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The structure of shift–modulation invariant spaces: The rational case

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Abstract

In this paper we study structural properties of shift–modulation invariant (SMI) spaces, also called Gabor subspaces, or Weyl–Heisenberg subspaces, in the case when shift and modulation lattices are rationally dependent. We prove the characterization of SMI spaces in terms of range functions analogous to the well-known description of shift-invariant spaces [C. de Boor, R. DeVore, A. Ron, The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$, J. Funct. Anal. 119 (1994) 37–78; M. Bownik, The structure of shift-invariant subspaces of $L_2(\mathbb{R}^n)$, J. Funct. Anal. 177 (2000) 282–309; H. Helson, Lectures on Invariant Subspaces, Academic Press, New York/London, 1964]. We also give a simple characterization of frames and Riesz sequences in terms on their behavior of the fibers of the range function. Next, we prove several orthogonal decomposition results of SMI spaces into simpler blocks, called principal SMI spaces. Then, this is used to characterize operators invariant under both shifts and modulations in terms of families of linear maps acting on the fibers of the range function. We also introduce the fundamental concept of the dimension function for SMI spaces. As a result, this leads to the classification of unitarily equivalent SMI spaces in terms of their dimension functions. Finally, we show several results illustrating our fiberization techniques to characterize dual Gabor frames.

Keywords: Gabor subspace; Weyl–Heisenberg subspace; Shift-modulation invariant space; Frame sequence; Riesz sequence; Dual frame; Dimension function; Range function; Range operator

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1. Introduction

The aim of this paper is to investigate the structure of shift–modulation invariant spaces. These are the subspaces of $L^2(\mathbb{R}^n)$ generated by Gabor systems, also called Weyl–Heisenberg systems. Gabor systems are a subject of the intensive study [4–6,9–17,19]. One of the fundamental problems in this area is to determine when two SMI spaces are unitarily equivalent, i.e., there exists a unitary operator between these spaces commuting both with shifts and modulatCCions. A similar problem in the context of shift-invariant (SI) spaces was settled by the author in [2]. It was proved that two SI spaces are unitarily equivalent if and only if their dimension functions coincide a.e. Recall from [1–3] that the dimension function of a SI space V is a $\mathbb{Z}^n$-periodic function $\dim_V : \mathbb{R}^n \to \mathbb{N} \cup \{0, \infty\}$, which measures the dimensions of the fibers of the range function corresponding to V.

The SMI spaces have, in general, a much more complex structure than their SI counterparts, since they must also obey modulation invariance. Obviously, every SMI space is also SI, hence every result about SI spaces can be applied in the shift–modulation setting. This might be a bit misleading, since SMI spaces are a very special kind of SI spaces. In particular, one can easily prove that their SI dimension functions take only two possible values: 0 or ∞. This is a consequence of the fact that every SMI space can be realized as a SI space with an infinite set of generators being modulations of each other. Therefore, general results about SI spaces have a limited applicability in the SMI setting and there is a need to develop a genuine shift–modulation theory.

The main goal of this work is to show that this is indeed possible if modulation and shift lattices are rationally dependent. The case when lattices are not rationally dependent requires a different set of techniques and it will not be treated here. Despite that our theory of SMI spaces is closely parallel to the shift-invariant theory, there are some significant differences setting them apart. To describe our results in some detail we need to recall a basic terminology.

Definition 1.1. Let Λ, Γ be two full rank lattices in $\mathbb{R}^n$, i.e., $\Lambda = P_0\mathbb{Z}^n$, $\Gamma = P_1\mathbb{Z}^n$ for some $n \times n$ non-singular matrices P_0, P_1 with real entries. Let $\mathcal{A} \subset L^2(\mathbb{R}^n)$ be a countable set of generators. The *Gabor system* $G(\mathcal{A}, \Lambda, \Gamma)$ is the set of translation and modulation shifts

$$G(\mathcal{A}, \Lambda, \Gamma) = \{M_\lambda T_\gamma \varphi \colon \lambda \in \Lambda,\ \gamma \in \Gamma,\ \varphi \in \mathcal{A}\}, \tag{1.1}$$

where $M_\lambda f(x) = e^{2\pi i \langle x, \lambda\rangle} f(x)$, $T_\gamma f(x) = f(x - \gamma)$. We say that a closed subspace $V \subset L^2(\mathbb{R}^n)$ is *shift–modulation invariant* (SMI) if

$$M_\lambda T_\gamma V \subset V \quad \text{for all } \lambda \in \Lambda,\ \gamma \in \Gamma. \tag{1.2}$$

The smallest SMI space generated by $\mathcal{A}$ is denoted by

$$S(\mathcal{A}, \Lambda, \Gamma) = \overline{\operatorname{span}} G(\mathcal{A}, \Lambda, \Gamma).$$

We say that two lattices Λ and Γ are *rationally dependent* if $\Lambda \cap \Gamma$ is a full rank lattice. Applying the standard dilation argument one can assume that the modulation lattice $\Lambda = \mathbb{Z}^n$, or alternatively that the shift lattice $\Gamma = \mathbb{Z}^n$. Then, any result involving Gabor systems with general lattices Λ, Γ can be deduced from a corresponding result when one of the lattices is $\mathbb{Z}^n$.

Therefore, for the sake of convenience and the ease of notation, we will always assume that the lattice of modulations $\Lambda = \mathbb{Z}^n$. Consequently, we will drop the dependence on Λ in the notation of a Gabor system $G(\mathcal{A}, \Gamma)$ and a Gabor subspace $S(\mathcal{A}, \Gamma)$. Moreover, the requirement that Λ and Γ are rationally dependent corresponds to the fact that Γ is a rational lattice, that is $\Gamma = P\mathbb{Z}^n$ for some $n \times n$ non-singular matrix P with rational entries.

Given a rational shift lattice Γ, define its *integral sub-lattice* $\Xi = \Xi(\Gamma)$ and its *extended super-lattice* $\Theta = \Theta(\Gamma)$ by

$$\Xi = \Gamma \cap \mathbb{Z}^n, \qquad \Theta = \Gamma + \mathbb{Z}^n. \tag{1.3}$$

The *dual lattice* to Ξ is given by

$$\Xi^* = \{k \in \mathbb{R}^n \colon \langle k, l\rangle \in \mathbb{Z} \text{ for all } l \in \Xi\}.$$

A *fundamental domain* of $\mathbb{R}^n/\Gamma$ is a set $I = I_\Gamma \subset \mathbb{R}^n$ such that $\{I + \gamma\colon \gamma \in \Gamma\}$ forms a partition of $\mathbb{R}^n$.

Let $L^2(\mathbb{R}^n/\Gamma)$ be the Hilbert space of all Γ-periodic measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|^2 = \int_{\mathbb{R}^n/\Gamma} |f(x)|^2\, dx < \infty.$$

In particular, $L^2(\mathbb{T}^n)$ is the usual space of $\mathbb{Z}^n$-periodic functions, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Naturally, $L^2(\mathbb{T}^n)$ can be identified with $L^2(I_n)$, where $I_n = [-1/2, 1/2)^n$ is the fundamental domain of $\mathbb{R}^n/\mathbb{Z}^n$.

In the close analogy to the shift-invariant case we establish a characterization of SMI spaces in terms of appropriate range functions. Unlike the SI case, the range function in the SMI setting is defined on the product domain $\mathbb{R}^n \times \mathbb{R}^n$ with values in subspaces of a finite-dimensional space $\mathbb{C}^p$ rather than the space $\ell^2(\mathbb{Z}^n)$ as in the SI case. It also satisfies rather complicated periodicity constraints, which are heavily dependent on the complexity of the rational dependence of shift and modulation lattices. However, for the sake of simplicity our characterization result can be stated as follows.

Theorem 1.1. *There is a* 1–1 *correspondence between SMI spaces V and measurable range functions*

$$J : I_\Theta \times I_{\Xi^*} \to \{E\colon E \text{ is a subspace of } \mathbb{C}^p\}. \tag{1.4}$$

More specifically, if $V = S(\mathcal{A}, \Gamma)$, then $J(x, \xi) = \operatorname{span} \mathcal{V}_{\mathcal{A}}(x, \xi)$, where

$$\mathcal{V}_{\mathcal{A}}(x, \xi) = \{\mathfrak{T}\varphi(x + l_j, \xi)\colon \varphi \in \mathcal{A},\ 1 \leqslant j \leqslant q\}, \tag{1.5}$$

Here, $\mathfrak{T}$ is the vector-valued Zak transform given by (3.4) *and $\{l_1, \dots, l_q\} \subset \Gamma$ are representatives of distinct cosets of $\Theta/\mathbb{Z}^n$.*

We should add that p and q above are the orders of the quotient groups $\Xi^*/\mathbb{Z}^n$ and $\Theta/\mathbb{Z}^n$, respectively. In particular, p and q have the same meaning as in Zibulski–Zeevi matrices for

1-dimensional Gabor system with time and frequency shift parameters a and b such that $ab = p/q \in \mathbb{Q}$, $\gcd(p,q) = 1$, see [13,24,30,31].

Theorem 1.1 enables us to introduce the concept of the dimension function for SMI spaces, which again is defined on the product domain $\mathbb{R}^n \times \mathbb{R}^n$ and takes only finite number of values unlike the SI case, where the dimension function has values in $\mathbb{N} \cup \{0, \infty\}$. The dimension function of an SMI space V is defined by $\dim_V(x,\xi) = \dim J(x,\xi)$. The rest of the paper is devoted to showing that this dimension function classifies unitarily equivalent SMI spaces.

Theorem 1.2. *Two SMI spaces V and W are unitarily equivalent if and only if*

$$\dim_V(x,\xi) = \dim_W(x,\xi) \quad \textit{for a.e. } (x,\xi) \in I_\Theta \times I_{\Xi^*}.$$

To achieve this goal we need two main ingredients. First, we demonstrate structural results for SMI spaces by showing decomposition theorems of general SMI spaces into simpler building blocks, called principal SMI spaces. The spectrum of an SMI space V is the set

$$\sigma(V) = \big\{(x,\xi)\colon \dim_V(x,\xi) > 0\big\}.$$

Then, our basic decomposition result can be stated in a simplified form as follows.

Theorem 1.3. *Every SMI space V can be decomposed as*

$$V = \bigoplus_{i=1}^{p} S(\varphi_i, \Gamma),$$

such that each φ_i is a principal generator of $S(\varphi_i,\Gamma)$, i.e., $\dim_{S(\varphi_i,\Gamma)} \leqslant 1$, and

$$\sigma\big(S(\varphi_1,\Gamma)\big) \supset \cdots \supset \sigma\big(S(\varphi_p,\Gamma)\big).$$

Second, we provide a description of morphisms between SMI spaces, that is operators commuting both with shifts and modulations. As a consequence of our techniques we establish several results on Gabor systems and SMI operators which provide the evidence for a phenomenon, which we should call, the fiberization paradigm for SMI spaces. This paradigm says that any reasonable property of an original Gabor system $G(\mathcal{A},\Gamma)$ (or an SMI operator) must propagate to the fibers of the corresponding finitely-dimensional systems $\mathcal{V}_{\mathcal{A}}(x,\xi)$ in the vector-valued Zak domain (or linear maps between fibers of the corresponding range functions). And vice versa, any reasonable property holding uniformly almost everywhere on fibers of the range function corresponds to the same property for the whole system. Simplifying things a bit we have the following two results.

Theorem 1.4. *The Gabor system $G(\mathcal{A},\Gamma) \subset L^2(\mathbb{R}^n)$ is a GGS $\Leftrightarrow$ $\frac{1}{\sqrt{p}}\mathcal{V}_{\mathcal{A}}(x,\xi) \subset \mathbb{C}^p$ are GGS uniformly for a.e. $(x,\xi) \in I_\Theta \times I_{\Xi^*}$, where $p = |\mathbb{Z}^n/\Xi|$.*

Here, a generic good system (GGS) represents any reasonable property of a collection of vectors in a Hilbert space such as: orthonormality, completeness, frame, frame sequence, Riesz basis, or Riesz sequence.

Theorem 1.5. *Suppose that V, W are SMI spaces and J, K are their corresponding range functions. There is a* 1–1 *correspondence between linear operators $L : V \to W$ commuting with shifts and modulations and measurable linear maps $R(x,\xi) : J(x,\xi) \to K(x,\xi)$, $(x,\xi) \in I_\Theta \times I_{\Xi^*}$.*

Moreover,

$$L \text{ has a GGP} \quad \Leftrightarrow \quad R(x,\xi) \text{ have GGP uniformly for a.e. } (x,\xi) \in I_\Theta \times I_{\Xi^*}.$$

Here, a generic good property (GGP) represents any reasonable property of a linear operator in a Hilbert space such as: being bounded, bounded from below, isometry, 1–1, onto, self-adjoint, etc.

Our paper is organized as follows. The starting point in Section 2 is the description of doubly invariant subspaces of $\ell^2(\mathbb{Z}^n)$ with higher multiplicity. In Section 3 we prove the characterization of SMI spaces in terms of range functions. In Section 4 we prove several manifestations of the fiberization paradigm for Gabor systems by characterizing Gabor frame and Riesz sequences. In Section 5 we establish fundamental decomposition results for SMI spaces as orthogonal sums of principal SMI spaces. In Section 6 we prove the characterization of operators commuting both with shifts and modulations in terms of range operators. As a consequence we classify unitarily equivalent SMI spaces in terms of their dimension functions. Finally, in Section 7 we show several results for dual Gabor systems using our fiberization techniques.

2. Doubly invariant subspaces with higher multiplicity shifts

The goal of this section is to provide a description of subspaces of $\ell^2(\mathbb{Z}^n)$, which are invariant under a certain subgroup of shifts. Recall from [20] that a closed subspace $V \subset \ell^2(\mathbb{Z}^n)$ is doubly invariant if $S_k V \subset V$ for all $k \in \mathbb{Z}^n$, where S_k is the shift operator by $k \in \mathbb{Z}^n$. The classical result of Wiener [20,25] says that every doubly invariant subspace $V \subset \ell^2(\mathbb{Z}^n)$ is of the form

$$V = \{a \in \ell^2(\mathbb{Z}^n): \operatorname{supp} \mathcal{F}a \subset W\} \tag{2.1}$$

for some Lebesgue measurable subset $W \subset \mathbb{T}^n$. Here, $\mathcal{F} : \ell^2(\mathbb{Z}^n) \to L^2(\mathbb{T}^n)$ is the Fourier transform given by

$$\mathcal{F}a(x) = \sum_{k \in \mathbb{Z}^n} a_k e^{2\pi i \langle k, x\rangle} \quad \text{for } a = (a_k) \in \ell^2(\mathbb{Z}^n). \tag{2.2}$$

Nevertheless, in the study of shift–modulation spaces we often encounter subspaces which are invariant under some specific subgroup of shifts, rather than invariant under all shifts. One can think of such spaces as doubly invariant with respect to higher multiplicity shifts. Despite the literature search we were not able to find an analogue of Wiener's theorem in this setting. Our main goal is to prove that such an analogue exists when phrased in the language of range functions, see Theorem 2.1.

Definition 2.1. Let Ξ be a full rank sub-lattice of $\mathbb{Z}^n$, i.e., $\Xi = P\mathbb{Z}^n$ for some $n \times n$ non-singular matrix P with integer entries. We say that a closed subspace $V \subset \ell^2(\mathbb{Z}^n)$ is doubly invariant with respect to the lattice Ξ, or simply *Ξ-invariant*, if

$$S_k V \subset V \quad \text{for all } k \in \Xi. \tag{2.3}$$

Here, $S_k : \ell^2(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n)$ is the shift operator by $k \in \mathbb{Z}^n$.

Given Ξ as above, let $\mathcal{D} = \{d_1, \dots, d_p\}$, where $p = |\det P|$, be representatives of distinct cosets of $\Xi^*/\mathbb{Z}^n$, where

$$\Xi^* = \{k \in \mathbb{R}^n\colon\ \langle k, l\rangle \in \mathbb{Z} \text{ for all } l \in \Xi\}$$

represents the dual lattice to Ξ. Equivalently, $\Xi^* = (P^*)^{-1}\mathbb{Z}^n$. The quotient group $\Xi^*/\mathbb{Z}^n$ induces a natural action on $\mathbb{C}^p$ given by

$$[k] \circ (z_1, \dots, z_p) = (z_{\nu(1)}, \dots, z_{\nu(p)}) \quad \text{for } (z_1, \dots, z_p) \in \mathbb{C}^p,$$

where for each $[k] \in \Xi^*/\mathbb{Z}^n$, ν is a unique permutation of $\{1, \dots, p\}$ satisfying

$$[k] + [d_i] = [d_{\nu(i)}] \quad \text{for } i = 1, \dots, p.$$

Definition 2.2. Let Ξ be a full rank sub-lattice of $\mathbb{Z}^n$, and hence Ξ^* is a super-lattice of $\mathbb{Z}^n$, i.e., $\mathbb{Z}^n \subset \Xi^*$. Define the space of Ξ^*-*quasi-periodic* functions as

$$L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p) = \{f \in L^2(\mathbb{T}^n, \mathbb{C}^p)\colon\ f(x+k) = [k] \circ f(x) \text{ for all } k \in \Xi^*\}.$$

In particular, any $f \in L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)$ must be $\mathbb{Z}^n$-periodic; that is, $f(x+k) = f(x)$ for all $k \in \mathbb{Z}^n$. A Hilbert space norm in $L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)$ is defined by

$$\|f\|_{L^2_{\Xi^*}(\mathbb{T}^n,\mathbb{C}^p)} = \Bigg(\int\limits_{\mathbb{R}^n/\Xi^*} \|f(x)\|^2_{\mathbb{C}^p}\, dx\Bigg)^{1/2} < \infty.$$

We note that the notion of Ξ^*-quasi-periodicity has a different meaning than the quasi-periodicity of the Zak transforms [16]. While a general $f \in L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)$ is not Ξ^*-periodic, the map $x \mapsto \|f(x)\|$ is Ξ^*-periodic, and hence the above norm is well defined. Moreover, any $f \in L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)$ is uniquely determined by its values on a fundamental domain of $\mathbb{R}^n/\Xi^*$. Once we fix such domain, say $I_{\Xi^*} = (P^*)^{-1} I_n$, we can identify $L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)$ with $L^2(I_{\Xi^*}, \mathbb{C}^p)$.

Proposition 2.1. *Let $\mathcal{T}_\Xi : \ell^2(\mathbb{Z}^n) \to L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)$ be a finite fiberization map given by*

$$\mathcal{T}_\Xi a(x) = (\mathcal{F}a(x+d_1), \dots, \mathcal{F}a(x+d_p)) \quad \textit{for } a \in \ell^2(\mathbb{Z}^n),$$

where $\mathcal{F}$ is the Fourier transform (2.2). *Then, $\mathcal{T}_\Xi$ is an isometric isomorphism.*

Proof. Clearly, $\mathcal{T}_\Xi$ is a composition of $\mathcal{F}$ and $T : L^2(\mathbb{T}^n) \to L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)$ defined as

$$Tf(x) = (f(x+d_1), \dots, f(x+d_p)) \quad \text{for } f \in L^2(\mathbb{T}^n).$$

A map T is isometry. It is clear that the family $\{I_{\Xi^*} + d_j\colon\ j = 1, \dots, p\}$ is a partition of a fundamental domain of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Hence, T is an isometric isomorphism. Consequently, so is $\mathcal{T}_\Xi$. $\square$

Definition 2.3. A Ξ*-range function* is any map

$$J:\mathbb{R}^n \to \{E \subset \mathbb{C}^p\colon\ E \text{ is a linear subspace}\},$$

which is compatible with the action of $\Xi^*/\mathbb{Z}^n$. That is:

$$[k]\circ J(x) = J(x+k) \quad \text{for any } k \in \Xi^*.$$

Let $P(x)$ be the orthogonal projection of $\mathbb{C}^p$ onto $J(x)$. We say that J is measurable if the map $x \mapsto P(x)$ is operator measurable.

