Contemporary Mathematics
Volume **00**, 1997

The spectral function of shift-invariant spaces on general lattices

Marcin Bownik and Ziemowit Rzeszotnik

10/2/2003

Abstract. We extend the notion of the spectral function of shift-invariant spaces introduced by the authors in [BRz] to the case of general lattices. The main feature is that the spectral function is not dependent on the choice of the underlying lattice with respect to which a space is shift-invariant. We also show that in general the spectral function is not additive on the orthogonal infinite sums of SI spaces with varying lattices.

1. Introduction

The shift-invariant (SI) spaces are closed subspaces of $L^2(\mathbb{R}^n)$ that are invariant under all shifts, i.e., integer translations. The theory of shift-invariant subspaces of $L^2(\mathbb{R}^n)$ plays an important role in many areas, most notably in the theory of wavelets, spline systems, Gabor systems, and approximation theory [BMM, BDR1, BDR2, BL, Bo1, BRz, Ji, RS1, RS2].

Given a SI space $V \subset L^2(\mathbb{R}^n)$, we would like to associate to V some kind of a function on $\mathbb{R}^n$ which encapsulates the most important properties of V, such as the "size" of V. One of such possible functions is the dimension function (or multiplicity function) of V, which measures the size of V by counting the dimensions of "fibers" of V. Another such possible function, which contains much more information than the dimension function, is the spectral function measuring "localized size" of V.

The goal of this work is to show the existence and the fundamental properties of the spectral function associated to SI spaces on general lattices. This extends the

2000 *Mathematics Subject Classification.* 42C40.

Key words and phrases. shift-invariant spaces, spectral function, dimension function, orthogonal wavelets.

The first author was partially supported by the NSF grant DMS-0200080.

spectral function for SI spaces with respect to the standard lattice $\mathbb{Z}^n$ introduced by the authors in [BRz]. Quite likely the most striking property of the spectral function shown in this work is independence of its definition with respect to the choice of the underlying lattice. This property is a unique feature of the spectral function, which is not shared by the dimension function.

The spectral function is not only interesting in itself, but it is also useful in studying wavelet and Gabor systems. Several applications of the spectral function were presented in [BRz]. For example, the spectral function can be employed to characterize approximation order of shift-invariant spaces [BRz, Section 2], dimension functions of refinable spaces and generalized multiresolution analyses [BRz, Section 3 and 4], and to provide a proof of Rieffel's incompleteness theorem for Gabor systems [BRz, Section 5]. In this work we concentrate on the relevance of the spectral function to the study of general shift-invariant systems with varying lattice of translations. Such systems were recently studied by Hernández, Labate, and Weiss [HLW].

In order to define the spectral function we need to recall a few basic facts about shift-invariant spaces.

Suppose $\Gamma \subset \mathbb{R}^n$ is a lattice of full rank, i.e., $\Gamma = P\mathbb{Z}^n$ for some $n \times n$ non-singular real matrix P. We say that a closed subspace $V \subset L^2(\mathbb{R}^n)$ is Γ-*shift-invariant* (Γ-SI) if for every function $f \in V$ we also have $T_\gamma f \in V$ when $\gamma \in \Gamma$, where $T_y f(x) = f(x-y)$ is the translation by a vector $y \in \mathbb{R}^n$. For any subset $\Phi \subset L^2(\mathbb{R}^n)$ let

$$\mathcal{S}^\Gamma(\Phi) = \overline{\operatorname{span}}\{T_\gamma \varphi : \varphi \in \Phi, \gamma \in \Gamma\}$$

be the Γ-SI space generated by Φ. A Γ-*principal shift-invariant* (Γ-PSI) space is a Γ-SI space V generated by a single function $\varphi \in L^2(\mathbb{R}^n)$, i.e., $V = \mathcal{S}^\Gamma(\{\varphi\}) = \mathcal{S}^\Gamma(\varphi)$.

A *range function* is any mapping

$$J : \mathbb{R}^n \to \{\text{closed subspaces of } \ell^2(\Gamma^*)\},$$

satisfying a consistency formula with respect to the lattice Γ^*,

$$J(\xi + \gamma) = S_\gamma(J(\xi)) \qquad \text{for all } \gamma \in \Gamma^*.$$

Here,

$$\Gamma^* = \{x \in \mathbb{R}^n : \langle x, k\rangle \in \mathbb{Z} \quad \text{for all } k \in \Gamma\}$$

is the dual lattice and $S_\gamma : \ell^2(\Gamma^*) \to \ell^2(\Gamma^*)$ is the shift operator given by

$$S_\gamma((a_k)_{k\in\Gamma^*}) = (a_{k-\gamma})_{k\in\Gamma^*}.$$

We say that J is *measurable* if the associated orthogonal projections $P_J(\xi)$ of $\ell^2(\Gamma^*)$ onto $J(\xi)$ are operator measurable, i.e., $\xi \mapsto P_J(\xi)v$ is measurable for any $v \in \ell^2(\Gamma^*)$. We note that the range function is uniquely determined by its values on the representatives of the cosets of $\mathbb{R}^n/\Gamma^*$. Therefore, it suffices to define the range function only on some fundamental domain of $\mathbb{R}^n/\Gamma^*$ and then extend it using the above consistency formula.

