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CONSTRUCTIVE APPROXIMATION

On a Problem of Daubechies

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Abstract. We solve a problem posed by Daubechies [12] by showing the nonexistence of orthonormal wavelet bases with good time-frequency localization associated with irrational dilation factors.

1. Introduction

The aim of this paper is to give a complete answer to a problem posed by Daubechies [12, Chapter 1]: "it is an open question whether there exist orthonormal wavelet bases (not necessarily associated with a multiresolution analysis), with good time-frequency localization, and with irrational a." Here a wavelet is a function $\psi \in L^2(\mathbb{R})$ such that the system of functions $\{a^{j/2}\psi(a^j x - bk) : j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. Without loss of generality, we can assume that $b = 1$ by a simple change of variables.

We are going to show that the answer is negative for any irrational a. Chui and Shi [10] have shown this for almost all irrational dilation factors a.

Theorem 1.1. *Suppose a dilation factor $a > 1$ is such that a^j is irrational for all integers $j \geq 1$. If ψ is an orthogonal wavelet associated with a, then ψ is a minimally supported frequency (MSF) wavelet, i.e., $|\hat{\psi}| = \mathbf{1}_K$ for some measurable $K \subset \mathbb{R}$.*

Theorem 1.1 can be easily deduced from Theorem 3.2, also due to Chui and Shi [10], or by a direct calculation based on the orthonormality, see Lemma 2.1. For a generalization of Theorem 1.1, see [4]. Therefore, a remaining difficulty in answering the Daubechies question lies in the set of "exceptional" irrational dilations a such that a^{j_0} is rational for some integer $j_0 \geq 2$.

We note that there are (multi-)wavelets with rational dilation factors which are well-localized both in time and frequency, i.e., in the Schwartz class, as was shown by Auscher in [1].

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Definition 1.2. We say that $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R})$ is a *multiwavelet* associated with the dilation factor a, $|a| \neq 0, 1$, if the *affine system*

$$X(\Psi) = \{\psi^l_{j,k} : j, k \in \mathbb{Z}, l = 1, \ldots, L\}$$

forms an orthonormal basis for $L^2(\mathbb{R})$. Here, for $\psi \in L^2(\mathbb{R})$, we set $\psi_{j,k}(x) = |a|^{j/2}\psi(a^j x - k)$ for $j, k \in \mathbb{Z}$.

Without loss of generality, we also assume that the dilation factor $|a| > 1$. In this work we consider more general multiwavelet bases, since for certain dilation factors there could be no well-localized single wavelets, but there are well-localized multiwavelets. Indeed, Auscher [2] has shown that if a is an integer, $|a| \geq 2$, and L is not divisible by $|a| - 1$, then any multiwavelet $\Psi = \{\psi^1, \ldots, \psi^L\}$ has to be poorly localized. Hence, if a is an integer and $|a| \geq 3$, then there are no (single) wavelets with good-time frequency localization, but there are well-localized (e.g., in the Schwartz class) multiwavelets Ψ with $L = |a| - 1$.

Our main goal is to show the following fact which answers Daubechies' question. For a more precise statement see Theorem 4.1.

Theorem 1.3. *If $\Psi = \{\psi^1, \ldots, \psi^L\}$ is an orthonormal multiwavelet associated with a* (*positive*) *irrational dilation factor, then at least one of the ψ^l's has a poor time-frequency localization.*

2. The Orthogonality Condition

In this section we are going to give necessary and sufficient conditions for $\{\psi^l_{j,k} : j, k \in \mathbb{Z}, l = 1, \ldots, L\}$ to be an orthonormal system (not necessarily complete) in terms of equations in the Fourier domain. These conditions take two different forms depending on whether some power of a is rational or not.

Lemma 2.1. *Suppose $|a| > 1$ is such that a^j is irrational for all integers $j \geq 1$. Then $\{\psi^l_{j,k} : j, k \in \mathbb{Z}, l = 1, \ldots, L\}$ is an orthonormal system if and only if*

$$\sum_{k\in\mathbb{Z}} \hat{\psi}^l(\xi + k)\overline{\hat{\psi}^{l'}(\xi + k)} = \delta_{l,l'} \qquad a.e. \quad \xi \in \mathbb{R}, \tag{2.1}$$

$$\hat{\psi}^l(\xi)\hat{\psi}^{l'}(a^j\xi) = 0 \qquad a.e. \quad \xi \in \mathbb{R}, \tag{2.2}$$

for all $l, l' = 1, \ldots, L$ and all $j \in \mathbb{Z}\backslash\{0\}$.

The condition (2.2) can also be written as

$$|a^j S \cap S| = 0 \qquad \text{for all} \quad j \in \mathbb{Z}\backslash\{0\}, \tag{2.3}$$

where

$$S = \bigcup_{l=1}^{L} \operatorname{supp} \hat{\psi}^l. \tag{2.4}$$

Proof. The orthonormality of $\{\psi^l_{j,k} : j, k \in \mathbb{Z}, l = 1, \dots, L\}$ can be stated as

$$\begin{aligned}(2.5)\qquad \delta_{l,l'}\delta_{j,0}\delta_{k,k'} = \langle \psi^l_{j,k}, \psi^{l'}_{0,k'}\rangle &= |a|^{j/2}\int_{\mathbb{R}} \psi^l(a^j x - k)\overline{\psi^{l'}(x-k')}\,dx \\ &= |a|^{j/2}\int_{\mathbb{R}} \psi^l(a^j x + a^j k' - k)\overline{\psi^{l'}(x)}\,dx,\end{aligned}$$

for all $j \in \mathbb{Z}$, $k, k' \in \mathbb{Z}$, $l, l' = 1, \dots, L$. By Plancherel's formula

$$\begin{aligned}(2.6)\qquad \delta_{l,l'}\delta_{j,0}\delta_{k,k'} &= \int_{\mathbb{R}} \overline{\hat\psi^{l'}(\xi)}\hat\psi^l(a^{-j}\xi)e^{2\pi i a^{-j}\xi(a^j k' - k)}\,d\xi \\ &= \int_{\mathbb{R}} \overline{\hat\psi^{l'}(\xi)}\hat\psi^l(a^{-j}\xi)e^{2\pi i \xi(k' - a^{-j}k)}\,d\xi.\end{aligned}$$

