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Atomic and molecular decompositions of anisotropic Besov spaces

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Abstract. In this work we develop the theory of weighted anisotropic Besov spaces associated with general expansive matrix dilations and doubling measures with the use of discrete wavelet transforms. This study extends the isotropic Littlewood-Paley methods of dyadic φ-transforms of Frazier and Jawerth [19,21] to non-isotropic settings.

Several results of isotropic theory of Besov spaces are recovered for weighted anisotropic Besov spaces. We show that these spaces are characterized by the magnitude of the φ-transforms in appropriate sequence spaces. We also prove boundedness of an anisotropic analogue of the class of almost diagonal operators and we obtain atomic and molecular decompositions of weighted anisotropic Besov spaces, thus extending isotropic results of Frazier and Jawerth [21].

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1. Introduction and statements of main results

Many classical function or distribution spaces occurring in analysis have been shown to admit decompositions into simpler elementary building blocks, often called atoms or molecules. Decomposition techniques have become a very useful tool in the study of a large class of function spaces and operators acting on them, starting with now-classical atomic decomposition of the Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p \leq 1$ of Coifman [13] and including atomic and molecular decomposition results for Triebel-Lizorkin and Besov spaces of Frazier and Jawerth [19–22].

There are several directions of extending classical function spaces arising in harmonic analysis of Euclidean spaces to other domains and non-isotropic settings.

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For example, the usual isotropic dyadic dilations can be replaced by more complicated non-isotropic dilation structures as in the study of parabolic Hardy spaces of Calderón and Torchinsky [11,12] or Hardy spaces on homogeneous groups of Folland and Stein [18]. The non-isotropic variants of Triebel-Lizorkin and Besov spaces for diagonal dilations have been studied by Besov, Il'in, and Nikol'skiĭ [1], Schmeisser and Triebel [32,34,36], Dintelmann [15,16], and Farkas [17]. The other direction is the study of weighted function spaces associated with general Muckenhoupt A_∞ weights. This direction of research for Besov and Triebel-Lizorkin spaces was carried over by Bui, Paluszyński, and Taibleson [6,7,9,10] and by Rychkov [30,31]. Recently, Roudenko [29] has studied matrix-weighted Besov spaces defined using matrix analogues of Muckenhoupt A_p weights introduced by Nazarov, Treil, and Volberg [26,33].

The goal of this work is to show that several aspects of the above mentioned developments can be extended to a larger class (than previously considered diagonal setting) of non-isotropic dilation structures associated with expansive dilations. In the context of Hardy spaces this goal was achieved by the author in [3], where it was demonstrated that significant portion of a real-variable isotropic H^p theory extends to such anisotropic setting. In the context of Triebel-Lizorkin spaces, analogous extension to weighted anisotropic setting was performed by the author and Ho [4]. The goal of this work is to show that one can also build a coherent theory of weighted anisotropic Besov spaces associated with expansive dilations with the use of the discrete φ-transforms of Frazier and Jawerth. Our formulation includes the previously studied classes of Besov spaces that corresponded to diagonal dilations. In what follows we summarize the results obtained in this work.

In this work we introduce and study Besov spaces on $\mathbb{R}^n$ associated with an expansive dilation A, that is an $n \times n$ real matrix all of whose eigenvalues λ satisfy $|\lambda| > 1$. The starting point is a basic representation formula for tempered distributions

$$f = \sum_{Q \in \mathcal{Q}} \langle f, \varphi_Q \rangle \psi_Q \qquad \text{where } \mathcal{Q} = \{A^{-j}([0,1]^n + k) : j \in \mathbb{Z},\ k \in \mathbb{Z}^n\},$$

where $\mathcal{Q}$ is the collection of all dilated cubes adapted to the action of a dilation A, and φ_Q and ψ_Q are translates and dilates of appropriate functions φ and ψ localized to Q. More precisely, functions φ and ψ must satisfy support and Calderón conditions (2.6) and (2.7). In particular, $\hat{\varphi}$ and $\hat{\psi}$ have to be smooth and compactly supported. In the case of dyadic dilation $A = 2Id$, this is a well-known reproducing formula for discrete φ-transforms of Frazier and Jawerth.

Following Frazier and Jawerth, we then define the φ-transform, which maps the distribution f to the sequence of its wavelet coefficients $S_\varphi f = \{\langle f, \varphi_Q \rangle\}_{Q \in \mathcal{Q}}$. For any sequence $s = \{s_Q\}_{Q \in \mathcal{Q}}$ of complex numbers, we define formally the inverse φ-transform, which maps s to a distribution $T_\psi s = \sum_{Q \in \mathcal{Q}} s_Q \psi_Q$. To guarantee meaningfulness and boundedness of these transforms, we need to introduce quantitative assumptions on distributions f and sequences s. We will assume that f belongs to anisotropic Besov space $\dot{\mathbf{B}}_p^{\alpha,q}$ (or its inhomogeneous counterpart $\mathbf{B}_p^{\alpha,q}$) and s belongs to its discrete variant $\dot{\mathbf{b}}_p^{\alpha,q}$ (or $\mathbf{b}_p^{\alpha,q}$).

Given $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, and a dilation A, we introduce the *anisotropic Besov space* $\dot{\mathbf{B}}_p^{\alpha,q}$ as the collection of all tempered distributions f (modulo polynomials) such that,

$$\|f\|_{\dot{\mathbf{B}}_p^{\alpha,q}} = \left(\sum_{j\in\mathbb{Z}} (|\det A|^{j\alpha} ||f * \varphi_j||_{L^p})^q \right)^{1/q} < \infty,$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies certain support conditions (3.2) and (3.3), and $\varphi_j(x) = |\det A|^j \varphi(A^j x)$. We show that this definition is independent of the choice of φ in a more general weighted setting, where $L^p(\mathbb{R}^n)$ is replaced by $L^p(\mathbb{R}^n, \mu)$ with μ a doubling measure on $\mathbb{R}^n$ corresponding to the action of a dilation A. The discrete Besov sequence space, $\dot{\mathbf{b}}_p^{\alpha,q}$ is defined as the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|s\|_{\dot{\mathbf{b}}_p^{\alpha,q}} = \left(\sum_{j\in\mathbb{Z}} \left(\sum_{Q\in\mathcal{Q},\ |Q|=|\det A|^{-j}} (|Q|^{-\alpha-1/2}|s_Q|)^p \right)^{q/p} \right)^{1/q} < \infty.$$

Then we have the following generalization of the fundamental result of Frazier and Jawerth [21,22] for φ-transforms on Besov spaces; see Theorem 3.5 for a rigorous statement. A similar result to Theorem 1.1 in the case of (unweighted) anisotropic spaces was obtained by Dintelmann, see [15].

Theorem 1.1. *The φ-transform $S_\varphi : \dot{\mathbf{B}}_p^{\alpha,q} \to \dot{\mathbf{b}}_p^{\alpha,q}$ and the inverse φ-transform $T_\psi : \dot{\mathbf{b}}_p^{\alpha,q} \to \dot{\mathbf{B}}_p^{\alpha,q}$ are bounded, and $T_\psi \circ S_\varphi$ is the identity on $\dot{\mathbf{B}}_p^{\alpha,q}$.*

This result will enable us to study operators on $\dot{\mathbf{B}}_p^{\alpha,q}$ by considering corresponding operators on $\dot{\mathbf{b}}_p^{\alpha,q}$, as it was done in the classical dyadic case by Frazier and Jawerth. This is because the $\dot{\mathbf{b}}_p^{\alpha,q}$ norm is generally easier to work with, since it is discrete and depends only on the magnitude of the sequence elements. A very useful sufficient condition for the boundedness of operators on $\dot{\mathbf{b}}_p^{\alpha,q}$ is the almost diagonal condition studied in great detail in [21,22]. We extend this class of operators to anisotropic setting associated with expansive dilations A and we show that the expected boundedness result, Theorem 4.2, holds for anisotropic Besov spaces. The corresponding result for Triebel-Lizorkin spaces was obtained by the author and Ho [4].

Next, we introduce notions of smooth atoms and molecules for anisotropic Besov spaces extending familiar isotropic atoms and molecules introduced in [19,21]. A smooth atom supported near the dilated cube $Q \in \mathcal{Q}$ must satisfy appropriate smoothness, compact support, and vanishing moments properties. Smooth molecules satisfy similar properties with the exception that support condition is relaxed by appropriate decay estimate. Theorems 5.5 and 5.7 establish fundamental synthesis $|| \sum_Q s_Q \Psi_Q ||_{\dot{\mathbf{B}}_p^{\alpha,q}} \leq C||s||_{\dot{\mathbf{b}}_p^{\alpha,q}}$, and analysis $||\{\langle f, \Phi_Q\rangle\}_Q||_{\dot{\mathbf{b}}_p^{\alpha,q}} \leq C||f||_{\dot{\mathbf{B}}_p^{\alpha,q}}$ estimates for smooth synthesis $\{\Psi_Q\}_Q$ and analysis molecules $\{\Phi_Q\}_Q$. Both of these results generalize the boundedness of the φ-transform and the inverse φ-transform in Theorem 1.1 to situations when neither $\{\Phi_Q\}_Q$ nor $\{\Psi_Q\}_Q$ are necessarily obtained by translates and dilates of a particular function in $\mathcal{S}$.

Finally, the above results are used to extend the fundamental smooth atomic decomposition results of Frazier and Jawerth [19] to a general setting of weighted anisotropic $\dot{\mathbf{B}}_p^{\alpha,q}$ spaces. Analogous results for inhomogeneous anisotropic Besov spaces $\mathbf{B}_p^{\alpha,q}$ are also outlined.

2. Some background tools

We start by recalling basic definitions and properties of non-isotropic Euclidean spaces associated with general expansive dilations.

2.1. Basic properties of quasi-norms ρ_A

A real $n \times n$ matrix A is an *expansive matrix*, often called a *dilation*, if $\min_{\lambda\in\sigma(A)} |\lambda|>1$, where $\sigma(A)$ is the set of all eigenvalues (the spectrum) of A. A fundamental notion in our study is a quasi-norm ρ_A associated with A, which induces a quasi-distance making $\mathbb{R}^n$ a space of homogeneous type. For rudimentary facts about spaces of homogeneous type we refer the reader to [14,23].

Definition 2.1. *A quasi-norm associated with an expansive matrix A is a measurable mapping $\rho_A : \mathbb{R}^n \to [0,\infty)$ satisfying*

$$\begin{aligned} &\rho_A(x) > 0, && \text{for } x \neq 0, \\ &\rho_A(Ax) = |\det A|\rho_A(x) && \text{for } x \in \mathbb{R}^n, \\ &\rho_A(x+y) \le H(\rho_A(x)+\rho_A(y)) && \text{for } x, y \in \mathbb{R}^n, \end{aligned} \tag{2.1}$$

where $H \geq 1$ is a constant.

Here, we will only list a few basic properties of quasi-norms ρ_A, which will be used subsequently. For more details we refer to [3,24]. We recall that all quasi-norms associated to a fixed dilation A are equivalent, see [3, Lemma 2.4]. Moreover, there always exist a quasi-norm ρ_A, which is C^∞ on $\mathbb{R}^n$ except the origin, see [24].

Proposition 2.1. *For any expansive matrix A and $\epsilon > 0$,*

$$\int_{B(0,1)} \rho_A(x)^{\epsilon-1}dx < \infty \qquad \textit{and} \qquad \int_{\mathbb{R}^n\setminus B(0,1)} \rho_A(x)^{-1-\epsilon}dx < \infty.$$

Lemma 2.2. *Suppose A is expansive matrix, and λ_- and λ_+ are any positive real numbers such that $1 < \lambda_- < \min_{\lambda\in\sigma(A)} |\lambda|$ and $\lambda_+ > \max_{\lambda\in\sigma(A)} |\lambda|$. Let*

$$\zeta_+ := \frac{\ln \lambda_+}{\ln|\det A|}, \qquad \zeta_- := \frac{\ln \lambda_-}{\ln|\det A|}.$$

Then for any quasi-norm ρ_A there exists a constant C such that,

$$C^{-1}\rho_A(x)^{\zeta_-} \le |x| \le C\rho_A(x)^{\zeta_+} \quad if \quad \rho_A(x) \ge 1 \tag{2.2}$$

and

$$C^{-1}\rho_A(x)^{\zeta_+} \le |x| \le C\rho_A(x)^{\zeta_-} \quad if \quad \rho_A(x) \le 1. \tag{2.3}$$

Furthermore, if A is diagonalizable over $\mathbb{C}$, then we may take $\lambda_- = \min_{\lambda\in\sigma(A)} |\lambda|$ and $\lambda_+ = \max_{\lambda\in\sigma(A)} |\lambda|$.

We will also need the following easy estimates

$$(1/c)\lambda_-^j|x| \le |A^j x| \le c\lambda_+^j|x| \quad \text{for } j \ge 0, \tag{2.4}$$

$$(1/c)\lambda_+^j|x| \le |A^j x| \le c\lambda_-^j|x| \quad \text{for } j \le 0, \tag{2.5}$$

for some constant $c > 0$, where λ_- and λ_+ are the same as in Lemma 2.2.

Proposition 2.3. $(\mathbb{R}^n, \rho_A, |\cdot|)$ *is a space of homogeneous type, where* ρ_A *is a quasi-norm associated with an expansive dilation* A*, and* $|\cdot|$ *is Lebesgue measure on* $\mathbb{R}^n$*.*

Next, we introduce the class of measures on $\mathbb{R}^n$, which are doubling with respect to an expansive matrix A.

Definition 2.2. *We say that a non-negative Borel measure* μ *on* $\mathbb{R}^n$ *is* ρ_A*-doubling, if there exists* $\beta > 0$ *such that*

$$\mu(B_{\rho_A}(x, |\det A| r)) \le |\det A|^\beta \mu(B_{\rho_A}(x, r)) \qquad \text{for all } x \in \mathbb{R}^n, r > 0,$$

where,

$$B_{\rho_A}(x, r) = \{y \in \mathbb{R}^n : \rho_A(x - y) < r\}.$$

The smallest such $\beta = \beta(\mu)$ *as above is called a* ρ_A*-doubling constant of* μ*.*

Remark 2.1. We remark that ρ_A-doubling measure μ does not have to be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. An example of a measure μ on $\mathbb{R}$, which is doubling and singular with respect to Lebesgue measure can be found in the work of Buckley and MacManus [5]. Moreover, it is not hard to show that the doubling constant β is always ≥ 1.

We also remark that any weight w in A_∞ (with respect to a quasi-distance ρ_A) defines a ρ_A-doubling measure μ by $d\mu(x) = w(x)dx$, see [4, Definition 2.2]. Unlike the case of Triebel-Lizorkin spaces [4], in this work we relax the assumption that a weight $w \in A_\infty$, since in the study of Besov spaces we do not have to use the Hardy-Littlewood maximal function nor weighted vector-valued Fefferman-Stein inequality.