Hence, any Ξ-range function J must be Ξ^*-quasi-periodic and in particular $\mathbb{Z}^n$-periodic, i.e., $J(x) = J(x+k)$ for all $k \in \mathbb{Z}^n$. Moreover, any Ξ-range function is uniquely determined by its values on a fundamental domain of $\mathbb{R}^n/\Xi^*$.

Remark 2.1. Note that the measurability of J is equivalent with $x \mapsto P(x)a$ being vector measurable for each $a \in \mathbb{C}^p$, which, in turn, is equivalent to $x \mapsto P(x)(\Phi(x))$ being vector measurable for each vector measurable $\Phi:\mathbb{T}^n \to \mathbb{C}^p$.

We are now ready to characterize Ξ-invariant spaces in terms of range functions. Theorem 2.1 is an analogue of the corresponding characterization result of shift-invariant spaces of $L^2(\mathbb{R}^n)$, see [2, Proposition 1.5], which dates back to Helson [20]. Theorem 2.1 is also a generalization of Wiener's Theorem, since in the usual doubly invariant case, $\mathcal{T}_\Xi = \mathcal{F}$, $J(x) = \{0\}$ or $\mathbb{C}$, and (2.4) is easily seen to be equivalent with (2.1).

Theorem 2.1. *A closed subspace $V \subset \ell^2(\mathbb{Z}^n)$ is Ξ-invariant if and only if*

$$V = \{a \in \ell^2(\mathbb{Z}^n)\colon\ \mathcal{T}_\Xi a(x) \in J(x) \textit{ for a.e. } x\}, \tag{2.4}$$

where J is a measurable Ξ-range function. The correspondence between V and J is 1–1 *under the convention that the range functions are identified if they are equal a.e. Furthermore, if V is generated by a finite or countable family $\mathcal{A} \subset \ell^2(\mathbb{Z}^n)$, i.e.,*

$$V = \overline{\operatorname{span}}\{S_k a\colon\ a \in \mathcal{A},\ k \in \Xi\}, \tag{2.5}$$

then the corresponding range function is given by

$$J(x) = \operatorname{span}\{\mathcal{T}_\Xi a(x)\colon\ a \in \mathcal{A}\}. \tag{2.6}$$

To prove Theorem 2.1 we will follow the same strategy as in the shift-invariant case [2]. Given a Ξ-range function J, define the space

$$M_J = \{\Phi \in L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)\colon\ \Phi(x) \in J(x) \text{ for a.e. } x \in \mathbb{T}^n\}. \tag{2.7}$$

Then, M_J is easily seen to be a closed subspace of $L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)$, regardless whether J is measurable or not. We need the following two adaptations of results due to Helson [20]. For their proof, see [2] or Section 3.

Lemma 2.1. *Let J be a measurable Ξ-range function with associated orthogonal projections $P(x)$, $x \in \mathbb{R}^n$. Let $\mathcal{P}$ be the orthogonal projection of $L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)$ onto M_J. Then for any $\Phi \in L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)$,*

$$(\mathcal{P}\Phi)(x) = P(x)\bigl(\Phi(x)\bigr) \quad \textit{for a.e. } x \in \mathbb{T}^n. \tag{2.8}$$

Corollary 2.1. *If $M_J = M_K$ for some measurable Ξ-range functions J and K, then $J(x) = K(x)$ for a.e. x.*

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Note that for any $a \in \ell^2(\mathbb{Z}^n)$ and $k \in \Xi$,

$$\mathcal{T}_\Xi(S_k a)(x) = \bigl(e^{2\pi i\langle k,x+d_1\rangle}\mathcal{F}a(x+d_1), \ldots, e^{2\pi i\langle k,x+d_p\rangle}\mathcal{F}a(x+d_p)\bigr) = e^{2\pi i\langle k,x\rangle}\mathcal{T}_\Xi a(x). \tag{2.9}$$

Therefore, a closed subspace $V \subset \ell^2(\mathbb{Z}^n)$ is Ξ-invariant if and only if $M = \mathcal{T}_\Xi V$ is a closed subspace of $L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)$ invariant under multiplication by exponentials in Ξ, i.e.,

$$\Phi(\cdot) \in M \quad \Rightarrow \quad e^{2\pi i\langle\cdot,k\rangle}\Phi(\cdot) \in M \quad \text{for all } k \in \Xi, \tag{2.10}$$

where $\cdot$ represents a generic variable.

Suppose that J is Ξ-range function. Then, the space V given by (2.4), or equivalently $V = (\mathcal{T}_\Xi)^{-1}M_J$, is Ξ-invariant by (2.9), regardless whether J is measurable or not. Conversely, suppose that V is Ξ-invariant generated by family $\mathcal{A} \subset \ell^2(\mathbb{Z}^n)$, that is (2.5). Let $J(x)$ be given by (2.6). Our goal is to show that J is a measurable Ξ-range function and that the space V can be recovered by (2.4).

Let $M = \mathcal{T}_\Xi V$. For any $\Phi \in M$, we can find a sequence $(\Phi_i)_{i\in\mathbb{N}}$ converging in norm to Φ and such that

$$(\mathcal{T}_\Xi)^{-1}\Phi_i \in \operatorname{span}\{S_k a\colon a \in \mathcal{A},\ k \in \Xi\}.$$

Hence, by (2.9), $\Phi_i(x) \in J(x)$ for all $i \in \mathbb{N}$ and all $x \in \mathbb{T}^n$. By choosing a subsequence of $(\Phi_i)_{i\in\mathbb{N}}$ we have pointwise a.e. convergence to Φ. Consequently, $\Phi(x) \in J(x)$ for a.e. x, and $M \subset M_J$, where M_J is given by (2.7).

To prove the converse inclusion, take any $\Psi \in M_J$, which is orthogonal to M. For any $\Phi \in \mathcal{T}_\Xi\mathcal{A}$ and $k \in \Xi$, we have $e^{2\pi i\langle\cdot,k\rangle}\Phi(\cdot) \in M$. Hence,

$$0 = \int_{I_{\Xi^*}} \bigl\langle e^{2\pi i\langle x,k\rangle}\Phi(x), \Psi(x)\bigr\rangle\, dx = \int_{I_{\Xi^*}} e^{2\pi i\langle x,k\rangle}\bigl\langle\Phi(x), \Psi(x)\bigr\rangle\, dx.$$

Since $\{e^{2\pi i\langle x,k\rangle}\}_{k\in\Xi}$ is an orthogonal basis of $L^2(I_{\Xi^*})$, the scalar function $x \mapsto \langle\Phi(x), \Psi(x)\rangle$ must vanish a.e. Therefore,

$$\bigl\langle\Phi(x), \Psi(x)\bigr\rangle = 0 \quad \text{for all } \Phi \in \mathcal{T}_\Xi\mathcal{A}, \text{ and a.e. } x \in \mathbb{T}^n,$$

that is $\Psi(x) \in J(x)^\perp$ for a.e. x. Hence, $\Psi = 0$. Since $M \subset M_J$ are closed, $M = M_J$ and (2.4) holds.

It remains to prove that $J(x)$ given by (2.6) is measurable. Let $\mathcal{P}$ be the orthogonal projection of $L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)$ onto $M = M_J$, and let $P(x)$ be the orthogonal projection of $\mathbb{C}^p$ onto $J(x)$. For any $\Psi \in L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)$, $(I - \mathcal{P})\Psi$ is orthogonal to M. By the above argument, $\Psi(x) - \mathcal{P}\Psi(x) \in J(x)^\perp$ for a.e. x. Combining this with the facts that $M = M_J$ and $\mathcal{P}\Psi(x) \in J(x)$ for a.e. x, we have

$$P(x)\big(\Psi(x)\big) = P(x)\big(\mathcal{P}\Psi(x)\big) = \mathcal{P}\Psi(x) \quad \text{for a.e. } x \in \mathbb{T}^n.$$

Since $\mathcal{P}\Psi(x)$ is vector measurable, so is $x \mapsto P(x)(\Psi(x))$ for any $\Psi \in L^2_{\Xi^*}(\mathbb{T}^n, \mathbb{C}^p)$. Consequently, J is a measurable Ξ-range function.

Finally, to prove that the correspondence between Ξ-invariant spaces and Ξ-measurable range functions is 1–1, we invoke Corollary 2.1. □

Definition 2.4. Let $V \subset \ell^2(\mathbb{Z}^n)$ be Ξ-invariant. The dimension function of V is a map

$$\dim_V : \mathbb{R}^n \to \{0, 1, \ldots, p\}, \quad \dim_V(x) = \dim J(x),$$

where $J(x)$ is a Ξ-range function from Proposition 2.1.

It follows immediately that $\dim_V$ is Ξ^*-periodic. However, it is much less immediate that the following result, which is an analogue of classification of unitarily equivalent shift-invariant spaces [2, Theorem 4.10], must also hold.

Proposition 2.2. *Let $V, W \subset \ell^2(\mathbb{Z}^n)$ be two Ξ-invariant spaces. Then V and W are unitarily equivalent, i.e., there exists a unitary operator $U : V \to W$ commuting with shifts $\{S_k\colon k \in \Xi\}$ if and only if*

$$\dim_V(x) = \dim_W(x) \quad \textit{for a.e. } x.$$

A direct proof of Proposition 2.2 is somewhat tedious, since it involves a decomposition result for Ξ-invariant spaces and a characterization of Ξ-invariant operators in terms of range operators as it was done in the shift-invariant case in [2]. Instead, we will deduce Proposition 2.2 as a consequence of an analogous classification for shift–modulation spaces in Section 6, see Example 6.1. Therefore, we now shift our attention to the more involved case of shift–modulation invariant spaces.

3. Characterization of shift–modulation invariant spaces

Our next goal is to characterize SMI spaces in terms of appropriate range functions which is analogous to the usual shift-invariant case [2, Proposition 1.5]. In order to do this we must introduce a necessary terminology.

Let $\mathcal{T}_1 : L^2(\mathbb{R}^n) \to L^2(I_n, \ell^2(\mathbb{Z}^n))$ be the isometric isomorphism

$$\mathcal{T}_1 f(x) = \big(f(x-k)\big)_{k \in \mathbb{Z}^n}. \tag{3.1}$$

The isometric isomorphism $\mathcal{T}_\Xi : \ell^2(\mathbb{Z}^n) \to L^2(I_{\Xi^*}, \mathbb{C}^p)$ induces then another isometric isomorphism

$$\mathcal{T}_2 : L^2\big(I_n, \ell^2\big(\mathbb{Z}^n\big)\big) \to L^2\big(I_n, L^2\big(I_{\Xi^*}, \mathbb{C}^p\big)\big),$$

defined by

$$\mathcal{T}_2(\Phi)(x) = \mathcal{T}_\Xi\big(\Phi(x)\big) \quad \text{for } \Phi \in L^2\big(I_n, \ell^2\big(\mathbb{Z}^n\big)\big),\ x \in I_{\Xi^*}.$$

However, we can identify $L^2(I_n, L^2(I_{\Xi^*}, \mathbb{C}^p))$ with $L^2(I_n \times I_{\Xi^*}, \mathbb{C}^p)$. Hence, by composing $\mathfrak{T} = \mathcal{T}_2 \circ \mathcal{T}_1$ we obtain an isometric isomorphism

$$\mathfrak{T} : L^2\big(\mathbb{R}^n\big) \to L^2\big(I_n \times I_{\Xi^*}, \mathbb{C}^p\big). \tag{3.2}$$

We will refer to $\mathfrak{T}$ as a *Ξ-Zak transform*. More explicitly, $\mathfrak{T}$ is defined as

$$\mathfrak{T}f(x,\xi) = \left(\sum_{k\in\mathbb{Z}^n} f(x-k)e^{2\pi i\langle k,\xi+d_1\rangle}, \ldots, \sum_{k\in\mathbb{Z}^n} f(x-k)e^{2\pi i\langle k,\xi+d_p\rangle}\right)$$
$$\text{for a.e. } (x,\xi) \in I_n \times I_{\Xi^*}, \tag{3.3}$$

where $\{d_1, \ldots, d_p\}$ are representatives of distinct cosets of $\Xi^*/\mathbb{Z}^n$, where $\Xi = \Gamma \cap \mathbb{Z}^n$. Naturally, the convergence of the above series is in L^2-norm, since the sequence $(f(x-k))_{k\in\mathbb{Z}^n}$ lies in $\ell^2(\mathbb{Z}^n)$ for a.e. x.

Therefore, we can simply define

$$\mathfrak{T}f(x,\xi) = \big(Zf(x,\xi+d_1), \ldots, Zf(x,\xi+d_p)\big), \tag{3.4}$$

where $Z : L^2(\mathbb{R}^n) \to L^2(I_n \times I_n)$ is the usual Zak transform given by

$$Zf(x,\xi) = \sum_{k\in\mathbb{Z}^n} f(x-k)e^{2\pi i\langle k,\xi\rangle}. \tag{3.5}$$

Remark 3.1. In the case when $\Gamma = \mathbb{Z}^n$, or more generally when $\mathbb{Z}^n \subset \Gamma$, Ξ-Zak transform is the usual Zak transform, i.e., $\mathfrak{T}f(x,\xi) = Zf(x,\xi)$. However, in general the above defined Ξ-Zak transform is vector-valued (with values in a finite-dimensional space). In this case $\mathfrak{T}$ is often called the vector-valued Zak transform [16, Chapter 8.3] or the piecewise Zak transform [30].

So far, the domain of Ξ-Zak transform $\mathfrak{T}f(x,\xi)$ was restricted to $(x,\xi) \in I_n \times I_{\Xi^*}$. Since it is often necessary to avoid such restrictions, we can extend the domain of $\mathfrak{T}f(x,\xi)$ for every $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ by using (3.4). Our next goal is to investigate periodicity properties of the resulting function.

Definition 3.1. Suppose $\Xi \subset \mathbb{Z}^n$ is an integral sub-lattice. Define the *Ξ-multiplex set M* as

$$M = \big(\mathbb{Z}^n/\Xi\big) \times \big(\Xi^*/\mathbb{Z}^n\big).$$

Define the action of M on the space $\mathbb{C}^p$ by

$$\big([l],[k]\big)\circ(v_1,\dots,v_p)=\big(e^{2\pi i\langle l,d_{\nu(j)}\rangle}v_{\nu(j)}\big)_{1\leqslant j\leqslant p}$$
$$\text{for } [l]\in\mathbb{Z}^n/\Xi,\ [k]\in\Xi^*/\mathbb{Z}^n,\ (v_1,\dots v_p)\in\mathbb{C}^p. \tag{3.6}$$

Here, ν is a unique permutation of $\{1,\dots,p\}$ such that

$$[k]+[d_i]=[d_{\nu(j)}]\quad\text{for } j=1,\dots,p. \tag{3.7}$$

In other words, each element $([l],[k])\in M$ defines a certain $p\times p$ unitary matrix, which is a certain composition of permutation and diagonal matrices. It may appear that M has a group structure, e.g. given by a semi-direct product, so that (3.6) is its unitary representation. However, one can easily see that this is not the case.

Definition 3.2. We say that $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}^p$ is *Ξ-multiplex-periodic* if for every $l\in\mathbb{Z}^n$ and $k\in\Xi^*$,

$$f(x+l,\xi+k)=e^{2\pi i\langle l,\xi\rangle}\big([l],[k]\big)\circ f(x,\xi)\quad\text{for } (x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n, \tag{3.8}$$

where $\circ$ represent the action of Ξ-multiplex set. In particular, any such f must be $\mathbb{Z}^n$-periodic in ξ-variable and, neglecting the phase term, it is also Ξ-periodic in x-variable. Define the Hilbert space $L^2_\Xi(\mathbb{R}^n\times\mathbb{T}^n,\mathbb{C}^p)$ of all Ξ-mutiplex-periodic f such that

$$\|f\|_{L^2_\Xi(\mathbb{R}^n\times\mathbb{T}^n,\mathbb{C}^p)}=\Bigg(\int\limits_{\mathbb{T}^n\times(\mathbb{R}^n/\Xi^*)}\big\|f(x,\xi)\big\|^2_{\mathbb{C}^p}\,d\xi\,dx\Bigg)^{1/2}<\infty.$$

The above norm is well defined since the map $(x,\xi)\mapsto\|f(x,\xi)\|$ is $\mathbb{Z}^n\times\Xi^*$-periodic. Moreover, any $f\in L^2_\Xi(\mathbb{R}^n\times\mathbb{T}^n,\mathbb{C}^p)$ is uniquely determined by its values on a fundamental domain of $\mathbb{T}^n\times(\mathbb{R}^n/\Xi^*)$. Once we fix such a domain, say $I_n\times I_{\Xi^*}$, we can identify this space with $L^2(I_n\times I_{\Xi^*},\mathbb{C}^p)$. Hence, we can deduce the following result.

Proposition 3.1. *The Ξ-Zak transform $\mathfrak{T}:L^2(\mathbb{R}^n)\to L^2_\Xi(\mathbb{R}^n\times\mathbb{T}^n,\mathbb{C}^p)$, given by* (3.4)*, is an isometric isomorphism.*

The only detail left to verify Proposition 3.1 is that for every $f\in L^2(\mathbb{R}^n)$, $\mathfrak{T}f$ is Ξ-multiplex periodic. Indeed, for $(l,k)\in\mathbb{Z}^n\times\Xi^*$,

$$\begin{aligned}\mathfrak{T}f(x+l,\xi+k)&=\big(Zf(x+l,\xi+k+d_j)\big)_{1\leqslant j\leqslant p}=\big(e^{2\pi i\langle l,\xi+k+d_j\rangle}Zf(x,\xi+k+d_j)\big)_{1\leqslant j\leqslant p}\\&=e^{2\pi i\langle l,\xi\rangle}\big(e^{2\pi i\langle l,d_{\nu(j)}\rangle}Zf(x,\xi+d_{\nu(j)})\big)_{1\leqslant j\leqslant p}=e^{2\pi i\langle l,\xi\rangle}\big([l],[k]\big)\circ\mathfrak{T}f(x,\xi),\end{aligned}$$

where the permutation ν is the same as in (3.7).

We are now ready to define range functions corresponding to shift–modulation spaces.

Definition 3.3. A *shift–modulation range function* $J=J(x,\xi)$ (with respect to the shift lattice Γ) is a mapping

$$J:\mathbb{R}^n\times\mathbb{T}^n\to\big\{E\subset\mathbb{C}^p\colon\ E \text{ is a linear subspace}\big\},$$

which is Γ-periodic in x variable and Ξ-multiplex-periodic. More precisely,

$$J(x+\gamma,\xi)=J(x,\xi) \quad \text{for } \gamma\in\Gamma, \tag{3.9}$$

$$J(x+l,\xi+k)=\big([l],[k]\big)\circ J(x,\xi) \quad \text{for } l\in\mathbb{Z}^n,\ k\in\Xi^*, \tag{3.10}$$

where $\circ$ represents the action of Ξ-multiplex set.