By the standard periodization argument the above identity becomes (2.1) when $j = 0$. Since $\mathbb{Z} - a^{-j}\mathbb{Z}$ is dense in $\mathbb{R}$ for all $j \in \mathbb{Z}\setminus\{0\}$, the above identity becomes (2.2) by the Fourier inversion formula. ■

An immediate consequence of Lemma 2.1 is Theorem 1.1. Indeed, suppose that $X(\{\psi\})$ is an orthonormal basis and let $W_j = \overline{\operatorname{span}}\{\psi_{j,k} : k \in \mathbb{Z}\}$. Since $W_j \subset \check{L}^2(a^j S)$ and $\bigoplus_{j\in\mathbb{Z}} W_j = L^2(\mathbb{R})$ we must have $W_j = \check{L}^2(a^j S)$, see Definition 3.1. Since the dimension function of the shift-invariant space W_0 satisfies

$$1 = \dim_{W_0}(\xi) = \dim_{\check{L}^2(S)}(\xi) = \sum_{k\in\mathbb{Z}} \mathbf{1}_S(\xi + k) \qquad \text{for a.e. } \xi,$$

see [3], [5], the support of $\hat\psi$ has measure $|S| = 1$, i.e., ψ is an MSF.

Suppose next that the dilation factor a does not satisfy the assumptions of Lemma 2.1, i.e., $a^j \in \mathbb{Q}$ for some $j \in \mathbb{Z}$. In this case the orthogonality conditions take a slightly more complicated form.

Lemma 2.2. *Suppose $|a| > 1$ is such that a^j is rational for some integer j. Let j_0 be the smallest integer ≥ 1 such that $a^{j_0} \in \mathbb{Q}$. Let $a^{j_0} = p/q$ for some relatively prime integers p, q. Then $\{\psi^l_{j,k} : j, k \in \mathbb{Z}, l = 1, \dots, L\}$ is an orthonormal system if and only if*

$$(2.7)\qquad \sum_{k\in\mathbb{Z}} \hat\psi^l(a^j(\xi + q^{j/j_0}k))\overline{\hat\psi^{l'}(\xi + q^{j/j_0}k)} = \delta_{l,l'}\delta_{j,0} \qquad \textit{a.e.}\quad \xi \in \mathbb{R},\quad j \in j_0\mathbb{Z},\quad j \geq 0,$$

$$(2.8)\qquad \hat\psi^l(\xi)\hat\psi^{l'}(a^j\xi) = 0 \qquad \textit{a.e.}\quad \xi \in \mathbb{R},\quad j \in \mathbb{Z}\setminus j_0\mathbb{Z},$$

for all $l, l' = 1, \dots, L$.

The condition (2.8) can also be written as

$$(2.9)\qquad |a^j S \cap S| = 0 \qquad \text{for all}\quad j \in \mathbb{Z}\setminus j_0\mathbb{Z},$$

where S is given by (2.4).

Proof. As in the proof of Lemma 2.1 the identity (2.6) reduces to (2.8) since $\mathbb{Z}-a^{-j}\mathbb{Z}$ is dense in $\mathbb{R}$ for all $j \in \mathbb{Z}\backslash j_0\mathbb{Z}$. By the scaling, the orthonormality of $\{\psi^l_{j,k} : j, k \in \mathbb{Z}, l = 1, \dots, L\}$ can be stated as (2.5) with the j's restricted to all integers ≤ 0 (or, alternatively, ≥ 0). If $j \le 0$ is a multiple of j_0, then $\mathbb{Z}-a^{-j}\mathbb{Z} = \mathbb{Z}+p^{-j/j_0}q^{j/j_0}\mathbb{Z} = q^{j/j_0}\mathbb{Z}$. Therefore for $j \le 0$ being a multiple of j_0, condition (2.6) becomes

$$\int_{\mathbb{R}} \overline{\hat{\psi}^{l'}(\xi)}\hat{\psi}^l(a^{-j}\xi)e^{2\pi i\xi q^{j/j_0}k}\,d\xi = \delta_{l,l'}\delta_{j,0}\delta_{k,0}.$$

Therefore by the standard periodization argument

$$\delta_{l,l'}\delta_{j,0}\delta_{k,0} = \int_0^{q^{-j/j_0}} \left(\sum_{m\in\mathbb{Z}} \overline{\hat{\psi}^{l'}(\xi+q^{-j/j_0}m)}\hat{\psi}^l(a^{-j}(\xi+q^{-j/j_0}m))\right) e^{2\pi i\xi q^{j/j_0}k}\,d\xi$$

for all $j \in j_0\mathbb{Z}$, $j \le 0$, $k \in \mathbb{Z}$, $l, l' = 1, \dots, L$. This in turn is equivalent to (2.7).