2.2. *Discrete wavelet transforms*

Suppose that $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\hat{\varphi}, \operatorname{supp}\hat{\psi} \subset [-\pi, \pi]^n \setminus \{0\} \tag{2.6}$$

$$\sum_{j\in\mathbb{Z}} \overline{\hat{\varphi}((A^*)^j\xi)}\hat{\psi}((A^*)^j\xi) = 1 \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \tag{2.7}$$

where A^* is the adjoint (transpose) of A. Here,

$$\operatorname{supp}\hat{\varphi} = \overline{\{\xi \in \mathbb{R}^n : \hat{\varphi}(\xi) \neq 0\}},$$

and the Fourier transform of f is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\langle x,\xi\rangle} dx.$$

For any $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let $Q_{j,k} = A^{-j}([0,1]^n + k)$ be the dilated cube, and $x_{Q_{j,k}} = A^{-j}k$ be its "lower-left corner". Let

$$\mathcal{Q} = \mathcal{Q}_A = \{Q_{j,k} : j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$$

be the collection of all dilated cubes. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, define

$$\varphi_j(x) = |\det A|^j \varphi(A^j x) \qquad \text{for } j \in \mathbb{Z},$$
$$\varphi_Q(x) = |\det A|^{j/2} \varphi(A^j x - k) = |Q|^{1/2} \varphi_j(x - x_Q) \qquad \text{for } Q = Q_{j,k} \in \mathcal{Q}.$$

It can be shown that the conditions (2.6), (2.7) imply that the *wavelet systems* $\{\varphi_Q : Q \in \mathcal{Q}\}$ and $\{\psi_Q : Q \in \mathcal{Q}\}$ form a pair of dual frames in $L^2(\mathbb{R}^n)$; for more details, see [2,4]. This means that $\{\varphi_Q : Q \in \mathcal{Q}\}$ and $\{\psi_Q : Q \in \mathcal{Q}\}$ are Bessel sequences and we have a reconstruction formula

$$f = \sum_{Q \in \mathcal{Q}} \langle f, \varphi_Q \rangle \psi_Q, \qquad \text{for all } f \in L^2(\mathbb{R}^n), \tag{2.8}$$

where the above series converges unconditionally in L^2.

The above formula has a counterpart in the form of basic reproducing identity (2.13) valid for the large class of tempered distributions modulo polynomials $\mathcal{S}'/\mathcal{P}$. For the basic properties of $\mathcal{S}'/\mathcal{P}$, we refer to [25, Section 3.3] or [34, Section 5.1]. Here, we only recall that $\mathcal{S}'/\mathcal{P}$ can be identified with the space of all continuous functionals on the closed subspace $\mathcal{S}_0(\mathbb{R}^n)$ of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ given by

$$\mathcal{S}_0(\mathbb{R}^n) = \{\varphi \in \mathcal{S} : \int \varphi(x) x^\alpha dx = 0 \quad \text{for all multi-indices } \alpha\}.$$

The counterpart of (2.8) for $\mathcal{S}'/\mathcal{P}$ follows from Lemmas 2.4 and 2.5, which show that any distribution f admits a kind of Littlewood-Paley decomposition and wavelet transform adapted to an expansive dilation A. Both of these results are anisotropic modifications of their well-known dyadic analogues, see [19,21,22]. The proofs of these lemmas can be found in [4].

Lemma 2.4. *Suppose that A is an expansive matrix and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is such that*

$$\sum_{j \in \mathbb{Z}} \hat{\varphi}((A^*)^j \xi) = 1 \qquad \textit{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \tag{2.9}$$

and supp $\hat{\varphi}$ *is compact and bounded away from the origin. Then for any $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$f = \sum_{j \in \mathbb{Z}} \varphi_j * f, \tag{2.10}$$

where $\varphi_j(x) = |\det A|^j \varphi(A^j x)$, and the convergence is in $\mathcal{S}'/\mathcal{P}$, where $\mathcal{P} \subset \mathcal{S}'$ is the class of all polynomials in $\mathbb{R}^n$.

More precisely, there exist a constant d depending only on the order of the distribution $\hat{f}$, a sequence of polynomials $\{P_k\}_{k=1}^{\infty} \subset \mathcal{P}$ with $\deg P_k \le d$, and $P \in \mathcal{P}$, such that

$$f = \lim_{k\to\infty} \left(\sum_{j=-k}^{\infty} \varphi_j * f + P_k \right) + P, \tag{2.11}$$

where the convergence is in $\mathcal{S}'$.

Lemma 2.5. *Suppose that A is an expansive matrix. If $g \in \mathcal{S}'(\mathbb{R}^n)$, $h \in \mathcal{S}(\mathbb{R}^n)$ and*

$$\operatorname{supp} \hat{g},\ \hat{h} \subset (A^*)^j [-\pi,\ \pi]^n \quad \textit{for some } j \in \mathbb{Z},$$

then,

$$(g * h)(x) = \sum_{k\in\mathbb{Z}^n} |\det A|^{-j} g(A^{-j}k) h(x - A^{-j}k), \tag{2.12}$$

with convergence in $\mathcal{S}'$. Consequently, if $\varphi, \psi \in \mathcal{S}'(\mathbb{R}^n)$ satisfy (2.6), (2.7), then

$$f = \sum_{Q\in\mathcal{Q}} \langle f, \varphi_Q\rangle \psi_Q, \qquad \textit{for any } f \in \mathcal{S}'/\mathcal{P}, \tag{2.13}$$

where the convergence of the above series, as well as the equality, is in $\mathcal{S}'/\mathcal{P}$. More precisely, there exists a sequence of polynomials $\{P_k\}_{k=1}^{\infty} \subset \mathcal{P}$ and $P \in \mathcal{P}$ such that

$$f = \lim_{k\to\infty} \left(\sum_{Q\in\mathcal{Q},\ |\det A|^{-k}\le|Q|\le|\det A|^k} \langle f, \varphi_Q\rangle \psi_Q + P_k \right) + P,$$

with convergence in $\mathcal{S}'$.

3. Besov spaces

In this section we investigate weighted anisotropic Besov spaces $\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ associated with general expansive dilation matrices using the Littlewood-Paley decomposition. Hence, in this work we will not be concerned with an alternative way of defining Besov spaces using L^p modulus of smoothness.

3.1. Homogeneous Besov spaces

Motivated by the classical definition of Besov spaces by Peetre [27], Triebel [34, 35], Frazier and Jawerth [21,22], and their weighted counterparts by Bui [6,8], we define anisotropic Besov spaces as follows.

Definition 3.1. *For $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ a ρ_A-doubling measure, we define the weighted anisotropic Besov space $\dot{\mathbf{B}}_p^{\alpha,q} = \dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ as the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that,*

$$\|f\|_{\dot{\mathbf{B}}_p^{\alpha,q}} = \left(\sum_{j \in \mathbb{Z}} (|\det A|^{j\alpha} ||f * \varphi_j||_{L^p(\mu)})^q \right)^{1/q} < \infty, \tag{3.1}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (3.2) and (3.3),

$$\operatorname{supp} \hat{\varphi} := \overline{\{\xi \in \mathbb{R}^n : \hat{\varphi}(\xi) \neq 0\}} \subset [-\pi, \pi]^n \setminus \{0\}, \tag{3.2}$$

$$\sup_{j \in \mathbb{Z}} |\hat{\varphi}((A^*)^j \xi)| > 0 \qquad \textit{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{3.3}$$

To emphasize the dependence on φ we will use the notation $\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)(\varphi)$ for (3.1). Later we will show that this definition is independent of φ.

The sequence space, $\dot{\mathbf{b}}_p^{\alpha,q} = \dot{\mathbf{b}}_p^{\alpha,q}(A, \mu)$ is the collection of all complex-valued sequences $s = \{s_Q\}_{Q \in \mathcal{Q}}$ such that

$$\|s\|_{\dot{\mathbf{b}}_p^{\alpha,q}} = \left(\sum_{j \in \mathbb{Z}} \left\| \sum_{Q \in \mathcal{Q},\ |Q| = |\det A|^{-j}} |Q|^{-\alpha} |s_Q| \tilde{\chi}_Q \right\|_{L^p(\mu)}^q \right)^{1/q} < \infty,$$

where $\tilde{\chi}_Q := |Q|^{-1/2} \chi_Q$ is the L^2-normalized characteristic function of the dilated cube Q.

Remark 3.1. The assumption (3.2) is made mostly for convenience and it can be relaxed by merely requiring that $\operatorname{supp} \hat{\varphi}$ is bounded and bounded away from the origin. It is clear that this leads to an equivalent definition of $\dot{\mathbf{B}}_p^{\alpha,q}$ spaces. We also remark that if $p = \infty$ then the Besov space $\dot{\mathbf{B}}_p^{\alpha,q}$ does not depend on the choice of the measure μ. However, it still depends on the choice of an expansive dilation A. Since the case of $p = \infty$ is also easier to deal with, in most arguments we will concentrate on the case $p < \infty$, leaving generally obvious verification of the case $p = \infty$ to the reader.

3.2. Completeness of $\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ spaces

We will use the following extension of a classical result of Plancherel and Pólya [19,28] to non-isotropic weighted Euclidean spaces.

Lemma 3.1. *Suppose K is a compact subset of $\mathbb{R}^n$, $0 < p < \infty$, and μ is a ρ_A-doubling measure on $\mathbb{R}^n$. Suppose $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\operatorname{supp} \hat{f} \subset (A^*)^j K$ for some $j \in \mathbb{Z}$. Then*

$$\left(\sum_{k \in \mathbb{Z}^n} \sup_{z \in Q_{j,k}} |f(z)|^p \mu(Q_{j,k}) \right)^{1/p} \leq C ||f||_{L^p(\mu)}, \tag{3.4}$$

where $Q_{j,k} = A^{-j}([0, 1]^n + k)$, and the constant $C = C(K, p, \mu)$ depends on K, p, and the doubling constant of μ.

Proof. Assume first that $K = [-1, 1]^n$. Let $\psi \in \mathcal{S}$ be such that $\operatorname{supp} \hat{\psi} \subset [-\pi, \pi]^n$ and $\hat{\psi}(\xi) = 1$ for $\xi \in [-1, 1]^n$. Let $g(x) = f(x + y)$, where $y \in \mathbb{R}^n$, and $h(x) = \psi_j(x) = |\det A|^j \psi(A^j x)$. Since $\operatorname{supp} \hat{g} = \operatorname{supp} \hat{f}$ and $\hat{h}(\xi) = 1$ for $\xi \in \operatorname{supp} \hat{f}$, we have $g * h = g$. Hence, by Lemma 2.5 we have

$$f(x+y) = g(x) = (g*h)(x) = |\det A|^{-j} \sum_{l\in\mathbb{Z}^n} g(A^{-j}l)h(x - A^{-j}l)$$
$$= \sum_{l\in\mathbb{Z}^n} f(A^{-j}l + y)\psi(A^j x - l).$$

Therefore, if we fix momentarily $k \in \mathbb{Z}^n$ and $y \in Q_{j,k}$, then

$$\sup_{z\in Q_{j,k}} |f(z)| \le \sup_{x\in A^{-j}([-1,1]^n)} |f(x+y)|$$
$$\le \sum_{l\in\mathbb{Z}^n} |f(A^{-j}l + y)| \sup_{x\in A^{-j}([-1,1]^n)} |\psi(A^j x - l)|.$$

Since $\psi \in \mathcal{S}(\mathbb{R}^n)$, for any $M > 0$ we have

$$\sup_{x\in A^{-j}([-1,1]^n)} |\psi(A^j x - l)| = \sup_{x\in[-1,1]^n} |\psi(x - l)| \le C_M(1 + |l|)^{-M}.$$

Hence, if $p > 1$, then by Hölder's inequality, $1/p + 1/p' = 1$,

$$\sup_{z\in Q_{j,k}} |f(z)|^p \le \left(\sum_{l\in\mathbb{Z}^n} |f(A^{-j}l+y)|^p(1+|l|)^{-n-1}\right)\left(\sum_{l\in\mathbb{Z}^n}(1+|l|)^{(n+1-M)p'}\right)^{p/p'}$$
$$\le C \sum_{l\in\mathbb{Z}^n} |f(A^{-j}l + y)|^p(1 + |l|)^{-n-1},$$

if we choose $M > 2n + 2$. Likewise, if $0 < p \le 1$, then by p-triangle inequality

$$\sup_{z\in Q_{j,k}} |f(z)|^p \le \sum_{l\in\mathbb{Z}^n} |f(A^{-j}l + y)|^p(1 + |l|)^{-Mp}$$
$$\le \sum_{l\in\mathbb{Z}^n} |f(A^{-j}l + y)|^p(1 + |l|)^{-n-1},$$

if we choose $M > (n + 1)/p$. Therefore, integrating the above estimate over $y \in Q_{j,k}$ with respect to μ, yields

$$\sup_{z\in Q_{j,k}} |f(z)|^p\mu(Q_{j,k}) \le C \sum_{l\in\mathbb{Z}^n}(1 + |l|)^{-n-1} \int_{Q_{j,k+l}} |f(x)|^p d\mu.$$

Summing over $k \in \mathbb{Z}^n$,

$$\sum_{k\in\mathbb{Z}^n} \sup_{z\in Q_{j,k}} |f(z)|^p\mu(Q_{j,k}) \le C||f||^p_{L^p(\mu)} \sum_{l\in\mathbb{Z}^n}(1 + |l|)^{-n-1} \le C||f||^p_{L^p(\mu)}, \tag{3.5}$$

which proves (3.4) under the constraint of $K = [-1, 1]^n$. We remark that the constant C in (3.5) is independent of μ and that the hypothesis of μ being a ρ_A-doubling measure was not used, yet.