Let $P(x,\xi)$ be the orthogonal projection of $\mathbb{C}^p$ onto $J(x,\xi)$. We say that J is *measurable* if the map $(x,\xi)\mapsto P(x,\xi)$ is operator measurable.

Remark 3.2. Let $\{l_1,\dots,l_q\}\subset\Gamma$ be representatives of distinct cosets of the quotient group $\Theta/\mathbb{Z}^n$. Define the group homomorphism $\rho:\Theta\to\mathbb{Z}^n/\Xi$ by

$$\rho(l)=[l-l_j] \quad \text{for } l\in\Theta,\ l\in l_j+\mathbb{Z}^n,\ j=1,\dots,q. \tag{3.11}$$

It is easy to show that ρ is well defined and that its definition is independent of the choice of representatives $\{l_1,\dots,l_q\}$ as long as they are elements of Γ. Then, conditions (3.9) and (3.10) can be combined into a single equivalent formula

$$J(x+l,\xi+k)=\big(\rho(l),[k]\big)\circ J(x,\xi) \quad \text{for } l\in\Theta, k\in\Xi^*. \tag{3.12}$$

In particular, (3.12) shows that any shift–modulation invariant range function J is uniquely determined by its values on the fundamental domain $I_\Theta\times I_{\Xi^*}$. Moreover, any such $J=J(x,\xi)$ is Γ-periodic in x-variable and $\mathbb{Z}^n$-periodic in ξ-variable.

Our goal is to characterize shift–modulation spaces in terms of shift–modulation range functions. More precisely, we have the following result.

Theorem 3.1. *Suppose $\Gamma\subset\mathbb{R}^n$ is a rational lattice. Define lattices Θ and Ξ by* (1.3)*. Then the following holds.*

(i) *A closed subspace $V\subset L^2(\mathbb{R}^n)$ is shift–modulation invariant* (*with respect to the shift lattice Γ*) *if and only if*

$$V=\big\{f\in L^2\big(\mathbb{R}^n\big):\ \mathfrak{Z}f(x,\xi)\in J(x,\xi) \text{ for a.e. } (x,\xi)\in\mathbb{R}^n\times\mathbb{T}^n\big\}, \tag{3.13}$$

where J is a measurable shift–modulation range function (*with respect to Γ*)*, and $\mathfrak{Z}$ denotes Ξ-Zak transform.*

(ii) *The correspondence between V and J is* 1–1 *under the convention that the range functions are identified if they are equal a.e.*

(iii) *Furthermore, if V is generated by a finite or countable family $\mathcal{A}\subset L^2(\mathbb{R}^n)$, i.e.,*

$$V=\overline{\operatorname{span}}\big\{M_kT_\gamma\varphi:\ \varphi\in\mathcal{A},\ k\in\mathbb{Z}^n,\ \gamma\in\Gamma\big\}, \tag{3.14}$$

then the corresponding range function is given by

$$J(x,\xi)=\operatorname{span}\big\{\mathfrak{Z}\varphi(x+l_j,\xi):\ j=1,\dots,q,\ \varphi\in\mathcal{A}\big\}, \tag{3.15}$$

where $\{l_1,\dots,l_q\}\subset\Gamma$ are representatives of distinct cosets of $\Theta/\mathbb{Z}^n$.

Equivalently, (3.13) can be written as

$$V = \{ f \in L^2(\mathbb{R}^n) \colon \mathfrak{T} f(x + l_j, \xi) \in J(x, \xi) \\ \text{for all } j = 1, \dots, q, \text{ and for a.e. } (x, \xi) \in I_\Theta \times I_{\Xi^*} \}. \tag{3.16}$$

Remark 3.3. Note that every shift–modulation space V is, in particular, modulation-invariant. Therefore, using the fiberization map $\mathcal{T}_1$ and [2, Proposition 1.5], V can be identified with the usual range function mapping I_n into closed subspaces $J(x)$ of $\ell^2(\mathbb{Z}^n)$. For general modulation-invariant spaces, $J(x)$ do not have to satisfy any additional properties with the exception of measurability. Since V is an SMI space, this imposes certain restrictions on the possible structure of spaces $J(x)$. In particular, it turns out that each $J(x) \subset \ell^2(\mathbb{Z}^n)$ must be Ξ-invariant, and hence it is characterized by Theorem 2.1. Heuristically, to prove Theorem 3.1 we have to apply first [2, Proposition 1.5] and then on each fiber of the resulting range function we should use Theorem 2.1. However, the actual argument is more complicated since we have to control both the measurability and multiplex-periodicity of the resulting shift–modulation range function. Consequently, the proof of Theorem 3.1 is, in a certain sense, a higher octane version of Theorem 2.1.

We start with a basic lemma describing SMI spaces in the Zak domain.

Lemma 3.1. *A closed subspace $V \subset L^2(\mathbb{R}^n)$ is shift–modulation invariant if and only if $M = \mathfrak{T}V \subset L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$ is a closed subspace invariant under multiplication by exponentials in x-variable,*

$$\Phi(x, \xi) \in M \quad \Rightarrow \quad e^{2\pi i \langle l, x \rangle} \Phi(x, \xi) \in M \quad \text{for all } l \in \mathbb{Z}^n, \tag{3.17}$$

and invariant under x-variable shifts by elements of Γ,

$$\Phi(x, \xi) \in M \quad \Rightarrow \quad \Phi(x - \gamma, \xi) \in M \quad \text{for all } \gamma \in \Gamma. \tag{3.18}$$

Proof. A direct calculation shows that for $l \in \mathbb{Z}^n$, $\gamma \in \Gamma$,

$$Z(M_l T_\gamma f)(x, \xi) = e^{2\pi i \langle l, x \rangle} Z f(x - \gamma, \xi).$$

Hence,

$$\mathfrak{T}(M_l T_\gamma f)(x, \xi) = e^{2\pi i \langle l, x \rangle} \mathfrak{T} f(x - \gamma, \xi) = \begin{cases} e^{2\pi i \langle l, x \rangle} \mathfrak{T} f(x, \xi) & \text{for } l \in \mathbb{Z}^n,\ \gamma = 0, \\ \mathfrak{T} f(x - \gamma, \xi) & \text{for } l = 0,\ \gamma \in \Gamma. \end{cases} \tag{3.19}$$

Therefore, V is shift–modulation invariant implies that (3.17) and (3.18) hold.

Conversely, if (3.17) and (3.18) hold, then by (3.19), $V = (\mathfrak{T}^{-1})M$ satisfies

$$f \in V \quad \Rightarrow \quad M_l f \in V \quad \text{for } l \in \mathbb{Z}^n,$$
$$f \in V \quad \Rightarrow \quad T_\gamma f \in V \quad \text{for } \gamma \in \Gamma,$$

which completes the proof of Lemma 3.1. □

Given a shift–modulation range function J, define the space

$$M_J := \{\Phi \in L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)\colon\ \Phi(x,\xi) \in J(x,\xi) \text{ for a.e. } (x,\xi) \in \mathbb{R}^n \times \mathbb{T}^n\}. \tag{3.20}$$

Since $e^{2\pi i\langle l,x\rangle} J(x,\xi) = J(x,\xi)$, the space $M = M_J$ satisfies (3.17). Likewise, (3.18) holds by (3.9). Furthermore, M_J is a closed subspace of $L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$, since every sequence of functions $(\Phi_j)_{j\in\mathbb{N}}$ converging in $L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$ norm has a pointwise almost everywhere convergent subsequence. Therefore, by Lemma 3.1, the space $V = (\mathfrak{T}^{-1})M_J$ is shift–modulation invariant regardless whether J is measurable or not. This already justifies one direction of Theorem 3.1(i).

To deal with the converse direction, we need the following extension of Lemma 2.1.

Lemma 3.2. *Let J be a measurable shift–modulation range function with associated orthogonal projections $P(x,\xi)$, $(x,\xi) \in \mathbb{R}^n \times \mathbb{T}^n$. Let $\mathcal{P}$ be the orthogonal projection of $L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$ onto M_J. Then for any $\Phi \in L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$,*

$$(\mathcal{P}\Phi)(x,\xi) = P(x,\xi)\big(\Phi(x,\xi)\big) \quad \textit{for a.e. } (x,\xi) \in \mathbb{R}^n \times \mathbb{T}^n. \tag{3.21}$$

Proof. Define an operator $\mathcal{P}'$ on $L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$ by

$$(\mathcal{P}'\Phi)(x,\xi) = P(x,\xi)\big(\Phi(x,\xi)\big) \quad \text{for a.e. } (x,\xi) \in \mathbb{R}^n \times \mathbb{T}^n. \tag{3.22}$$

It is clear that $\mathcal{P}'\Phi$ is Ξ-multiplex-periodic, since both $P(x,\xi)$ and $\Phi(x,\xi)$ are. Also, since $\|P(x,\xi)\| \leqslant 1$, the right-hand side of (3.22) is a measurable vector function, which belongs to $L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$. Moreover, $(\mathcal{P}')^2 = \mathcal{P}'$ and $(\mathcal{P}')^* = \mathcal{P}$ since $P(x,\xi)$ is an orthogonal projection for a.e. (x,ξ). Let M' be the range of the orthogonal projection $\mathcal{P}'$. To show (3.21), it remains to prove that $M' = M_J$. Since the inclusion $M' \subset M_J$ is trivial, it suffices to show that $M_J \subset M'$.

Take any $\Psi \in M_J$, which is orthogonal to M'. Then, for all $\Phi \in L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$,

$$0 = \int\limits_{I_n \times I_{\Xi^*}} \big\langle P(x,\xi)\big(\Phi(x,\xi)\big), \Psi(x,\xi)\big\rangle_{\mathbb{C}^p}\, dx\, d\xi = \int\limits_{I_n \times I_{\Xi^*}} \big\langle \Phi(x,\xi), P(x,\xi)\big(\Psi(x,\xi)\big)\big\rangle_{\mathbb{C}^p}\, dx\, d\xi.$$

Since $\Psi(x,\xi) \in J(x,\xi)$, we have $\Psi(x,\xi) = P(x,\xi)(\Psi(x,\xi)) = 0$ for a.e. $(x,\xi) \in I_n \times I_{\Xi^*}$. Thus, $\Psi = 0$, which shows $M_J = M'$, since $M' \subset M_J$ are closed. □

Corollary 3.1. *If $M_J = M_K$ for some measurable shift–modulation range functions J and K, then $J(x,\xi) = K(x,\xi)$ for a.e. (x,ξ).*

Proof. Let $\Phi \in L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$ be initially defined by $\Phi(x,\xi) = e_j$ for $(x,\xi) \in I_n \times I_{\Xi^*}$ and then extended to a Ξ-multiplex-periodic function on $\mathbb{R}^n \times \mathbb{T}^n$. Here, $\{e_j\colon\ j = 1,\dots,p\}$ is the standard orthonormal basis of $\mathbb{C}^p$. Then we apply Lemma 3.2 for such defined Φ,

$$P(x,\xi)e_j = Q(x,\xi)e_j \quad \text{for all } j = 1,\dots,p, \text{ and a.e. } x \in I_n \times I_{\Xi^*},$$

where $P(x,\xi)$, $Q(x,\xi)$ are orthogonal projections onto $J(x,\xi)$ and $K(x,\xi)$, respectively. Therefore, $P(x,\xi)=Q(x,\xi)$ for a.e. $(x,\xi)\in\mathbb{R}^n\times\mathbb{T}^n$. $\square$

Finally, we are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. We have already proved that whenever J is shift–modulation range function, then the space V given by (3.13), or equivalently $V=\mathfrak{T}^{-1}M_J$, is SMI, regardless whether J is measurable or not. Conversely, suppose that the space V is SMI generated by a family $\mathcal{A}\subset L^2(\mathbb{R}^n)$, that is (3.14). Let $J(x,\xi)$ be given by (3.15). Our goal is to show that J is a measurable shift–modulation range function and that the space V can be recovered by (3.13).

First, note that the definition (3.15) does not depend on the choice of representatives $\{l_1,\dots,l_q\}\subset\Gamma$, since $\mathfrak{T}\varphi$ is Ξ-periodic in x-variable. Moreover, $(x,\xi)\mapsto\mathfrak{T}\varphi(x+l_j,\xi)$ is Ξ-multiplex periodic implies that $J(x,\xi)$ is, too. Take any $\gamma\in\Gamma$ and write it as $\gamma=l_{j_0}+l$, where $j_0=1,\dots,q$, and $l\in\mathbb{Z}^n$. Since $l\in\Gamma$, $l\in\Xi$. Hence, by Ξ-periodicity of $\mathfrak{T}\varphi$ in x-variable,

$$\begin{aligned}J(x+\gamma,\xi)&=\operatorname{span}\bigl\{\mathfrak{T}\varphi(x+l+l_{j_0}+l_j,\xi)\colon\ j=1,\dots,q,\ \varphi\in\mathcal{A}\bigr\}\\&=\operatorname{span}\bigl\{\mathfrak{T}\varphi(x+l_{j_0}+l_j,\xi)\colon\ j=1,\dots,q,\ \varphi\in\mathcal{A}\bigr\}=J(x,\xi),\end{aligned}$$

since $\{l_{j_0}+l_j\colon\ j=1,\dots,q\}\subset\Gamma$ are also representatives of distinct cosets of $\Theta/\mathbb{Z}^n$. This proves that J is a shift–modulation range function.

To prove that (3.13) holds, let $M=\mathfrak{T}V$. For any $\Phi\in M$, we can find a sequence $(\Phi_j)_{j\in\mathbb{N}}$ converging in norm to Φ and such that

$$\mathfrak{T}^{-1}\Phi_j\in\operatorname{span}\bigl\{M_kT_\gamma\varphi\colon\ \varphi\in\mathcal{A},\ k\in\mathbb{Z}^n,\ \gamma\in\Gamma\bigr\}.$$

Hence, by (3.19) and Γ-periodicity of J in x-variable, $\Phi_j(x,\xi)\in J(x,\xi)$ for all $j\in\mathbb{N}$ and all $(x,\xi)\in\mathbb{R}^n\times\mathbb{T}^n$. By choosing a subsequence of $(\Phi_j)_{j\in\mathbb{N}}$, we have pointwise a.e. convergence to Φ. Consequently, $\Phi(x,\xi)\in J(x,\xi)$ for a.e. (x,ξ), and $M\subset M_J$, where M_J is given by (3.20).

To prove the converse inclusion, take any $\Psi\in M_J$, which is orthogonal to M. For any $\Phi\in\mathfrak{T}\mathcal{A}$, using (3.17), (3.18), and Ξ-multiplex-periodicity of Φ, we have

$$e^{2\pi i\langle x,l\rangle}\Phi(x+k,\xi)=e^{2\pi i(\langle x,l\rangle+\langle\xi,k\rangle)}\Phi(x,\xi)\in M\quad\text{for all }l\in\mathbb{Z}^n,\ k\in\Xi.$$

Hence,

$$\begin{aligned}0&=\int\limits_{I_n\times I_{\Xi^*}}\bigl\langle e^{2\pi i(\langle x,l\rangle+\langle\xi,k\rangle)}\Phi(x,\xi),\Psi(x,\xi)\bigr\rangle\,dx\,d\xi\\&=\int\limits_{I_n\times I_{\Xi^*}}e^{2\pi i(\langle x,l\rangle+\langle\xi,k\rangle)}\bigl\langle\Phi(x,\xi),\Psi(x,\xi)\bigr\rangle\,dx\,d\xi.\end{aligned}$$

Since $\{e^{2\pi i(\langle x,l\rangle+\langle\xi,k\rangle)}\}_{l\in\mathbb{Z}^n,\ k\in\Xi}$ is an orthogonal basis of $L^2(I_n\times I_{\Xi^*})$, the scalar function $(x,\xi)\mapsto\langle\Phi(x,\xi),\Psi(x,\xi)\rangle$ must vanish a.e. Therefore,

$$\bigl\langle\Phi(x,\xi),\Psi(x,\xi)\bigr\rangle=0\quad\text{for all }\Phi\in\mathfrak{T}\mathcal{A},\ \text{and a.e. }(x,\xi)\in\mathbb{R}^n\times\mathbb{T}^n.$$

That is $\Psi(x,\xi) \in J(x,\xi)^\perp$ for a.e. (x,ξ). Hence, $\Psi = 0$. Since $M \subset M_J$ are closed, $M = M_J$ and (3.13) holds.

It remains to prove that J given by (3.15) is measurable. Let $\mathcal{P}$ be the orthogonal projection of $L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$ onto $M = M_J$, and let $P(x,\xi)$ be the orthogonal projection of $\mathbb{C}^p$ onto $J(x,\xi)$. For any $\Psi \in L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$, $(I - \mathcal{P})\Psi$ is orthogonal to M. By the above argument, $\Psi(x,\xi) - \mathcal{P}\Psi(x,\xi) \in J(x,\xi)^\perp$ for a.e. (x,ξ). Combining this with the fact that $M = M_J$, $\mathcal{P}\Psi(x,\xi) \in J(x,\xi)$ for a.e. (x,ξ), and we have

$$P(x,\xi)\big(\Psi(x,\xi)\big) = P(x,\xi)\big(\mathcal{P}\Psi(x,\xi)\big) = \mathcal{P}\Psi(x,\xi) \quad \text{for a.e. } (x,\xi) \in \mathbb{R}^n \times \mathbb{T}^n.$$

Since $\mathcal{P}\Psi(x,\xi)$ is vector measurable, so is $(x,\xi) \mapsto P(x,\xi)(\Psi(x,\xi))$ for any $\Psi \in L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$. Consequently, J is a measurable shift–modulation range function. Finally, to prove that the correspondence between SMI spaces and measurable shift–modulation range functions is 1–1, we invoke Corollary 3.1. □

Theorem 3.1 enables us to introduce the notion of the dimension function for SMI spaces.

Definition 3.4. Let $V \subset L^2(\mathbb{R}^n)$ be an SMI space. The dimension function of V is a map

$$\dim_V : \mathbb{R}^n \times \mathbb{R}^n \to \{0, 1, \dots, p\}, \quad \dim_V(x,\xi) = \dim J(x,\xi),$$

where $J(x,\xi)$ is a shift–modulation range function from Theorem 3.1. The *spectrum* of V is defined as

$$\sigma(V) = \big\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \colon\ J(x,\xi) \neq \{0\}\big\}.$$

It follows immediately from (3.12) that $\dim_V$ is $\Theta \times \Xi^*$-periodic, hence its values are uniquely determined on the fundamental domain of $\Theta \times \Xi^*$, e.g. the set $I_\Theta \times I_{\Xi^*}$.

Later, we will prove that the dimension function classifies unitary equivalence of SMI spaces. For now, note that if we have two orthogonal SMI spaces V and W, then their corresponding shift–modulation range functions J and K must be pointwise orthogonal $J(x,\xi) \perp K(x,\xi)$ for a.e. (x,ξ). By Lemma 3.2 and Theorem 3.1, the range function corresponding to $V \oplus W$ is simply $L = L(x,\xi) = J(x,\xi) \oplus K(x,\xi)$. Hence, the dimension function is additive with respect to orthogonal sums

$$\dim_{V \oplus W} = \dim_V + \dim_W.$$

Obviously, the additivity is also true with respect to countable orthogonal sums.

4. Gabor frame and Riesz sequences

In this section our aim is to give a simple characterization of Gabor frame and Riesz sequences using the fiberization techniques introduced in the previous section. Hence, our goal is to establish a fiberization paradigm for Gabor systems claiming that any reasonable property of the original Gabor system $G(\mathcal{A}, \Gamma)$ is equivalent to the same property holding uniformly over the fibers in the Zak domain. This is analogous to the fiberization paradigm for SI systems established by the author in [2].