Define the Hilbert space

$$L^2(\mathbb{R}^n/\Gamma^*, \ell^2(\Gamma^*)) = \{\Phi : \mathbb{R}^n \to \ell^2(\Gamma^*) : \quad \Phi(\xi+\gamma) = S_\gamma \Phi(\xi) \text{ for all } \gamma \in \Gamma^*, \\ \text{and } ||\Phi||^2 := \int_{\mathbb{R}^n/\Gamma^*} ||\Phi(\xi)||^2_{\ell^2(\Gamma^*)} d\xi < \infty\}.$$

Let $\mathcal{T} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n, \ell^2(\Gamma^*))$ be an isometric isomorphism defined for $f \in L^2(\mathbb{R}^n)$ by

$$\mathcal{T}f : \mathbb{R}^n \to \ell^2(\Gamma^*), \qquad \mathcal{T}f(\xi) = (\hat{f}(\xi+\gamma))_{\gamma\in\Gamma^*},$$

where $\hat{f}(\xi) = \int f(x)e^{-2\pi i\langle x,\xi\rangle}\, dx$.

The following proposition, due to Helson [He, Theorem 8, p. 59], plays an important role in the theory of general SI spaces in $L^2(\mathbb{R}^n)$. A proof of Proposition 1.1 for $\mathbb{Z}^n$-SI spaces can be also found in [Bo1, Proposition 1.5]. A general case of Γ-SI is an immediate consequence of $\mathbb{Z}^n$-SI case.

Proposition 1.1. *A closed subspace $V \subset L^2(\mathbb{R}^n)$ is Γ-SI if and only if*

$$V = \{f \in L^2(\mathbb{R}^n) : \mathcal{T}f(\xi) \in J(\xi) \text{ for a.e. } \xi \in \mathbb{R}^n\}, \tag{1.1}$$

where J is a measurable range function. The correspondence between V and J is one-to-one under the convention that the range functions are identified if they are equal a.e. Furthermore, if $V = \mathcal{S}^\Gamma(\Phi)$ for some countable $\Phi \subset L^2(\mathbb{R}^n)$, then

$$J(\xi) = \overline{\operatorname{span}}\{\mathcal{T}\varphi(\xi) : \varphi \in \Phi\}. \tag{1.2}$$

The *dimension function* of a Γ-SI space $V \subset L^2(\mathbb{R}^n)$ is a mapping $\dim_V^\Gamma : \mathbb{R}^n \to \mathbb{N} \cup \{0,\infty\}$ given by $\dim_V^\Gamma(\xi) = \dim J(\xi)$, where J is the range function corresponding to V. Alternatively, the dimension function of V can be introduced as the *multiplicity function* of the projection-valued measure coming from the representation of the lattice Γ on V via translations by Stone's Theorem, see [Ba, BMM].

Note also that $V = \check{L}^2(E)$ is Γ-SI for any lattice Γ, where E is a measurable subset of $\mathbb{R}^n$ and

$$\check{L}^2(E) = \{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \hat{f} \subset E\}, \tag{1.3}$$

moreover its dimension function is given by

$$\dim_V^\Gamma(\xi) = \sum_{k\in\Gamma^*} \mathbf{1}_E(\xi+k). \tag{1.4}$$

2. The spectral function

In this section we show the basic properties of the spectral function associated to general shift-invariant spaces with respect to an arbitrary lattice $\Gamma \subset \mathbb{R}^n$. The spectral function for shift-invariant spaces with respect to the standard lattice $\mathbb{Z}^n$ was introduced and investigated by the authors in [BRz].

The spectral function for $\mathbb{Z}^n$-SI spaces.

An interesting feature of the spectral function for $\mathbb{Z}^n$-SI spaces is that it can be defined in several equivalent ways. Indeed, it was shown in [BRz] that the spectral function of $\mathbb{Z}^n$-SI space V can be defined using either of the following methods:

(a) orthogonal projections of the unit standard vectors of $\ell^2(\mathbb{Z}^n)$ onto the range function [BRz, Definition 2.1],

(b) decompositions of V into the orthogonal sum of PSI spaces [BRz, Proposition 2.2],

(c) generators of SI systems forming tight frame (with constant 1) for V [BRz, Lemma 2.3],

(d) diagonal terms of dual Gramians for SI systems as in (c) [BRz, Remark after Lemma 2.3],

(e) density formula based on the Lebesgue differentiation theorem [BRz, formula (2.15)].

More precisely, the following proposition can serve as the definition of the spectral function.