Finally, to show (3.4) for a general compact K, let $j_0 \le 0$ be such that $K \subset (A^*)^{-j_0}([-1, 1]^n)$. Define $f_{j_0}(x) = f(A^{j_0}x)$ and the measure μ_{j_0} satisfying $\mu_{j_0}(E) = \mu(A^{j_0}E)$ for Borel sets $E \subset \mathbb{R}^n$. Since $\operatorname{supp} \hat{f}_{j_0} = (A^*)^{j_0}(\operatorname{supp} \hat{f}) \subset [-1, 1]^n$, we have by (3.5)

$$\sum_{k\in\mathbb{Z}^n}\sup_{z\in Q_{j-j_0,k}}|f(z)|^p\mu(Q_{j-j_0,k})=\sum_{k\in\mathbb{Z}^n}\sup_{z\in Q_{j,k}}|f_{j_0}(z)|^p\mu_{j_0}(Q_{j,k})$$
$$\leq C||f_{j_0}||^p_{L^p(\mu_{j_0})}=C||f||^p_{L^p(\mu)}. \quad (3.6)$$

Since μ is ρ_A-doubling, it is not hard to show that there exists $c=c(\beta)>0$, depending only on ρ_A-doubling constant β, such that for any $j\in\mathbb{Z}$ and $k,l\in\mathbb{Z}^n$,

$$Q_{j-j_0,k}\cap Q_{j,l}\neq\emptyset \implies \mu(Q_{j,l})\leq c|\det A|^{-j_0\beta}\mu(Q_{j-j_0,k}). \quad (3.7)$$

For any $l\in\mathbb{Z}^n$, let $z_l\in Q_{j,l}$ be a point where $\sup_{z\in Q_{j,l}}|f(z)|$ is achieved. Let $k=k(l)\in\mathbb{Z}^n$ be such that $z_l\in Q_{j-j_0,k}$. Then by (3.7),

$$\sup_{z\in Q_{j,l}}|f(z)|^p\mu(Q_{j,l})=\sup_{z\in Q_{j-j_0,k}}|f(z)|^p\mu(Q_{j,l})$$
$$\leq c|\det A|^{-j_0\beta}\sup_{z\in Q_{j-j_0,k}}|f(z)|^p\mu(Q_{j-j_0,k}).$$

Summing the above over $l\in\mathbb{Z}^n$ and using the observation that each dilated cube $Q_{j-j_0,k}$ can be chosen at most 2^n times, by (3.6) we have

$$\sum_{l\in\mathbb{Z}^n}\sup_{z\in Q_{j,l}}|f(z)|^p\mu(Q_{j,l})\leq 2^nc|\det A|^{-j_0\beta}\sum_{k\in\mathbb{Z}^n}\sup_{z\in Q_{j-j_0,k}}|f(z)|^p\mu(Q_{j-j_0,k})$$
$$\leq C||f||^p_{L^p(\mu)},$$

which shows (3.4). □

As a consequence of Lemma 3.1 we have the following corollary, which is a refinement of [4, Lemma 3.1].

Corollary 3.2. *Suppose K is a compact subset of $\mathbb{R}^n$, $0<p<\infty$, and μ is a ρ_A-doubling measure on $\mathbb{R}^n$. Then there exist $c,N>0$ such that for all $j\geq 0$,*

$$\sup_{x\in\mathbb{R}^n}\frac{|f(x)|}{(1+|x|)^N}\leq c^{j+1}||f||_{L^p(\mu)} \qquad \text{for all } f\in\mathcal{S}' \text{ with } \operatorname{supp}\hat{f}\subset(A^*)^jK. \quad (3.8)$$

Proof. Initially, suppose that $j=0$. Take any $f\in\mathcal{S}'$ with $\operatorname{supp}\hat{f}\subset K$. By Lemma 3.1, there exists a constant $C>0$ such that

$$\sup_{k\in\mathbb{Z}^n}\sup_{z\in Q_{0,k}}|f(z)|^p\mu(Q_{0,k})\leq C||f||^p_{L^p(\mu)}.$$

Since μ is ρ_A-doubling, we have $\mu(B_{\rho_A}(k,1))\leq C\mu(Q_{0,k})$ for some $C>0$ depending on the doubling constant of μ. Hence,

$$\sup_{k\in\mathbb{Z}^n}\sup_{z\in Q_{0,k}}|f(z)|^p\mu(B_{\rho_A}(k,1))\leq C||f||_{L^p(\mu)}, \quad (3.9)$$

Again, by ρ_A-doubling of μ, there exist constants $C,M>0$ such that

$$\mu(B_{\rho_A}(0,1))\leq C(1+\rho_A(k))^M\mu(B_{\rho_A}(k,1)) \qquad \text{for all } k\in\mathbb{Z}^n. \quad (3.10)$$

Moreover, by Lemma 2.2, we have

$$1+\rho_A(k) \le C \inf_{z\in Q_{0,k}} (1+\rho_A(z)) \le C \inf_{z\in Q_{0,k}} (1+|z|)^{1/\zeta_-}. \tag{3.11}$$

Combining (3.9)–(3.11),

$$\begin{aligned}\sup_{x\in\mathbb{R}^n} \frac{|f(x)|^p}{(1+|x|)^{M/\zeta_-}} &\le C \sup_{k\in\mathbb{Z}^n} \sup_{z\in Q_{0,k}} \frac{|f(z)|^p}{(1+\rho_A(k))^M} \\ &\le C \sup_{k\in\mathbb{Z}^n} \sup_{z\in Q_{0,k}} |f(z)|^p \frac{\mu(B_{\rho_A}(k,1))}{\mu(B_{\rho_A}(0,1))} \le C \frac{\|f\|^p_{L^p(\mu)}}{\mu(B_{\rho_A}(0,1))},\end{aligned} \tag{3.12}$$

where the constant C depends only on p, K, and the ρ_A-doubling constant of μ. Hence, (3.8) holds for $j=0$ with $N=M/(\zeta_- p)$.

To show (3.8) for arbitrary $j\ge 0$ we apply a scaling argument. Suppose $f\in\mathcal{S}'$ with $\operatorname{supp}\hat{f} \subset (A^*)^j K$. Define $f_{-j}(x)=f(A^{-j}x)$ and the measure μ_{-j} by $\mu_{-j}(E)=\mu(A^{-j}E)$ for Borel sets $E\subset\mathbb{R}^n$. We remark that ρ_A-doubling constants of μ and μ_{-j} are the same. Using (2.4) and (3.12),

$$\begin{aligned}\sup_{x\in\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^N} &= \sup_{x\in\mathbb{R}^n} \frac{|f_{-j}(x)|}{(1+|A^{-j}x|)^N} \le C(\lambda_+)^{jN} \sup_{x\in\mathbb{R}^n} \frac{|f_{-j}(x)|}{(1+|x|)^N} \\ &\le C(\lambda_+)^{jN} \frac{\|f_{-j}\|_{L^p(\mu_{-j})}}{\mu_{-j}(B_{\rho_A}(0,1))^{1/p}} = \frac{C(\lambda_+)^{jN}\|f\|_{L^p(\mu)}}{\mu(B_{\rho_A}(0,|\det A|^{-j}))^{1/p}} \\ &\le \frac{C((\lambda_+)^N|\det A|^{\beta/p})^j}{\mu(B_{\rho_A}(0,1))^{1/p}} \|f\|_{L^p(\mu)} \le c^{j+1}\|f\|_{L^p(\mu)},\end{aligned} \tag{3.13}$$

where in the penultimate step we again used ρ_A-doubling of μ. Here, $c>0$ is a constant independent of f and $j\ge 0$. This completes the proof of Corollary 3.2. □

As one of the consequences of Corollary 3.2 we can conclude the completeness of $\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ spaces.

Proposition 3.3. *The inclusion map $i : \dot{\mathbf{B}}_p^{\alpha,q} = \dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu) \hookrightarrow \mathcal{S}'/\mathcal{P}$ is continuous. Moreover, $\dot{\mathbf{B}}_p^{\alpha,q}$ equipped with $\|\cdot\|_{\dot{\mathbf{B}}_p^{\alpha,q}}$ is a quasi-Banach space, i.e., $\dot{\mathbf{B}}_p^{\alpha,q}$ is a complete quasi-normed space.*

Proof. We recall that $\mathcal{S}_0(\mathbb{R}^n)$ can be defined as a collection of $\phi\in\mathcal{S}$ such that semi-norms

$$\|\phi\|_M = \sup_{|\beta|\le M} \sup_{\xi\in\mathbb{R}^n} |\partial^\beta \hat{\phi}(\xi)|(|\xi|^M + |\xi|^{-M}) < \infty \qquad \text{for any } M\in\mathbb{N}. \tag{3.14}$$

Moreover, semi-norms $\|\cdot\|_M$ generate a topology of a locally convex space on $\mathcal{S}_0(\mathbb{R}^n)$.

As a consequence of Lemma 3.2 applied for $f*\varphi_j$ and $K=\bigcup_{j\le 0}(A^*)^j(\operatorname{supp}\hat\varphi)$, there exists c, $N>0$ such that for any $j\in\mathbb{Z}$ and $\phi\in\mathcal{S}$,

$$\begin{aligned}|\langle f*\varphi_j,\phi\rangle| &\le c^{\max(0,j)+1}||f*\varphi_j||_{L^p(\mu)}||(1+|x|)^N\phi(x)||_\infty\\ &\le c^{|j|+1}|\det A|^{-j\alpha}||f||_{\dot{\mathbf{B}}_p^{\alpha,q}}||(1+|x|)^N\phi(x)||_\infty\\ &\le c^{|j|+1}|\det A|^{|j||\alpha|}||f||_{\dot{\mathbf{B}}_p^{\alpha,q}}\sup_{|\beta|\le n+1,|\gamma|\le N}||\hat\phi||_{\beta,\gamma},\end{aligned}$$

where $||\phi||_{\beta,\gamma}=\sup_{x\in\mathbb{R}^n}|x^\beta||\partial^\gamma\phi(x)|$ denotes the usual semi-norm in $\mathcal{S}(\mathbb{R}^n)$. Let $h\in\mathcal{S}(\mathbb{R}^n)$ and $r>0$ be such that $\hat h(\xi)=1$ for all $\xi\in\operatorname{supp}\hat\varphi$ and $\operatorname{supp}\hat h\subset\{\xi:1/r<|\xi|<r\}$ by (3.2). Hence, using $\operatorname{supp}\widehat{\varphi_j}\subset\operatorname{supp}\widehat{h_j}=(A^*)^j(\operatorname{supp}\hat h)$, where $h_j(x)=|\det A|^j h(A^jx)$, we have

$$\begin{aligned}|\langle f*\varphi_j,\phi\rangle| &= |\langle f*\varphi_j,h_j*\phi\rangle|\\ &\le c^{|j|+1}|\det A|^{|j||\alpha|}||f||_{\dot{\mathbf{B}}_p^{\alpha,q}}\sup_{|\beta|\le n+1,|\gamma|\le N}||\hat h((A^*)^{-j}\cdot)\hat\phi||_{\beta,\gamma}\end{aligned}$$

Using growth estimates (2.4) and (2.5) and simple but tedious support techniques, one can control the above by semi-norms $||\cdot||_M$ as in (3.14). That is, for any $s_1>0$ there exists $M>0$ such that

$$\begin{aligned}&\sup_{|\beta|\le n+1,|\gamma|\le N}||\hat h((A^*)^{-j}\cdot)\hat\phi(\cdot)||_{\beta,\gamma}\\ &\quad\le C|\det A|^{-|j|s_1}\sup_{|\beta|\le M}\sup_{\xi\in\mathbb{R}^n}|\partial^\beta\hat\phi(\xi)|(|\xi|^M+|\xi|^{-M})\\ &\quad= C|\det A|^{-|j|s_1}||\phi||_M \qquad \text{for any } j\in\mathbb{Z}.\end{aligned}\tag{3.15}$$

Indeed, (3.15) follows from the simple observation that $\hat h((A^*)^{-j}\cdot)\hat\phi(\cdot)$ is supported in the dilated annulus $(A^*)^j(\{\xi:1/r<|\xi|<r\})$, and therefore the decay of $\hat\phi$ at the origin and at infinity can be used to control any kind of growth produced by taking derivatives of $h((A^*)^{-j}\cdot)$ in the computation of the semi-norms $||\hat h((A^*)^{-j}\cdot)\hat\phi(\cdot)||_{\beta,\gamma}$. For more details of these kinds of estimates, see the proof of [4, Lemma 2.6]. Combining the last two estimates we deduce the existence of $M>0$ and $s_2>0$ such that

$$|\langle f*\varphi_j,\phi\rangle|\le C||f||_{\dot{\mathbf{B}}_p^{\alpha,q}}|\det A|^{-|j|s_2}||\phi||_M.$$

Hence, by Lemma 2.4

$$\begin{aligned}|\langle f,\phi\rangle| &\le \sum_{j\in\mathbb{Z}}|\langle f*\varphi_j,\phi\rangle|\le C\sum_{j\in\mathbb{Z}}||f||_{\dot{\mathbf{B}}_p^{\alpha,q}}|\det A|^{-|j|s_2}||\phi||_M\\ &\le C||f||_{\dot{\mathbf{B}}_p^{\alpha,q}}||\phi||_M,\end{aligned}$$

which shows that $i:\dot{\mathbf{B}}_p^{\alpha,q}\hookrightarrow\mathcal{S}'/\mathcal{P}$ is continuous.

Once the continuity of the inclusion map i is established, the completeness of $\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)$ is immediate by Fatou's Lemma and Lemma 2.4, if φ satisfies (2.9) in addition to (3.2). In the case of general φ satisfying only (3.2) and (3.3) one must use a variant of Lemma 2.4, where (2.10) is replaced by

$$f = \sum_{j\in\mathbb{Z}} f * \varphi_j * \tilde{\psi}_j, \qquad \text{convergence in } \mathcal{S}'/\mathcal{P},$$

where ψ is as in Lemma 3.6 and $\tilde{\psi}(x) = \overline{\psi(-x)}$. This completes the proof of Proposition 3.3. □

3.3. Wavelet transforms for $\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$

Suppose that $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ are such that supp $\hat{\varphi}$, supp $\hat{\psi}$ are compact and bounded away from the origin.