We now recall the basic definitions.

Definition 4.1. A sequence $(f_i)_{i\in I}$ of vectors in a Hilbert space $\mathcal{H}$ is a *frame sequence*, if there exist constants $0 < c_0 \leqslant c_1 < \infty$ such that

$$c_0\|f\|^2 \leqslant \sum_{i\in I}\bigl|\langle f, f_i\rangle\bigr|^2 \leqslant c_1\|f\|^2 \quad \text{for all } f \in \overline{\operatorname{span}}\{f_i\colon\, i\in I\}. \tag{4.1}$$

In addition, if $\overline{\operatorname{span}}\{f_i\colon\, i\in I\} = \mathcal{H}$, then $(f_i)_{i\in I}$ is a *frame* for $\mathcal{H}$. If only the upper bound holds in (4.1), then $(f_i)_{i\in I}$ is said to be a *Bessel sequence*. We say that $(f_i)_{i\in I}$ is a *tight frame* for $\mathcal{H}$, if (4.1) holds for equal constants $c_0 = c_1$, and for all $f \in \mathcal{H}$.

A sequence $(f_i)_{i\in I} \subset \mathcal{H}$ is a *Riesz sequence*, if

$$c_0\|a\|^2 \leqslant \Bigl\|\sum_{i\in I} a_i f_i\Bigr\|^2 \leqslant c_1\|f\|^2 \quad \text{for all } a = (a_i)_i \in \ell^2(I).$$

In addition, if $\overline{\operatorname{span}}\{f_i\colon\, i\in I\} = \mathcal{H}$, then $(f_i)_{i\in I}$ is a *Riesz basis* for $\mathcal{H}$.

Theorem 4.1. *Suppose that $\Gamma \subset \mathbb{R}^n$ is a rational lattice, Θ, Ξ are given by* (1.3)*, and $\{l_1,\dots,l_q\} \subset \Gamma$ are representatives of distinct cosets of $\Theta/\mathbb{Z}^n$. Suppose also that $\mathcal{A} \subset L^2(\mathbb{R}^n)$ is countable, $0 < c_0 \leqslant c_1 < \infty$.*

Then $G(\mathcal{A},\Gamma)$ is a GGS with bounds c_0, c_1 if and only if

$$\mathcal{V}_{\mathcal{A}}(x,\xi) = \bigl\{\mathfrak{T}\varphi(x + l_j, \xi)\colon\, 1 \leqslant j \leqslant q,\ \varphi \in \mathcal{A}\bigr\} \subset \mathbb{C}^p \tag{4.2}$$

are GGS with bounds pc_0, pc_1 for a.e. $(x,\xi) \in I_\Theta \times I_{\Xi^*}$.

Here, a generic good system (*GGS*) *is either*:

(i) *Bessel sequence* (*when the lower bound $c_0 = 0$*),
(ii) *frame sequence,*
(iii) *frame,*
(iv) *Riesz sequence,*
(v) *Riesz basis.*

One should note that several authors have used the Zak transform techniques to characterize Gabor frames. Characterizations of Gabor frames in terms of the vector-valued Zak transform in one dimension were obtained by Janssen [21–23], Zeevi and Zibulski [30,31]. An analogous characterization for Gabor frame sequences was obtained by Gabardo and Han [13]. Some higher-dimensional results were also obtained by Ron and Shen [28]. Therefore, certain parts of Theorem 4.1 could be deduced from earlier works. Nevertheless, both the level of the generality and the formulation of Theorem 4.1 in the context of Gabor systems appear to be original. The following two elementary observations will be helpful in the proof of Theorem 4.1.

Remark 4.1. If the system $\mathcal{V}_{\mathcal{A}}(x,\xi)$ satisfies one of (i)–(v) for a.e. $(x,\xi) \in I_\Theta \times I_{\Xi^*}$ for a certain choice of representatives, then it satisfies the same property for all other choices. Indeed, suppose that $\{l'_1,\dots,l'_q\} \subset \Gamma$ is another choice of representatives of $\Theta/\mathbb{Z}^n$. By rearrangement, we can assume that l_j and l'_j represent the same coset for $1 \leqslant j \leqslant q$. Hence, $l_j - l'_j \in \mathbb{Z}^n \cap \Gamma = \Xi$. Consequently, by Ξ-multiplex periodicity,

$$\mathfrak{T}\varphi\big(x+l'_j,\xi\big)=e^{2\pi i\langle l_j-l'_j,\xi\rangle}\mathfrak{T}\varphi(x+l_j,\xi),$$

which proves our assertion.

Remark 4.2. Likewise, if the system $\mathcal{V}_{\mathcal{A}}(x,\xi)$ satisfies one of (i)–(v) for a.e. $(x,\xi)\in I_\Theta\times I_{\Xi^*}$, then it satisfies the same property for a.e. $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$. Indeed, take any $(l,k)\in\Theta\times\Xi$ and represent $l=l_{j_0}+l'$ for some $1\leqslant j_0\leqslant q$ and $l'\in\mathbb{Z}^n$. Again, by Ξ-multiplex periodicity,

$$\mathfrak{T}\varphi(x+l+l_j,\xi+k)=\mathfrak{T}\varphi(x+l+l_j+l_{j_0},\xi+k)=e^{2\pi i\langle l',\xi\rangle}\big([l'],[k]\big)\circ\mathfrak{T}\varphi(x+l_j+l_{j_0},\xi).$$

Since multiplex action is unitary, and $\{l_{j_0}+l_j\colon 1\leqslant j\leqslant q\}$ are also representatives of distinct cosets of $\Theta/\mathbb{Z}^n$, $\{\mathfrak{T}\varphi(x+l+l_j,\xi+k)\colon 1\leqslant j\leqslant q,\varphi\in\mathcal{A}\}$ satisfies one of (i)–(v) as the system (4.2) does. Since $(l,k)\in\Theta\times\Xi$ is arbitrary, the system (4.2) must satisfy the same property for a.e. $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$.

Proof of (i), (ii). Take any $\varphi\in\mathcal{A}$ and $f\in L^2(\mathbb{R}^n)$. Then by Proposition 3.1 and (3.19)

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n}\sum_{\gamma\in\Gamma}\big|\langle M_kT_\gamma\varphi,f\rangle\big|^2\\
&\quad=\sum_{k\in\mathbb{Z}^n}\sum_{\gamma\in\Gamma}\big|\langle\mathfrak{T}M_kT_\gamma\varphi,\mathfrak{T}f\rangle\big|^2\\
&\quad=\sum_{k\in\mathbb{Z}^n}\sum_{\gamma\in\Gamma}\bigg|\int_{I_n\times I_{\Xi^*}}e^{2\pi i\langle k,x\rangle}\big\langle\mathfrak{T}\varphi(x+\gamma,\xi),\mathfrak{T}f(x,\xi)\big\rangle\,dx\,d\xi\bigg|^2\\
&\quad=\sum_{k\in\mathbb{Z}^n}\sum_{j=1}^{q}\sum_{l\in\Xi}\bigg|\int_{I_n\times I_{\Xi^*}}e^{2\pi i\langle k,x\rangle}e^{2\pi i\langle l,\xi\rangle}\big\langle\mathfrak{T}\varphi(x+l_j,\xi),\mathfrak{T}f(x,\xi)\big\rangle\,dx\,d\xi\bigg|^2\\
&\quad=\frac{1}{p}\sum_{j=1}^{q}\int_{I_n\times I_{\Xi^*}}\big|\big\langle\mathfrak{T}\varphi(x+l_j,\xi),\mathfrak{T}f(x,\xi)\big\rangle\big|^2\,dx\,d\xi.
\end{aligned}$$

In the penultimate step we used (3.8) and the fact that every $\gamma\in\Gamma$ has a unique decomposition as $\gamma=l_j+l$ for some $1\leqslant j\leqslant q$ and $l\in\Xi$. In the last step we used the fact that $\{p^{1/2}e^{2\pi i(\langle k,x\rangle+\langle l,\xi\rangle)}\}_{k\in\mathbb{Z}^n,\,l\in\Xi}$ is an orthonormal basis of $L^2(I_n\times I_{\Xi^*})$. Summing the above formula over $\varphi\in\mathcal{A}$, we have

$$\sum_{\varphi\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n}\sum_{\gamma\in\Gamma}\big|\langle M_kT_\gamma\varphi,f\rangle\big|^2=\frac{1}{p}\sum_{\varphi\in\mathcal{A}}\sum_{j=1}^{q}\int_{I_n\times I_{\Xi^*}}\big|\big\langle\mathfrak{T}\varphi(x+l_j,\xi),\mathfrak{T}f(x,\xi)\big\rangle\big|^2\,dx\,d\xi. \tag{4.3}$$

Let J be the shift–modulation range function associated with $S(\mathcal{A},\Gamma)$, which is given by (3.15) by Theorem 3.1.

Suppose that the system (4.2) is a frame sequence, or Bessel sequence (when $c_0=0$), with bounds pc_0, pc_1 for a.e. $(x,\xi)\in I_\Theta\times I_{\Xi^*}$. Then, by Remark 4.2

$$pc_0\|v\|^2 \leqslant \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} \left|\left\langle \mathfrak{T}\varphi(x+l_j,\xi), v\right\rangle\right|^2 \leqslant pc_1\|v\|^2 \quad \text{for all } v\in J(x,\xi), \text{ a.e. } (x,\xi). \tag{4.4}$$

By Theorem 3.1, if $f\in S(\mathcal{A},\Gamma)$, then $v=\mathfrak{T}f(x,\xi)\in J(x,\xi)$ for a.e. (x,ξ). Integrating (4.4) for $v=\mathfrak{T}f(x,\xi)$ over $I_n\times I_{\Xi^*}$, (4.3) shows that $G(\mathcal{A},\Gamma)$ is a frame sequence with bounds c_0, c_1.

Conversely, suppose that $G(\mathcal{A},\Gamma)$ is a frame sequence with bounds c_0, c_1. Let $D\subset\mathbb{C}^p$ be a countable dense subset. To prove (4.4), it suffices to show that for any $v\in D$

$$pc_0\left\|P(x,\xi)v\right\|^2 \leqslant \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} \left|\left\langle \mathfrak{T}\varphi(x+l_j,\xi), P(x,\xi)v\right\rangle\right|^2 \leqslant pc_1\left\|P(x,\xi)v\right\|^2 \quad \text{a.e. } (x,\xi), \tag{4.5}$$

where $P(x,\xi)$ is the orthogonal projection of $\mathbb{C}^p$ onto $J(x,\xi)$. Assume on the contrary that (4.5) fails. Since D is countable, there exists a measurable set $E\subset\mathbb{R}^n\times\mathbb{T}^n$, with $|E|>0$, $v_0\in D$, and $\varepsilon>0$, such that at least one of the following two happens (in the Bessel case only (4.6)):

$$\sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} \left|\left\langle \mathfrak{T}\varphi(x+l_j,\xi), P(x,\xi)v_0\right\rangle\right|^2 \geqslant (pc_1+\varepsilon)\left\|P(x,\xi)v_0\right\|^2 \quad \text{a.e. } (x,\xi)\in E, \tag{4.6}$$

$$\sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} \left|\left\langle \mathfrak{T}\varphi(x+l_j,\xi), P(x,\xi)v_0\right\rangle\right|^2 \leqslant (pc_0-\varepsilon)\left\|P(x,\xi)v_0\right\|^2 \quad \text{a.e. } (x,\xi)\in E. \tag{4.7}$$

Suppose that (4.6) happens. Without loss of generality, we can also assume that $E\subset I$ is a subset of a fundamental domain I of $\mathbb{R}^n/\mathbb{Z}^n\times\mathbb{R}^n/\Xi^*$, since at least one of the sets $E\cap((l+I_n)\times(k+I_{\Xi^*}))$, $l\in\mathbb{Z}^n$, $k\in\Xi^*$, has a positive measure. Let $\Phi\in L^2_{\Xi}(\mathbb{R}^n\times\mathbb{T}^n,\mathbb{C}^p)$ be initially defined on I by

$$\Phi(x,\xi)=\begin{cases} P(x,\xi)v_0 & \text{for } (x,\xi)\in E,\\ 0 & \text{for } (x,\xi)\in I\setminus E,\end{cases}$$

and then uniquely extended to Ξ-multiplex periodic function on $\mathbb{R}^n\times\mathbb{T}^n$. Finally, define $f\in L^2(\mathbb{R}^n)$ by $\mathfrak{T}f=\Phi$. By Theorem 3.1, $f\in S(\mathcal{A},\Gamma)$. Moreover, (4.3) holds if $I_n\times I_{\Xi^*}$ is replaced by any other fundamental domain of $\mathbb{R}^n/\mathbb{Z}^n\times\mathbb{R}^n/\Xi^*$. Hence, by (4.6)

$$\begin{aligned}
\sum_{\varphi\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n}\sum_{\gamma\in\Gamma}\left|\langle M_kT_\gamma\varphi, f\rangle\right|^2 &= \frac{1}{p}\sum_{\varphi\in\mathcal{A}}\sum_{j=1}^{q}\int_I \left|\left\langle \mathfrak{T}\varphi(x+l_j,\xi), \mathfrak{T}f(x,\xi)\right\rangle\right|^2\,dx\,d\xi\\
&= \frac{1}{p}\sum_{\varphi\in\mathcal{A}}\sum_{j=1}^{q}\int_E \left|\left\langle \mathfrak{T}\varphi(x+l_j,\xi), P(x,\xi)v_0\right\rangle\right|^2\,dx\,d\xi\\
&\geqslant (c_1+\varepsilon/p)\int_E \left\|P(x,\xi)v_0\right\|^2\,dx\,d\xi\\
&= (c_1+\varepsilon/p)\int_I \left\|\Phi(x,\xi)\right\|^2\,dx\,d\xi = (c_1+\varepsilon/p)\|f\|^2,
\end{aligned}$$

which is a contradiction with c_1 being upper bound of $G(\mathcal{A},\Gamma)$. Likewise, (4.7) leads to a contradiction with the lower bound of $G(\mathcal{A},\Gamma)$. This shows (4.4) and completes the proof of (i) and (ii). $\square$

Proof of (iv). Let $(a_{\varphi,k,\gamma})_{(\varphi,k,\gamma)\in\mathcal{A}\times\mathbb{Z}^n\times\Gamma}$ be any sequence with all but finitely many zero terms. For each $\varphi\in\mathcal{A}$, $j=1,\dots,q$, define a complex exponential polynomial

$$p_{\varphi,j}(x,\xi)=\sum_{k\in\mathbb{Z}^n}\sum_{l\in\Xi}a_{\varphi,k,l_j+l}e^{2\pi i(\langle k,x\rangle+\langle l,\xi\rangle)}.$$

Recall that any $\gamma\in\Gamma$ can be uniquely decomposed as $\gamma=l_j+l$ for some $1\leqslant j\leqslant q$ and $l\in\Xi$. Hence, by Proposition 3.1 and (3.19)

$$\begin{aligned}
&\Big\|\sum_{(\varphi,k,\gamma)\in\mathcal{A}\times\mathbb{Z}^n\times\Gamma}a_{\varphi,k,\gamma}M_kT_\gamma\varphi\Big\|^2\\
&\quad=\Big\|\sum_{(\varphi,k,\gamma)\in\mathcal{A}\times\mathbb{Z}^n\times\Gamma}a_{\varphi,k,\gamma}e^{2\pi i\langle k,x\rangle}\mathfrak{T}\varphi(x+\gamma,\xi)\Big\|^2_{L^2(I_n\times I_{\Xi^*})}\\
&\quad=\Big\|\sum_{(\varphi,k,l)\in\mathcal{A}\times\mathbb{Z}^n\times\Xi}\sum_{j=1}^{q}a_{\varphi,k,l_j+l}e^{2\pi i(\langle k,x\rangle+\langle l,\xi\rangle)}\mathfrak{T}\varphi(x+l_j,\xi)\Big\|^2_{L^2(I_n\times I_{\Xi^*})}\\
&\quad=\int_{I_n\times I_{\Xi^*}}\Big\|\sum_{\varphi\in\mathcal{A}}\sum_{j=1}^{q}p_{\varphi,j}(x,\xi)\mathfrak{T}\varphi(x+l_j,\xi)\Big\|^2_{\mathbb{C}^p}\,dx\,d\xi. \qquad (4.8)
\end{aligned}$$

On the other hand, by the Plancherel formula

$$\sum_{(\varphi,k,\gamma)\in\mathcal{A}\times\mathbb{Z}^n\times\Gamma}|a_{\varphi,k,\gamma}|^2=p\sum_{\varphi\in\mathcal{A}}\sum_{j=1}^{q}\int_{I_n\times I_{\Xi^*}}\big|p_{\varphi,j}(x,\xi)\big|^2\,dx\,d\xi. \qquad (4.9)$$

Suppose that the system (4.2) is a Riesz sequence with bounds pc_0, pc_1 for a.e. $(x,\xi)\in I_\Theta\times I_{\Xi^*}$. In particular, by Remark 4.2 for a.e. $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$,

$$pc_0\sum_{\varphi\in\mathcal{A}}\sum_{j=1}^{q}\big|p_{\varphi,j}(x,\xi)\big|^2\leqslant\Big\|\sum_{\varphi\in\mathcal{A}}\sum_{j=1}^{q}p_{\varphi,j}(x,\xi)\mathfrak{T}\varphi(x+l_j,\xi)\Big\|^2\leqslant pc_1\sum_{\varphi\in\mathcal{A}}\sum_{j=1}^{q}\big|p_{\varphi,j}(x,\xi)\big|^2. \qquad (4.10)$$

Integrating (4.4) over $I_n\times I_{\Xi^*}$ and using (4.8) and (4.9) shows that $G(\mathcal{A},\Gamma)$ is a Riesz sequence with bounds c_0, c_1.

Conversely, suppose that $G(\mathcal{A},\Gamma)$ is a frame sequence with bounds c_0, c_1. By (4.8) and (4.9) this is equivalent to

$$pc_0 \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} \int_{I_n\times I_{\Xi^*}} \left|p_{\varphi,j}(x,\xi)\right|^2 dx\,d\xi \leqslant \int_{I_n\times I_{\Xi^*}} \left\| \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} p_{\varphi,j}(x,\xi)\mathfrak{T}\varphi(x+l_j,\xi)\right\|_{\mathbb{C}^p}^2 dx\,d\xi$$

$$\leqslant pc_1 \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} \int_{I_n\times I_{\Xi^*}} \left|p_{\varphi,j}(x,\xi)\right|^2 dx\,d\xi, \tag{4.11}$$

where only finite number of polynomials $p_{\varphi,j}$ are non-zero.