Note also that condition (2.7) can be equivalently stated as

$$\sum_{k\in\mathbb{Z}} \hat{\psi}^l(a^{-j}(\xi+p^{j/j_0}k))\overline{\hat{\psi}^{l'}(\xi+p^{j/j_0}k)} = \delta_{l,l'}\delta_{j,0} \qquad \text{a.e. } \xi \in \mathbb{R}, \quad j \in j_0\mathbb{Z}, \quad j \ge 0,$$

for all $l, l' = 1, \dots, L$. ■

3. The Characterization of Affine Tight Frames

If the dilation factor is an integer, then a great deal is known about the properties of the affine system $X(\Psi)$. For example, one can give different characterizations for $X(\Psi)$ to be a tight frame or an orthonormal basis, see [6], [9], [11], [13], [15]. For noninteger dilation factors significant progress has been made recently by Chui and Shi [10] who gave a characterization of pairs of dual frame wavelets. Their method of proof yields in fact a slightly stronger result which is of great importance in this work. The motivation behind this development is Auscher's proof [2] of the nonexistence of well-localized wavelets in $\mathbb{H}^2(\mathbb{R}) = \check{L}^2(0, \infty)$, see also [13, Chapter 7.6].

Definition 3.1. Suppose a is real and $|a| > 1$. We say that a measurable subset E of $\mathbb{R}$ is *a-multiplicatively invariant* if $aE = E$ modulo sets of measure zero. Given such E we introduce the closed subspace $\check{L}^2(E) \subset L^2(\mathbb{R})$ by

$$\check{L}^2(E) = \{f \in L^2(\mathbb{R}) : \operatorname{supp}\hat{f} = \{\xi : \hat{f}(\xi) \neq 0\} \subset E\}.$$

We say that $\Psi = \{\psi^1, \dots, \psi^l\} \subset \check{L}^2(E)$ is a *multiwavelet* for $\check{L}^2(E)$ associated with a if $\{\psi^l_{j,k} : l = 1, \dots, L, j \in \mathbb{Z}, k \in \mathbb{Z}\}$ forms an orthonormal basis for $\check{L}^2(E)$.

Suppose $|a| > 1$ and E is an a-multiplicatively invariant set. If $\psi \in \check{L}^2(E)$ then, clearly, $\psi_{j,k}(x) = |a|^{j/2}\psi(a^jx-k) \in \check{L}^2(E)$ for any $j, k \in \mathbb{Z}$. Hence, it is meaningful to say that $X(\Psi)$ forms an orthonormal basis (Bessel family or frame) for $\check{L}^2(E)$.

Theorem 3.2. *Suppose that* $\Psi = \{\psi^1, \dots, \psi^L\}$, $\Phi = \{\varphi^1, \dots, \varphi^L\} \subset \check{L}^2(E)$. *If* $X(\Psi)$ *and* $X(\Phi)$ *are Bessel families which satisfy*

$$\sum_{l=1}^{L} \sum_{\substack{(j,m)\in\mathbb{Z}\times\mathbb{Z},\\ \alpha=a^{-j}m}} \hat{\varphi}^l(a^j\xi)\overline{\hat{\psi}^l(a^j(\xi+\alpha))} = \delta_{\alpha,0}\mathbf{1}_E(\xi) \qquad \text{for a.e.} \quad \xi \in \mathbb{R}, \tag{3.1}$$

and for all $\alpha \in \Lambda$, *then* $X(\Psi)$ *and* $X(\Phi)$ *form a pair of dual frames in* $\check{L}^2(E)$. *Here* Λ *denotes the set of all* a*-adic numbers*, *i.e.*,

$$\Lambda = \{\alpha \in \mathbb{R} : \alpha = a^{-j}m \text{ for some } (j,m) \in \mathbb{Z}\times\mathbb{Z}\}. \tag{3.2}$$

Conversely, *if* $X(\Psi)$ *and* $X(\Phi)$ *are a pair of dual frames in* $\check{L}^2(E)$, *then* (3.1) *holds*.

In fact, we are going to be interested in Theorem 3.2 only if the dilation factor a is rational. Chui and Shi [10] have also shown that, for $a \in \mathbb{Z}$, condition (3.1) can be rewritten as

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \hat{\varphi}^l(a^j\xi)\overline{\hat{\psi}^l(a^j\xi)} = \mathbf{1}_E(\xi) \qquad \text{for a.e. } \xi, \tag{3.3}$$

$$\sum_{l=1}^{L} \sum_{j=0}^{\infty} \hat{\varphi}^l(a^j\xi)\overline{\hat{\psi}^l(a^j(\xi+t))} = 0 \qquad \text{for all} \quad t \in \mathbb{Z}\backslash a\mathbb{Z}, \text{ a.e. } \xi. \tag{3.4}$$

If $a \in \mathbb{Q}\backslash\mathbb{Z}$, $a = p/q$ for $p, q \in \mathbb{Z}$ with $\gcd(p,q) = 1$, then (3.1) can be stated as (3.3) and

$$\sum_{l=1}^{L} \sum_{j=0}^{s} \hat{\varphi}^l(a^j\xi)\overline{\hat{\psi}^l(a^j(\xi+q^s t))} = 0 \qquad \text{for a.e. } \xi, \tag{3.5}$$

for all $s = 0, 1, 2, \dots$ and all $t \in \mathbb{Z}$ not divisible by p nor q ($p, q \nmid t$), see [10].