Proposition 2.1. *Suppose $V \subset L^2(\mathbb{R}^n)$ is $\mathbb{Z}^n$-SI. Then for a.e. $\xi \in \mathbb{R}^n$ the following expressions are equal:*

$$\text{(i)} \qquad ||P_J(\xi)e_0||^2,$$

where $P_J(\xi)$ is the orthogonal projection onto the range function $J(\xi)$ corresponding to V and given by Proposition 1.1,

$$\text{(ii)} \qquad \sum_{\varphi\in\Phi} |\hat{\varphi}(\xi)|^2,$$

where $\Phi \subset V$ is such that its translates by $\mathbb{Z}^n$,

$$E^{\mathbb{Z}^n}(\Phi) = \{T_k\varphi : k \in \mathbb{Z}^n, \varphi \in \Phi\},$$

form a tight frame with constant 1 for V; moreover, this sum is independent of the choice of such Φ,

$$\text{(iii)} \qquad \lim_{r\to 0^+} \frac{||P_V(\check{\mathbf{1}}_{B(\xi,r)})||^2}{|B(\xi,r)|},$$

where P_V is the orthogonal projection of $L^2(\mathbb{R}^n)$ onto V and $B(\xi, r)$ denotes the ball with center ξ and radius r.

The spectral function of V is defined as mapping $\sigma_V : \mathbb{R}^n \to \mathbb{R}$, where $\sigma_V(\xi)$ is given by either one of (i)–(iii).

The next proposition lists the most important properties of the spectral function, see [BRz, Proposition 2.6]. In particular, Proposition 2.2 shows that the spectral function behaves nicely with respect to the action of modulations and dilations. This is relevant in the study of Gabor systems and wavelets. Recall, that the *modulation* by a vector $a \in \mathbb{R}^n$ of $f \in L^2(\mathbb{R}^n)$ is given by

$$M_a(f)(x) = e^{2\pi i \langle a,x \rangle} f(x).$$

The *dilation* by $n \times n$ non-singular matrix A of $f \in L^2(\mathbb{R}^n)$ is given by

$$D_A f(x) = |\det A|^{\frac{1}{2}} f(Ax).$$

In Proposition 2.2(g), we restrict our attention to dilations A preserving the lattice $\mathbb{Z}^n$ because this is exactly when in general we can expect that $D_A V$ is $\mathbb{Z}^n$-SI if V is $\mathbb{Z}^n$-SI. Later we will see that this property holds for a general non-singular matrix A.

PROPOSITION 2.2. *Let $\mathfrak{S}$ be the collection of all $\mathbb{Z}^n$-SI subspaces of $L^2(\mathbb{R}^n)$. The spectral function satisfies the following properties: ($V, W \in \mathfrak{S}$)*
(a) $\sigma_V : \mathbb{R}^n \to [0, 1]$ is a measurable function,
(b) $V = \bigoplus_{i\in\mathbb{N}} V_i$, where $V_i \in \mathfrak{S} \implies \sigma_V(\xi) = \sum_{i\in\mathbb{N}} \sigma_{V_i}(\xi)$,
(c) $V \subset W \implies \sigma_V(\xi) \le \sigma_W(\xi)$,
(d) $V \subset W \implies (V = W \iff \sigma_V(\xi) = \sigma_W(\xi))$,
(e) $\sigma_V(\xi) = \mathbf{1}_E(\xi) \iff V = \check{L}^2(E)$,
(f) $\sigma_{M_a(V)}(\xi) = \sigma(\xi - a)$, where M_a is a modulation by $a \in \mathbb{R}^n$,
(g) $\sigma_{D_A V}(\xi) = \sigma_V((A^*)^{-1}\xi)$, where D_A is a dilation by non-singular integer matrix A,
(h) $\dim_V^{\mathbb{Z}^n}(\xi) = \sum_{k\in\mathbb{Z}^n} \sigma_V(\xi + k)$.

Finally, the following lemma will be very useful in our considerations, see [BRz, Lemma 2.8].

LEMMA 2.3. *Suppose V is $\mathbb{Z}^n$-SI and $K \subset \mathbb{R}^n$ is a measurable set such that $|K \cap (l + K)| = 0$ for all $l \in \mathbb{Z}^n \setminus \{0\}$. Then*

$$||P_V(\check{\mathbf{1}}_K)||^2 = \int_K \sigma_V(\xi) d\xi, \tag{2.1}$$

where P_V is an orthogonal projection of $L^2(\mathbb{R}^n)$ onto V.

The spectral function for general SI spaces.

Using Propositions 2.1 and 2.2 it is not hard to give a definition of the spectral function σ_V^Γ for a general lattice Γ. Possibly the slickest way of defining the spectral function of a Γ-SI space V, where $\Gamma = P\mathbb{Z}^n$ for some $n \times n$ non-singular matrix P is suggested by Proposition 2.2(g). In fact, we could set

$$\sigma_V^\Gamma(\xi) = \sigma_{D_P V}(P^*\xi) \qquad \text{for } \xi \in \mathbb{R}^n,$$

because $D_P V$ is $\mathbb{Z}^n$-SI and therefore has the usual spectral function. One can show that this definition does not depend on the choice of a matrix P such that $\Gamma = P\mathbb{Z}^n$. Indeed, given two matrices P_1 and P_2 with $\Gamma = P_1\mathbb{Z}^n = P_2\mathbb{Z}^n$ we can verify using Proposition 2.2(g) that

$$\sigma_{D_{P_2}V}(\xi) = \sigma_{D_{P_1^{-1}P_2}D_{P_1}V}(\xi) = \sigma_{D_{P_1}V}((((P_1^{-1}P_2)^*)^{-1}\xi) = \sigma_{D_{P_1}V}(P_1^*(P_2^*)^{-1}\xi),$$

since $P_1^{-1}P_2$ has integer entries and $D_{P_1}V$ is $\mathbb{Z}^n$-SI.