Suppose that $m_{\varphi,j} \in L^\infty(I_n \times I_{\Xi^*})$, $\varphi \in \mathcal{A}$, $j = 1,\dots,q$. By Lusin's theorem, we can find a sequence of $\mathbb{Z}^n \times \Xi$-periodic, complex exponential polynomials $(p^i)_{i\in\mathbb{N}}$, depending on $\varphi \in \mathcal{A}$, $j = 1,\dots,q$, such that

$$\left\|p^i\right\|_\infty \leqslant \|m_{\varphi,j}\|_\infty \quad \text{and} \quad p^i(x,\xi) \to m_{\varphi,j}(x,\xi) \quad \text{as } i\to\infty \text{ for a.e. } (x,\xi).$$

By the Lebesgue Dominated Convergence theorem (4.11) can be strengthened to

$$pc_0 \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} \int_{I_n\times I_{\Xi^*}} \left|m_{\varphi,j}(x,\xi)\right|^2 dx\,d\xi \leqslant \int_{I_n\times I_{\Xi^*}} \left\| \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} m_{\varphi,j}(x,\xi)\mathfrak{T}\varphi(x+l_j,\xi)\right\|_{\mathbb{C}^p}^2 dx\,d\xi$$

$$\leqslant pc_1 \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} \int_{I_n\times I_{\Xi^*}} \left|m_{\varphi,j}(x,\xi)\right|^2 dx\,d\xi, \tag{4.12}$$

where only a finite number of $m_{\varphi,j} \in L^\infty(I_n \times I_{\Xi^*})$ are non-zero. Let $D \subset \ell^2(\mathcal{A} \times \{1,\dots,q\})$ be a countable dense set with the property that each $d = (d_{\varphi,j})_{\varphi,j} \in D$ has a finite number of non-zero coordinates. To complete the proof of (iv), it suffices to show that for every $d \in D$,

$$pc_0 \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} |d_{\varphi,j}|^2 \leqslant \left\| \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} d_{\varphi,j}\mathfrak{T}\varphi(x+l_j,\xi)\right\|^2$$

$$\leqslant pc_1 \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} |d_{\varphi,j}|^2 \quad \text{a.e. } (x,\xi). \tag{4.13}$$

On the contrary, if (4.13) fails, then there exists a measurable set $E \subset \mathbb{R}^n \times \mathbb{T}^n$, with $|E| > 0$, $d \in D$, and $\varepsilon > 0$, such that at least one of the following happens:

$$\left\| \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} d_{\varphi,j}\mathfrak{T}\varphi(x+l_j,\xi)\right\|^2 \geqslant (pc_1+\varepsilon) \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} |d_{\varphi,j}|^2 \quad \text{a.e. } (x,\xi)\in E, \tag{4.14}$$

$$\left\| \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} d_{\varphi,j}\mathfrak{T}\varphi(x+l_j,\xi)\right\|^2 \leqslant (pc_0-\varepsilon) \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} |d_{\varphi,j}|^2 \quad \text{a.e. } (x,\xi)\in E. \tag{4.15}$$

Without loss of generality, we can assume that E is a subset of a fundamental domain of $\mathbb{R}^n/\mathbb{Z}^n \times \mathbb{R}^n/\Xi^*$. Define $m_{\varphi,j} \in L^\infty(I)$ by $m_{\varphi,j} = d_{\varphi,j}\mathbf{1}_E$. Since (4.12) is also valid if the fundamental domain $I_n \times I_{\Xi^*}$ is replaced by I, then

$$\begin{aligned}\int_I \Bigl\| \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} m_{\varphi,j}(x,\xi)\mathfrak{T}\varphi(x+l_j,\xi)\Bigr\|_{\mathbb{C}^p}^2 dx\,d\xi &= \int_E \Bigl\| \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} d_{\varphi,j}\mathfrak{T}\varphi(x+l_j,\xi)\Bigr\|_{\mathbb{C}^p}^2 dx\,d\xi \\ &\geqslant (pc_1+\varepsilon)|E| \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} |d_{\varphi,j}|^2 \\ &= (pc_1+\varepsilon) \sum_{\varphi\in\mathcal{A}} \sum_{j=1}^{q} \int_I \bigl|m_{\varphi,j}(x,\xi)\bigr|^2 dx\,d\xi.\end{aligned}$$

Hence, (4.14) contradicts (4.12). Likewise, (4.15) contradicts (4.12). Therefore, (4.13) must hold and (4.2) is a Riesz sequence for a.e. (x,ξ). □

Proof of (iii) and (v). By Theorem 3.1, the Gabor system $G(\mathcal{A},\Gamma)$ is complete, that is its closed linear span equals $L^2(\mathbb{R}^n)$, if and only if its corresponding shift–modulation range function $J(x,\xi)=\mathbb{C}^p$ for a.e. (x,ξ). By (3.15), this is equivalent to the property that the system (4.2) is complete in $\mathbb{C}^p$ for a.e. (x,ξ). Since a frame (or Riesz basis) is simply a frame sequence (or Riesz sequence) which is complete, then (ii) and (iv) imply (iii) and (v), respectively. □

As an immediate corollary of Theorem 4.1 we have

Theorem 4.2. *Suppose that $\Gamma\subset\mathbb{R}^n$ is a rational lattice, and let $|\mathbb{R}^n/\Gamma|$ be the Lebesgue measure of a fundamental domain of $\mathbb{R}^n/\Gamma$. Suppose that $\mathcal{A}\subset L^2(\mathbb{R}^n)$ is countable.*
Then the following are true:

(i) *If $G(\mathcal{A},\Gamma)$ is complete, then $\mathcal{A}$ has at least $\lceil|\mathbb{R}^n/\Gamma|\rceil$ elements.*
(ii) *If $G(\mathcal{A},\Gamma)$ is a Riesz sequence, then $\mathcal{A}$ has at most $\lfloor|\mathbb{R}^n/\Gamma|\rfloor$ elements.*
(iii) *If $G(\mathcal{A},\Gamma)$ is a Riesz basis, then $|\mathbb{R}^n/\Gamma|$ is an integer and $\mathcal{A}$ has exactly $|\mathbb{R}^n/\Gamma|$ elements.*

Here, $\lfloor x\rfloor$ and $\lceil x\rceil$ denote the floor and ceiling functions, respectively. Theorem 4.2 has a very rich history and is known either as the Density Theorem, or part (i) as the Incompleteness Theorem for Gabor systems, see [9,16,19]. The same result also holds for general (not necessarily rational) lattices Γ. Part (i) of Theorem 4.2 is a consequence of Rieffel's result on von Neumann algebras associated with lattices [26]. An alternative proof of (i), which does not use von Neumann algebras, was given by Rzeszotnik and the author [3]. Here, we merely indicate that Theorem 4.2 easily follows from our results.

Proof. As usual Θ, Ξ are given by (1.3), p is the order of $\mathbb{Z}^n/\Xi$, and q is the order of $\Theta/\mathbb{Z}^n$. Note that $|\mathbb{R}^n/\Xi|=|\mathbb{R}^n/\Gamma||\Gamma/\Xi|$. Since the order of Γ/Ξ is the same as $\Theta/\mathbb{Z}^n$, which equals q, then we have $|\mathbb{R}^n/\Gamma|=p/q$.

If $G(\mathcal{A},\Gamma)$ is complete, then by Theorem 3.1, the number of vectors in (4.2) is at least the dimension of $\mathbb{C}^p$. Hence, $q|\mathcal{A}|\geqslant p$, which shows (i). Likewise, if $G(\mathcal{A},\Gamma)$ is a Riesz sequence, then by Theorem 4.1, the number of vectors in (4.2) is at most the dimension of $\mathbb{C}^p$. Hence, $q|\mathcal{A}|\leqslant p$, which shows (i). Finally, if $G(\mathcal{A},\Gamma)$ is a Riesz basis, then $q|\mathcal{A}|=p$, proving (iii). □

Another immediate consequence of Theorem 4.1 is the following result of Han and Wang [18, Lemma 3.2].

Corollary 4.1. *Suppose that $\Gamma \subset \mathbb{R}^n$ is a rational lattice, and the Gabor system $G(\mathcal{A}, \Gamma)$ is a tight frame with bound* 1. *Then*

$$\sum_{\varphi \in \mathcal{A}} \|\varphi\|^2 = \left|\mathbb{R}^n/\Gamma\right|.$$

Proof. Since the system (4.2) is a tight frame with bound p, we have

$$\sum_{\varphi \in \mathcal{A}} \sum_{j=1}^{q} \left\|\mathfrak{T}\varphi(x + l_j, \xi)\right\|^2 = p \dim \mathbb{C}^p = p^2.$$

Integrating the above over $(x, \xi) \in I_\Theta \times I_{\Xi^*}$ yields

$$\sum_{\varphi \in \mathcal{A}} \|\varphi\|^2 = p^2 |I_\Theta \times I_{\Xi^*}| = p/q = \left|\mathbb{R}^n/\Gamma\right|. \qquad \square$$

5. Decomposition of shift–modulation invariant spaces

The main goal of this section is to prove the existence of a decomposition of an SMI space as an orthogonal sum of much simpler SMI spaces. More precisely, we say that an SMI space V is *principal*, if its dimension function $\dim_V \leqslant 1$. Then, Theorem 5.1 shows that every SMI space enjoys an orthogonal decomposition into principal SMI spaces.

The concept of principal SMI is borrowed from the theory of shift-invariant spaces, where the space is called principal if it is generated by a single generator. However, in the context of SMI spaces it is no longer true in general that if an SMI space V is generated by a single generator, then V must be principal. By Theorem 3.1, we can only claim that the dimension function $\dim_V \leqslant q$, and it is not difficult to see that the equality may happen. Hence, we need to introduce the concept of a principal generator for SMI spaces.

Definition 5.1. We say that $\varphi \in L^2(\mathbb{R}^n)$ is a *principal generator* if

$$\dim_{S(\varphi,\Gamma)} \leqslant 1. \tag{5.1}$$

Despite its simplicity, the above concept is too broad and we need to impose a more restrictive conditions on the size of the support of a principal generator φ in the Zak domain.

Definition 5.2. We say that $\varphi \in L^2(\mathbb{R}^n)$ is a *minimal principal generator* if for a.e. $(x, \xi) \in \mathbb{R}^n \times \mathbb{T}^n$, there exists $j_0 = j_0(x, \xi)$, $0 \leqslant j_0 \leqslant q$, such that

$$\left\|\mathfrak{T}\varphi(x + l_j, \xi)\right\| = \delta_{j,j_0} \quad \text{for all } 1 \leqslant j \leqslant q. \tag{5.2}$$

In other words, if $j_0 = 0$, then $\mathfrak{T}\varphi(x + l_j, \xi) = 0$ for all values of $1 \leqslant j \leqslant q$. Otherwise, if $1 \leqslant j_0 \leqslant q$, then $\|\mathfrak{T}\varphi(x + l_j, \xi)\| = 0$ for all values of $1 \leqslant j \leqslant q$, with the exception of a single value $1 \leqslant j_0 \leqslant q$, for which $\|\mathfrak{T}\varphi(x + l_{j_0}, \xi)\| = 1$. Here, as usual $\{l_1, \ldots, l_q\} \subset \Gamma$ are representatives of distinct cosets of $\Theta/\mathbb{Z}^n$. It is also convenient to assume that $l_1 = 0$.

Remark 5.1. It is clear that every minimal principal generator φ generates a principal SMI space $V = S(\varphi, \Gamma)$, since by Theorem 3.1, $\dim_V(x, \xi) \leqslant 1$ for a.e. (x, ξ). The converse is also true, which is a consequence of the decomposition Theorem 5.1, see Corollary 5.1. Moreover, by Theorem 4.1, the Gabor system $G(\varphi, \Gamma)$ generated by a minimal principal generator φ is a tight frame with bound 1.

Lemma 5.1. *Suppose $V \subset L^2(\mathbb{R}^n)$ is an SMI space. Then there exists a minimal principal generator φ such that*

$$S(\varphi, \Gamma) \subset V, \quad \textit{and} \quad \sigma\big(S(\varphi, \Gamma)\big) = \sigma(V). \tag{5.3}$$

Furthermore, φ can be chosen so that

$$\operatorname{supp} \mathfrak{T}\varphi \subset \bigcup_{l \in \mathbb{Z}^n} (l + I_\Theta) \times \mathbb{R}^n, \tag{5.4}$$

where I_Θ is a fundamental domain of $\mathbb{R}^n/\Theta$. In particular, we have

$$\dim_{S(\varphi,\Gamma)}(x, \xi) = \mathbf{1}_{\sigma(S(\varphi,\Gamma))}(x, \xi) = \sum_{j=1}^{q} \big\| \mathfrak{T}\varphi(x + l_j, \xi)\big\|. \tag{5.5}$$

Proof. Let J be the range function corresponding to V. We claim that we can find a measurable function $\Phi : I_\Theta \times I_{\Xi^*} \to \mathbb{C}^p$, such that

$$\Phi(x, \xi) \in J(x, \xi), \quad \text{and} \quad \big\|\Phi(x, \xi)\big\| = \mathbf{1}_{\sigma(V)}(x, \xi) \quad \text{a.e. } (x, \xi) \in I_\Theta \times I_{\Xi^*}, \tag{5.6}$$

where $\sigma(V)$ is the spectrum of V. Let $P(x, \xi)$ be the orthogonal projection onto $J(x, \xi)$ and $\{e_1, \dots, e_p\}$ be the standard basis of $\mathbb{C}^p$. To show the existence of such Φ, it suffices to consider measurable functions

$$\Phi_j(x, \xi) = \begin{cases} P(x, \xi)e_j / \|P(x, \xi)e_j\| & \text{if } P(x, \xi)e_j \neq 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 \leqslant j \leqslant q,$$

and notice that their supports cover $\sigma(V)$ (modulo null sets). Hence, it suffices to glue them together on their respective supports to get a single function Φ satisfying (5.6). More precisely, let $\{E_j\}_{j=1}^p$ be a partition of $\sigma(V) \cap (I_\Theta \times I_{\Xi^*})$ such that $E_j \subset \operatorname{supp} \Phi_j$ for all $1 \leqslant j \leqslant p$. Define

$$\Phi(x, \xi) = \Phi_j(x, \xi) \quad \text{if } (x, \xi) \in E_j,\ 1 \leqslant j \leqslant p,$$

and $\Phi(x, \xi) = 0$ elsewhere on $I_\Theta \times I_{\Xi^*}$. Then, Φ satisfies (5.6).

Next, we extend Φ to a larger domain $\bigcup_{j=1}^q (l_j + I_\Theta) \times I_{\Xi^*}$, $l_1 = 0$, by

$$\Phi(x, \xi) = 0 \quad \text{for all } (x, \xi) \in \bigcup_{j=2}^{q} (l_j + I_\Theta) \times I_{\Xi^*}. \tag{5.7}$$

Since $\bigcup_{j=1}^{q}(l_j + I_\Theta) \times I_{\Xi^*}$ is a fundamental domain of $\mathbb{T}^n \times \mathbb{R}^n/\Xi^*$, we can uniquely extend Φ to a Ξ-multiplex-periodic function on $\mathbb{R}^n \times \mathbb{T}^n$. It is clear that

$$\Phi \in L^2_\Xi\big(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p\big), \qquad \Phi(x,\xi) \in J(x,\xi) \quad \text{a.e. } (x,\xi).$$

Let $\varphi = \mathfrak{T}^{-1}\Phi$. Combining (5.6) and (5.7) implies (5.2), and φ is a minimal principal generator. By Theorem 3.1 and (5.6), we must have (5.3). Furthermore, (5.7) implies that (5.4) holds. Finally, (5.5) is a consequence of the fact that φ is a minimal principal generator and Theorem 3.1. $\square$

Theorem 5.1. *Suppose that V is an SMI space with respect to a rational lattice Γ. Then, V can be decomposed as an orthogonal sum*

$$V = \bigoplus_{i=1}^{p} S(\varphi_i, \Gamma), \tag{5.8}$$

where each φ_i is a minimal principal generator of $S(\varphi_i, \Gamma)$, and

$$\big\{(x,\xi)\colon \dim_V(x,\xi) \geqslant i\big\} = \sigma\big(S(\varphi_i,\Gamma)\big) \quad \textit{for all } 1 \leqslant i \leqslant p. \tag{5.9}$$

Moreover, we can choose φ_i's such that

$$\operatorname{supp} \mathfrak{T}\varphi_i \subset \bigcup_{l \in \mathbb{Z}^n} (l + I_\Theta) \times \mathbb{R}^n \quad \textit{for all } 1 \leqslant i \leqslant p, \tag{5.10}$$

where I_Θ is a fundamental domain of $\mathbb{R}^n/\Theta$. In particular, we have

$$\dim_V(x,\xi) = \sum_{i=1}^{p} \dim_{S(\varphi_i,\Gamma)}(x,\xi) = \sum_{i=1}^{p}\sum_{j=1}^{q} \big\|\mathfrak{T}\varphi_i(x + l_j, \xi)\big\|. \tag{5.11}$$

Note that by (5.9), minimal principal generators $\varphi_i = 0$ for $i > \operatorname{ess\,sup} \dim_V$. Consequently, the orthogonal sum (5.8) may effectively consists of fewer terms than p. However, for notational convenience we will pretend that we have always p minimal principal generators despite the fact the some of them could be zero. Moreover, (5.9) implies that the spectra of principal spaces $S(\varphi_i, \Gamma)$ are nested,

$$\sigma(V) = \sigma\big(S(\varphi_1,\Gamma)\big) \supset \cdots \supset \sigma\big(S(\varphi_p,\Gamma)\big).$$

Proof. To prove Theorem 5.1, we apply inductively Lemma 5.1. Let φ_1 be a minimal principal generator guaranteed by Lemma 5.1. Assume that we have minimal principal generators $\varphi_1, \dots, \varphi_k$, for some $1 \leqslant k \leqslant q-1$, such that the each φ_i is a minimal principal generator of $S(\varphi_i, \Gamma)$, $1 \leqslant i \leqslant k$, and these spaces are mutually orthogonal. Assume also that (5.9) and (5.10) hold for $1 \leqslant i \leqslant k$. Applying Lemma 5.1 for the space

$$V' = V \ominus \left(\bigoplus_{i=1}^{k} S(\varphi_i, \Gamma)\right)$$

yields a minimal principal generator φ_{k+1} such that $S(\varphi_{k+1},\Gamma)$ is orthogonal to the previous spaces $S(\varphi_i,\Gamma)$. Since the dimension function is additive with respect to orthogonal sums,

$$\dim_V = \dim_{V'} + \sum_{i=1}^{k} \dim_{S(\varphi_i,\Gamma)} = \dim_{V'} + \sum_{i=1}^{k} \mathbf{1}_{\sigma(S(\varphi_i,\Gamma))}.$$

Hence, $\dim_V(x,\xi) \geqslant k+1$ implies that $(x,\xi)\in\sigma(S(\varphi_i,\Gamma))$, $1\leqslant i\leqslant k$, and $(x,\xi)\in\sigma(V') = \sigma(S(\varphi_{k+1},\Gamma))$. Conversely, $\dim_V(x,\xi)\leqslant k$ implies by induction hypothesis (5.9) valid for $1\leqslant i\leqslant k$, that

$$\dim_V(x,\xi) = \sum_{i=1}^{k} \mathbf{1}_{\sigma(S(\varphi_i,\Gamma))}(x,\xi),$$

and consequently $(x,\xi)\notin\sigma(V')=\sigma(S(\varphi_{k+1},\Gamma))$. This shows (5.9) for $i=k+1$. To prove (5.8) note that (5.9) implies that

$$\dim_V = \sum_{i=1}^{p} \mathbf{1}_{\sigma(S(\varphi_i,\Gamma))} = \sum_{i=1}^{p} \dim_{S(\varphi_i,\Gamma)} = \dim_{\bigoplus_{i=1}^{p} S(\varphi_i,\Gamma)}.$$

Since $\bigoplus_{i=1}^{p} S(\varphi_i,\Gamma)\subset V$, we must have equality. Finally, (5.11) is an immediate consequence of (5.5). □

As an immediate corollary of Theorem 5.1 we have

Corollary 5.1. *Suppose $V\subset L^2(\mathbb{R}^n)$ is a principal SMI space. Then there exists a minimal principal generator φ, such that $V=S(\varphi,\Gamma)$. Furthermore, φ can be chosen so that*

$$\operatorname{supp}\mathfrak{T}\varphi \subset \bigcup_{l\in\mathbb{Z}^n}(l+I_\Theta)\times\mathbb{R}^n, \tag{5.12}$$

where I_Θ is a fundamental domain of $\mathbb{R}^n/\Theta$.