As a special case of Theorem 3.2 we obtain the following corollary:

Corollary 3.3. *Suppose that* $\Psi = \{\psi^1, \dots, \psi^L\} \subset \check{L}^2(E)$ *and* $X(\Psi)$ *forms a tight frame with constant* 1 *for* $\check{L}^2(E)$, *where* $E \subset \mathbb{R}$ *is some measurable* a*-multiplicatively invariant set*. *Then*

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}^l(a^j\xi)|^2 = \mathbf{1}_E(\xi) \qquad \text{for a.e.} \quad \xi \in \mathbb{R}. \tag{3.6}$$

Since we do not need the full strength of Theorem 3.2, and since the original argument of Chui and Shi [10] appears to be incomplete, we will present the proof of Corollary 3.3 in some detail. Consider

$$\mathcal{D} = \{f \in L^2(\mathbb{R}) : \hat{f} \in L^\infty(\mathbb{R}), \operatorname{supp} \hat{f} \subset K \text{ for some compact } K \subset \mathbb{R}\backslash\{0\}\},$$

which is a dense subspace of $L^2(\mathbb{R})$. We will use the following result, see [7, Lemma 3.1] which was originally shown for dilations with integer entries.

Lemma 3.4. *Suppose* $\psi \in L^2(\mathbb{R})$. *Then*

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(a^j\xi)|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}\setminus\{0\}) \quad \Leftrightarrow \quad \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 < \infty \qquad \textit{for all} \quad f \in \mathcal{D}.$$

If this is the case, then, for any $f \in \mathcal{D}$,

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 = M(f) + R(f),$$

where

$$M(f) = \sum_{j\in\mathbb{Z}} |a|^j \int_{\mathbb{R}} |\hat{f}(a^j\xi)|^2 |\hat{\psi}(\xi)|^2 \, d\xi,$$

$$R(f) = \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}\setminus\{0\}} |a|^j \int_{\mathbb{R}} \overline{\hat{f}(a^j\xi)}\, \hat{f}(a^j(\xi+m))\hat{\psi}(\xi)\overline{\hat{\psi}(\xi+m)}\, d\xi.$$

The convergence in $R(f)$ is absolute in the sense that the above series converges if we take absolute values inside the integrals. Therefore, we can freely exchange the summation with integration. The proof of Lemma 3.4 follows verbatim the proof of [7, Lemma 3.1] with the help of the following elementary lemma, see [7, Lemma 2.4].

Lemma 3.5. *Suppose* $g \in L^\infty(\mathbb{R})$ *is such that* $\operatorname{supp} g \subset (\xi_0 - |a|^{j_0}/2, \xi_0 + |a|^{j_0}/2)$ *for some* $\xi_0 \in \mathbb{R}$ *and* $j_0 \in \mathbb{Z}$ *satisfying* $2|\xi_0| > |a|^{j_0}$. *Then*

$$\sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}\setminus\{0\}} |a|^j |g(a^j\xi)g(a^j(\xi+m))| \le 2|a|^{j_0} \left\lceil \log_{|a|}\left(\frac{2|\xi_0| + |a|^{j_0}}{2|\xi_0| - |a|^{j_0}}\right)\right\rceil \|g\|_\infty^2 \mathbf{1}_\Upsilon(\xi),$$

where $\Upsilon = \Upsilon(\xi_0, j_0) = \bigcup_{j<j_0} a^{-j}(\xi_0 - |a|^{j_0}/2, \xi_0 + |a|^{j_0}/2)$.

Proof of Corollary 3.3. Given $k \in \mathbb{Z}$, let $H_k = (-|a|^k/2, |a|^k/2)$. Let $0 \neq \xi_0 \in E$ be an arbitrary point of density of E. By the Lebesgue differentiation theorem almost every point of E is its point of density. We will show that

$$\lim_{k\to-\infty} \frac{1}{|H_k|} \int_{\xi_0+H_k} \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}^l(a^j\xi)|^2 \, d\xi = 1. \tag{3.7}$$

By the Lebesgue differentiation theorem this would imply (3.6). To show (3.7) consider the sequence of test functions $(f_k)_{k<0} \subset \check{L}^2(E)$ given by

$$\hat{f}_k(\xi) = |H_k|^{-1/2} \mathbf{1}_{(\xi_0+H_k)\cap E}(\xi).$$

Since

$$\lim_{k\to-\infty} \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} |\langle f_k, \psi^l_{j,m}\rangle|^2 = \lim_{k\to-\infty} \|f_k\|_2^2 = 1,$$

by Lemma 3.4, it suffices to show that $R(f_k) \to 0$ as $k \to -\infty$, where

$$R(f) = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}\setminus\{0\}} |a|^j \int_{\mathbb{R}} \overline{\hat{f}(a^j\xi)} \hat{f}(a^j(\xi+m)) \hat{\psi}^l(\xi) \overline{\hat{\psi}^l(\xi+m)}\, d\xi.$$

By Lemma 3.5, for k sufficiently small,

$$|R(f_k)| \le \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{|a|^j}{2} \int_{\mathbb{R}} |\hat{f}_k(a^j\xi)\hat{f}_k(a^j(\xi+m))|(|\hat{\psi}^l(\xi)|^2 + |\hat{\psi}^l(\xi+m)|^2)\, d\xi,$$

$$\sum_{l=1}^{L} \int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{|a|^j}{2} (|\hat{f}_k(a^j\xi)\hat{f}_k(a^j(\xi+m))| + |\hat{f}_k(a^j\xi)\hat{f}_k(a^j(\xi-m))|)|\hat{\psi}^l(\xi)|^2\, d\xi$$

$$\le \sum_{l=1}^{L} 4|a|^k \|\hat{f}_k\|_\infty^2 \int_{\bigcup_{j<k} a^{-j}(\xi_0+H_k)} |\hat{\psi}^l(\xi)|^2 \to 0 \qquad \text{as} \quad k \to -\infty,$$

by the Lebesgue dominated convergence theorem. Therefore, $M(f_k) \to 1$ as $k \to -\infty$ which shows (3.7) and completes the proof. ■

4. The Main Result

We are now to ready to show the main theorem which completely answers the question posed by Daubechies.