However, for the purposes of a systematic development of the spectral function it is better to use the following formal definition (we will see later in (2.5) that it coincides with the above formulation).

Definition 2.4. Suppose $V \subset L^2(\mathbb{R}^n)$ is Γ-SI with the range function $J(\xi)$ and the corresponding projection $P_J(\xi)$. The *spectral function* of V is a measurable mapping $\sigma_V^\Gamma : \mathbb{R}^n \to [0,1]$ given by

$$\sigma_V^\Gamma(\xi) = ||P_J(\xi)e_0||^2 \qquad \text{for } \xi \in \mathbb{R}^n, \tag{2.2}$$

where $\{e_k\}_{k\in\Gamma^*}$ denotes the standard basis of $\ell^2(\Gamma^*)$.

Note that there is a very simple relationship between the spectral and the dimension function

$$\dim_V^\Gamma(\xi) = \sum_{k\in\Gamma^*} \sigma_V^\Gamma(\xi + k). \tag{2.3}$$

Indeed,

$$\sum_{k\in\Gamma^*} \sigma_V^\Gamma(\xi+k) = \sum_{k\in\Gamma^*} ||P_J(\xi+k)e_0||^2 = \sum_{k\in\Gamma^*} ||P_J(\xi)e_k||^2 = \dim \mathrm{Ran}(P_J(\xi))$$
$$= \dim_V^\Gamma(\xi).$$

Since the above definition is rather abstract we can give an alternative description of the spectral function of V in terms of a family generating a tight frame for V.

LEMMA 2.5. *Suppose $V \subset L^2(\mathbb{R}^n)$ is Γ-SI space and $\Phi \subset V$ is a countable family such that $E^\Gamma(\Phi) = \{T_k\varphi : k \in \Gamma, \varphi \in \Phi\}$ forms a tight frame with constant 1 for the space V. Then*

$$\sigma_V^\Gamma(\xi) = \frac{1}{|\mathbb{R}^n/\Gamma|} \sum_{\varphi\in\Phi} |\hat{\varphi}(\xi)|^2, \tag{2.4}$$

where $|\mathbb{R}^n/\Gamma|$ is the Lebesgue measure of the fundamental domain of $\mathbb{R}^n/\Gamma$, i.e., $|\mathbb{R}^n/\Gamma| = |\det P|$, if $\Gamma = P\mathbb{Z}^n$ for some $n \times n$ nonsingular matrix P.

In particular, (2.4) does not depend on the choice of Φ as long as $E^\Gamma(\Phi)$ is a tight frame with constant 1 for V.

PROOF. Let $\Gamma = P\mathbb{Z}^n$ for some $n \times n$ nonsingular matrix P. As we noticed before, D_PV is $\mathbb{Z}^n$-SI. We claim that

$$\sigma_V^\Gamma(\xi) = \sigma_{D_PV}^{\mathbb{Z}^n}(P^*\xi) \qquad \text{a.e. } \xi \in \mathbb{R}^n. \tag{2.5}$$

Indeed, let $I : \ell^2(\Gamma^*) \to \ell^2(\mathbb{Z}^n)$ be an isometric isomorphism given by $I((a_k)_{k\in\Gamma^*}) = (a_{P^*k})_{k\in\mathbb{Z}^n}$. Let $J(\xi)$ be the range function of Γ-SI space V. Then using Proposition 1.1, it is easy to verify that $\tilde{J}(\xi) = I(J((P^*)^{-1}\xi))$ is the range function of $\mathbb{Z}^n$-SI space D_PV, which shows the claim.

It is also clear that $E^\Gamma(\Phi)$ is a tight frame with constant 1 for V if and only if $E^{\mathbb{Z}^n}(D_P\Phi)$ is a tight frame with constant 1 for D_PV. Since Lemma 2.5 is valid for the standard lattice $\mathbb{Z}^n$, hence

$$\sigma_V^\Gamma(\xi) = \sigma_{D_PV}^{\mathbb{Z}^n}(P^*\xi) = \sum_{\phi\in D_P\Phi} |\hat{\phi}(P^*\xi)|^2 = \frac{1}{|\det P|} \sum_{\varphi\in\Phi} |\hat{\varphi}(\xi)|^2,$$

which completes the proof of Lemma 2.5.

THEOREM 2.6. *Suppose $V \subset L^2(\mathbb{R}^n)$ is a Γ-SI space for some lattice Γ. Then*

$$\sigma_V^\Gamma(\xi) = \lim_{r\to 0^+} \frac{||P_V(\check{\mathbf{1}}_{B(\xi,r)})||^2}{|B(\xi,r)|} \qquad \textit{for a.e. } \xi \in \mathbb{R}^n. \tag{2.6}$$

PROOF. Suppose that $K \subset \mathbb{R}^n$ is a measurable set such that $|K \cap (l + K)| = 0$ for all $l \in \Gamma^* \setminus \{0\}$. Then combining Lemma 2.3 and (2.5),

$$\begin{aligned} ||P_V(\check{\mathbf{1}}_K)||^2 &= ||P_{D_PV}(D_P(\check{\mathbf{1}}_K))||^2 = \frac{1}{|\det P|}||P_{D_PV}(\check{\mathbf{1}}_{P^*K})||^2 \\ &= \frac{1}{|\det P|}\int_{P^*K} \sigma_{D_PV}^{\mathbb{Z}^n}(\xi)d\xi = \int_K \sigma_V^\Gamma(\xi)d\xi. \end{aligned}$$

Applying the Lebesgue Differentiation Theorem yields (2.6).