Finally, it is very useful to introduce the concept of a maximal principal generator.

Definition 5.3. We say that $\varphi\in L^2(\mathbb{R}^n)$ is a *maximal principal generator* if for a.e. $(x,\xi)\in\mathbb{R}^n\times\mathbb{T}^n$,

$$\|\mathfrak{T}\varphi(x,\xi)\| = 0 \text{ or } 1, \tag{5.13}$$

and

$$\mathfrak{T}\varphi(x+l_j,\xi) = c_j\mathfrak{T}\varphi(x,\xi) \quad \text{for } 1\leqslant j\leqslant q, \tag{5.14}$$

for some constant $c_j=c_j(x,\xi)$ with $|c_j|=1$. Here, as usual $\{l_1,\dots,l_q\}\subset\Gamma$ are representatives of distinct cosets of $\Theta/\mathbb{Z}^n$. It is also convenient to assume that $l_1=0$.

Remark 5.2. It is clear that every maximal principal generator φ generates a principal SMI space $V = S(\varphi, \Gamma)$, since by Theorem 3.1, $\dim_V(x, \xi) \leqslant 1$ for a.e. (x, ξ). The converse is also true, which is a consequence of Lemma 5.2. Moreover, by Theorem 4.1, the Gabor system $G(\varphi, \Gamma)$ generated by a maximal principal generator φ is a tight frame with bound q.

The next result provides a simple method of moving between minimal and maximal principal generators.

Lemma 5.2. *If φ is a minimal principal generator, then the function $\tilde{\varphi}$ given by*

$$\mathfrak{T}\tilde{\varphi}(x, \xi) = \begin{cases} \mathfrak{T}\varphi(x - l_j, \xi), & 1 \leqslant j \leqslant q \text{ and } (x - l_j, \xi) \in \operatorname{supp} \mathfrak{T}\varphi, \\ 0, & \text{otherwise}, \end{cases} \tag{5.15}$$

is a maximal principal generator. Conversely, if $\tilde{\varphi}$ is a maximal principal generator, then the function φ given by

$$\mathfrak{T}\varphi = \mathbf{1}_E \mathfrak{T}\tilde{\varphi} \tag{5.16}$$

is a minimal principal generator, where E is any measurable set such that $\{(-l_j, 0) + E\colon 1 \leqslant j \leqslant q\}$ is a partition (modulo null sets) of $\mathbb{R}^n \times \mathbb{T}^n$. In either case, φ and $\tilde{\varphi}$ generate the same principal space

$$S(\varphi, \Gamma) = S(\tilde{\varphi}, \Gamma). \tag{5.17}$$

Proof. Suppose φ is a minimal principal generator. By (5.1), at most one of the points $(x - l_j, \xi)$, $1 \leqslant j \leqslant q$, belongs to $\operatorname{supp} \mathfrak{T}\varphi$, and $\mathfrak{T}\tilde{\varphi}(x, \xi)$ is well defined. Moreover, we have

$$\mathfrak{T}\tilde{\varphi}(x + l_1, \xi) = \cdots = \mathfrak{T}\tilde{\varphi}(x + l_q, \xi),$$

and (5.14) holds with constants $c_j = 1$, $1 \leqslant j \leqslant q$. Hence, $\tilde{\varphi}$ is a maximal principal generator.

Conversely, suppose $\tilde{\varphi}$ is a maximal principal generator and let E satisfy the hypotheses of Lemma 5.2. Then $(x + l_j, \xi) \in E$ for at exactly one $1 \leqslant j \leqslant q$ for a.e. $(x, \xi) \in \mathbb{R}^n \times \mathbb{T}^n$. Hence, φ given by (5.16) satisfies (5.1), and therefore, φ is a minimal principal generator.

The property (5.17) is an immediate consequence of Theorem 3.1. □

Note that Lemma 5.2 also holds if one chooses a smaller set E such that $\{(-l_j, 0) + E\colon 1 \leqslant j \leqslant q\}$ is a partition (modulo null sets) of $\operatorname{supp} \mathfrak{T}\tilde{\varphi}$, instead of $\mathbb{R}^n \times \mathbb{T}^n$, since the definition of φ by (5.16) is unaffected.

As a corollary of Lemma 5.2, we have the following variant of Theorem 5.1.

Theorem 5.2. *Suppose that V is an SMI space with respect to a rational lattice Γ. Then, V can be decomposed as an orthogonal sum*

$$V = \bigoplus_{i=1}^{p} S(\varphi_i, \Gamma), \tag{5.18}$$

where each φ_i is a maximal principal generator of $S(\varphi_i, \Gamma)$, and

$$\big\{(x,\xi)\colon \dim_V(x,\xi)\geqslant i\big\}=\sigma\big(S(\varphi_i,\Gamma)\big)=\big\{(x,\xi)\colon \mathfrak{T}\varphi_i(x,\xi)\neq 0\big\},\quad 1\leqslant i\leqslant p. \tag{5.19}$$

In particular, we have

$$\dim_V(x,\xi)=\sum_{i=1}^{p}\dim_{S(\varphi_i,\Gamma)}(x,\xi)=\sum_{i=1}^{p}\big\|\mathfrak{T}\varphi_i(x,\xi)\big\|. \tag{5.20}$$

Finally, we prove a simple description of principal SMI spaces in terms of their maximal principal generators.

Theorem 5.3. *Suppose $V\subset L^2(\mathbb{R}^n)$ is a principal SMI space and φ is its maximal principal generator, i.e., $V=S(\varphi,\Gamma)$. Then*

$$V=\big\{f\in L^2\big(\mathbb{R}^n\big)\colon \mathfrak{T}f=m\mathfrak{T}\varphi \text{ for some } m\in L^2\big(\mathbb{T}^n\times\mathbb{R}^n/\Xi^*\big)\big\}. \tag{5.21}$$

Moreover, if we require in (5.21) *that* $\operatorname{supp} m\subset\operatorname{supp}\mathfrak{T}\varphi$*, then*

$$\|f\|_{L^2(\mathbb{R}^n)}=\|m\|_{L^2(\mathbb{T}^n\times\mathbb{R}^n/\Xi^*)}. \tag{5.22}$$

Proof. Let J be the range function corresponding to the SMI space V. Let V' be the space given by the right-hand side of (5.21). Since for $f\in V'$

$$\mathfrak{T}f(x,\xi)\in\operatorname{span}\big\{\mathfrak{T}\varphi(x,\xi)\big\}\subset J(x,\xi),$$

hence we have $V'\subset V$. Conversely, take $f\in V$. Since

$$\mathfrak{T}f(x,\xi)\in J(x,\xi)=\operatorname{span}\big\{\mathfrak{T}\varphi(x+l_j,\xi)\colon 1\leqslant j\leqslant q\big\}=\operatorname{span}\big\{\mathfrak{T}\varphi(x,\xi)\big\}\quad \text{a.e. }(x,\xi),$$

we define $m(x,\xi)$ as a unique constant such that

$$\mathfrak{T}f(x,\xi)=m(x,\xi)\mathfrak{T}\varphi(x,\xi)$$

if $(x,\xi)\in\operatorname{supp}\mathfrak{T}\varphi$, and $m(x,\xi)=0$, otherwise. Employing (5.13) we have

$$\begin{aligned}\|f\|^2&=\int_{\mathbb{T}^n\times\mathbb{R}^n/\Xi^*}\|\mathfrak{T}f\|^2=\int_{\operatorname{supp}\mathfrak{T}\varphi\cap(I_n\times I_{\Xi^*})}\|\mathfrak{T}f\|^2=\int_{\operatorname{supp}\mathfrak{T}\varphi\cap(I_n\times I_{\Xi^*})}|m|^2\|\mathfrak{T}\varphi\|^2\\&=\int_{\operatorname{supp}\mathfrak{T}\varphi\cap(I_n\times I_{\Xi^*})}|m|^2=\|m\|^2_{L^2(\mathbb{T}^n\times\mathbb{R}^n/\Xi^*)},\end{aligned}$$

which proves (5.22). □

6. Shift–modulation invariant operators

The goal of this section is to provide a description of the class of shift–modulation invariant operators, i.e., operators commuting both with shifts and modulations. In the language of the category theory, such operators are simply morphisms between SMI spaces. Since SMI spaces can be described in terms of range functions, it seems plausible to expect that morphisms between SMI spaces must correspond to linear maps between fibers of the corresponding range functions. The precise formulation of this relationship uses the concept of the range operator and it is stated in Theorem 6.1. The analogous correspondence for shift-invariant spaces was established by the author in [2, Section 4]. As a consequence we prove several results manifesting the fiberization paradigm for SMI operators. More precisely, we postulate that any reasonable property of an SMI operator is equivalent to the same property holding uniformly over the linear maps of the corresponding range operator. As a consequence of these techniques we deduce Theorem 6.6 which provides a classification of unitarily equivalent SMI spaces in terms of their dimension functions.

Definition 6.1. Suppose that $V \subset L^2(\mathbb{R}^n)$ is an SMI space. We say that a bounded linear operator $L : V \to L^2(\mathbb{R}^n)$ is shift–modulation invariant (SMI) if L commutes with shifts and modulations,

$$LM_kT_\gamma = M_kT_\gamma L \quad \text{for all } k \in \mathbb{Z}^n,\ \gamma \in \Gamma. \tag{6.1}$$

Lemma 6.1. *Assume that $\varphi \in L^2(\mathbb{R}^n)$ satisfies the condition* (5.13). *Suppose that $L : S(\varphi, \Gamma) \to L^2(\mathbb{R}^n)$ is a bounded SMI operator. Then for every $m \in L^2(\mathbb{T}^n \times \mathbb{R}^n/\Xi^*)$,*

$$\mathfrak{T}L\mathfrak{T}^{-1}(m\Phi)(x,\xi) = m(x,\xi)\mathfrak{T}L\mathfrak{T}^{-1}(\Phi)(x,\xi) \quad \textit{a.e. } (x,\xi) \in \mathbb{R}^n \times \mathbb{T}^n, \tag{6.2}$$

where $\Phi = \mathfrak{T}\varphi$. Moreover,

$$\left\|\mathfrak{T}L\mathfrak{T}^{-1}(\Phi)(x,\xi)\right\| \leqslant \|L\| \left\|\Phi(x,\xi)\right\| \quad \textit{a.e. } (x,\xi) \in \mathbb{R}^n \times \mathbb{T}^n. \tag{6.3}$$

Proof. Recall that for any $f \in L^2(\mathbb{R}^n)$,

$$\mathfrak{T}(M_kT_lf)(x,\xi) = e^{2\pi i\langle k,x\rangle}e^{2\pi i\langle l,\xi\rangle}\mathfrak{T}f(x,\xi) \quad \text{for } k \in \mathbb{Z}^n,\ l \in \Xi = \Gamma \cap \mathbb{Z}^n.$$

Hence, modulations and shifts on $L^2(\mathbb{R}^n)$ correspond to multiplications by complex exponentials on $L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$, that is

$$M_kT_lf = \mathfrak{T}^{-1}\left(e^{2\pi i\langle k,\cdot_x\rangle}e^{2\pi i\langle l,\cdot_\xi\rangle}\mathfrak{T}f\right) \quad \text{for } k \in \mathbb{Z}^n,\ l \in \Xi.$$

Since L is an SMI operator

$$\mathfrak{T}L\mathfrak{T}^{-1}\left(e^{2\pi i\langle k,\cdot_x\rangle}e^{2\pi i\langle l,\cdot_\xi\rangle}\Phi\right) = \mathfrak{T}LM_kT_l\varphi = \mathfrak{T}M_kT_lL\varphi = e^{2\pi i\langle k,\cdot_x\rangle}e^{2\pi i\langle l,\cdot_\xi\rangle}\mathfrak{T}L\mathfrak{T}^{-1}\Phi.$$

Therefore, by linearity of L, (6.2) holds for all polynomials

$$p(x,\xi) = \sum_{k\in\mathbb{Z}^n,\, l\in\Xi} a_{k,l}e^{2\pi i\langle k,x\rangle}e^{2\pi i\langle l,\xi\rangle} \in L^2\left(\mathbb{T}^n \times \mathbb{R}^n/\Xi^*\right).$$

Since $\mathfrak{T}$ is an isometric isomorphism, we have $\|\mathfrak{T}L\mathfrak{T}^{-1}\| = \|L\| < \infty$, and

$$\begin{aligned}\int_{I_n\times I_{\Xi^*}} \left|p(x,\xi)\right|^2 \left\|\mathfrak{T}L\mathfrak{T}^{-1}\Phi(x,\xi)\right\|^2 dx\,d\xi &= \int_{I_n\times I_{\Xi^*}} \left\|\mathfrak{T}L\mathfrak{T}^{-1}(p\Phi)(x,\xi)\right\|^2 dx\,d\xi \\ &= \left\|\mathfrak{T}L\mathfrak{T}^{-1}(p\Phi)\right\|^2_{L^2_\Xi} \leqslant \|L\|^2\|p\Phi\|^2_{L^2_\Xi} \\ &= \|L\|^2 \int_{I_n\times I_{\Xi^*}} \left|p(x,\xi)\right|^2 \left\|\Phi(x,\xi)\right\|^2 dx\,d\xi. \end{aligned} \tag{6.4}$$

As in the proof of Theorem 3.1(iv), for any $r \in L^\infty(I_n \times I_{\Xi^*})$, we can find a sequence of polynomials $(p_i)_{i\in\mathbb{N}}$, such that

$$\|p_i\|_\infty \leqslant \|r\|_\infty \quad \text{and} \quad p_i(x,\xi) \to r(x,\xi) \quad \text{as } i\to\infty \text{ for a.e. } (x,\xi).$$

By the Lebesgue Dominated Convergence theorem (6.4) can be strengthened to

$$\int_{I_n\times I_{\Xi^*}} \left|r(x,\xi)\right|^2 \left\|\mathfrak{T}L\mathfrak{T}^{-1}\Phi(x,\xi)\right\|^2 dx\,d\xi \leqslant \|L\|^2 \int_{I_n\times I_{\Xi^*}} \left|r(x,\xi)\right|^2 \left\|\Phi(x,\xi)\right\|^2 dx\,d\xi. \tag{6.5}$$

Since $r \in L^\infty(I_n \times I_{\Xi^*})$ is arbitrary (6.5) yields (6.3).

Finally, take a sequence of polynomials $(p_i)_{i\in\mathbb{N}}$ converging to m in $L^2(\mathbb{T}^n \times \mathbb{R}^n/\Xi^*)$ norm. Then, $p_i\Phi \to m\Phi$ in $L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$ norm, since $\|\Phi\|_\infty \leqslant 1$. By choosing a subsequence, we can assume that for a.e. (x,ξ),

$$p_i(x,\xi) \to m(x,\xi) \quad \text{and} \quad \mathfrak{T}L\mathfrak{T}^{-1}(p_i\Phi)(x,\xi) \to \mathfrak{T}L\mathfrak{T}^{-1}(m\Phi)(x,\xi) \quad \text{as } i\to\infty. \tag{6.6}$$

Since (6.2) holds for polynomials, by (6.6) the same must hold for a general $m \in L^2(\mathbb{T}^n \times \mathbb{R}^n/\Xi^*)$. $\square$

Remark 6.1. Note that Lemma 6.1 holds, in particular, if φ is either a minimal or maximal principal generator. Furthermore, in the latter case every $f \in S(\varphi,\Gamma)$ must be of the form $f = \mathfrak{T}^{-1}(m\Phi)$ for some $m \in L^2(\mathbb{T}^n \times \mathbb{R}^n/\Xi^*)$ by Theorem 5.3. Hence, if φ is a maximal principal generator, then (6.2) provides a very simple description of the action of L on the entire space $S(\varphi,\Gamma)$.

The assumption (5.13) in Lemma 6.1 is merely for the convenience.

Corollary 6.1. *Suppose that $V \subset L^2(\mathbb{R}^n)$ is SMI and a bounded linear operator $L : V \to L^2(\mathbb{R}^n)$ is also SMI. Then for every $\Phi \in \mathfrak{T}V$, and a measurable $\mathbb{Z}^n \times \Xi^*$-periodic function such that $m\Phi \in L^2_\Xi(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$, we have that $m\Phi \in \mathfrak{T}V$ and both* (6.2) *and* (6.3) *hold.*

Proof. Corollary 6.1 is an immediate consequence of Theorem 3.1 and Lemma 6.1. Indeed, given $\Phi \in \mathfrak{T}V$, define $\Phi_0 \in \mathfrak{T}V$ by

$$\Phi_0(x,\xi) = \begin{cases} \Phi(x,\xi)/\|\Phi(x,\xi)\| & \text{for } (x,\xi) \in \operatorname{supp}\Phi, \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 3.1, $\Phi_0 \in \mathfrak{T}V$ satisfies

$$\|\Phi_0(x,\xi)\| = 0 \text{ or } 1 \quad \text{for a.e. } (x,\xi),$$

and by Lemma 6.1, (6.2) and (6.3) hold for Φ_0. Moreover, since $m\Phi \in L^2$, we have that m_0, $mm_0 \in L^2(\mathbb{T}^n \times \mathbb{R}^n/\Xi^*)$, where $m_0(x,\xi) = \|\Phi(x,\xi)\|$. Hence,

$$\begin{aligned}\mathfrak{T}L\mathfrak{T}^{-1}(m\Phi)(x,\xi) &= \mathfrak{T}L\mathfrak{T}^{-1}(mm_0\Phi_0)(x,\xi)\\ &= m(x,\xi)m_0(x,\xi)\mathfrak{T}L\mathfrak{T}^{-1}(\Phi_0)(x,\xi)\\ &= m(x,\xi)\mathfrak{T}L\mathfrak{T}^{-1}(\Phi)(x,\xi),\end{aligned}$$

which proves (6.2). The proof of (6.3) is similar, or one could use the original argument in the proof of Lemma 6.1 since it did not use the assumption (5.13). □

We are now ready to introduce the concept of a shift–modulation range operator as a collection of linear maps defined on fibers of the range function and satisfying the natural periodicity and measurability conditions. This concept complements the notion of the range function in the sense that it provides the description of the morphisms between SMI spaces (that is SMI operators) on the Zak domain analogous to the description of SMI spaces by range functions.

Definition 6.2. Suppose V is an SMI space and J is its corresponding range function as in Theorem 3.1. A *shift–modulation range operator* on J is a mapping

$$R : \mathbb{R}^n \times \mathbb{T}^n \to \{T\colon\ T \text{ is a linear map defined on a subspace of } \mathbb{C}^p\},$$

such that:

(i) the domain of $R(x,\xi)$ equals $J(x,\xi)$ for a.e. (x,ξ),
(ii) R is Γ-periodic in x-variable

$$R(x+\gamma,\xi) = R(x,\xi) \quad \text{for } \gamma \in \Gamma, \tag{6.7}$$

(iii) R is Ξ-multiplex periodic meaning that

$$R(x+l,\xi+k) = \big([l],[k]\big) \circ R(x,\xi) \circ \big([l],[k]\big)^{-1} \quad \text{for } l \in \mathbb{Z}^n,\ k \in \Xi^*, \tag{6.8}$$

where $([k],[l])$ is the $p \times p$ unitary matrix given by (3.6) and $\circ$ represents the composition of linear maps.