Theorem 4.1. *Suppose a is an irrational dilation factor $a > 1$. If $\Psi = \{\psi^1, \dots, \psi^L\}$ is an orthonormal multiwavelet associated with a, then at least one of the ψ^l's is poorly localized in time. More precisely, there exists $l = 1, \dots, L$ such that, for any $\delta > 0$,*

$$\limsup_{|x|\to\infty} |\psi^l(x)||x|^{1+\delta} = \infty. \tag{4.1}$$

Proof. By contradiction, suppose there is an orthonormal multiwavelet $\Psi = \{\psi^1, \dots, \psi^L\}$ associated with an irrational dilation factor a and satisfying

$$|\psi^l(x)| = O(|x|^{-1-\delta}) \qquad \text{as} \quad |x| \to \infty, \tag{4.2}$$

for all $l = 1, \dots, L$. Note that by (4.2) we can assume that each $\hat{\psi}^l(\xi)$ is continuous on $\mathbb{R}$.

For the sake of simplicity, and to better illustrate the idea of the proof, let us assume in addition that

$$|\hat{\psi}^l(\xi)| = O(|\xi|^{-\delta}) \qquad \text{as} \quad |\xi| \to \infty, \tag{4.3}$$

for all $l = 1, \dots, L$. Later we will show how to eliminate this assumption using some facts about Sobolev spaces.

If a is such that a^j is irrational for all $j \geq 1$, then by the work of Chui and Shi [10] for $L = 1$ and [4] for general $L \geq 1$, Ψ has to be a combined MSF multiwavelet, i.e., the union of the supports of the $\hat{\psi}^l$'s has minimal (Lebesgue) measure. This means that

$$\sum_{l=1}^{L} |\hat{\psi}^l(\xi)|^2 = \mathbf{1}_K(\xi) \qquad \text{for a.e. } \xi, \tag{4.4}$$

for some multiwavelet set of order K, i.e., $\sum_{j\in\mathbb{Z}} \mathbf{1}_K(a^j\xi) = 1$ and $\sum_{k\in\mathbb{Z}} \mathbf{1}_K(\xi + k) = L$ for a.e. ξ. However, since the $\hat{\psi}^l$'s are continuous (4.4) cannot hold. Indeed, it suffices to consider an open set $\{\xi : \frac{1}{3} < \sum_{l=1}^{L} |\hat{\psi}^l(\xi)|^2 < \frac{2}{3}\}$.

Suppose now that a is such that $a, \dots, a^{j_0-1}$ are irrational and that a^{j_0} is rational for some $j_0 \geq 2$. Let $E = \bigcup_{j\in j_0\mathbb{Z}} a^j S$, where $S = \bigcup_{l=1}^{L} \operatorname{supp} \hat{\psi}^l$. By Lemma 2.2 the sets $E, aE, \dots, a^{j_0-1}E$ are pairwise disjoint (modulo sets of measure zero). Furthermore, they are a^{j_0}-multiplicatively invariant and (modulo sets of measure zero)

$$E \cup aE \cup \cdots \cup a^{j_0-1}E = \mathbb{R}. \tag{4.5}$$

Let $\tilde{W}_j = \overline{\operatorname{span}}\{\psi^l_{j+ij_0,k} : i, k \in \mathbb{Z},\ l = 1, \dots, L\}$ for $j \in \mathbb{Z}$. Since $\tilde{W}_j \subset \check{L}^2(a^jE)$ and $\tilde{W}_0 \oplus \cdots \oplus \tilde{W}_{j_0-1} = L^2(\mathbb{R})$, we actually have $\tilde{W}_j = \check{L}^2(a^jE)$ for all $j \in \mathbb{Z}$.

In particular, $\Psi = \{\psi^1, \dots, \psi^L\}$ becomes an orthogonal multiwavelet associated with the rational dilation factor $\tilde{a} = a^{j_0}$ on $\check{L}^2(E)$. Therefore, by Corollary 3.3,

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}^l(\tilde{a}^j\xi)|^2 = \mathbf{1}_E(\xi) \qquad \text{for a.e.} \quad \xi \in \mathbb{R}. \tag{4.6}$$

We claim that a function

$$s(\xi) = \sum_{l=1}^{L} \sum_{j=-\infty}^{\infty} |\hat{\psi}^l(\tilde{a}^j\xi)|^2 \tag{4.7}$$

is continuous everywhere, except possibly 0. Indeed, it suffices to show that the above series converges uniformly on compact intervals of $\mathbb{R}\backslash\{0\}$. By (4.6) we necessarily have $\hat{\psi}^l(0) = 0$ for all $l = 1, \dots, L$. Hence, by (4.2),

$$\frac{|\hat{\psi}^l(\xi)|}{|\xi|^{\delta/2}} = \left| \int_{\mathbb{R}} \frac{1 - e^{-2\pi i x\xi}}{|x\xi|^{\delta/2}} |x|^{\delta/2} \psi^l(x)\, dx \right| \leq C \int_{\mathbb{R}} |\psi^l(x)||x|^{\delta/2}\, dx \leq C',$$

since $y \mapsto |1 - e^{-2\pi i y}|/|y|^{\delta/2}$ is bounded on $\mathbb{R}$ (and without loss of generality $0 < \delta < 1$). Therefore

$$|\hat{\psi}^l(\xi)| = O(|\xi|^{\delta/2}) \qquad \text{as} \quad |\xi| \to 0. \tag{4.8}$$

By (4.3), (4.8), and splitting the summation over positive and negative j's we conclude that the series (4.7) converges uniformly on intervals $[-R, -1/R]$ and $[1/R, R]$ for any $R > 1$. This shows that $s(\xi)$ is continuous on $\mathbb{R}\backslash\{0\}$.

