)$ *if and only if* (4.4), (4.5) *hold and* $\|\psi^l\| = 1$ *for* $l = 1, \ldots, L$.

Proof. By Lemma 3.1 (4.3) implies that

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}^l(B^j\xi)|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) \qquad \text{for } l = 1, \ldots, L,$$

so we can apply Corollary 3.3 with $\Psi = \Phi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n))$ to obtain (4.4) and (4.5). Conversely, assume (4.4) and (4.5); then by the same corollary we have

$$\|f\|^2 = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\langle f, \psi^l_{j,k}\rangle|^2 \qquad \text{for all } f \in \mathcal{D}.$$

By the well-known result about abstract tight frames from Chapter 7 of [14] we have the above for all $f \in L^2(\mathbb{R}^n)$. Furthermore, $X(\Psi)$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ if $\|\psi^l\| \geq 1$ for $l = 1, \ldots, L$. ∎

5. FINAL REMARKS

It is relatively easy to construct an affine tight frame for an arbitrary dilation. Here we present a simple construction of such a frame which is generated by a single function ψ which is in the Schwartz class and $\hat{\psi}$ is C^∞ with compact support.

EXAMPLE. For $0 < a < (4\|B\|)^{-1}$ consider η: $\mathbb{R}^n \to \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ of class C^∞ such that

$$\operatorname{supp} \eta = \{\xi \in \mathbb{R}^n : a < |\xi| < 2a\|B\|\}.$$

It is not hard to give an explicit example of such function. Since the set $\{j \in \mathbb{Z} : a < \|B^j\xi\| < 2\|B\|a\}$ has at least one element for all $\xi \in \mathbb{R}^n\backslash\{0\}$ we conclude that $\tilde{\eta}(\xi) = \sum_{j\in\mathbb{Z}} \eta(B^j\xi) > 0$ for all $\xi \neq 0$ and $\tilde{\eta}$ is C^∞ on $\mathbb{R}^n\backslash\{0\}$.

Define $\psi \in L^2(\mathbb{R}^n)$ by $\hat{\psi}(\xi) = \sqrt{\eta(\xi)/\tilde{\eta}(\xi)}$. Clearly

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(B^j\xi)|^2 = \sum_{j\in\mathbb{Z}} \eta(B^j\xi)/\tilde{\eta}(B^j\xi) = \sum_{j\in\mathbb{Z}} \eta(B^j\xi)/\tilde{\eta}(\xi) = 1.$$

To guarantee

$$\hat{\psi}(B^j\xi)\hat{\psi}(B^j(\xi + s)) = 0 \qquad \text{for all } \xi \in \mathbb{R}^n,\ j \geq 0,\ s \in \mathbb{S}$$

we have must have $B^{-j} \operatorname{supp} \eta \cap (B^{-j} \operatorname{supp} \eta - s) = \emptyset$, so $\operatorname{supp} \eta \cap (\operatorname{supp} \eta - B^j s) = \emptyset$; that is $B^j s \notin (\operatorname{supp} \eta - \operatorname{supp} \eta) \subset \{\xi \in \mathbb{R}^n : |\xi| < 4a\|B\|\}$, which is true since $s \in \mathbb{S}$, and $j \geq 0$. Therefore (4.4) and (4.5) hold and by Theorem 4.2 $\{\psi_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ forms a tight frame with constant 1 in $L^2(\mathbb{R}^n)$. Note that this frame is not an orthogonal basis since $\|\psi\| < 1$. A different approach of constructing tight frames having an MRA-like structure is presented in [2].

The above example yields a function from the Schwartz class with compact support in the Fourier domain generating a tight frame. It is less obvious how to find smooth generators of tight frames with compact support in the direct space. In the recent paper [12] Gröchenig and Ron have shown how to construct (for arbitrary dilation A) tight frames $X(\Psi)$ with functions Ψ of class C^r with compact support for any $r < \infty$. In their construction the number of functions in Ψ grows with r—the level of desired smoothness.