Remark. Theorem 2.6 is the key result of the paper and has interesting implications. It suggests that for any closed subspace $V \subset L^2(\mathbb{R}^n)$ we can consider

$$\sigma_V^+(\xi) = \limsup_{r\to 0^+} \frac{||P_V(\check{\mathbf{1}}_{B(\xi,r)})||^2}{|B(\xi,r)|}, \qquad \text{and} \qquad \sigma_V^-(\xi) = \liminf_{r\to 0^+} \frac{||P_V(\check{\mathbf{1}}_{B(\xi,r)})||^2}{|B(\xi,r)|}$$

Of course both these functions are zero if V is "small", e.g., V has a finite dimension. Therefore, σ_V^+ and σ_V^- are going to be interesting only if V is "big" in some sense, for example in the case when V is a Γ-SI space. In this case we can also see that $\sigma_V^+ = \sigma_V^-$.

A surprising consequence of Theorem 2.6 is the following

Corollary 2.7. *Suppose $V \subset L^2(\mathbb{R}^n)$ is both Γ_1-SI and Γ_2-SI space for two lattices Γ_1 and Γ_2. Then,*

$$\sigma_V^{\Gamma_1}(\xi) = \sigma_V^{\Gamma_2}(\xi) \qquad \textit{for a.e. } \xi \in \mathbb{R}^n.$$

Consequently, the spectral function is independent of the underlying lattice.

Corollary 2.7 shows a huge advantage of the spectral function over the dimension function. Namely, in order to talk meaningfully about the dimension function of a certain SI subspace V of $L^2(\mathbb{R}^n)$, one needs to specify a lattice Γ with respect to which V is SI. However, to talk about the spectral function of V, one doesn't need to specify the lattice with respect to which V is SI, since the resulting $\sigma_V(\xi)$ will be the same in each case. This suggests that $\sigma_V(\xi)$ is even more inherent notion of "size" of a SI space than the dimension function. It is also a more subtle notion, since one can always easily recover the dimension function from the spectral function using

$$\dim_V^\Gamma(\xi) = \sum_{k\in\Gamma^*} \sigma_V(\xi+k).$$

The following example illustrates the basic difference between dimension and spectral functions.

Example. Let $V = \mathcal{S}^{\mathbb{Z}}(\varphi)$, where φ is a function in $L^2(\mathbb{R})$ whose integer shifts are orthonormal. Then $\sigma_V(\xi) = |\hat\varphi(\xi)|^2$ and $\dim_V(\xi) = 1$. If we consider the standard "dilation by 2" operator $Df(x) = \sqrt{2}f(2x)$, then the space DV is $\mathbb{Z}$-SI and also $\frac{1}{2}\mathbb{Z}$-SI. As we can see, $\sigma_{DV}^\Gamma(\xi) = |\hat\varphi(\frac{\xi}{2})|^2$ if Γ is $\mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}$, while $\dim_{DV}^\Gamma(\xi) = 2$ if $\Gamma = \mathbb{Z}$ and $\dim_{DV}^\Gamma(\xi) = 1$ if $\Gamma = \frac{1}{2}\mathbb{Z}$.

If for a given closed subspace $V \subset L^2(\mathbb{R}^n)$ the functions σ_V^+ and σ_V^- coincide then we can define $\sigma_V = \sigma_V^+ = \sigma_V^-$. This function satisfies some obvious properties like additivity on orthogonal sums and $0 \le \sigma_V \le 1$. If we restrict our attention to general SI spaces we obtain the following analogue of Proposition 2.2

PROPOSITION 2.8. *Let $\mathfrak{S} = \mathfrak{S}_{\text{all}}$ be the collection of all possible SI subspaces of $L^2(\mathbb{R}^n)$, i.e., $V \in \mathfrak{S}$ if and only if there exists a lattice Γ such that V is Γ-SI. The spectral function satisfies the following properties: ($V, W \in \mathfrak{S}$)*

(a) $\sigma_V : \mathbb{R}^n \to [0,1]$ *is a measurable function,*

(b) $V = \bigoplus_{i=1}^N V_i$, *where* $V_i \in \mathfrak{S} \implies \sigma_V(\xi) = \sum_{i=1}^N \sigma_{V_i}(\xi)$,

(c) $V = \bigoplus_{i\in\mathbb{N}} V_i$, *where* V_i *is* Γ-*SI for a fixed lattice* $\Gamma \implies \sigma_V(\xi) = \sum_{i\in\mathbb{N}} \sigma_{V_i}(\xi)$,