Since $s(\xi) = \mathbf{1}_E(\xi)$ for a.e. $\xi \in \mathbb{R}$ we necessarily have that $s(\xi)$ is constantly equal to 0 or 1 for all $\xi \in (0, \infty)$. The same is true for the interval $(-\infty, 0)$. Therefore, $E \cap (0, \infty)$ is either $\emptyset$ or $(0, \infty)$ (modulo sets of measure zero). Also, $E \cap (-\infty, 0)$ is either $\emptyset$ or $(-\infty, 0)$. Since $a > 0$ we have $E = aE$, which is a contradiction with $|E \cap aE| = 0$. This ends the proof of Theorem 4.1 under the additional assumption (4.3). ■

In order to eliminate (4.3) we will need a lemma due to Meyer [14, Chapter 2.4, Lemma 6] which reduces the "global" Sobolev H^m-norm to H^m-norms "localized" about the points $k \in \mathbb{Z}$. Recall that, given $m \geq 0$, we define the *Sobolev space* $H^m(\mathbb{R})$ as

$$H^m(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \|f\|_{H^m} = \left(\int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^m \, d\xi \right)^{1/2} < \infty \right\}.$$

Lemma 4.2. *Suppose that w is a C^∞-function with compact support. If $g \in H^m(\mathbb{R})$, then the sequence $(\omega_k)_{k\in\mathbb{Z}}$, where $\omega_k = \|g(\cdot)w(\cdot - k)\|_{H^m}$, belongs to $\ell^2(\mathbb{Z})$.*

The idea of the proof of Lemma 4.2, in the case when m is an integer, is outlined in [14]. Since we need this result for noninteger m's we present its proof.

Proof. By a direct calculation,

$$(g(\cdot)w(\cdot - k))\hat{}\,(\xi) = (\hat{g}(\cdot) * e^{-2\pi i k \cdot} \hat{w}(\cdot))(\xi) = \int_{\mathbb{R}} e^{-2\pi i k \eta} \hat{g}(\xi - \eta)\hat{w}(\eta) \, d\eta.$$

By the Poisson summation formula and the Cauchy–Schwarz inequality we have, for any $N > 0$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \left| \int_{\mathbb{R}} e^{-2\pi i k \eta} \hat{g}(\xi - \eta)\hat{w}(\eta)\, d\eta \right|^2 \\
&= \sum_{k\in\mathbb{Z}} \left| \int_0^1 e^{-2\pi i k \eta} \sum_{l\in\mathbb{Z}} \hat{g}(\xi - \eta + l)\hat{w}(\eta - l)\, d\eta \right|^2 = \int_0^1 \left| \sum_{l\in\mathbb{Z}} \hat{g}(\xi - \eta + l)\hat{w}(\eta - l) \right|^2 d\eta \\
&\leq \int_0^1 \left(\sum_{l\in\mathbb{Z}} \frac{|\hat{g}(\xi - \eta + l)|^2}{(1 + |\eta - l|^2)^N} \right) \left(\sum_{l\in\mathbb{Z}} |\hat{w}(\eta - l)|^2 (1 + |\eta - l|^2)^N \right) d\eta \\
&\leq C \left(\sup_{\eta\in[0,1]} \sum_{l\in\mathbb{Z}} \frac{1}{1 + |\eta - l|^2} \right) \int_{\mathbb{R}} \frac{|\hat{g}(\xi - \eta)|^2}{(1 + |\eta|^2)^N}\, d\eta = C' \int_{\mathbb{R}} \frac{|\hat{g}(\xi - \eta)|^2}{(1 + |\eta|^2)^N}\, d\eta,
\end{aligned}$$

since $|\hat{w}(\eta)|^2 \leq C(1 + |\eta|^2)^{-N-1}$ for some $C = C(N) > 0$. Therefore, by a direct estimate, we have that, for $N > m + \frac{1}{2}$,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \|g(\cdot)w(\cdot - k)\|_{H^m}^2 &\leq C' \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(1 + |\xi|^2)^m |\hat{g}(\xi - \eta)|^2}{(1 + |\eta|^2)^N}\, d\eta\, d\xi \\
&= C' \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(1 + |\xi + \eta|^2)^m}{(1 + |\eta|^2)^N} d\eta\, |\hat{g}(\xi)|^2\, d\xi \\
&\leq C'' \int_{\mathbb{R}} (1 + |\xi|^2)^m |\hat{g}(\xi)|^2\, d\xi = C'' \|g\|_{H^m}^2. \qquad \blacksquare
\end{aligned}$$

Proof of a General Case of Theorem 4.1. To eliminate the additional assumption (4.3), it remains to show that the function $s(\xi)$, given by (4.7), is continuous everywhere, possibly except 0, i.e., the series $\sum_{l=1}^{L} \sum_{j=0}^{\infty} |\hat{\psi}^l(\tilde{a}^j \xi)|^2$ converges uniformly on compact intervals of $\mathbb{R}\backslash\{0\}$.