Let $P(x,\xi)$ be the orthogonal projection of $\mathbb{C}^p$ onto $J(x,\xi)$. We say that R is *measurable* if the map $(x,\xi) \mapsto R(x,\xi) \circ P(x,\xi)$ is operator measurable.

Note that linear maps appearing in equalities (6.7) and (6.8) have identical domains due to properties (3.9) and (3.10) of shift–modulation range functions.

Remark 6.2. Let $\rho : \Theta \to \mathbb{Z}^n/\Xi$ be the group homomorphism defined by (3.11). Analogously to the range function case, the conditions (6.7) and (6.8) for the range operator R can be combined into a single equivalent formula

$$R(x+l, \xi+k) = \big(\rho(l), [k]\big) \circ R(x,\xi) \circ \big(\rho(l), [k]\big)^{-1} \quad \text{for } l \in \Theta,\ k \in \Xi^*. \tag{6.9}$$

In particular, (6.9) shows that any shift–modulation range operator R is uniquely determined by its values on the fundamental domain $I_\Theta \times I_{\Xi^*}$. Moreover, any such $R = R(x,\xi)$ is Γ-periodic in x-variable and $\mathbb{Z}^n$-periodic in ξ-variable.

Our goal is to characterize SMI operators in terms of shift–modulation range functions. More precisely, we have the following result.

Theorem 6.1. *Suppose $V \subset L^2(\mathbb{R}^n)$ is an SMI space and J is its corresponding range function. Then the following holds.*

(i) *For every bounded SMI operator $L : V \to L^2(\mathbb{R}^n)$, there exists a measurable shift–modulation range operator R on J such that*

$$\mathfrak{T}Lf(x,\xi) = R(x,\xi)\big(\mathfrak{T}f(x,\xi)\big) \quad \textit{for a.e. } (x,\xi) \in \mathbb{R}^n \times \mathbb{T}^n, \textit{ and } f \in V. \tag{6.10}$$

Moreover,

$$\|L\| = \operatorname{ess\,sup}_{(x,\xi)\in\mathbb{R}^n\times\mathbb{T}^n} \big\|R(x,\xi)\big\| < \infty. \tag{6.11}$$

(ii) *Conversely, for every measurable shift–modulation range operator R on J, such that the essential supremum in* (6.11) *is finite, there exists a bounded SMI operator L such that* (6.10) *holds.*

(iii) *The correspondence between L and R is* 1–1 *under the usual convention that the range operators are identified if they are equal a.e.*

Proof. First, suppose that we have a bounded SMI operator $L : V \to L^2(\mathbb{R}^n)$. By Theorem 5.2, we can decompose $V = \bigoplus_{i=1}^p V_i$ into principal SMI space $V_i = S(\varphi_i, \Gamma)$, where each φ_i is a maximal principal generator of V_i. By Theorem 3.1, the range function J of V is given by

$$J(x,\xi) = \operatorname{span}\big\{\mathfrak{T}\varphi_i(x+l_j, \xi):\ j = 1,\dots,q,\ i = 1,\dots,p\big\},$$

where $\{l_1,\dots,l_q\} \subset \Gamma$ are representatives of distinct cosets of $\Theta/\mathbb{Z}^n$. By (5.14)

$$J(x,\xi) = \operatorname{span}\big\{\mathfrak{T}\varphi_i(x,\xi):\ i = 1,\dots,p\big\}. \tag{6.12}$$

Moreover, the orthogonality $V_i \perp V_{i'}$ for $i \neq i'$ implies the orthogonality of the corresponding range functions, and consequently

$$\mathfrak{T}\varphi_i(x,\xi) \perp \mathfrak{T}\varphi_{i'}(x,\xi) \quad \text{for } i \neq i' \text{ and a.e. } (x,\xi). \tag{6.13}$$

For simplicity, define $\Phi_i = \mathfrak{T}\varphi_i$, $i = 1, \ldots, p$. Combining (5.13), (6.12), and (6.13), implies that the collection of non-zero vectors $\{\Phi_1(x,\xi), \ldots, \Phi_p(x,\xi)\} \setminus \{0\}$ forms an orthonormal basis of $J(x,\xi)$ for a.e. (x,ξ).

Given $(x,\xi) \in \mathbb{R}^n \times \mathbb{T}^n$, define $R(x,\xi) : J(x,\xi) \to \mathbb{C}^p$ by

$$R(x,\xi)\left(\sum_{i=1}^{p} \alpha_i \Phi_i(x,\xi)\right) = \sum_{i=1}^{p} \alpha_i \mathfrak{T}L\mathfrak{T}^{-1}(\Phi_i)(x,\xi) \tag{6.14}$$

for any scalars $\alpha_1, \ldots, \alpha_p \in \mathbb{C}$. By (6.2), we have

$$\Phi_i(x,\xi) = 0 \quad \Rightarrow \quad \mathfrak{T}L\mathfrak{T}^{-1}(\Phi_i)(x,\xi) = 0 \quad \text{a.e.}$$

and $R(x,\xi)$ is well defined. It remains to show that R is a shift–modulation range operator.

Take any $f \in V$ and decompose it as $f = f_1 + \cdots + f_p$, where $f_i \in V_i$. Then by Theorem 5.3

$$\mathfrak{T}f = \mathfrak{T}f_1 + \cdots + \mathfrak{T}f_p = m_1\Phi_1 + \cdots + m_p\Phi_p,$$

for some $m_i \in L^2(\mathbb{T}^n \times \mathbb{R}^n/\Xi^*)$, $i = 1, \ldots, p$. By Lemma 6.1

$$\begin{aligned}
\mathfrak{T}Lf(x,\xi) &= \mathfrak{T}L\mathfrak{T}^{-1}\left(\sum_{i=1}^{p} m_i\Phi_i\right)(x,\xi) = \sum_{i=1}^{p} m_i(x,\xi)\mathfrak{T}L\mathfrak{T}^{-1}(\Phi_i)(x,\xi) \\
&= \sum_{i=1}^{p} m_i(x,\xi)R(x,\xi)\big(\Phi_i(x,\xi)\big) = R(x,\xi)\left(\sum_{i=1}^{p} m_i(x,\xi)\Phi_i(x,\xi)\right) \\
&= R(x,\xi)\big(\mathfrak{T}f(x,\xi)\big).
\end{aligned} \tag{6.15}$$

Hence, (6.10) holds.

To see that R is measurable, take any $\Phi \in L^2_{\Xi}(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$. Let $\mathcal{P}$ be the orthogonal projection of $L^2_{\Xi}(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$ onto $M_J = \mathfrak{T}V$, and let $P(x,\xi)$ be the orthogonal projection of $J(x,\xi)$ onto $\mathbb{C}^p$. Then, by Lemma 3.2 and (6.15) applied to $f = \mathfrak{T}^{-1}\mathcal{P}\Phi$,

$$(x,\xi) \mapsto R(x,\xi) \circ P(x,\xi)\big(\Phi(x,\xi)\big) = R(x,\xi)\big(\mathcal{P}\Phi(x,\xi)\big) = \mathfrak{T}L\mathfrak{T}^{-1}(\Phi)(x,\xi)$$

is operator measurable. Since Φ is arbitrary, the map $(x,\xi) \mapsto R(x,\xi)$ is measurable.

Next, we prove that R is Ξ-multiplex periodic. Take any $f \in V$. Since both $\mathfrak{T}f$ and $\mathfrak{T}Lf$ are Ξ-multiplex periodic, then by (6.15)

$$\begin{aligned}
&R(x+l, \xi+k)\big(\mathfrak{T}f(x+k, \xi+l)\big) \\
&\quad = \mathfrak{T}Lf(x+k, \xi+l) \\
&\quad = e^{2\pi i\langle l,\xi\rangle}\big([k],[l]\big) \circ R(x,\xi)\big(\mathfrak{T}f(x,\xi)\big) \\
&\quad = e^{2\pi i\langle l,\xi\rangle}\big([k],[l]\big) \circ R(x,\xi)\big(e^{-2\pi i\langle l,\xi\rangle}\big([k],[l]\big)^{-1} \circ \mathfrak{T}f(x+l, \xi+k)\big) \\
&\quad = \big([k],[l]\big) \circ R(x,\xi) \circ \big([k],[l]\big)^{-1}\big(\mathfrak{T}f(x+l, \xi+k)\big)
\end{aligned}$$

for any $(l,k) \in \mathbb{Z}^n \times \Xi^*$. In particular, if $f = \varphi_i$, $i = 1, \ldots, p$, then we have

$$R(x+l, \xi+k)\big(\Phi_i(x+k, \xi+l)\big) = \big([k],[l]\big) \circ R(x,\xi) \circ \big([k],[l]\big)^{-1}\big(\Phi_i(x+l, \xi+k)\big) \quad \text{a.e. } (x,\xi).$$

Since the vectors $\{\Phi_1(x+l,\xi+k), \ldots, \Phi_p(x+l,\xi+k)\}$ span $J(x+l,\xi+k)$, (6.8) follows.

Finally, we demonstrate that R is Γ-periodic in x-variable. Take any $f \in V$. Since L commutes with T_γ, $\gamma \in \Gamma$, then by (3.19) and (6.15)

$$\begin{aligned} R(x-\gamma,\xi)\big(\mathfrak{T}f(x-\gamma,\xi)\big) &= \mathfrak{T}Lf(x-\gamma,\xi) = \mathfrak{T}T_\gamma Lf(x,\xi) = \mathfrak{T}LT_\gamma f(x,\xi) \\ &= R(x,\xi)\big(\mathfrak{T}T_\gamma f(x,\xi)\big) = R(x,\xi)\big(\mathfrak{T}f(x-\gamma,\xi)\big) \end{aligned}$$

for any $\gamma \in \Gamma$. In particular, if $f = \varphi_i$, $i = 1, \ldots, p$, then we have

$$R(x-\gamma,\xi)\big(\Phi_i(x-\gamma,\xi)\big) = R(x,\xi)\big(\Phi_i(x-\gamma,\xi)\big) \quad \text{a.e. } (x,\xi).$$

Since the vectors $\{\Phi_1(x-\gamma,\xi), \ldots, \Phi_p(x-\gamma,\xi)\}$ span $J(x-\gamma,\xi) = J(x,\xi)$, (6.7) follows.

Consequently, R is a measurable shift-invariant range operator on J satisfying (6.10). Finally, to prove (6.11) we employ (6.3) and (6.10). Let

$$C = \operatorname{ess\,sup}_{(x,\xi)\in\mathbb{R}^n\times\mathbb{T}^n} \big\|R(x,\xi)\big\|. \tag{6.16}$$

Then

$$\begin{aligned} \|L\| &= \sup_{\|f\|_{L^2(\mathbb{R}^n)}\leqslant 1} \|Lf\| = \sup_{\|f\|\leqslant 1} \bigg(\int_{I_n\times I_{\Xi^*}} \big\|\mathfrak{T}Lf(x,\xi)\big\|^2\,dx\,d\xi \bigg)^{1/2} \\ &= \sup_{\|f\|\leqslant 1} \bigg(\int_{I_n\times I_{\Xi^*}} \big\|R(x,\xi)\big(\mathfrak{T}f(x,\xi)\big)\big\|^2\,dx\,d\xi \bigg)^{1/2} \\ &\leqslant C \sup_{\|f\|\leqslant 1} \bigg(\int_{I_n\times I_{\Xi^*}} \big\|\mathfrak{T}f(x,\xi)\big\|^2\,dx\,d\xi \bigg)^{1/2} = C. \end{aligned}$$

To prove the converse estimate, we will show that for any $s \in S^{p-1} = \{s \in \mathbb{C}^p\colon \|s\| = 1\}$,

$$\operatorname{ess\,sup}_{(x,\xi)\in\mathbb{R}^n\times\mathbb{T}^n} \big\|R(x,\xi)\big(\Psi_s(x,\xi)\big)\big\| \leqslant \|L\|, \quad \text{where } \Psi_s = \sum_{i=1}^p s_i\Phi_i, \tag{6.17}$$

and Φ_i's are the same as before. On the contrary, if (6.17) fails, then there would exist $\varepsilon > 0$ and a measurable set $D \subset \mathbb{R}^n \times \mathbb{T}^n$ with $|D| > 0$, such that

$$\big\|R(x,\xi)\big(\Psi_s(x,\xi)\big)\big\| > \|L\| + \varepsilon \quad \text{for } (x,\xi) \in D.$$

Without loss of generality, we can assume that D is invariant under shifts in $\mathbb{Z}^n \times \varXi^*$, namely $D \subset \mathbb{T}^n \times (\mathbb{R}^n/\varXi^*)$. Consider $\Psi = \Psi_s \mathbf{1}_D$, which, by Theorem 3.1, is an element of $\mathfrak{T}V$. Consequently, $\psi = \mathfrak{T}^{-1}\Psi \in V$ and by (6.10)

$$\|\mathfrak{T}L\psi\|^2 = \left(\int\limits_{I_n \times I_{\varXi^*}} \big\| R(x,\xi)\big(\Psi(x,\xi)\big)\big\|^2 \,dx\,d\xi \right)^{1/2} = \left(\int\limits_D \big\| R(x,\xi)\big(\Psi_s(x,\xi)\big)\big\|^2 \,dx\,d\xi \right)^{1/2}$$

$$\geqslant \big(\|L\|+\varepsilon\big)\left(\int\limits_D \big\|\Psi_s(x,\xi)\big\|^2 \,dx\,d\xi \right)^{1/2} = \big(\|L\|+\varepsilon\big)\|\Psi\|^2 = \big(\|L\|+\varepsilon\big)\|\psi\|^2,$$

which is a contradiction, since $\mathfrak{T}$ is an isometry. Hence, (6.17) holds. Finally, let S be a countable dense subset of S^{p-1}. By Theorem 3.1, $\{\Psi_s(x,\xi)\colon s \in S^{p-1}\}$ contains a unit sphere in $J(x,\xi)$ for a.e. (x,ξ), and by (6.17)

$$\operatorname{ess\,sup}_{(x,\xi)\in\mathbb{R}^n\times\mathbb{T}^n} \big\|R(x,\xi)\big\| = \operatorname{ess\,sup}_{(x,\xi)\in\mathbb{R}^n\times\mathbb{T}^n} \sup_{s\in S\subset S^{p-1}} \big\|R(x,\xi)\big(\Psi_s(x,\xi)\big)\big\| \leqslant \|L\|,$$

which shows (6.11) and completes the proof of (i).

To show the converse statement (ii), assume that R is a measurable shift–modulation range operator on J, such that C in (6.16) is finite. Take any $f \in V$. Since the map $(x,\xi) \mapsto \mathfrak{T}f(x,\xi)$ is measurable, $F(x,\xi) = R(x,\xi)(\mathfrak{T}f(x,\xi))$ is also measurable. Moreover, F is $\varXi$-multiplex periodic since for every $l \in \mathbb{Z}^n$ and $k \in \varXi^*$,

$$\begin{aligned} F(x+l,\xi+k) &= R(x+l,\xi+k)\big(\mathfrak{T}f(x+l,\xi+k)\big) \\ &= e^{2\pi i\langle l,\xi\rangle} R(x+l,\xi+k)\big(\big([l],[k]\big)\circ \mathfrak{T}f(x,\xi)\big) \\ &= e^{2\pi i\langle l,\xi\rangle}\big([l],[k]\big)\circ R(x,\xi)\big(\mathfrak{T}f(x,\xi)\big) = e^{2\pi i\langle l,\xi\rangle}\big([l],[k]\big)\circ F(x,\xi), \end{aligned}$$

where in the penultimate step we used (6.8). Therefore, $F \in L^2_{\varXi}(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$, since

$$\|F\|^2 = \int\limits_{I_n\times I_{\varXi^*}} \big\|F(x,\xi)\big\|^2\,dx\,d\xi \leqslant C^2 \int\limits_{I_n\times I_{\varXi^*}} \big\|\mathfrak{T}f(x,\xi)\big\|^2\,dx\,d\xi = C^2\|f\|^2 < \infty.$$

Define the operator $L\colon V \to L^2(\mathbb{R}^n)$ by $Lf = \mathfrak{T}^{-1}F$. Then L is linear and bounded $\|Lf\| \leqslant C\|f\|$. Using (3.19) and (6.7),

$$\begin{aligned} \mathfrak{T}LM_lT_\gamma f(x,\xi) &= R(x,\xi)\big(\mathfrak{T}M_lT_\gamma f(x,\xi)\big) = R(x,\xi)\big(e^{2\pi i\langle l,x\rangle}\mathfrak{T}f(x-\gamma,\xi)\big) \\ &= e^{2\pi i\langle l,x\rangle}R(x,\xi)\big(\mathfrak{T}f(x-\gamma,\xi)\big) = e^{2\pi i\langle l,x\rangle}R(x-\gamma,\xi)\big(\mathfrak{T}f(x-\gamma,\xi)\big) \\ &= e^{2\pi i\langle l,x\rangle}\mathfrak{T}Lf(x-\gamma,\xi) = \mathfrak{T}M_lT_\gamma Lf(x,\xi) \quad \text{a.e. } (x,\xi), \end{aligned}$$

where $l \in \mathbb{Z}^n$, $\gamma \in \Gamma$. Hence, L is an SMI operator which satisfies (6.10) by the virtue of its definition.