(d) $V \subset W \implies \sigma_V(\xi) \le \sigma_W(\xi)$,

(e) $V \subset W \implies (V = W \iff \sigma_V(\xi) = \sigma_W(\xi))$,

(f) $\sigma_V(\xi) = \mathbf{1}_E(\xi) \iff V = \check{L}^2(E)$,

(g) $\sigma_{M_a(V)}(\xi) = \sigma(\xi - a)$, *where* M_a *is a modulation by* $a \in \mathbb{R}^n$,

(h) $\sigma_{D_A V}(\xi) = \sigma_V((A^*)^{-1}\xi)$, *where* D_A *is a dilation by a non-singular* **real** *matrix* A,

(i) if V *is* Γ-*SI then* $\dim_V^\Gamma(\xi) = \sum_{k\in\Gamma^*} \sigma_V(\xi + k)$.

PROOF. The proof of Proposition 2.8 is a routine. Part (a) is immediate by Definition 2.4. Parts (b), (d) are a consequence of Theorem 2.6. Parts (c), (f), (g), and (i) follow from the corresponding properties in Proposition 2.2 and (2.5) by considering $\mathbb{Z}^n$-SI space $D_P V$.

It is not hard to see that (2.5) implies (h) for V being $\mathbb{Z}^n$-SI. If V is Γ-SI, where $\Gamma = P\mathbb{Z}^n$, it suffices to use this observation together with

$$\sigma_{D_A V}(\xi) = \sigma_{D_{P^{-1}A} D_P V}(\xi) = \sigma_{D_P V}(P^*(A^*)^{-1}\xi) = \sigma_V((A^*)^{-1}\xi).$$

Finally, to show the least trivial property (e), assume that V is Γ_1-SI, W is Γ_2-SI, and $V \subset W$. If $V = W$ then $\sigma_V = \sigma_W$ by Theorem 2.6. To show the converse, suppose that $\sigma_V = \sigma_W$. Let V' be the smallest Γ_2-SI space containing V, i.e., $V' = \mathcal{S}^{\Gamma_2}(V)$. Clearly $V \subset V' \subset W$ thus by (d) we must have $\sigma_{V'} = \sigma_W$. Since V' and W are both Γ_2-SI, we have $V' = W$ by Proposition 2.2 (d). On the other hand, V' is also Γ_1-SI (the space is spanned by Γ_2 translations of Γ_1-SI space and translations commute) and $\sigma_V = \sigma_{V'}$. Hence, $V = V'$ again by Proposition 2.2 (d). This shows $V = W$ and completes the proof of Proposition 2.8.

We remark that unlike Proposition 2.2, Proposition 2.8(b) does not hold in general for **countable** collection of V_i's such that all V_i's are not necessarily SI with respect to some common lattice Γ. This will be illustrated by examples in Section 3. In general, we can only expect that if $V = \bigoplus_{i=1}^\infty V_i$, where $V_i \in \mathfrak{S}$ and $\mathfrak{S}$ is the same as in Proposition 2.8, then we always have

$$\sigma_V(\xi) \ge \sum_{i=1}^\infty \sigma_{V_i}(\xi) \qquad \text{a.e. } \xi \in \mathbb{R}^n.$$

Indeed, this follows easily from Proposition 2.8 (b) and (d).

We would like to point out that in some interesting cases Proposition 2.8(b) does hold for infinite sums. For example, if $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$ is an orthogonal wavelet with respect to an expansive dilation A then the Calderón condition

$$\sum_{j\in\mathbb{Z}} \sum_{\psi\in\Psi} |\hat{\psi}((A^*)^j \xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n \tag{2.7}$$

holds. Recall that $n \times n$ matrix A is expansive if all eigenvalues λ of A satisfy $|\lambda| > 1$. Condition (2.7), originally discovered by Meyer, was investigated by many authors as one of the equations characterizing wavelets. That it is equivalent to the completeness of the corresponding orthogonal affine system was proven in [Bo2, Bo3, Rz]. Recently it was shown that (2.7) holds not only for expansive dilations (with real entries), but also for wider classes of dilations expanding on subspaces, see [HLW, La]. Since the Calderón condition (2.7) can be written as

$$\sum_{j\in\mathbb{Z}} \sigma_{D_{A^j}(\mathcal{S}(\Psi))} = \sigma_{L^2(\mathbb{R}^n)}$$

and $\bigoplus_{j\in\mathbb{Z}} D_{A^j}(\mathcal{S}(\Psi)) = L^2(\mathbb{R}^n)$ we recognize it as Proposition 2.8(b) holding for this infinite sum.

3. Examples

In this section we show that the spectral function is not additive on infinite orthogonal sums of SI spaces with respect to varying lattices.

Example 3.1. Suppose that $\varphi \in L^2(\mathbb{R})$ and $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal sequence.