Definition 3.2. *The φ-transform S_φ, often called the analysis transform, is the map taking each $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ to the sequence $S_\varphi f = \{(S_\varphi f)_Q\}_{Q\in\mathcal{Q}}$ defined by $(S_\varphi f)_Q = \langle f, \varphi_Q\rangle$. This is well-defined, since $\int x^\gamma \varphi_Q(x)dx = 0$ for any multi-index γ. Here, we follow the pairing convention which is consistent with the usual scalar product in $L^2(\mathbb{R}^n)$, i.e., $\langle f, \varphi\rangle = f(\overline{\varphi})$ for $f \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$. The inverse φ-transform, T_ψ, often called the synthesis transform, is the map taking the sequence $s = \{s_Q\}_{Q\in\mathcal{Q}}$ to $T_\psi s = \sum_{Q\in\mathcal{Q}} s_Q \psi_Q$. We will show later that $T_\psi s$ is well-defined for $s \in \dot{\mathbf{b}}_p^{\alpha,q}$.*

Lemma 3.4. *Suppose $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is a ρ_A-doubling measure. Then there exist constants $C, \lambda > 0$ such that*

$$\|s_\lambda^*\|_{\dot{\mathbf{b}}_p^{\alpha,q}(A,\mu)} \le C\|s\|_{\dot{\mathbf{b}}_p^{\alpha,q}(A,\mu)} \qquad \textit{for all } s = \{s_Q\}_Q, \tag{3.16}$$

where the sequence $s_\lambda^* = \{(s_\lambda^*)_Q\}_{Q\in\mathcal{Q}}$ is given by

$$(s_\lambda^*)_Q = \sum_{P\in\mathcal{Q},\ |P|=|Q|} \frac{|s_P|}{(1+|Q|^{-1}\rho_A(x_Q - x_P))^\lambda}.$$

Proof. We start by showing that for any $L > \beta$, where β is ρ_A-doubling constant of μ, there exists $C > 0$ such that

$$\sum_{Q\in\mathcal{Q},\ |P|=|Q|} \frac{\mu(Q)}{(1+|Q|^{-1}\rho_A(x_Q - x_P))^L} \le C\mu(P) \qquad \text{for any } P \in \mathcal{Q}. \tag{3.17}$$

First, we will show that (3.17) holds for $|P| = 1$ with a constant C depending only on β. Then by ρ_A-doubling of μ

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^n} \frac{\mu(Q_{0,k})}{(1+\rho_A(k-l))^L} &\le C \int_{\mathbb{R}^n} \frac{d\mu(x)}{(1+\rho_A(x-l))^L} \\
&= C \int_{B_{\rho_A}(l,1)} \frac{d\mu(x)}{(1+\rho_A(x-l))^L} \\
&\quad + C \sum_{j=0}^{\infty} \int_{B_{\rho_A}(l,|\det A|^{j+1})\setminus B_{\rho_A}(l,|\det A|^j)} \frac{d\mu(x)}{(1+\rho_A(x-l))^L}
\end{aligned}$$

$$\le C\mu(B_{\rho_A}(l,1)) + C\sum_{j=0}^{\infty}\mu(B_{\rho_A}(l,|\det A|^{j+1}))|\det A|^{-jL}$$
$$\le C\mu(B_{\rho_A}(l,1))\left(1+\sum_{j=0}^{\infty}|\det A|^{(j+1)\beta-jL}\right) \le C\mu(B_{\rho_A}(l,1)) \le C\mu(Q_{0,l}).$$

To show (3.17) for a general $P \in \mathcal{Q}$ with $\mathcal{P} = |\det A|^{-j}$, it suffices to consider a measure μ_{-j} given by $\mu_{-j}(E) = \mu(A^{-j}E)$ for Borel sets $E \subset \mathbb{R}^n$ and use a scaling argument.

If $p > 1$ then by Hölder's inequality, $1/p + 1/p' = 1$, $|(s^*_\lambda)_Q|^p$ can be bounded by

$$\sum_{P\in\mathcal{Q},\ |P|=|Q|}\frac{|s_P|^p}{(1+|Q|^{-1}\rho_A(x_Q-x_P))^\lambda}$$
$$\times\left(\sum_{P\in\mathcal{Q},\ |P|=|Q|}\frac{1}{(1+|Q|^{-1}\rho_A(x_Q-x_P))^\lambda}\right)^{p/p'}$$
$$\le C\sum_{P\in\mathcal{Q},\ |P|=|Q|}\frac{|s_P|^p}{(1+|Q|^{-1}\rho_A(x_Q-x_P))^\lambda}, \tag{3.18}$$

since the sum in parenthesis is finite by a discrete version of Proposition 2.1. More precisely, $\sum_{k\in\mathbb{Z}^n}(1+\rho_A(k))^{-\lambda} < \infty$ for $\lambda > 1$. If $p \le 1$ then by p-triangle inequality

$$|(s^*_\lambda)_Q|^p \le \sum_{P\in\mathcal{Q},\ |P|=|Q|}\frac{|s_P|^p}{(1+|Q|^{-1}\rho_A(x_Q-x_P))^{p\lambda}}. \tag{3.19}$$

Hence, if we choose $\lambda > \beta\max(1,1/p)$, then combining (3.17) with (3.18) or (3.19) yields (3.16) by a straightforward calculation. □

The next result is a generalization of the fundamental result of Frazier and Jawerth [19,22] on the boundedness of φ-transforms on Besov spaces.

Theorem 3.5. *Suppose $\alpha \in \mathbb{R}$, $0 < p,q \le \infty$, μ is ρ_A-doubling, and $\varphi,\psi \in \mathcal{S}(\mathbb{R}^n)$ are such that* supp $\hat\varphi$, supp $\hat\psi$ *are compact and bounded away from the origin. Then the operators $S_\varphi : \dot{\mathbf{B}}^{\alpha,q}_p(\mathbb{R}^n,A,\mu)(\tilde\varphi) \to \dot{\mathbf{b}}^{\alpha,q}_p(A,\mu)$ and $T_\psi : \dot{\mathbf{b}}^{\alpha,q}_p(A,\mu) \to \dot{\mathbf{B}}^{\alpha,q}_p(\mathbb{R}^n,A,\mu)(\varphi)$ are bounded, where $\tilde\varphi(x) = \overline{\varphi(-x)}$. In addition, if φ,ψ satisfy (2.6), (2.7) then $T_\psi \circ S_\varphi$ is the identity on $\dot{\mathbf{B}}^{\alpha,q}_p(\mathbb{R}^n,A,\mu)(\varphi) = \dot{\mathbf{B}}^{\alpha,q}_p(\mathbb{R}^n,A,\mu)(\tilde\varphi)$.*

Proof. We will only prove the case of $p,q < \infty$ and leave details of the easier case of $p = \infty$ or $q = \infty$ to the reader.

First, we will prove the boundedness of T_ψ. Take any $s = \{s_Q\}_Q \in \dot{\mathbf{b}}^{\alpha,q}_p$ with finite support. Since the supports of $\hat\varphi$ and $\hat\psi$ are bounded and bounded away from the origin, there is an integer M such that supp $\widehat{\varphi_j} \cap$ supp $\widehat{\psi_i} = \emptyset$ for $|i-j| > M$. Therefore,

$$(\varphi_j * f)(x) = \sum_{i=j-M}^{i=j+M}\sum_{|P|=|\det A|^{-i}} s_P(\varphi_j * \psi_P)(x).$$

By direct calculations, see [4, Theorem 3.5], one can show that for any $\lambda > 1$, there is a constant $C = C(\lambda) > 0$ such that

$$|\varphi_j * f(x)| \le C \sum_{i=j-M}^{i=j+M} \sum_{|Q|=|\det A|^{-i}} (s^*_\lambda)_Q \tilde{\chi}_Q(x).$$

Hence, by choosing λ as in Lemma 3.4, we have

$$\begin{aligned} \|T_\psi s\|_{\dot{\mathbf{B}}^{\alpha,q}_p} &\le C\Bigg(\sum_{j\in\mathbb{Z}} \Bigg\| \sum_{l=-M}^{M} \sum_{|Q|=|\det A|^{-j+l}} |\det A|^{j\alpha} (s^*_\lambda)_Q \tilde{\chi}_Q \Bigg\|^q_{L^p(\mu)}\Bigg)^{1/q} \\ &\le C\Bigg(\sum_{j\in\mathbb{Z}} \sum_{l=-M}^{M} \Bigg\| \sum_{|Q|=|\det A|^{-j}} |\det A|^{(j+l)\alpha} (s^*_\lambda)_Q \tilde{\chi}_Q \Bigg\|^q_{L^p(\mu)}\Bigg)^{1/q} \\ &\le C\|s^*_\lambda\|_{\dot{\mathbf{b}}^{\alpha,q}_p} \le C\|s\|_{\dot{\mathbf{b}}^{\alpha,q}_p}. \end{aligned} \tag{3.20}$$

To show that the same estimate holds for arbitrary $s \in \dot{\mathbf{b}}^{\alpha,q}_p$, we apply the above argument for some special φ satisfying additionally (2.9) and (3.2). Then by Proposition 3.3 and (3.20), $T_\psi s = \sum_Q s_Q \psi_Q$ is a well-defined element of $\mathcal{S}'/\mathcal{P}$, since sequences with finite support are dense in $\dot{\mathbf{b}}^{\alpha,q}_p$ for $p, q < \infty$. Hence, by a limiting argument, the above estimate must also hold for arbitrary $s \in \dot{\mathbf{b}}^{\alpha,q}_p$, which shows the boundedness of T_ψ.

Next, we will prove the boundedness of S_φ. Suppose that $f \in \dot{\mathbf{B}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)(\tilde{\varphi})$ and $s_Q = \langle f, \varphi_Q\rangle$. Since, for any $Q \in \mathcal{Q}$,

$$|\langle f, \varphi_Q\rangle| = |Q|^{1/2} |(\tilde{\varphi}_j * f)(x_Q)| \le |Q|^{1/2} \sup_{z\in Q} |(\tilde{\varphi}_j * f)(z)|,$$

hence, by Lemma 3.1 applied for $\tilde{\varphi}_j * f$,

$$\Bigg\| \sum_{|Q|=|\det A|^{-j}} |Q|^{-\alpha} |s_Q| \tilde{\chi}_Q \Bigg\|_{L^p(\mu)} \le C|\det A|^{j\alpha} \|\tilde{\varphi}_j * f\|_{L^p(\mu)}.$$

Summing the above raised to the power of q over $j \in \mathbb{Z}$ yields $\|S_\varphi f\|_{\dot{\mathbf{b}}^{\alpha,q}_p} = \|s\|_{\dot{\mathbf{b}}^{\alpha,q}_p} \le C\|f\|_{\dot{\mathbf{B}}^{\alpha,q}_p(\mathbb{R}^n,A,\mu)(\tilde{\varphi})}$.

Finally, if we assume additionally that φ and ψ satisfy (2.6) and (2.7), then by Lemma 2.5, $T_\psi \circ S_\varphi$ is the identity on $\dot{\mathbf{B}}^{\alpha,q}_p$. More precisely, $\dot{\mathbf{B}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)(\tilde{\varphi}) \hookrightarrow \dot{\mathbf{B}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)(\varphi)$ is a bounded inclusion. Hence, by reversing the roles of φ and $\tilde{\varphi}$ we have

$$\dot{\mathbf{B}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)(\tilde{\varphi}) = \dot{\mathbf{B}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)(\varphi),$$

which completes the proof of Theorem 3.5. □

3.4. Basic properties of $\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$

Next, we will show that the definition of Besov spaces $\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ does not depend on the choice of φ satisfying (3.2) and (3.3). We will use the following elementary lemma, see [4].

Lemma 3.6. *Suppose $\varphi \in \mathcal{S}$ satisfies (3.2) and (3.3). Then there exists $\psi \in \mathcal{S}$ such that (2.6) and (2.7) are satisfied.*

Corollary 3.7. *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is ρ_A-doubling measure. Then the space $\dot{\mathbf{B}}_p^{\alpha,q}$ is well-defined in the sense that, for any φ^1 and φ^2 satisfying (3.2) and (3.3), their associated quasi-norms in $\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)(\varphi^i)$, $i = 1, 2$, are equivalent, i.e., there exist constants $C_1, C_2 > 0$ such that*

$$C_1\|f\|_{\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)(\varphi^1)} \le \|f\|_{\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)(\varphi^2)} \le C_2\|f\|_{\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)(\varphi^1)}. \tag{3.21}$$

Proof. Suppose that φ^1 and φ^2 each satisfy (3.2) and (3.3), then by Lemma 3.6, it is possible to find ψ^1 and ψ^2 so that (2.6) and (2.7) are satisfied for each pair φ^i, ψ^i, $i = 1, 2$. Then by Lemma 2.5,

$$\begin{aligned}\|f\|_{\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)(\varphi^1)} &= \|(T_{\tilde{\psi}^2} \circ S_{\tilde{\varphi}^2})f\|_{\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)(\varphi^1)} \le C\|S_{\tilde{\varphi}^2} f\|_{\dot{\mathbf{b}}_p^{\alpha,q}(A,\mu)} \\ &\le C\|f\|_{\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)(\varphi^2)},\end{aligned}$$

by the boundedness of $S_{\tilde{\varphi}^2}$ and $T_{\tilde{\psi}^2}$, since the pair $\tilde{\varphi}^2$, $\tilde{\psi}^2$ satisfies (2.6) and (2.7). Reversing the roles of φ^1 and φ^2, yields (3.21). □

Remark 3.2. It follows from the above argument and Theorem 3.5 that we have a more general estimate

$$||f||_{\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)(\varphi)} \le C||f||_{\dot{\mathbf{B}}_p^{\alpha,q}} \qquad \text{for all } f \in \dot{\mathbf{B}}_p^{\alpha,q},$$

where $\varphi \in \mathcal{S}$ is such that $\operatorname{supp}\hat{\varphi}$ is compact and bounded away from the origin. Hence, φ may not necessarily satisfy (3.3) and consequently $\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)(\varphi)$ may not be a complete quasi-normed space.

We will also need the following very useful fact, which resolves all sorts of issues caused by the fact the elements of $\dot{\mathbf{B}}_p^{\alpha,q}$ are equivalence classes of tempered distributions $\mathcal{S}'$ modulo polynomials $\mathcal{P}$. This result guarantees the existence of canonical representatives of elements in $\dot{\mathbf{B}}_p^{\alpha,q}$ modulo polynomials of degree $\le L = \lfloor \alpha/\zeta_- \rfloor$. Proposition 3.8 is a generalization of [21, Remark B.4] and [27, pp. 52–56] in the unweighted case and [8, Proposition 1.1] in the weighted case. The analogous result for weighted anisotropic Triebel-Lizorkin spaces was shown by the author and Ho [4].

Proposition 3.8. *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, μ is ρ_A-doubling measure, and $f \in \dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$. For any $\varphi^1 \in \mathcal{S}(\mathbb{R}^n)$ such that* $\operatorname{supp}\widehat{\varphi^1}$ *is compact* *and bounded away from the origin, and (2.9) holds, there exists a sequence of polynomials* $\{P_k^1\}_{k=1}^{\infty}$ *with* $\deg P_k^1 \le L = \lfloor \alpha/\zeta_- \rfloor$ *such that*

$$g^1 := \lim_{k\to\infty} \left(\sum_{j=-k}^{\infty} (\varphi^1)_j * f + P_k^1 \right) \tag{3.22}$$

exists in $\mathcal{S}'$. *Moreover, if* g^2 *is the corresponding limit in (3.22) for some other* $\varphi^2 \in \mathcal{S}(\mathbb{R}^n)$ *such that* $\operatorname{supp}\widehat{\varphi^2}$ *is compact and bounded away from the origin, and (2.9) holds, then*

$$g^1 - g^2 \in \mathcal{P} \quad \textit{and} \quad \deg(g^1 - g^2) \le L. \tag{3.23}$$

In particular, Proposition 3.8 asserts that if $\alpha < 0$ then the polynomials $\{P_k^1\}$ are simply not needed in (3.22) and the limit $g = \lim_{k\to\infty} \sum_{j=-k}^{\infty} (\varphi^1)_j * f$ exists in $\mathcal{S}'$ and is independent of the choice of φ^1 as above.

For the sake of completeness we include the proof of Proposition 3.8, which is a minor modification of the corresponding $\dot{\mathbf{F}}_p^{\alpha,q}$ result [4, Proposition 3.8].