Let w be a C^{∞}-function with compact support such that $w(\xi) = 1$ for $\xi \in [0,1]$. By (4.2), we have $\hat{\psi}^l \in H^{(1+\delta)/2}(\mathbb{R})$ for $l = 1, \dots, L$. Hence, by Lemma 4.2 and by Sobolev's embedding theorem, $H^{(1+\delta)/2}(\mathbb{R}) \subset L^{\infty}(\mathbb{R})$,

$$\begin{aligned}(4.9)\quad \sum_{k\in\mathbb{Z}} \|\hat{\psi}^l \mathbf{1}_{[k,k+1]}\|_{\infty}^2 &\le \sum_{k\in\mathbb{Z}} \|\hat{\psi}^l(\cdot) w(\cdot - k)\|_{\infty}^2 \\ &\le C \sum_{k\in\mathbb{Z}} \|\hat{\psi}^l(\cdot) w(\cdot - k)\|_{H^{(1+\delta)/2}}^2 \le C' \|\hat{\psi}^l\|_{H^{(1+\delta)/2}}^2 < \infty.\end{aligned}$$

It now suffices to show that for any $R > 1$, $\sum_{j=0}^{\infty} |\hat{\psi}^l(\tilde{a}^j \xi)|^2$ converges uniformly on intervals $[-R, -1/R]$ and $[1/R, R]$. Indeed, for any $\xi \in [1/R, R]$ and integers J satisfying $\tilde{a}^J(\tilde{a} - 1) > R$, by (4.9),

$$\begin{aligned}\sum_{j=J}^{\infty} |\hat{\psi}^l(\tilde{a}^j \xi)|^2 &\le \sum_{j=J}^{\infty} \|\hat{\psi}^l \mathbf{1}_{[\lfloor \tilde{a}^j \xi \rfloor, \lfloor \tilde{a}^j \xi \rfloor + 1]}\|_{\infty}^2 \\ &\le \sum_{k \ge \lfloor \tilde{a}^J / R \rfloor} \|\hat{\psi}^l \mathbf{1}_{[k,k+1]}\|_{\infty}^2 \to 0 \qquad \text{as} \quad J \to \infty,\end{aligned}$$

since $(\lfloor \tilde{a}^j \xi \rfloor)_{j \ge J}$ is a strictly increasing sequence of positive integers for any $\xi \in [1/R, R]$, where $\lfloor x \rfloor$ denotes the greatest integer $\le x$. Likewise, we have uniform convergence on $[-R, -1/R]$. Since $R > 1$ is arbitrary this shows that $\sum_{l=1}^{L} \sum_{j=0}^{\infty} |\hat{\psi}^l(\tilde{a}^j \xi)|^2$ and, hence, $s(\xi)$ given by (4.7), is continuous on $\mathbb{R}\setminus\{0\}$. This completes the proof of a general case of Theorem 4.1. ■

We conclude this section with several remarks regarding the applicability of Theorem 4.1 to more general situations.

Remark 1. Note that the proof of Theorem 4.1 also works for all negative dilation factors a such that both a and a^2 are irrational. Indeed, in this case j_0, being the smallest integer $j \ge 1$ such that a^j is rational, is ≥ 3 and E, being either $(0, \infty)$ or $(-\infty, 0)$, is a contradiction with $|E \cap a^2 E| = 0$. Finally, suppose that $a < 0$ is irrational and that a^2 is rational. In this case, if Ψ is an orthonormal multiwavelet associated with a and satisfying (4.2), then Ψ is also an orthonormal multiwavelet associated with a^2 for either $\check{L}^2(0, \infty)$ or $\check{L}^2(-\infty, 0)$. Without loss of generality, we can assume that Ψ is a multiwavelet for $\mathbb{H}^2(\mathbb{R}) = \check{L}^2(0, \infty)$. Therefore, the existence of well-localized multiwavelets for this class of negative dilation factors is equivalent to the problem of the existence of well-localized multiwavelets for $\mathbb{H}^2(\mathbb{R})$ posed by Meyer in [14]. This problem was answered (negatively) by Auscher [2] for all integer dilation factors with the use of a wavelet dimension function. Recently, Speegle and this author [8] have shown that the same conclusion also holds for rational dilation factors. We also note that an easy modification of the proof of Theorem 4.1 shows that there are no well-localized multiwavelets for $\mathbb{H}^2(\mathbb{R})$ associated with irrational dilation factors. Therefore, the Hardy space $\mathbb{H}^2(\mathbb{R})$ does not admit wavelets with good time-frequency localization for any (positive) real dilation factor.

Remark 2. Using the ideas in the proof of Lemma 2.2 and Theorem 4.1 it is not hard to show that even biorthogonal multiwavelets cannot be well-localized in time for (positive) irrational dilation factors. Indeed, suppose that a is an "exceptional" irrational dilation factor, i.e., $a \in \mathbb{R}\backslash\mathbb{Q}$ is such that $a^{j_0} \in \mathbb{Q}$ for some $j_0 \geq 2$; the other case is covered in [4]. If $\Psi = \{\psi^1, \ldots, \psi^L\}$, $\Phi = \{\varphi^1, \ldots, \varphi^L\}$ is a biorthogonal multiwavelet associated with a then, necessarily,

$$E = \bigcup_{j \in j_0\mathbb{Z}} a^j \left(\bigcup_{l=1}^{L} \operatorname{supp} \hat{\psi}^l \right) = \bigcup_{j \in j_0\mathbb{Z}} a^j \left(\bigcup_{l=1}^{L} \operatorname{supp} \hat{\varphi}^l \right),$$

by a biorthogonal variant of Lemma 2.2. Furthermore, $E, aE, \ldots, a^{j_0-1}E$ are pairwise disjoint (modulo sets of measure zero). The rest of the proof is the same as before with the exception that we consider

$$s(\xi) = \sum_{l=1}^{L} \sum_{j=-\infty}^{\infty} \hat{\varphi}^l(\tilde{a}^j\xi)\overline{\hat{\psi}^l(\tilde{a}^j\xi)},$$

where $\tilde{a} = a^{j_0}$, and use a biorthogonal variant of Corollary 3.3.