Let $N \geq 3$ be a fixed integer. Define the sequence of nested lattices $\Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \dots$, where

$$\Gamma_i = N^i \mathbb{Z} \qquad \text{for } i = 0, 1, 2, \dots \tag{3.1}$$

It is not hard to see that there exists a sequence $\{t_i\}_{i=1}^\infty \subset \mathbb{Z}$ such that

$$\begin{gathered} \bigcup_{i=1}^{\infty} (t_i + \Gamma_i) = \mathbb{Z} \\ (t_i + \Gamma_i) \cap (t_j + \Gamma_j) = \emptyset \qquad \text{for } i \neq j. \end{gathered} \tag{3.2}$$

Indeed, one can define $\{t_i\}$ by induction as follows. Let $t_1 = 0$. Once $t_1, \dots, t_i$ are defined, let t_{i+1} be an integer t with the minimal absolute value $|t|$ such that

$$(t + \Gamma_{i+1}) \cap \left(\bigcup_{j=1}^{i} (t_j + \Gamma_j) \right) = \emptyset.$$

Define the sequence of spaces $\{V_i\}$ by

$$V_i = \mathcal{S}^{\Gamma_i}(T_{t_i}\varphi) = \overline{\operatorname{span}}\{\varphi(x - t_i - k) : k \in \Gamma_i\}.$$

Obviously, V_i is Γ_i-SI for $i = 1, 2, \dots$. Moreover, by (3.2),

$$\bigoplus_{i=1}^{\infty} V_i = \mathcal{S}^{\mathbb{Z}}(\varphi). \tag{3.3}$$

Since

$$\sigma_{V_i}(\xi) = \frac{1}{N^i}|\hat{\varphi}(\xi)|^2 = \frac{1}{N^i}\sigma_{\mathcal{S}^{\mathbb{Z}}(\varphi)}(\xi),$$

hence

$$\sum_{i=1}^{\infty} \sigma_{V_i}(\xi) = \frac{1}{N-1}\sigma_{\mathcal{S}^{\mathbb{Z}}(\varphi)}(\xi). \tag{3.4}$$

Therefore, for $N \geq 3$,

$$\sum_{i=1}^{\infty} \sigma_{V_i}(\xi) \neq \sigma_{\mathcal{S}^{\mathbb{Z}}(\varphi)}(\xi),$$

despite (3.3) and the fact that V_i is Γ_i-SI for $i = 1, 2, \dots$.

The next example is a refinement of Example 3.1.

Example 3.2. Define the sequence of functions $\{\varphi_k\}_{k\in\mathbb{Z}}$ by $\hat{\varphi}_k = \mathbf{1}_{[k,k+1]}$. Let

$$V_i^k = \mathcal{S}^{\Gamma_i}(T_{t_i}\varphi_k) \qquad \text{for } k \in \mathbb{Z},\ i = 1, 2, \dots,$$

where $\{\Gamma_i\}$ and $\{t_i\}$ are the same as in Example 3.1. Clearly,

$$\bigoplus_{k\in\mathbb{Z}} \bigoplus_{i=1}^{\infty} V_i^k = \bigoplus_{k\in\mathbb{Z}} \check{L}^2(k, k+1) = L^2(\mathbb{R}),$$

but

$$\sum_{k\in\mathbb{Z}} \sum_{i=1}^{\infty} \sigma_{V_i^k}(\xi) = \sum_{k\in\mathbb{Z}} \frac{1}{N-1}\mathbf{1}_{[k,k+1]}(\xi) = \frac{1}{N-1} \qquad \text{for a.e. } \xi \in \mathbb{R}. \tag{3.5}$$

Shift-invariant systems with varying lattices.

Example 3.2 shows difficulties one may encounter when trying to characterize shift-invariant systems with varying lattices which form orthonormal bases (or more generally Riesz bases or frames) for $L^2(\mathbb{R}^n)$. In one such study, Hernández, Labate, and Weiss, consider general shift-invariant systems of the form

$$\{T_{C_p k} g_p : k \in \mathbb{Z}^n,\ p \in P\}, \tag{3.6}$$

where $\{C_p\}_{p\in P}$ is a collection of non-singular $n\times n$ matrices and $\{g_p\}_{p\in P}$ is a collection of functions in $L^2(\mathbb{R}^n)$. In their main result [HLW, Theorem 2.1], the authors characterize when the above system (3.6) forms a tight frame for $L^2(\mathbb{R}^n)$ under a delicate technical assumption of the local integrability, see [HLW, (2.7)]. In particular, they show that if (3.6) forms a tight frame with constant 1 for $L^2(\mathbb{R}^n)$ then necessarily

$$\sum_{p\in P}\frac{1}{|\det C_p|}|\hat g_p(\xi)|^2=1 \qquad \text{a.e. } \xi\in\mathbb{R}^n. \tag{3.7}$$

Example 3.2 shows that the local integrability condition of [HLW] can not be removed. Indeed, by a simple change of indexing set, the system in Example 3.2 can be easily written in the form (3.6). Since $\sigma_{V_i^k}(\xi)=\frac{1}{N^i}|\hat\varphi_k(\xi)|^2$, (3.5) implies that

$$\sum_{p\in P}\frac{1}{|\det C_p|}|\hat g_p(\xi)|^2=\sum_{k\in\mathbb{Z}}\sum_{i=1}^{\infty}\frac{1}{N^i}|\hat\varphi_k(\xi)|^2=\frac{1}{N-1} \qquad \text{for a.e. } \xi\in\mathbb{R}.$$

This obviously contradicts (3.7) and shows that the local integrability condition is essential.