Proof. Note that Lemma 2.4 already guarantees the existence of polynomials $\{P_k^1\}_{k=1}^{\infty}$ with $\deg P_k^1 \le d$ for some $d \ge 0$ such that (3.22) holds. However, it is not clear why d could be chosen to be $\le L = \lfloor \alpha/\zeta_- \rfloor$ and why (3.23) holds. Nevertheless, by Lemma 2.4 we know that $\sum_{j=0}^{\infty} (\varphi^1)_j * f$ converges in $\mathcal{S}'$, see also [4].

Let $c, N > 0$ be the constants guaranteed by Corollary 3.2 for a compact set $K = \bigcup_{j<0} (A^*)^j (\operatorname{supp}\widehat{\varphi^1})$. Then, for any $j < 0$ and a multi-index β, by Remark 3.2 and Lemma 3.2,

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} \frac{|\partial^\beta((\varphi^1)_j * f)(x)|}{(1+|x|)^N} &\le c||\partial^\beta((\varphi^1)_j * f)||_{L^p(\mu)} = c||(\partial^\beta(\varphi^1)_j) * f||_{L^p(\mu)} \\
&\le C \sum_{|\gamma|=|\beta|} |a_{\gamma,j}| ||(\partial^\gamma \varphi^1)_j * f||_{L^p(w)} \\
&\le C(\lambda_-)^{j|\beta|} |\det A|^{-j\alpha} \sum_{|\gamma|=|\beta|} ||f||_{\dot{\mathbf{B}}_p^{\alpha,\infty}(\mathbb{R}^n,A,\mu)(\partial^\gamma\varphi^1)} \\
&\le C|\det A|^{j(|\beta|\zeta_- - \alpha)} \sum_{|\gamma|=|\beta|} ||f||_{\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)(\partial^\gamma\varphi^1)} \\
&\le C|\det A|^{j(|\beta|\zeta_- - \alpha)} ||f||_{\dot{\mathbf{B}}_p^{\alpha,q}}.
\end{aligned} \tag{3.24}$$

Here, we used that for any $\varphi \in \mathcal{S}$ and a multi-index β, there exists a constant $C > 0$ such that for all $j \ge 0$, we have

$$\partial^\beta \varphi_j(x) = \sum_{|\gamma|=|\beta|} a_{\gamma,j} (\partial^\gamma \varphi)_j(x), \qquad \text{where } |a_{\gamma,j}| \le C(\lambda_-)^{j|\gamma|}. \tag{3.25}$$

This follows from the chain rule and the estimate $||A^j||^{|\gamma|} \le C(\lambda_-)^{j|\gamma|}$ for $j \le 0$, see also [3, the proof of Lemma 5.2].

Therefore, by (3.24), $\sum_{j<0} \partial^\beta((\varphi^1)_j * f)$ converges in $\mathcal{S}'$ for any $|\beta| > L$, since $|\beta|\zeta_- - \alpha > 0$. Consequently, [4, Proposition 2.7] yields polynomials $\{P_k^1\}_{k=1}^{\infty}$ with $\deg P_k^1 \le L$ and $g^1 \in \mathcal{S}'$ such that (3.22) holds.

To show (3.23), let φ^2 be another function satisfying hypotheses of Proposition 3.8, and let $g^2 \in \mathcal{S}'$ be the corresponding limit of (3.22) for some sequence of polynomials $\{P_k^2\}_{k=1}^\infty$ with $\deg P_k^2 \le L$. Since g^1 and g^2 represent the same equivalence class in $\mathcal{S}'/\mathcal{P}$ of $f \in \dot{\mathbf{B}}_p^{\alpha,q}$, $g^1 - g^2$ is a polynomial. Let $K = \bigcup_{j<0}(A^*)^j(\operatorname{supp}\widehat{\varphi^1} \cup \operatorname{supp}\widehat{\varphi^2})$. Then, by a simple support argument and (2.9),

$$\operatorname{supp}\left(\sum_{j=-k}^{\infty}((\varphi^1)_j * f - (\varphi^2)_j * f)\right)^{\widehat{}} \subset (A^*)^{-k}K \qquad \text{for any } k \in \mathbb{Z}. \tag{3.26}$$

Let φ be given by

$$\hat{\varphi}(\xi) = \sum_{j=0}^{\infty}\widehat{\varphi^1}((A^*)^{-j}\xi) - \widehat{\varphi^2}((A^*)^{-j}\xi). \tag{3.27}$$

Then it is not hard to verify that $\hat{\varphi}$ is C^∞ and that the support of $\hat{\varphi}$ is bounded and bounded away from the origin. Moreover,

$$\sum_{j=-k}^{\infty}((\varphi^1)_j * f - (\varphi^2)_j * f) = \varphi_{-k} * f \qquad \text{for any } k \in \mathbb{Z}, \tag{3.28}$$

where the series converges in $\mathcal{S}'$, see [4]. Hence, by (3.24) and (3.28), for any $\phi \in \mathcal{S}$ and $|\beta| > L$,

$$\begin{aligned}
|\langle\partial^\beta(g^1 - g^2), \phi\rangle| &= \left|\lim_{k\to\infty}\left\langle\sum_{j=-k}^{\infty}\partial^\beta((\varphi^1)_j * f + P_k^1 - (\varphi^2)_j * f - P_k^2), \phi\right\rangle\right| \\
&= \lim_{k\to\infty}\left|\left\langle\sum_{j=-k}^{\infty}\partial^\beta((\varphi^1)_j * f - (\varphi^2)_j * f), \phi\right\rangle\right| = \lim_{k\to\infty}|\langle\partial^\beta(\varphi_{-k} * f), \phi\rangle| \\
&\le \lim_{k\to\infty}\sup_{x\in\mathbb{R}^n}\frac{|\partial^\beta(\varphi_{-k} * f)(x)|}{(1+|x|)^N}\int_{\mathbb{R}^n}(1+|x|)^N|\phi(x)|dx \\
&\le C\lim_{k\to\infty}|\det A|^{-k(|\beta|\zeta_- - \alpha)}||f||_{\dot{\mathbf{B}}_p^{\alpha,q}} = 0.
\end{aligned}$$

This shows (3.23) and completes the proof of Theorem 3.8. □

As an immediate corollary of Lemma 2.5 and Proposition 3.8, we have

Corollary 3.9. *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, μ is ρ_A-doubling measure, and $f \in \dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$. Given $\varphi^1, \psi^1 \in \mathcal{S}$ satisfying (2.6) and (2.7), there exists a sequence of polynomials $\{P_k^1\}_{k=1}^\infty$ with* $\deg P_k^1 \le L = \lfloor \alpha/\zeta_- \rfloor$ *such that*

$$g^1 := \lim_{k\to\infty}\left(\sum_{Q\in\mathcal{Q},\ |\det A|^{-k}\le|Q|\le|\det A|^k}\langle f, (\varphi^1)_Q\rangle(\psi^1)_Q + P_k^1\right) \tag{3.29}$$

exists in $\mathcal{S}'$. Moreover, if g^2 is the corresponding limit in (3.29) for some other $\varphi^2, \psi^2 \in \mathcal{S}$ satisfying (2.6) and (2.7), then (3.23) holds.

3.5. Inhomogeneous Besov spaces

In this subsection we briefly describe basic properties of inhomogeneous Besov spaces $\mathbf{B}_p^{\alpha,q}$. Most of them are straightforward modifications of the corresponding homogeneous results and therefore we will only outline required changes.

Definition 3.3. *For $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ a ρ_A-doubling measure, we define the weighted inhomogeneous anisotropic Besov space $\mathbf{B}_p^{\alpha,q} = \mathbf{B}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ as the collection of all $f \in \mathcal{S}'$ such that,*

$$\|f\|_{\mathbf{B}_p^{\alpha,q}} = \|f * \Phi\|_{L^p(\mu)} + \left(\sum_{j=1}^{\infty} (|\det A|^{j\alpha} \|f * \varphi_j\|_{L^p(\mu)})^q \right)^{1/q} < \infty,$$

where $\Phi \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (3.2), (3.30), and (3.31),

$$\operatorname{supp} \hat{\Phi} \subset [-\pi, \pi]^n, \tag{3.30}$$

$$\sup_{j \ge 1}\{|\hat{\varphi}((A^*)^{-j}\xi)|, |\hat{\Phi}(\xi)|\} > 0 \qquad \text{for all } \xi \in \mathbb{R}^n, \tag{3.31}$$

As in the homogeneous case, this definition is independent of Φ and φ as above.

Let $\mathcal{Q}_0 = \{Q \in \mathcal{Q} : |Q| \le 1\}$. The sequence space, $\mathbf{b}_p^{\alpha,q} = \mathbf{b}_p^{\alpha,q}(A, \mu)$ is the collection of all complex-valued sequences $s = \{s_Q\}_{Q \in \mathcal{Q}_0}$ such that

$$\|s\|_{\mathbf{b}_p^{\alpha,q}} = \left(\sum_{j=0}^{\infty} \left\| \sum_{Q \in \mathcal{Q}, |Q| = |\det A|^{-j}} |Q|^{-\alpha} |s_Q| \tilde{\chi}_Q \right\|_{L^p(\mu)}^q \right)^{1/q} < \infty,$$

where $\tilde{\chi}_Q = |Q|^{-1/2}\chi_Q$ is the L^2-normalized characteristic function of the dilated cube Q.

Since $\mathbf{b}_p^{\alpha,q}$ is trivially isometrically imbedded in $\dot{\mathbf{b}}_p^{\alpha,q}$, virtually all results for $\dot{\mathbf{b}}_p^{\alpha,q}$ have immediate analogues for $\mathbf{b}_p^{\alpha,q}$. In particular, it is immediate that Lemma 3.4 holds for $\mathbf{b}_p^{\alpha,q}$.

We can also define φ-transform S_φ and the inverse φ-transform T_ψ corresponding to the inhomogeneous setting.

Definition 3.4. *Suppose that $\Phi, \Psi \in \mathcal{S}(\mathbb{R}^n)$, $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (3.2), (3.3), and (3.31). Define the inhomogeneous φ-transform $S_\varphi = S_{\Phi,\varphi}$ to be the map taking each $f \in \mathcal{S}'(\mathbb{R}^n)$ to the sequence $S_\varphi f = \{(S_\varphi f)_Q\}_{Q \in \mathcal{Q}_0}$ defined by*

$$(S_\varphi f)_Q = \langle f, \Phi_Q \rangle \quad \text{if} \quad |Q| = 1, \qquad (S_\varphi f)_Q = \langle f, \varphi_Q \rangle \quad \text{if} \quad |Q| < 1.$$

The inhomogeneous inverse φ-transform $T_\psi = T_{\Psi,\psi}$ is the map taking the sequence $s = \{s_Q\}_{Q \in \mathcal{Q}_0}$ to $T_\psi s = \sum_{|Q|=1} s_Q \Psi_Q + \sum_{|Q|<1} s_Q \psi_Q$.

Given a pair $\Phi, \varphi \in \mathcal{S}$ satisfying (3.2), (3.30), and (3.31) one can show that there exists another pair $\Psi, \psi \in \mathcal{S}$ satisfying the same properties such that

$$\overline{\hat{\Phi}(\xi)}\hat{\Psi}(\xi) + \sum_{j=1}^{\infty} \overline{\hat{\varphi}((A^*)^{-j}\xi)}\hat{\psi}((A^*)^{-j}\xi) = 1 \qquad \text{for all } \xi \in \mathbb{R}^n. \tag{3.32}$$

Moreover, we have the following representation formula,

$$f = \sum_{|Q|=1} \langle f, \Phi_Q \rangle \Psi_Q + \sum_{|Q|<1} \langle f, \varphi_Q \rangle \psi_Q, \tag{3.33}$$

for any $f \in \mathcal{S}'(\mathbb{R}^n)$ with convergence in $\mathcal{S}'$.

One can then show that Theorem 3.5 adapts directly to inhomogeneous setting. That is, S_φ is a bounded operator from $\mathbf{B}_p^{\alpha,q}$ to $\mathbf{b}_p^{\alpha,q}$ and T_ψ is a bounded operator from $\mathbf{b}_p^{\alpha,q}$ to $\mathbf{B}_p^{\alpha,q}$. Moreover, if $\Phi, \Psi \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (3.2), (3.30), and (3.31), then $T_\psi \circ S_\varphi$ is the identity operator on $\mathbf{B}_p^{\alpha,q}$. Consequently, an analogue of Corollary 3.7 also holds for $\mathbf{B}_p^{\alpha,q}$. Note that technical convergence issues in $\mathcal{S}'/\mathcal{P}$ covered by Proposition 3.8 and Corollary 3.9 are non-existent in inhomogeneous case thanks to (3.33).

4. Almost diagonal operators

In this section we study the class of almost diagonal operators on $\dot{\mathbf{b}}_p^{\alpha,q}(A,\mu)$, which was introduced in the dyadic case by Frazier and Jawerth [21]. Almost diagonal operators on $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces for expansive dilations were introduced by the author and Ho [4]. The interest of these operators on $\dot{\mathbf{b}}_p^{\alpha,q}$ spaces arises from their close connection to operators on Besov $\dot{\mathbf{B}}_p^{\alpha,q}$ spaces.

Definition 4.1. *Suppose $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ a ρ_A-doubling measure. Let $J = \beta/p + \max(0, 1-1/p)$, where β is ρ_A-doubling constant of μ. We say that a matrix $\{a_{QP}\}_{Q,P\in\mathcal{Q}}$ is almost diagonal, if there exists an $\epsilon > 0$ such that,*

$$\sup_{Q,P\in\mathcal{Q}} |a_{QP}|/\kappa_{QP}(\epsilon) < \infty \tag{4.1}$$

where

$$\kappa_{QP}(\epsilon) = \left(\frac{|Q|}{|P|}\right)^{\alpha} \left(1 + \frac{\rho_A(x_Q - x_P)}{\max(|P|,|Q|)}\right)^{-J-\epsilon} \min\left[\left(\frac{|Q|}{|P|}\right)^{\frac{1+\epsilon}{2}}, \left(\frac{|P|}{|Q|}\right)^{\frac{1+\epsilon}{2}+J-1}\right].$$

Consequently, we say that $\mathcal{A}$ is an almost diagonal operator on $\dot{\mathbf{b}}_p^{\alpha,q}(A,\mu)$, if it is associated with an almost diagonal matrix $\{a_{QP}\}_{Q,P\in\mathcal{Q}}$. That is, $(\mathcal{A}s)_Q = \sum_{Q\in\mathcal{Q}} a_{QP}s_P$ for every $s = \{s_P\}_P \in \dot{\mathbf{b}}_p^{\alpha,q}$.

Remark 4.1. We remark that almost diagonal condition for $\dot{\mathbf{b}}_p^{\alpha,q}$ and $\dot{\mathbf{f}}_p^{\alpha,q}$ are identical, with the exception of the method of determining the decay parameter J, see [4, Definition 4.1]. In the unweighted case the corresponding $J = \max(1, 1/p, 1/q)$ for $\dot{\mathbf{f}}_p^{\alpha,q}$ versus $J = \max(1, 1/p)$ for $\dot{\mathbf{b}}_p^{\alpha,q}$. Moreover, since the doubling constant always satisfies $\beta \ge 1$, we have $J \ge 1$.