Remark 3. The conclusion (4.1) in Theorem 4.1 is optimal in the sense that it may fail when $\delta = 0$. Indeed, for any real dilation factor a, Speegle has constructed an MSF wavelet ψ associated with a such that the support of $\hat{\psi}$ consists of at most three intervals, see the example below. Therefore, such a wavelet ψ satisfies $|\psi(x)| = O(|x|^{-1})$ as $|x| \to \infty$.

Example (Speegle). Suppose that a dilation factor $a > 1$. Define the set

$$K = \left[-\frac{la(a^3-1)}{a^4-1}, -\frac{l(a^3-1)}{a^4-1} \right] \cup \left[\frac{1}{a^4-1}, \frac{l(a-1)}{a^4-1} \right] \cup \left[\frac{la^3(a-1)}{a^4-1}, \frac{a^4}{a^4-1} \right],$$

where $l = \lceil 1/(a-1) \rceil$ is the smallest integer $\geq 1/(a-1)$. An elementary calculation shows that K is a wavelet set, i.e.,

$$\sum_{j\in\mathbb{Z}} \mathbf{1}_K(a^j\xi) = 1 \qquad \text{and} \qquad \sum_{k\in\mathbb{Z}} \mathbf{1}_K(\xi + k) = 1 \qquad \text{for a.e.} \quad \xi \in \mathbb{R}. \tag{4.10}$$

Therefore $\psi \in L^2(\mathbb{R})$, given by $\hat{\psi} = \mathbf{1}_K$, is an MSF wavelet associated with the positive a that satisfies $|\psi(x)| \leq C(1 + |x|)^{-1}$ for some constant $C > 0$. Likewise, if a dilation factor $a < -1$, then $\tilde{K} = [a^2/(1-a^2), 1/(1-a^2)]$ satisfies (4.10), and hence ψ, given by $\hat{\psi} = \mathbf{1}_{\tilde{K}}$, is also an MSF wavelet associated with a.

Remark 4. Finally we note that Theorem 4.1 easily generalizes to several dimensions in the case of the isotropic dilations of the form $a\,\mathrm{Id}$. In this case, we say that $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$ is an orthonormal *multiwavelet* if the affine system

$$\psi^l_{j,k}(x) = |a|^{nj/2}\psi(a^jx - k) \qquad \text{for} \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^n, \quad l = 1, \ldots, L,$$

is an orthonormal basis for $L^2(\mathbb{R}^n)$. Naturally, condition (4.1) has to be replaced by

$$\limsup_{|x|\to\infty} |\psi^l(x)||x|^{n+\delta} = \infty.$$

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References

1. P. Auscher (1992): *Wavelet bases for $L^2(\mathbb{R})$ with rational dilation factor*. In: Wavelets and their Applications. Boston: Jones and Bartlett, pp. 439–451.
2. P. Auscher (1995): *Solution of two problems on wavelets*. J. Geom. Anal., **5**:181–236.
3. C. de Boor, R. A. DeVore, A. Ron (1994): *The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$*. J. Funct. Anal., **119**:37–78.
4. M. Bownik (to appear): *Combined MSF multiwavelets*. J. Fourier Anal. Appl.
5. M. Bownik (2000): *The structure of shift invariant subspaces of $L^2(\mathbb{R}^n)$*. J. Funct. Anal., **177**:282–309.
6. M. Bownik (2001): *On characterizations of multiwavelets in $\mathbb{R}^n$*. Proc. Amer. Math. Soc., **129**:3265–3274.
7. M. Bownik (2000): *A characterization of affine dual frames in $\mathbb{R}^n$*. Appl. Comput. Harmonic Anal., **8**:203–221.
8. M. Bownik, D. Speegle (2002): *The wavelet dimension function for real dilations and dilations admitting non-MSF wavelets*. In: Approximation Theory X: Wavelets, Splines, and Applications. Nashville, TN: Vanderbilt University Press, pp. 63–85.
9. A. Calogero (2000): *A characterization of wavelets on general lattices*. J. Geom. Anal., **10**:597–622.
10. C. K. Chui, X. Shi (2000): *Orthonormal wavelets and tight frames with arbitrary real dilations*. Appl. Comput. Harmonic Anal., **9**:243–264.
11. C. K. Chui, X. Shi, J. Stöckler (1998): *Affine frames, quasi-affine frames, and their duals*. Adv. in Comput. Math., **8**:1–17.
12. I. Daubechies (1992): Ten Lectures on Wavelets. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM).
13. E. Hernández, G. Weiss (1996): *A first course on wavelets*. In: Studies in Advanced Mathematics. Boca Raton, FL: CRC Press.
14. Y. Meyer (1992): Wavelets and Operators. Cambridge: Cambridge University Press.
15. A. Ron, Z. Shen (1997): *Affine systems in $L_2(\mathbb{R}^d)$: The analysis of the analysis operator*. J. Funct. Anal., **148**:408–447.

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