Not much is known about the existence of "nice" orthogonal wavelets in higher dimensions. In [7] Dai *et al.* have shown the existence of orthogonal basis $X(\{\psi\})$ generated by a single function $\psi \in L^2(\mathbb{R}^n)$; see also [19]. Even though ψ itself is smooth it decays slowly at infinity since $\hat{\psi}$ is the characteristic function of some set. Strichartz presented a method of obtaining r-regular wavelets $\Psi = \{\psi^1, \ldots, \psi^{q-1}\}$, $q = |\det A|$ for dilations which admit Haar-type basis; see [20]. Since not all dilations have this property (see [15, 16]) one needs-special argument to prove the existence of r-regular wavelets for arbitrary dilations; see [3]. Finally, for some specific dilations in $\mathbb{R}^2$ Belogay and Wang in [1] constructed nonseparable C^r wavelets with compact support the size of which depends on r.

REFERENCES

1. E. Belogay and Y. Wang, Arbitrarily smooth orthogonal nonseparable wavelets in $\mathbb{R}^2$, *SIAM J. Math. Anal.* **30** (1999), 678–697.
2. M. Bownik, Tight frames of multidimensional wavelets, *J. Fourier Anal. Appl.* **3** (1997), 525–542.
3. M. Bownik, The construction of r-regular wavelets for arbitrary dilations, preprint, 1999.
4. A. Calogero, A characterization of wavelets on general lattices, *J. Geom. Anal.*, to appear.
5. A. Calogero and G. Garrigós, A characterization of wavelet families arising from biorthogonal MRA's of multiplicity d, preprint, 1999.
6. C. K. Chui, X. Shi, and J. Stöckler, Affine frames, quasi-affine frames, and their duals, *Adv. Comput. Math.* **8** (1998), 1–17.
7. X. Dai, D. R. Larson, and D. M. Speegle, Wavelet sets in $\mathbb{R}^n$, *J. Fourier Anal. Appl.* **3** (1997), 451–456.
8. M. Frazier, G. Garrigós, K. Wang, and G. Weiss, A characterization of functions that generate wavelet and related expansion, Proceedings of the conference dedicated to Professor Miguel de Guzmán, El Escorial, 1996, *J. Fourier Anal. Appl.* **3** (1997), 883–906.
9. M. Frazier and B. Jawerth, The ϕ-transform and applications to distribution spaces, *in* "Function Spaces and Applications, Lund, 1986," pp. 223–246, Springer Verlag, Berlin/New York, 1988.
10. M. Frazier, B. Jawerth, and G. Weiss, "Littlewood–Paley Theory and the Study of Function Spaces," Conference Board of the Mathematical Sciences, Washington, DC, 1991.
11. G. Gripenberg, A necessary and sufficient condition for the existence of a father wavelet, *Studia Math.* **114** (1995), 207–226.
12. K. Gröchenig and A. Ron, Tight compactly supported wavelet frames of arbitrarily high smoothness, *Proc. Amer. Math. Soc.* **126** (1998), 1101–1107.
13. B. Han, On dual wavelet tight frames, *Appl. Comput. Harmon. Anal.* **4** (1997), 380–413.
14. E. Hernández and G. Weiss, "A First Course on Wavelets," Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1996.
15. J. C. Lagarias and Y. Wang, Haar bases for $L^2(\mathbb{R}^n)$ and algebraic number theory, *J. Number Theory* **57** (1996), 181–197.
16. J. C. Lagarias and Y. Wang, Corrigendum and addendum to: Haar bases for $L^2(\mathbb{R}^n)$ and algebraic number theory, *J. Number Theory* **76** (1999), 330–336.
17. A. Ron and Z. Shen, Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator, *J. Funct. Anal.* **148** (1997), 408–447.
18. A. Ron and Z. Shen, Affine systems in $L_2(\mathbb{R}^d)$. II. Dual systems, *J. Fourier Anal. Appl.* **3** (1997), 617–637.

19. P. Soardi and D. Weiland, Single wavelets in n-dimensions, *J. Fourier Anal. Appl.* **4** (1998), 299–315.

20. R. S. Strichartz, Wavelets and self-affine tilings, *Constr. Approx.* **9** (1993), 327–346.

21. X. Wang, "The Study of Wavelets from the Properties of their Fourier Transforms," Ph. D. Thesis, Washington University, 1995.

22. P. Wojtaszczyk, "A Mathematical Introduction to Wavelets," Cambridge University Press, Cambridge, UK, 1997.