Lemma 4.1. *Suppose μ is a ρ_A–doubling measure, and $L > \beta$, the doubling constant of μ. Then for any $i \le j \in \mathbb{Z}$ and $Q \in \mathcal{Q}$ with $|Q| = |\det A|^{-j}$, we have*

$$\sum_{P\in\mathcal{Q},\ |P|=|\det A|^{-i}} \frac{\mu(P)}{(1+\rho_A(x_Q - x_P)/|P|)^L} \le C|\det A|^{(j-i)\beta}\mu(Q), \tag{4.2}$$

where the constant C depends only on L and β.

Proof. Let $P' \in \mathcal{Q}$ be such that $x_Q \in P'$ and $|P'| = |\det A|^{-i}$. Since μ is ρ_A-doubling, it is not hard to show that there exists $C > 0$ (independent of i, j, and Q) such that $\mu(P') \le C|\det A|^{(j-i)\beta}\mu(Q)$, see (3.7). Therefore, by (3.17),

$$\begin{aligned}\sum_{P\in\mathcal{Q},\ |P|=|\det A|^{-i}} & \frac{\mu(P)}{(1+\rho_A(x_Q-x_P)/|P|)^L} \\ &\le C \sum_{P\in\mathcal{Q},\ |P|=|\det A|^{-i}} \frac{\mu(P)}{(1+\rho_A(x_{P'}-x_P)/|P|)^L} \\ &\le C\mu(P') \le C|\det A|^{(j-i)\beta}\mu(Q),\end{aligned}$$

which shows (4.2). □

Theorem 4.2. *Suppose $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ a ρ_A-doubling measure. An almost diagonal operator $\mathcal{A}$ is bounded as a linear operator on $\dot{\mathbf{b}}_p^{\alpha,q}(A,\mu)$.*

Proof. We will first show that the proof of Theorem 4.2 can be reduced to the case of $\alpha = 0$. This is not surprising, since analogous reducing technique works also for $\dot{\mathbf{f}}_p^{\alpha,q}$, see [21, Theorem 3.3].

Suppose that Theorem 4.2 is true in the case $\alpha = 0$. Let $\mathcal{A}$ be an almost diagonal operator on $\dot{\mathbf{b}}_p^{\alpha,q}$ with matrix $\{a_{QP}\}_{Q,P}$. Let $\mathcal{B}$ be a linear operator on $\dot{\mathbf{b}}_p^{0,q}$ with matrix $\{b_{QP}\}_{Q,P}$ defined by

$$b_{QP} = (|P|/|Q|)^\alpha a_{QP}.$$

It is easy to see that $\mathcal{B}$ satisfies the almost diagonal condition (4.1) with $\alpha = 0$. Given $\{s_P\}_P \in \dot{\mathbf{b}}_p^{\alpha,q}$, define $\{t_P\}_P \in \dot{\mathbf{b}}_p^{0,q}$ by $t_P = |P|^{-\alpha}s_P$. By Theorem 4.2 for $\alpha = 0$,

$$\begin{aligned}\left\|\left\{\sum_P a_{QP}s_P\right\}_Q\right\|_{\dot{\mathbf{b}}_p^{\alpha,q}} &= \left\|\left\{\sum_P |Q|^{-\alpha}a_{QP}s_P\right\}_Q\right\|_{\dot{\mathbf{b}}_p^{0,q}} = \left\|\left\{\sum_P b_{QP}t_P\right\}_Q\right\|_{\dot{\mathbf{b}}_p^{0,q}} \\ &\le C\|\{t_P\}_P\|_{\dot{\mathbf{b}}_p^{0,q}} = C\|\{s_P\}_P\|_{\dot{\mathbf{b}}_p^{\alpha,q}},\end{aligned}$$

which shows the reduction to the case $\alpha = 0$.

Next, it remains to show Theorem 4.2 for $\alpha = 0$. Again, we consider only the case of $p, q < \infty$ leaving details of the special case of $p = \infty$ or $q = \infty$ to the reader.

Let $\mathcal{A}$ be an almost diagonal operator on $\dot{\mathbf{b}}_p^{0,q}$ with matrix $\{a_{QP}\}_{Q,P}$ satisfying condition (4.1). We write $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$, with

$$(\mathcal{A}_0 s)_Q = \sum_{P\in\mathcal{Q},\ |P|\ge|Q|} a_{QP}s_P \qquad \text{and} \qquad (\mathcal{A}_1 s)_Q = \sum_{P\in\mathcal{Q},\ |P|<|Q|} a_{QP}s_P$$

for $s = \{s_P\}_P \in \dot{\mathbf{b}}_p^{0,q}$. For $Q \in \mathcal{Q}$, $|Q| = |\det A|^{-j}$, we have

$$\begin{aligned}|(\mathcal{A}_0 s)_Q| &\le C \sum_{|P|\ge|Q|} \kappa_{QP}(\epsilon)|s_P| \\ &\le C \sum_{|P|\ge|Q|} \left(\frac{|Q|}{|P|}\right)^{(1+\epsilon)/2} \frac{|s_P|}{(1+|P|^{-1}\rho_A(x_P-x_Q))^{J+\epsilon}}\end{aligned}$$

$$\leq C \sum_{i\in\mathbb{Z},\ i\leq j} \sum_{|P|=|\det A|^{-i}} |\det A|^{(i-j)(1+\epsilon)/2} \frac{|s_P|}{(1+|P|^{-1}\rho_A(x_P-x_Q))^{J+\epsilon}}.$$

Hence, if $p > 1$ then by Hölder's inequality, $1/p + 1/p' = 1$,

$$|(\mathcal{A}_0 s)_Q|^p \leq C\bigg(\sum_{i\leq j} \sum_{|P|=|\det A|^{-i}} \frac{|\det A|^{(i-j)p'\epsilon/4}}{(1+|P|^{-1}\rho_A(x_P-x_Q))^{1+p'\epsilon/2}}\bigg)^{p/p'}$$
$$\times \sum_{i\leq j} \sum_{|P|=|\det A|^{-i}} |\det A|^{(i-j)p(1/2+\epsilon/4)} \frac{|s_P|^p}{(1+|P|^{-1}\rho_A(x_P-x_Q))^{\beta+p\epsilon/2}}.$$

By Proposition 2.1 and by a simple geometric series estimate, the double sum in the above parenthesis is finite, and hence

$$|(\mathcal{A}_0 s)_Q|^p \leq C \sum_{i\leq j} \sum_{|P|=|\det A|^{-i}} \frac{|\det A|^{(i-j)p(1/2+\epsilon/4)}|s_P|^p}{(1+|P|^{-1}\rho_A(x_P-x_Q))^{\beta+p\epsilon/2}}. \tag{4.3}$$

Analogously, if $p \leq 1$ then by p-triangle inequality we also obtain (4.3). Since,

$$\rho_A(x_P - x) \leq H(\rho_A(x_P - x_Q) + \rho_A(x_Q - x)) \leq H(\rho_A(x_P - x_Q) + |Q|c)$$
$$\text{for any } x \in Q,$$

hence, by (4.3),

$$\|\mathcal{A}_0 s\|^q_{\dot{\mathbf{b}}^{0,q}_p} \leq C \sum_{j\in\mathbb{Z}} \bigg(\int_{\mathbb{R}^n} \sum_{i\leq j} \sum_{\substack{|P|=|\det A|^{-i}\\ |Q|=|\det A|^{-j}}} \frac{|\det A|^{(i-j)p(1/2+\epsilon/4)}|s_P|^p}{(1+|P|^{-1}\rho_A(x_P-x))^{\beta+p\epsilon/2}} \frac{\chi_Q(x)}{|Q|^{p/2}} d\mu(x)\bigg)^{q/p}$$
$$= C \sum_{j\in\mathbb{Z}} \bigg(\sum_{i\leq j} \sum_{|P|=|\det A|^{-i}} \int_{\mathbb{R}^n} \frac{|\det A|^{(i-j)p(1/2+\epsilon/4)}|s_P|^p}{(1+|P|^{-1}\rho_A(x_P-x))^{\beta+p\epsilon/2}} \frac{d\mu(x)}{|\det A|^{-jp/2}}\bigg)^{q/p}$$
$$\leq C \sum_{j\in\mathbb{Z}} \bigg(\sum_{i\leq j} |\det A|^{(i-j)p\epsilon/4} \sum_{|P|=|\det A|^{-i}} |s_P|^p \frac{\mu(P)}{|P|^{p/2}}\bigg)^{q/p}.$$

Here, we used that for any $L > \beta$ and $P \in \mathcal{Q}$,

$$\int_{\mathbb{R}^n} \frac{d\mu(x)}{(1+\rho_A(x_P-x)/|P|)^L} \leq C\mu(P),$$

which is a continuous analogue of (3.17). Thus, if $q/p > 1$ then by Hölder's inequality, $p/q + 1/r = 1$, the last expression in parenthesis is bounded by

$$\bigg(\sum_{i\leq j} |\det A|^{(i-j)pr\epsilon/8}\bigg)^{1/r} \bigg(\sum_{i\leq j} |\det A|^{(i-j)q\epsilon/8} \bigg(\sum_{|P|=|\det A|^{-i}} |s_P|^p \frac{\mu(P)}{|P|^{p/2}}\bigg)^{q/p}\bigg)^{p/q}$$
$$\leq C\bigg(\sum_{i\leq j} |\det A|^{(i-j)q\epsilon/8} \bigg\|\sum_{|P|=|\det A|^{-i}} s_P \tilde{\chi}_P\bigg\|^q_{L^p(\mu)}\bigg)^{p/q}$$

Likewise, if $q/p \le 1$, then by q/p-triangle inequality we arrive at the same estimate. Therefore,

$$\begin{aligned}\|\mathcal{A}_0 s\|^q_{\dot{\mathbf{b}}^{0,q}_p} &\le C \sum_{j\in\mathbb{Z}} \sum_{i\le j} |\det A|^{(i-j)q\epsilon/8} \Big\| \sum_{|P|=|\det A|^{-i}} s_P \tilde{\chi}_P \Big\|^q_{L^p(\mu)} \\ &\le C \sum_{i\in\mathbb{Z}} \Big\| \sum_{|P|=|\det A|^{-i}} s_P \tilde{\chi}_P \Big\|^q_{L^p(\mu)} = C\|s\|^q_{\dot{\mathbf{b}}^{0,q}_p}.\end{aligned}$$

To complete the proof of Theorem 4.2, it remains to show similar bounds for $\mathcal{A}_1 s$. For $Q \in \mathcal{Q}$, $|Q| = |\det A|^{-j}$, we have

$$\begin{aligned}|(\mathcal{A}_1 s)_Q| \le C \sum_{i\in\mathbb{Z},\ i>j} \sum_{|P|=|\det A|^{-i}} &|\det A|^{(j-i)(\epsilon/2+J-1/2)} \\ &\times \frac{|s_P|}{(1+|Q|^{-1}\rho_A(x_P - x_Q))^{J+\epsilon}}.\end{aligned}$$

Thus, by similar calculations as for $\mathcal{A}_0 s$ and by Hölder's inequality or by p-triangle inequality,

$$|(\mathcal{A}_1 s)_Q|^p \le C \sum_{i>j} \sum_{|P|=|\det A|^{-i}} \frac{|\det A|^{(j-i)p(\beta/p-1/2+\epsilon/4)}|s_P|^p}{(1+|Q|^{-1}\rho_A(x_P-x_Q))^{\beta+p\epsilon/2}}. \tag{4.4}$$

Hence, by Lemma 4.1 and by summing first over Q,

$$\begin{aligned}\|\mathcal{A}_1 s\|^q_{\dot{\mathbf{b}}^{0,q}_p} &\le C \sum_{j\in\mathbb{Z}} \Bigg(\sum_{i>j} \sum_{\substack{|P|=|\det A|^{-i}\\ |Q|=|\det A|^{-j}}} \frac{|\det A|^{(j-i)p(\beta/p-1/2+\epsilon/4)}|s_P|^p}{(1+|Q|^{-1}\rho_A(x_P-x_Q))^{\beta+p\epsilon/2}} \frac{\mu(Q)}{|Q|^{p/2}} \Bigg)^{q/p} \\ &\le C \sum_{j\in\mathbb{Z}} \Bigg(\sum_{i>j} |\det A|^{(j-i)p(\beta/p+\epsilon/4)+(i-j)\beta} \sum_{|P|=|\det A|^{-i}} |s_P|^p \frac{\mu(P)}{|P|^{p/2}} \Bigg)^{q/p} \\ &\le C \sum_{j\in\mathbb{Z}} \Bigg(\sum_{i>j} |\det A|^{(j-i)p\epsilon/4} \sum_{|P|=|\det A|^{-i}} |s_P|^p \frac{\mu(P)}{|P|^{p/2}} \Bigg)^{q/p}.\end{aligned}$$

Again, by Hölder's inequality (if $q/p > 1$) or by q/p-triangle inequality (if $q/p \le 1$),

$$\begin{aligned}\|\mathcal{A}_1 s\|^q_{\dot{\mathbf{b}}^{0,q}_p} &\le C \sum_{j\in\mathbb{Z}} \sum_{i>j} |\det A|^{(j-i)q\epsilon/8} \Big\| \sum_{|P|=|\det A|^{-i}} s_P \tilde{\chi}_P \Big\|^q_{L^p(\mu)} \\ &\le C \sum_{i\in\mathbb{Z}} \Big\| \sum_{|P|=|\det A|^{-i}} s_P \tilde{\chi}_P \Big\|^q_{L^p(\mu)} = C\|s\|^q_{\dot{\mathbf{b}}^{0,q}_p}.\end{aligned}$$

This completes the proof of Theorem 4.2. □

5. Atomic and molecular decompositions

5.1. Smooth molecules

Our goal is to define smooth molecules on Besov spaces adapted to anisotropic setting of expansive dilation matrices considered in this work. To achieve this we will adapt the usual notion of smooth molecules associated with dyadic dilations by Frazier and Jawerth [19,21,22] to a non-isotropic situation. For the motivation behind Definition 5.1 we refer to [4, Section 5.1].