Orthogonal wavelets for non-expansive dilations.

One of the longer standing problems in the theory of wavelets is the following question. For which non-singular $n\times n$ matrices A, does there exist a wavelet $\Psi=\{\psi^1,\dots,\psi^L\}\subset L^2(\mathbb{R}^n)$ associated with A such that

$$\{D_{A^j}T_k\psi : j\in\mathbb{Z},\ k\in\mathbb{Z}^n,\ \psi\in\Psi\}$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$?

Only a few necessary and sufficient conditions for such A are known despite a very interesting progress obtained recently by Speegle [Sp]. Speegle's work suggests that an initial step toward answering this question might involve showing that the Calderón condition (2.7) must hold. At the present time, this is known to be true only for the class of dilations expanding on subspaces introduced by Hernández, Labate, and Weiss [HLW]. Whether (2.7) must hold for all other dilations is an open problem.

This in turn may be considered as a part of a bigger problem of understanding for which families of orthogonal SI spaces $\{W_i\}$ (with varying lattices) the spectral function is additive, i.e., $\sum\sigma_{W_i}=\sigma_{\oplus W_i}$. Indeed, in the special case of a wavelet Ψ and

$$W_i=\overline{\operatorname{span}}\{D_{A^i}T_k\psi : k\in\mathbb{Z}^n,\ \psi\in\Psi\},$$

the additivity of the spectral function is equivalent to (2.7). Example 3.2 merely suggests that the answer to the last problem must necessarily take into account the dilation structure of spaces $\{W_i\}$.

References

[Ba] L. W. Baggett, *An abstract interpretation of the wavelet dimension function using group representations*, J. Funct. Anal. **173** (2000), 1–20.

[BMM] L. W. Baggett, H. A. Medina, and K. D. Merrill, *Generalized multi-resolution analyses and a construction procedure for all wavelet sets in* $\mathbb{R}^n$, J. Fourier Anal. Appl. **5** (1999), 563–573.

[BDR1] C. de Boor, R. A. DeVore, and A. Ron, *The structure of finitely generated shift-invariant spaces in* $L_2(\mathbb{R}^d)$, J. Funct. Anal. **119** (1994), 37–78.

[BDR2] C. de Boor, R. A. DeVore, and A. Ron, *Approximation from shift-invariant subspaces of* $L_2(\mathbb{R}^d)$, Trans. Amer. Math. Soc. **341** (1994), 787–806.

[BL] J. J. Benedetto and S. Li, *The theory of multiresolution analysis frames and applications to filter banks*, Appl. Comput. Harmon. Anal. **5** (1998), 389–427.

[Bo1] M. Bownik, *The structure of shift-invariant subspaces of* $L^2(\mathbb{R}^n)$, J. Funct. Anal. **177** (2000), 282–309.

[Bo2] M. Bownik, *On characterizations of multiwavelets in* $L^2(\mathbb{R}^n)$, Proc. Amer. Math. Soc. **129** (2001), 3265–3274.

[Bo3] M. Bownik, *Quasi-affine systems and the Calderòn condition*, Harmonic Analysis at Mount Holyoke, Contemp. Math. **320** (2003), 29–43.

[BRz] M. Bownik and Z. Rzeszotnik, *The spectral function of shift-invariant systems*, Michigan Math. J. (to appear).

[BRS] M. Bownik, Z. Rzeszotnik, and D. Speegle, *A characterization of dimension functions of wavelets*, Appl. Comput. Harmon. Anal. **10** (2001), 71–92.

[Da] I. Daubechies, *Ten lectures on wavelets*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[He] H. Helson, *Lectures on invariant subspaces*, Academic Press, New York-London, 1964.

[HW] E. Hernández and G. Weiss, *A first course on wavelets*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1996.

[HWL] E. Hernández, D. Labate, and G. Weiss, *A unified characterization of reproducing systems generated by a finite family, II*, J. Geom. Anal. **12** (2002), 615–662.

[Ji] R.-Q. Jia, *Shift-invariant spaces and linear operator equations*, Israel J. Math. **103** (1998), 259–288.

[La] R. Laugesen, *Translational averaging for completeness, characterization and oversampling of wavelets*, Collect. Math. **53** (2002), 211–249.

[RS1] A. Ron and Z. Shen, *Frames and stable bases for shift-invariant subspaces of* $L_2(\mathbb{R}^d)$, Canad. J. Math. **47** (1995), 1051–1094.

[RS2] A. Ron and Z. Shen, *Affine systems in* $L_2(\mathbb{R}^d)$*: the analysis of the analysis operator*, J. Funct. Anal. **148** (1997), 408–447.

[Rz] Z. Rzeszotnik, *Calderón's condition and wavelets*, Collect. Math. **52** (2001), 181–191.

[Sp] D. Speegle, *Dilation and translation tilings of* $\mathbb{R}^n$ *for non-expansive dilations*, Collect. Math. (to appear).

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109–1109
E-mail address: marbow@umich.edu

Department of Mathematics, University of Texas, Austin, TX 78712–1082
E-mail address: zioma@mail.ma.utexas.edu