Definition 5.1. *Suppose $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and β is a ρ_A -doubling constant of measure μ. Let $J = \beta/p + \max(0, 1-1/p)$ and $N = \max(\lfloor (J-\alpha-1)/\zeta_- \rfloor, -1)$.*

We say that $\Psi_Q(x)$ is a smooth synthesis molecule for $\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ supported near $Q \in \mathcal{Q}$ with $|Q| = |\det A|^{-j}$ and $j \in \mathbb{Z}$, if there exist $M > J$ such that

$$|\partial^\gamma [\Psi_Q(A^{-j}\cdot)](x)| \le \frac{|\det A|^{j/2}}{(1+\rho_A(x - A^j x_Q))^M} \qquad \text{for } |\gamma| \le \lfloor \alpha/\zeta_- \rfloor + 1, \tag{5.1}$$

$$|\Psi_Q(x)| \le \frac{|\det A|^{j/2}}{(1+\rho_A(A^j(x - x_Q)))^{\max(M,(M-\alpha)\zeta_+/\zeta_-)}}, \tag{5.2}$$

$$\int x^\gamma \Psi_Q(x)dx = 0 \qquad \text{for } |\gamma| \le N. \tag{5.3}$$

We say that a collection $\{\Psi_Q\}_{Q\in\mathcal{Q}}$ is a family of smooth synthesis molecules, if each Ψ_Q is a smooth synthesis molecule supported near Q.

We say that $\Phi_Q(x)$ is a smooth analysis molecule for $\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ supported near $Q \in \mathcal{Q}$ with $|Q| = |\det A|^{-j}$ and $j \in \mathbb{Z}$, if there exists $M > J$ such that

$$|\partial^\gamma [\Phi_Q(A^{-j}\cdot)](x)| \le \frac{|\det A|^{j/2}}{(1+\rho_A(x - A^j x_Q))^M} \qquad \text{for } |\gamma| \le N + 1, \tag{5.4}$$

$$|\Phi_Q(x)| \le \frac{|\det A|^{j/2}}{(1+\rho_A(A^j(x - x_Q)))^{\max(M,1+\alpha\zeta_+/\zeta_- + M - J)}}, \tag{5.5}$$

$$\int x^\gamma \Phi_Q(x)dx = 0 \qquad \text{for } |\gamma| \le \lfloor \alpha/\zeta_- \rfloor. \tag{5.6}$$

We say that $\{\Phi_Q\}_{Q\in\mathcal{Q}}$ is a family of smooth analysis molecules, if each Φ_Q is a smooth analysis molecule supported near Q.

The following remark is needed to clarify the above definition.

Remark 5.1. In Definition 5.1, Φ_Q and Ψ_Q should be understood as a function indexed by $Q \in \mathcal{Q}$, which is not necessarily equal to the usual convention $\Psi_Q(x) = |\det A|^{j/2}\Psi(A^j x - k)$ used throughout Section 3. Conditions (5.1) and (5.4) should be understood as follows. Let D_A be the *dilation operator* given by $D_A f(x) = f(Ax)$. Then the left hand side of (5.1) is simply $|\partial^\gamma (D_{A^{-j}}\Psi_Q)(x)|$ and similarly for (5.4). Moreover, to avoid any ambiguity, (5.1) and (5.4) require that Ψ_Q and Φ_Q have continuous partial derivatives of order $\lfloor \alpha/\zeta_- \rfloor + 1$ and $N + 1$, respectively.

Remark 5.2. If $\alpha < 0$ then the smoothness condition (5.1) is void. Likewise, if $\alpha > J - 1$ then $N = -1$ and the vanishing moment condition (5.3) is void. Furthermore, if $\alpha = 0$, $0 < p \leq 1$ and μ is the Lebesgue measure, then $N = \lfloor (J-1)/\zeta_- \rfloor = \lfloor (1/p - 1)/\zeta_- \rfloor$, and (5.3) coincides with the vanishing moment condition for atoms in the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$, see [3, Section 4]. Similar comments are applicable for smooth analysis molecules.

Remark 5.3. Analogous definition of smooth molecules for Triebel-Lizorkin spaces was introduced by the author and Ho [4]. The definitions of smooth molecules for both $\dot{\mathbf{B}}_p^{\alpha,q}$ and $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces are virtually identical with the only exception being the method of calculating the decay parameter J. Hence, all properties of smooth molecules derived in [4] can be readily applied to the case of Besov spaces.

The main motivation behind somewhat non-obvious orders of decay and smoothness imposed on smooth molecules is revealed in the following lemma, which is a non-isotropic variant of [21, Corollary B.3].

Lemma 5.1. *Suppose $\{\Phi_Q\}_Q$ and $\{\Psi_Q\}_Q$ are families of smooth analysis and synthesis molecules for $\dot{\mathbf{B}}_p^{\alpha,q}$, respectively. Then the matrix $\{a_{QP}\}$, given by $a_{QP} = \langle \Psi_P, \Phi_Q \rangle$, is almost diagonal on $\dot{\mathbf{b}}_p^{\alpha,q}$. More precisely, there exist $C > 0$ and $\epsilon > 0$, such that*

$$|\langle \Psi_P, \Phi_Q \rangle| \leq C \kappa_{QP}(\epsilon) \qquad \text{for all } Q, P \in \mathcal{Q}.$$

The proof of Lemma 5.1 for $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces can be found in [4]. Since the notion of almost diagonal matrices is identical for both $\dot{\mathbf{b}}_p^{\alpha,q}$ and $\dot{\mathbf{f}}_p^{\alpha,q}$ (with the exception of the way the decay parameter J is computed) then by Remark 5.3, exactly the same proof as in [4] yields the corresponding result for $\dot{\mathbf{B}}_p^{\alpha,q}$.

We also have two immediate consequences of Lemma 5.1 and the following approximation result, the proof of which can be found in [4].

Corollary 5.2. *Suppose $\{\Psi_Q\}_Q$ is a family of smooth synthesis molecules for $\dot{\mathbf{B}}_p^{\alpha,q}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $0 \notin \operatorname{supp} \hat{\varphi}$. Then the matrix $\{a_{QP}\}$, given by $a_{QP} = \langle \Psi_P, \varphi_Q \rangle$, is almost diagonal.*

Corollary 5.3. *Suppose $\{\Phi_Q\}_Q$ is a family of smooth analysis molecules for $\dot{\mathbf{B}}_p^{\alpha,q}$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $0 \notin \operatorname{supp} \hat{\psi}$. Then the matrix $\{a_{QP}\}$, given by $a_{QP} = \langle \psi_P, \Phi_Q \rangle$, is almost diagonal.*

Lemma 5.4. *Suppose that Φ is a smooth analysis (or synthesis) molecule of supported near $Q \in \mathcal{Q}$. Then there exists a sequence $\{\phi_k\}_{k=1}^{\infty} \subset \mathcal{S}$ and $c > 0$ such that $c\phi_k$ is a smooth analysis (or synthesis) molecule supported near Q for every k, and $\phi_k(x) \to \Phi(x)$ uniformly on $\mathbb{R}^n$ as $k \to \infty$.*

5.2. Smooth molecular decompositions

We are now ready to show generalizations of Theorem 3.5 in the situation when the usual wavelet families of translates and dilates $\{\varphi_Q\}_{Q \in \mathcal{Q}}$ and $\{\psi_Q\}_{Q \in \mathcal{Q}}$ are replaced by families of smooth analysis $\{\Phi_Q\}_{Q \in \mathcal{Q}}$ and synthesis molecules $\{\Psi_Q\}_{Q \in \mathcal{Q}}$.

Theorem 5.5 (Smooth Molecular Synthesis)**.** *Suppose A is an expansive matrix and μ is a ρ_A-doubling measure. Then, there exists a constant $C > 0$, such that for $f = \sum_{Q\in\mathcal{Q}} s_Q \Psi_Q$ and $\{\Psi_Q\}_Q$ a family of smooth synthesis molecules for $\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$, we have*

$$\|f\|_{\dot{\mathbf{B}}_p^{\alpha,q}} \le C\|\{s_Q\}_Q\|_{\dot{\mathbf{b}}_p^{\alpha,q}} \qquad \textit{for all } \{s_Q\}_Q \in \dot{\mathbf{b}}_p^{\alpha,q}.$$

Proof. By Lemma 2.5, we can write $\Psi_P = \sum_Q \langle \Psi_P, \varphi_Q\rangle \psi_Q$ with the convergence in $\mathcal{S}'/\mathcal{P}$. By Theorem 4.2 and Corollary 5.2, $\mathcal{A}$ given by the matrix $\{a_{QP}\}_{Q,P} = \{\langle \Psi_P, \varphi_Q\rangle\}_{Q,P}$ is a bounded operator on $\dot{\mathbf{b}}_p^{\alpha,q}(A,\mu)$. Since,

$$\mathrm{T}_\psi \mathcal{A}s = \sum_Q \sum_P a_{QP} s_P \psi_Q = \textstyle\sum_P s_P \sum_Q \langle \Psi_P, \varphi_Q\rangle \psi_Q = \sum_P s_P \Psi_P = f$$

then by Theorem 3.5,

$$\|f\|_{\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)} = \|\mathrm{T}_\psi \mathcal{A}s\|_{\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)} \le C\|\mathcal{A}s\|_{\dot{\mathbf{b}}_p^{\alpha,q}(A,\mu)} \le C\|s\|_{\dot{\mathbf{b}}_p^{\alpha,q}(A,\mu)}.$$

□

If $0 < p \le 1$ then the conclusion of Theorem 5.7 still holds when smooth molecules $\{\Psi_Q\}_Q$ are not necessarily indexed by the usual family of dilated cubes $\mathcal{Q}$, but instead they can be indexed by less structured families of cubes. More precisely, one can allow families of cubes of the form

$$\mathcal{Q}' = \{Q' = A^{-j}([0,1]^n + x_{j,k}) : j \in \mathbb{Z},\ x_{j,k} \in \mathbb{R}^n\}$$

where for each $j \in \mathbb{Z}$, $\{x_{j,k}\}_k$ is an arbitrary sequence of points in $\mathbb{R}^n$. Then we have the following result.

Theorem 5.6. *Suppose that $0 < p \le 1$ and $\{\Psi_{Q'}\}_{Q'\in\mathcal{Q}'}$ is a family of smooth synthesis molecules supported near cubes $Q' \in \mathcal{Q}'$, where $\mathcal{Q}'$ is as above. Then, for $f = \sum_{Q'\in\mathcal{Q}'} s_{Q'}\Psi_{Q'}$, we have*

$$\|f\|_{\dot{\mathbf{B}}_p^{\alpha,q}} \le C\Bigg(\sum_{j\in\mathbb{Z}}\Bigg(\sum_{Q'\in\mathcal{Q}',\ |Q'|=|\det A|^{-j}} (|Q'|^{-\alpha-1/2}|s_{Q'}|)^p \mu(Q')\Bigg)^{q/p}\Bigg)^{1/q}. \tag{5.7}$$

Proof. Since we cannot use almost diagonality argument as in Theorem 5.5, we must resort to elementary, but tedious calculations yielding

$$|\varphi_j * \Psi_{Q'}(x)| \le C \begin{cases} |\det A|^{j/2-(j-i)(\alpha+\varepsilon+1/2)}(1+\rho_A(A^i(x-A^{-i}x_{i,k})))^{-M} & \text{for } j \ge i, \\ |\det A|^{j/2-(i-j)(J-\alpha+\varepsilon-1/2)}(1+\rho_A(A^j(x-A^{-i}x_{i,k})))^{-M} & \text{for } j < i, \end{cases}$$

where φ satisfies (3.2) and (3.3), $Q' \in \mathcal{Q}'$ with $|Q'| = |\det A|^{-i}$, $M > J = \beta/p$ and $\varepsilon > 0$. Alternatively, one can show the above by applying [4, Lemmas 6.3 and 6.4]. Hence, if we write

$$\varphi_j * f = \Bigg(\sum_{i\le j} + \sum_{i>j}\Bigg) \sum_{|Q'|=|\det A|^{-i}} s_{Q'}\varphi_j * \Psi_{Q'},$$

and use a continuous version of estimate (4.2) and p-triangle inequality, we obtain

$$\begin{aligned}||f||^q_{\dot{\mathbf{B}}^{\alpha,q}_p} \le C \sum_{j\in\mathbb{Z}} \Bigg(\sum_{i\le j} |\det A|^{ip(\alpha+1/2)-(j-i)p\varepsilon} \sum_{|Q'|=|\det A|^{-i}} |s_{Q'}|^p \mu(Q') \Bigg)^{q/p} \\ +C \sum_{j\in\mathbb{Z}} \Bigg(\sum_{i>j} |\det A|^{ip(\alpha+1/2)-(i-j)p(J+\varepsilon)} \\ \times \sum_{|Q'|=|\det A|^{-i}} |s_{Q'}|^p |\det A|^{\beta(i-j)} \mu(Q') \Bigg)^{q/p}.\end{aligned}$$

Thus, by Hölder's inequality if $q \ge p$ and by q/p-triangle inequality if $q < p$, we have (5.7). □

Next, we will show the converse of Theorem 5.5.

Theorem 5.7 (Smooth Molecular Analysis)**.** *Suppose A is an expansive matrix and μ a ρ_A-doubling measure. There exists a constant $C > 0$, such that, if $\{\Phi_Q\}_Q$ is a family of smooth analysis molecules, then*

$$\|\{\langle f, \Phi_Q\rangle\}_Q\|_{\dot{\mathbf{b}}^{\alpha,q}_p} \le C\|f\|_{\dot{\mathbf{B}}^{\alpha,q}_p} \qquad \textit{for all } f \in \dot{\mathbf{B}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu).$$

The main technical difficulty in the proof Theorem 5.7 is to justify the meaningfulness of the pairing $\langle f, \Phi_Q\rangle$, since $f \in \dot{\mathbf{B}}^{\alpha,q}_p$ is an equivalence class in $\mathcal{S}'/\mathcal{P}$, and Φ_Q may not even belong to $\mathcal{S}$. Therefore, we need to show a precise pairing procedure, which is a consequence of Proposition 3.8.

Lemma 5.8. *Suppose $f \in \dot{\mathbf{B}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)$ and Φ_Q is a smooth analysis molecule for $\dot{\mathbf{B}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)$ supported near $Q \in \mathcal{Q}$. Then for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (2.6) and (2.7), the series*

$$\langle f, \Phi_Q\rangle := \sum_{j\in\mathbb{Z}} \langle \tilde{\varphi}_j * \psi_j * f, \Phi_Q\rangle = \sum_{P\in\mathcal{Q}} \langle f, \varphi_P\rangle\langle \psi_P, \Phi_Q\rangle \tag{5.8}$$

converges absolutely and its value is independent of the choice of φ and ψ satisfying (2.6) and (2.7).

The proof of Lemma 5.8 is virtually the same as the proof of the analogous result for Triebel-Lizorkin spaces, see [4, Lemma 5.7].

Proof. First, note that for any $f \in \dot{\mathbf{B}}^{\alpha,q}_p$, there exists a matrix $\{b_{QP}\}_{Q,P\in\mathcal{Q}}$ such that $b_{QP} \ge 0$ and

$$|\langle f, \varphi_P\rangle||\langle \psi_P, \phi\rangle| \le b_{QP} \quad \text{and} \quad \sum_P b_{QP} < \infty, \tag{5.9}$$

whenever ϕ is a smooth analysis molecule supported near Q. Indeed, (5.9) follows by Corollary 5.3 and Theorems 3.5 and 4.2. This shows the absolute convergence of the series in (5.8).

To show independence of the choice of φ and ψ, let $\{\phi_l\}_{l=1}^{\infty} \subset \mathcal{S}$ be the sequence of (constant multiples of) smooth analysis molecules supported near Q and converging uniformly to Φ_Q guaranteed by Lemma 5.4. By Proposition 3.8 and Corollary 3.9, there exists a sequence of polynomials $\{P_k\}_{k=1}^{\infty}$ with deg $P_k \le L = \lfloor \alpha/\zeta_- \rfloor$ such that $\sum_{j=-k}^{\infty} \tilde{\varphi}_j * \psi_j * f + P_k$ converges in $\mathcal{S}'$ as $k \to \infty$. Therefore, for each l, we can define

$$\begin{aligned}\langle f, \phi_l \rangle &:= \left\langle \lim_{k\to\infty} \sum_{j=-k}^{\infty} \tilde{\varphi}_j * \psi_j * f + P_k, \phi_l \right\rangle = \lim_{k\to\infty} \sum_{j=-k}^{\infty} \langle \tilde{\varphi}_j * \psi_j * f, \phi_l \rangle \\ &= \lim_{k\to\infty} \sum_{P\in\mathcal{Q}, |P|\ge |\det A|^{-k}} \langle f, \varphi_P \rangle \langle \psi_P, \phi_l \rangle = \sum_{P\in\mathcal{Q}} \langle f, \varphi_P \rangle \langle \psi_P, \phi_l \rangle,\end{aligned}$$

since the above series converges absolutely by (5.9). Moreover, by (3.23) in Proposition 3.8 and (5.6), this definition does not depend on the choice of φ and ψ. Since $\langle \psi_P, \phi_l \rangle \to \langle \psi_P, \Phi_Q \rangle$ as $l \to \infty$, by (5.8) and the Lebesgue Dominated Convergence Theorem,

$$\sum_{P\in\mathcal{Q}} \langle f, \varphi_P \rangle \langle \psi_P, \phi_l \rangle \to \sum_{P\in\mathcal{Q}} \langle f, \varphi_P \rangle \langle \psi_P, \Phi_Q \rangle \qquad \text{as } l \to \infty.$$

By the above reasoning, this limit is independent of φ and ψ satisfying (2.6) and (2.7). This shows that $\langle f, \Phi_Q \rangle$ is well-defined by (5.8) and completes the proof of Lemma 5.8. □

Proof of Theorem 5.7. Once Lemma 5.8 is shown, the proof of Theorem 5.7 is trivial. Recall that by Lemma 5.8,

$$\langle f, \Phi_Q \rangle := \sum_P \langle f, \varphi_P \rangle \langle \psi_P, \Phi_Q \rangle.$$

By Theorem 4.2 and Corollary 5.2, the operator $\mathcal{A}$ given by the matrix $\{a_{QP}\}_{Q,P} = \{\langle \psi_P, \Phi_Q \rangle\}_{Q,P}$ is bounded on $\dot{\mathbf{b}}_p^{\alpha,q}(A,\mu)$. Since $\langle f, \Phi_Q \rangle = \sum_P \langle f, \varphi_P \rangle a_{QP}$, by Theorem 3.5, we have

$$\|\{\langle f, \Phi_Q \rangle\}\|_{\dot{\mathbf{b}}_p^{\alpha,q}(A,\mu)} = \|\mathcal{A} S_\varphi f\|_{\dot{\mathbf{b}}_p^{\alpha,q}(A,\mu)} \le C \|f\|_{\dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)}. \qquad □$$

5.3. Smooth atomic decompositions

In this subsection we introduce smooth atoms for anisotropic Besov spaces and show their atomic decomposition. This extends the classical smooth atoms for dyadic dilations introduced by Frazier and Jawerth [19].

Definition 5.2. *A function $a_Q(x)$ is said to be a smooth atom supported near a dilated cube $Q = Q_{j,k} = A^{-j}([0,1]^n + k) \in \mathcal{Q}$ if it satisfies*

$$\operatorname{supp} a_Q \subset A^{-j}([-\delta_0, 1+\delta_0]^n + k), \tag{5.10}$$

where $\delta_0 > 0$ is some fixed constant, and

$$|\partial^\gamma [a_Q(A^{-j}\cdot)](x)| \le |Q|^{-1/2} \qquad \text{for } |\gamma| \le \tilde{K}, \tag{5.11}$$

$$\int_{\mathbb{R}^n} x^\gamma a_Q(x)dx = 0 \qquad \text{for } |\gamma| \le \tilde{N}, \tag{5.12}$$

where $\tilde{N} \ge N$ is the same as in Definition 5.1 and $\tilde{K} \ge \max(\lfloor \alpha/\zeta_- \rfloor + 1, 0)$. Recall that

$$N = \max(\lfloor (J-\alpha-1)/\zeta_- \rfloor, -1), \quad \textit{where } J = \beta/p + \max(0, 1-1/p).$$

We say that $\{a_Q\}_{Q\in\mathcal{Q}}$ is a family of smooth atoms, if each function a_Q is a smooth atom supported near Q.

Remark 5.4. It is clear that every smooth atom a_Q is always some fixed constant multiple of a smooth synthesis molecule supported near Q. Moreover, this constant multiple depends only on $\delta_0 > 0$, which controls the relative size of the support of a_Q. Indeed, the support condition (5.10) together with (5.11) imply the decay conditions (5.1) and (5.2) for any value of $M > J$.

Theorem 5.9 (Smooth Atomic Decomposition)**.** *Suppose A is an expansive matrix, $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ a ρ_A-doubling measure. For any $f \in \dot{\mathbf{B}}_p^{\alpha,q}$ there exists a family of smooth atoms $\{a_Q\}$ and a sequence of coefficients $s = \{s_Q\} \in \dot{\mathbf{b}}_p^{\alpha,q}$, such that,*

$$f = \sum_{Q\in\mathcal{Q}} s_Q a_Q, \qquad \textit{and} \qquad \|s\|_{\dot{\mathbf{b}}_p^{\alpha,q}} \le C\|f\|_{\dot{\mathbf{B}}_p^{\alpha,q}}, \tag{5.13}$$

where the above series converges unconditionally in $\dot{\mathbf{B}}_p^{\alpha,q}$. Conversely, for any family of smooth atoms $\{a_Q\}$,

$$\left\| \sum_Q s_Q a_Q \right\|_{\dot{\mathbf{B}}_p^{\alpha,q}} \le C\|s\|_{\dot{\mathbf{b}}_p^{\alpha,q}}. \tag{5.14}$$

Proof. The converse direction (5.14) follows immediately from Theorem 5.5 and Remark 5.4. Let $\theta \in \mathcal{S}$ be such that $\operatorname{supp}\theta \subset B(0,\delta_0)$, and

$$\int x^\gamma \theta(x)dx = 0 \qquad \text{for all } |\gamma| \le \tilde{N}, \tag{5.15}$$

$$|\hat{\theta}(\xi)| \ge c > 0 \qquad \text{for all } (2||A||)^{-1} \le |\xi| \le 1. \tag{5.16}$$

The construction of such θ can be found in [19, Theorem 2.6]. Then, one can show, see [4, Theorem 5.8], that there exists $\varphi \in \mathcal{S}$ satisfying (3.2), (3.3), and

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}((A^*)^j\xi)\hat{\theta}((A^*)^j\xi) = 1 \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

Therefore, by Lemma 2.4 we can expand $f \in \dot{\mathbf{B}}_p^{\alpha,q}$ as

$$f = \sum_{j\in\mathbb{Z}} \theta_j * \varphi_j * f,$$

where the equality and convergence is in $\mathcal{S}'/\mathcal{P}$. Consequently,

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{Q},\ |Q|=|\det A|^{-j}} \int_Q \theta_j(x-y)(\varphi_j * f)(y)dy. \tag{5.17}$$

We will show the above series converges in $\dot{\mathbf{B}}_p^{\alpha,q}$, and consequently, by Lemma 2.5 and Proposition 3.3, it must converge to f in $\mathcal{S}'/\mathcal{P}$. Indeed, for $Q \in \mathcal{Q}$ with $|Q| = |\det A|^{-j}$, we define

$$s_Q = |Q|^{1/2} \sup_{y\in Q} |(\varphi_j * f)(y)|,$$

$$a_Q(x) = \begin{cases} s_Q^{-1} \int_Q \theta_j(x-y)(\varphi_j * f)(y)dy, & \text{if } s_Q \neq 0, \\ 0 & \text{if } s_Q = 0. \end{cases}$$

Hence, we can rewrite (5.17) as $f = \sum_{Q\in\mathcal{Q}} s_Q a_Q$.

Therefore, by Lemma 3.1 (see also the proof of the boundedness of S_φ in Theorem 3.5) we have $s = \{s_Q\}_Q \in \dot{\mathbf{b}}_p^{\alpha,q}$ and $||s||_{\dot{\mathbf{b}}_p^{\alpha,q}} \le C||f||_{\dot{\mathbf{B}}_p^{\alpha,q}}$. Hence, to guarantee, that the series (5.17) converges in $\dot{\mathbf{B}}_p^{\alpha,q}$, by Theorem 5.5 and Remark 5.4, it suffices to verify that each a_Q is a smooth atom. It is immediate that a_Q satisfies (5.10) and (5.12). Finally, to verify (5.11) it may be necessary to re-normalize $\{a_Q\}$ and $\{s_Q\}$ by some fixed multiplicative factor depending only θ. The verification of this is a routine and can be found in [4]. This completes the proof of Theorem 5.9. □

5.4. Atomic and molecular decompositions of $\mathbf{B}_p^{\alpha,q}$

The above smooth atomic and molecular decompositions results for $\dot{\mathbf{B}}_p^{\alpha,q}$ spaces can be easily adapted to inhomogeneous anisotropic Besov spaces $\mathbf{B}_p^{\alpha,q}$ introduced in Section 3.5. Here, we only outline modifications that need to be done to achieve this.

Definition 5.3. *Suppose* $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, *and* μ *a* ρ_A*-doubling measure on* $\mathbb{R}^n$. *Let* $\mathcal{Q}_0 = \{Q \in \mathcal{Q} : |Q| \le 1\}$.

We say that $\Psi_Q(x)$ *is an inhomogeneous smooth synthesis (or analysis) molecule for* $\mathbf{B}_p^{\alpha,q}$ *supported near* $Q \in \mathcal{Q}_0$, *if it satisfies (5.1)–(5.3) (or (5.4)–(5.6)) if* $|Q| < 1$, *and (5.1) (or (5.4)) only if* $|Q| = 1$. *Hence, we do not assume that* Ψ_Q *has any vanishing moments if* $|Q| = 1$. *A collection* $\{\Psi_Q\}_{Q\in\mathcal{Q}_0}$ *is a family of inhomogeneous smooth synthesis (or analysis) molecules, if each* Ψ_Q *is a smooth synthesis (or analysis) molecule supported near* Q.

We say that an operator $\mathcal{A}$ with a matrix $\{a_{PQ}\}_{P,Q\in\mathcal{Q}_0}$ is *almost diagonal* for $\mathbf{b}_p^{\alpha,q}$ if there exists an $\epsilon > 0$ such that

$$\sup_{P,Q\in\mathcal{Q}_0} |a_{QP}|/\kappa_{QP}(\epsilon) < \infty. \tag{5.18}$$

As a consequence of Theorem 4.2 and the observation that $\mathbf{b}_p^{\alpha,q} \hookrightarrow \dot{\mathbf{b}}_p^{\alpha,q}$ is an isometric embedding, any almost diagonal operator $\mathcal{A}$ on $\mathbf{b}_p^{\alpha,q}$ is bounded.

Suppose that $\{\Psi_Q\}$ and $\{\Phi_Q\}$ are families of inhomogeneous smooth synthesis and analysis molecules, respectively. Then the inhomogeneous analogue of Lemma 5.1 holds, i.e., the matrix $\{a_{PQ}\}_{P,Q} = \{\langle \Psi_P, \Phi_Q\rangle\}_{P,Q}$ is almost diagonal on $\mathbf{b}_p^{\alpha,q}$. The proof of this fact is a slight modification of the homogeneous case. As a consequence, we have inhomogeneous analogues of smooth molecular decompositions of Theorems 5.5 and 5.7.

For the inhomogeneous analogue of smooth atomic decomposition of Theorem 5.9 we need the following definition.

Definition 5.4. *A function $a_Q(x)$ is said to be a inhomogeneous smooth atom supported near a dilated cube $Q \in \mathcal{Q}_0$ if it satisfies (5.10), (5.11), and (5.12) if $|Q| < 1$ and (5.10) and (5.11) only if $|Q| = 1$. We say that $\{a_Q\}_{Q\in\mathcal{Q}_0}$ is a family of inhomogeneous smooth atoms, if each function a_Q is a smooth atom supported near Q.*

Theorem 5.10. *Suppose A is an expansive matrix and μ a ρ_A-doubling measure. For any $f \in \mathbf{B}_p^{\alpha,q}$ there exists a family of inhomogeneous smooth atoms $\{a_Q\}$ and a sequence of coefficients $s = \{s_Q\} \in \mathbf{b}_p^{\alpha,q}$, such that,*

$$f = \sum_{Q\in\mathcal{Q}_0} s_Q a_Q, \qquad \textit{and} \qquad \|s\|_{\mathbf{b}_p^{\alpha,q}} \le C\|f\|_{\mathbf{B}_p^{\alpha,q}},$$

where the above series converges unconditionally in $\mathbf{B}_p^{\alpha,q}$. Conversely, for any family of inhomogeneous smooth atoms $\{a_Q\}$,

$$\Big\| \sum_{Q\in\mathcal{Q}_0} s_Q a_Q \Big\|_{\mathbf{B}_p^{\alpha,q}} \le C\|s\|_{\mathbf{b}_p^{\alpha,q}}.$$

The proof of Theorem 5.10 is a direct modification of the proof of Theorem 5.9 with the help of reproducing formula,

$$f = \Theta * \Phi * f + \sum_{j\ge 1} \theta_j * \varphi_j * f,$$

where $\theta, \Theta, \varphi, \Phi \in \mathcal{S}$ satisfy (3.2), (3.30), (3.31), (5.15), (5.16), $\operatorname{supp}\theta$, $\operatorname{supp}\Theta \subset B(0,\delta_0)$, $|\hat{\Theta}(\xi)| \ge c > 0$ for $|\xi| \le 1$ and

$$\hat{\Phi}(\xi)\hat{\Theta}(\xi) + \sum_{j=1}^{\infty} \hat{\varphi}((A^*)^{-j}\xi)\hat{\theta}((A^*)^{-j}\xi) = 1 \qquad \text{for all } \xi \in \mathbb{R}^n. \tag{5.19}$$

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