Finally, the uniqueness of the correspondence between L and R is shown using the same method as Corollary 3.1. Indeed, suppose we have an SMI operator L and let R_1 and R_2 be two corresponding range operators both satisfying (6.10). Then for any $s = (s_1,\dots,s_p) \in S^{p-1}$,

$$\sum_{i=1}^{p} s_i R_1(x,\xi)\big(\Phi_i(x,\xi)\big) = R_1(x,\xi)\big(\Psi_s(x,\xi)\big) = \mathfrak{T} L \mathfrak{T}^{-1} \Psi_s(x,\xi)$$

$$= R_2(x,\xi)\big(\Psi_s(x,\xi)\big) = \sum_{i=1}^{p} s_i R_2(x,\xi)\big(\Phi_i(x,\xi)\big),$$

where Ψ_s is the same as in (6.17). Since $s \in S^{p-1}$ is arbitrary and $\{\Phi_i(x,\xi)\colon i = 1,\dots,p\}$ spans $J(x,\xi)$, we have

$$R_1(x,\xi) = R_2(x,\xi) \quad \text{a.e. } (x,\xi). \tag{6.18}$$

Conversely, if we have two range operators R_1 and R_2 satisfying hypotheses of (ii) and (6.18), then they lead to the same SMI operator L due to (6.10). □

Theorem 6.2. *Suppose L is an SMI operator on V and R is its corresponding shift–modulation operator on J as in Theorem* 6.1. *Then L is bounded from below by a constant $c > 0$,*

$$\|Lf\| \geqslant c\|f\| \quad \textit{for } f \in V, \tag{6.19}$$

if and only if

$$\big\|R(x,\xi)a\big\| \geqslant c\|a\| \quad \textit{for } a \in J(x,\xi) \textit{ and a.e. } (x,\xi). \tag{6.20}$$

Proof. By (6.10)

$$\|Lf\|^2 = \int_{I_n \times I_{\Xi^*}} \big\|R(x,\xi)\big(\mathfrak{T}f(x,\xi)\big)\big\|^2\,dx\,d\xi \quad \text{for all } f \in V. \tag{6.21}$$

Hence, if (6.20) holds then

$$\|Lf\|^2 \geqslant c^2 \int_{I_n \times I_{\Xi^*}} \big\|\mathfrak{T}f(x,\xi)\big\|^2\,dx\,d\xi \geqslant c^2\|f\|^2.$$

Conversely, assume (6.19). We will show that for any $s \in S^{p-1} = \{s \in \mathbb{C}^p\colon \|s\| = 1\}$,

$$\big\|R(x,\xi)\big(\Psi_s(x,\xi)\big)\big\| \geqslant c\big\|\Psi_s(x,\xi)\big\|, \quad \text{where } \Psi_s = \sum_{i=1}^{p} s_i \Phi_i, \tag{6.22}$$

and Φ_i's are the same as in the proof of Theorem 6.1. On the contrary, if (6.22) fails, then there would exist $\varepsilon > 0$ and a measurable set $D \subset \mathbb{R}^n \times \mathbb{T}^n$ with $|D| > 0$, such that

$$\big\|R(x,\xi)\big(\Psi_s(x,\xi)\big)\big\| \leqslant (c-\varepsilon)\big\|\Psi_s(x,\xi)\big\| \quad \text{for } (x,\xi) \in D.$$

Without loss of generality, we can assume that D is invariant under shifts in $\mathbb{Z}^n \times \Xi^*$, namely $D \subset \mathbb{T}^n \times (\mathbb{R}^n/\Xi^*)$. Consider $\Psi = \Psi_s \mathbf{1}_D$, which by Theorem 3.1, is an element of $\mathfrak{T}V$. Consequently, $\psi = \mathfrak{T}^{-1}\Psi \in V$ and by (6.21)

$$\|L\psi\|^2 = \int_{I_n \times I_{\Xi^*}} \big\|R(x,\xi)\big(\Psi(x,\xi)\big)\big\|^2\,dx\,d\xi = \int_D \big\|R(x,\xi)\big(\Psi_s(x,\xi)\big)\big\|^2\,dx\,d\xi$$
$$\leqslant (c-\varepsilon)^2 \int_D \big\|\Psi_s(x,\xi)\big\|^2\,dx\,d\xi = (c-\varepsilon)^2\|\Psi\|^2 = (c-\varepsilon)^2\|\psi\|^2,$$

which is a contradiction, since $\mathfrak{T}$ is an isometry. Hence, (6.22) holds. Finally, let S be a countable dense subset of S^{p-1}. By Theorem 3.1, $\{\Psi_s(x,\xi)\colon\ s \in S^{p-1}\}$ contains a unit sphere in $J(x,\xi)$ for a.e. (x,ξ), and by (6.22)

$$\big\|R(x,\xi)\big\| = \sup_{s\in S\subset S^{p-1},\,\Psi_s(x,\xi)\neq 0} \big\|R(x,\xi)\big(\Psi_s(x,\xi)\big)\big\|/\big\|\Psi_s(x,\xi)\big\| \geqslant c \quad \text{for a.e. } (x,\xi)$$

which shows (6.20) and completes the proof of Theorem 6.2. □

As an immediate consequence of Theorems 6.1 and 6.2 we have

Corollary 6.2. *An SMI operator $L : V \to L^2(\mathbb{R}^n)$ is an isometry if and only if its corresponding range operator $R(x,\xi)$ is an isometry for a.e. (x,ξ).*

Next, we investigate properties of the dimension function of an SMI space under the action of an SMI operator.

Theorem 6.3. *Suppose $V \subset L^2(\mathbb{R}^n)$ is an SMI space and $L : V \to L^2(\mathbb{R}^n)$ is an SMI operator. Then $V' = \overline{L(V)}$ is SMI and its range function J' satisfies*

$$J'(x,\xi) = R(x,\xi)\big(J(x,\xi)\big) \quad \text{a.e. } (x,\xi), \tag{6.23}$$

where J is the range function of V and R is the range operator of L. In particular, we have

$$\dim_{V'}(x,\xi) \leqslant \dim_V(x,\xi) \quad \text{for a.e. } (x,\xi). \tag{6.24}$$

Proof. By Theorem 5.2, we can decompose V as an orthogonal sum of principal spaces generated by maximal principal generators φ_i, $i = 1,\dots,p$. Since $V' = S(\{L\varphi_1,\dots,L\varphi_p\},\Gamma)$, then by Theorem 3.1, (6.7), and (6.10), the range function of V' satisfies

$$\begin{aligned} J'(x,\xi) &= \operatorname{span}\big\{\mathfrak{T}L\varphi_i(x+l_j,\xi)\colon\ j=1,\dots,q,\ i=1,\dots,p\big\} \\ &= \operatorname{span}\big\{R(x+l_j,\xi)\big(\mathfrak{T}\varphi_i(x+l_j,\xi)\big)\colon\ i=1,\dots,p\big\} \\ &= \operatorname{span}\big\{R(x,\xi)\big(\mathfrak{T}\varphi_i(x+l_j,\xi)\big)\colon\ j=1,\dots,q,\ i=1,\dots,p\big\} = R(x,\xi)\big(J(x,\xi)\big). \end{aligned}$$

Here, $\{l_1,\dots,l_q\} \subset \Gamma$ are representatives of distinct cosets of $\Theta/\mathbb{Z}^n$ and R is the range operator corresponding to L. Therefore, $\dim J'(x,\xi) \leqslant \dim J(x,\xi)$, which proves (6.24). □

Lemma 6.2. *Suppose $V, W \subset L^2(\mathbb{R}^n)$ are two SMI spaces, $L : V \to W$ is an SMI operator, and R is its corresponding range operator. Then, the adjoint operator $L^* : W \to V$ is also SMI and its corresponding range operator R^* is given by*

$$R^*(x,\xi) = \big(R(x,\xi)\big)^* \quad \textit{for a.e. } (x,\xi). \tag{6.25}$$

Proof. Let J and J' be the range functions of V and W, respectively, and R be the range operator of L. Consequently, $R(x,\xi)$ is a linear map between $J(x,\xi)$ and $J'(x,\xi)$ for a.e. (x,ξ). Clearly, R^* is a measurable range operator satisfying

$$\operatorname{ess\,sup}_{(x,\xi)\in\mathbb{R}^n\times\mathbb{T}^n} \big\|R^*(x,\xi)\big\| = \operatorname{ess\,sup}_{(x,\xi)\in\mathbb{R}^n\times\mathbb{T}^n} \big\|R(x,\xi)\big\| = \|L\| < \infty.$$

Hence, by Theorem 6.1 there exists a corresponding SMI operator $\tilde{L} : W \to V$ satisfying

$$\mathfrak{T}\tilde{L}g(x,\xi) = R(x,\xi)^*\big(\mathfrak{T}g(x,\xi)\big) \quad \text{for a.e. } (x,\xi) \in \mathbb{R}^n \times \mathbb{T}^n, \text{ and } g \in W.$$

Take any $f \in V$ and $g \in W$. Then

$$\begin{aligned}\langle Lf, g\rangle = \langle \mathfrak{T}Lf, \mathfrak{T}g\rangle &= \int\limits_{I_n\times I_{\Xi^*}} \big\langle R(x,\xi)\big(\mathfrak{T}f(x,\xi)\big), \mathfrak{T}g(x,\xi)\big\rangle\, dx\, d\xi \\ &= \int\limits_{I_n\times I_{\Xi^*}} \big\langle \mathfrak{T}f(x,\xi), R^*(x,\xi)\big(\mathfrak{T}g(x,\xi)\big)\big\rangle\, dx\, d\xi = \langle \mathfrak{T}f, \mathfrak{T}\tilde{L}g\rangle = \langle f, \tilde{L}g\rangle.\end{aligned}$$

Hence, $\tilde{L} = L^*$. $\square$

As a corollary of Lemma 6.2 and Theorem 6.3 we have

Theorem 6.4. *Suppose $V, W \subset L^2(\mathbb{R}^n)$ are two SMI spaces and $L : V \to W$ is an SMI operator. Then the following are true*:

(i) *If L is* 1–1, *then*

$$\dim_V(x,\xi) \leqslant \dim_W(x,\xi) \quad \textit{for a.e. } (x,\xi). \tag{6.26}$$

(ii) *If L is onto, then*

$$\dim_V(x,\xi) \geqslant \dim_W(x,\xi) \quad \textit{for a.e. } (x,\xi). \tag{6.27}$$

(iii) *If L is an isomorphism, then*

$$\dim_V(x,\xi) = \dim_W(x,\xi) \quad \textit{for a.e. } (x,\xi). \tag{6.28}$$

Proof. Let J and J' be the range functions of V and W, respectively, and R be the range operator of L. Consequently, $R(x,\xi)$ is a linear map between $J(x,\xi)$ and $J'(x,\xi)$ for a.e. (x,ξ).

First, suppose that L is 1–1. We claim that the linear maps $R(x,\xi)$ are also 1–1 for a.e. (x,ξ). Otherwise, we would have a set of positive measure $D \subset \mathbb{T}^n \times (\mathbb{R}^n/\Xi^*)$ such that $\ker R(x,\xi) = \{0\}$ for all $(x,\xi) \in D$. Then, one can find a function $0 \neq \Phi \in L^2(\mathbb{R}^n \times \mathbb{T}^n, \mathbb{C}^p)$ such that $\Phi(x,\xi) \in R(x,\xi)$ a.e. By (6.10) this implies that $L\mathfrak{T}^{-1}\Phi = 0$, which is a contradiction. Note that the converse implication is also trivially true. Namely, if $R(x,\xi)$ is 1–1 for a.e. (x,ξ), then L is also 1–1. Hence, (6.26) follows immediately from (6.23).

Next, suppose that $L : V \to W$ is onto. Recall that this is equivalent to the fact that $L^* : W \to V$ is 1–1. Hence, (6.27) follows. Finally, (6.28) is immediate from (6.26) and (6.27). $\square$

Theorem 6.4 has the following converse.

Theorem 6.5. *Suppose $V, W \subset L^2(\mathbb{R}^n)$ are two SMI spaces. Then the following are true*:

(i) *If* (6.26) *holds, then there exists an isometry of V* **into** *W.*
(ii) *If* (6.27) *holds, then there exists a partial isometry of V* **onto** *W.*
(iii) *If* (6.28) *holds, then there exists an isometric isomorphism of V and W.*

Proof. By Theorem 5.1, we can decompose V and W as orthogonal sums

$$V = \bigoplus_{i=1}^{p} S(\varphi_i, \Gamma), \qquad W = \bigoplus_{i=1}^{p} S(\varphi_i', \Gamma),$$

where each φ_i and φ_i' is a minimal principal generator. Thus, we can also require that

$$\operatorname{supp}\mathfrak{T}\varphi_i, \operatorname{supp}\mathfrak{T}\varphi_i' \subset \bigcup_{l\in\mathbb{Z}^n} (l + I_\Theta) \times \mathbb{R}^n \quad \text{for all } 1 \leqslant i \leqslant p, \tag{6.29}$$

where I_Θ is a fundamental domain of $\mathbb{R}^n/\Theta$. Furthermore, if (6.26) holds, then (3.15) and (5.9) imply that for $1 \leqslant i \leqslant p$,

$$\bigcup_{j=1}^{p} (-l_j, 0) + \operatorname{supp}\mathfrak{T}\varphi_i = \sigma\big(S(\varphi_i, \Gamma)\big) = \big\{(x,\xi)\colon \dim_V(x,\xi) \geqslant i\big\} \subset \big\{(x,\xi)\colon \dim_W(x,\xi) \geqslant i\big\}$$
$$= \sigma\big(S\big(\varphi_i', \Gamma\big)\big) = \bigcup_{j=1}^{p} (-l_j, 0) + \operatorname{supp}\mathfrak{T}\varphi_i'.$$

By (6.29) the above unions are disjoint. Therefore, (6.29) implies that we have inclusions

$$\operatorname{supp}\mathfrak{T}\varphi_i \subset \operatorname{supp}\mathfrak{T}\varphi_i' \quad \text{for all } 1 \leqslant i \leqslant p. \tag{6.30}$$

Given $(x,\xi) \in E := \bigcup_{l\in\mathbb{Z}^n} (l + I_\Theta) \times \mathbb{R}^n$, define $R(x,\xi) : J(x,\xi) \to J'(x,\xi)$ by

$$R(x,\xi)\left(\sum_{i=1}^{p} \alpha_i \mathfrak{T}\varphi_i(x,\xi)\right) = \sum_{i=1}^{p} \alpha_i \mathfrak{T}\varphi_i'(x,\xi), \tag{6.31}$$

where $\alpha_1, \dots, \alpha_p \in \mathbb{C}$, and J and J' are range functions of V and W, respectively. Recall that the non-zero vectors $\{\mathfrak{T}\varphi_1(x,\xi), \dots, \mathfrak{T}\varphi_p(x,\xi)\} \setminus \{0\}$ form an orthonormal basis of $J(x,\xi)$, and the analogous statement holds for $J'(x,\xi)$. Hence, $R(x,\xi)$ is a well-defined isometry of $J(x,\xi)$ into $J'(x,\xi)$ by (6.30). Then, a simple calculation as in the proof of Theorem 6.1 shows that R is a Ξ-multiplex periodic function on E, and hence, R is Ξ-periodic in x-variable. Next, extend the definition of $R(x,\xi)$ to arbitrary $(x,\xi) \in \mathbb{R}^n \times \mathbb{T}^n$ by setting

$$R(x,\xi) = R(x - l_j, \xi) \quad \text{if } 1 \leqslant j \leqslant q \text{ and } (x - l_j, \xi) \in E. \tag{6.32}$$

The above definition assures that R is Γ-periodic in x-variable as a function on $\mathbb{R}^n \times \mathbb{T}^n$. Hence, the Ξ-multiplex periodicity on E extends to the entire domain $\mathbb{R}^n \times \mathbb{T}^n$. Clearly, R is also measurable. Therefore, R is a measurable shift–modulation range operator such that each $R(x,\xi)$ is an isometry. By Corollary 6.2, R corresponds to an isometry $L : V \to W$, which proves (i).

Note that one could define L more directly by setting $L(\varphi_i) = \varphi_i'$ for all $1 \leqslant i \leqslant p$ and extending it to the unique SMI operator from V to W. That is, we set

$$L(M_k T_\gamma \varphi_i) = M_k T_\gamma \varphi_i' \quad \text{for } k \in \mathbb{Z}^n, \gamma \in \Gamma,\ 1 \leqslant i \leqslant p,$$

and extend it to the whole V by linearity and density. However, the proof that such L is indeed well defined, though more direct, would be as involved as the above argument.

Recall that $L : V \to W$ is a partial isometry if and only if $L^* L$ is an orthogonal projection. Moreover, recall that $L : V \to W$ is an onto partial isometry if and only if $L^* : W \to V$ is an isometry. Hence, (ii) is an immediate consequence of (i).

Finally, assume that (6.28) holds. Then, the isometry $L : V \to W$ given by (i) must be onto, since each linear map $R(x,\xi)$ is an isometry of $J(x,\xi)$ onto $J'(x,\xi)$ by (6.23). This completes the proof of Theorem 6.5. □

As an immediate consequence of Theorems 6.4 and 6.5 we have

Theorem 6.6. *Let $V, W \subset L^2(\mathbb{R}^n)$ be two SMI spaces. Then V and W are unitarily equivalent, in the sense that there exists a unitary SMI operator $L : V \to W$, if and only if*

$$\dim_V(x,\xi) = \dim_W(x,\xi) \quad \textit{for a.e. } (x,\xi).$$

As a consequence of Theorem 6.6, we can easily deduce Proposition 2.2.

Example 6.1. Suppose that the space V is Ξ-invariant in the sense of Definition 2.1. Then we can canonically associate to V an SMI space $\tilde{V}$ with respect to the shift lattice $\Gamma = \Xi$ by

$$\tilde{V} = \{ f \in L^2(\mathbb{R}^n) \colon \mathcal{T}_1 f(x) \in V \text{ for a.e. } x \in I_n \}, \tag{6.33}$$

where $\mathcal{T}_1$ is given by (3.1). An easy argument shows that $\tilde{V}$ is an SMI space, since V is Ξ-invariant. Alternatively, let J be the range function of V such that (2.4) in Theorem 2.1 holds. Then, we can define the SMI space $\tilde{V}$ as

$$\tilde{V} = \{ f \in L^2(\mathbb{R}^n) \colon \mathfrak{T} f(x,\xi) \in J(\xi) \text{ for a.e. } (x,\xi) \in \mathbb{R}^n \times \mathbb{T}^n \}. \tag{6.34}$$

In other words, $\tilde{V}$ is the unique SMI space whose shift–modulation range function $\tilde{J}(x,\xi) = J(\xi)$. An easy verification shows that both definitions of $\tilde{V}$ are equivalent.

In addition, suppose that we have another Ξ-invariant space W and $L: V \to W$ is a bounded linear operator commuting with shifts in Ξ. Then the operator L can be lifted to an SMI operator $\tilde{L}: \tilde{V} \to \tilde{W}$ defined fiberwise by

$$\mathcal{T}_1 \tilde{L} f(x) = L\big(\mathcal{T}_1 f(x)\big) \quad \text{a.e. } x \in I_n. \tag{6.35}$$

Again, it is not difficult to verify that $\tilde{L}$ is an SMI operator. Consequently, if L is unitary, so is $\tilde{L}$ and

$$\dim_V(\xi) = \dim_{\tilde{V}}(x,\xi) = \dim_{\tilde{W}}(x,\xi) = \dim_W(\xi) \quad \text{a.e. } (x,\xi). \tag{6.36}$$

This proves one implication of Proposition 2.2.

To show the other one, take any Ξ-invariant spaces V and W satisfying (6.36). As usual, let $\tilde{V}$ and $\tilde{W}$ be their canonical SMI spaces. Finally, let $\tilde{L}: \tilde{V} \to \tilde{W}$ be the unitary SMI operator guaranteed by Theorem 6.6. Then by Remark 3.3 and [2, Theorem 4.5], $\tilde{L}$ must be of the form

$$\mathcal{T}_1 \tilde{L} f(x) = R(x)\big(\mathcal{T}_1 f(x)\big) \quad \text{a.e. } x \in I_n, \tag{6.37}$$

where R is a measurable range operator, i.e., $R(x)$ is a bounded linear map from V to W, and $x \mapsto R(x)$ is operator measurable. Furthermore, [2, Corollary 4.7] implies that $R(x)$ must be a unitary map for a.e. x. Since $\tilde{L}$ is an SMI operator, $R(x)$ must commute with shifts in Ξ. Hence, we can choose any x outside a null measure set in I_n, and define $L = R(x)$. Then $L: V \to W$ is the required unitary operator commuting with shifts in Ξ. This shows that Proposition 2.2 is indeed a direct consequence of Theorem 6.6.

We conclude this section by some observations about functional calculus for SMI operators. We shall concentrate on two basic forms of functional calculus listed below.

Definition 6.3. Suppose that T is a bounded operator acting on a Banach space B. Let $\operatorname{sp}(T)$ be the spectrum of T. Then, for any holomorphic function h defined on some neighborhood Ω of $\operatorname{sp}(T)$, define

$$h(T) = \frac{1}{2\pi i} \int_\gamma h(\lambda)(\lambda I - T)^{-1}\, d\lambda, \tag{6.38}$$

where γ is any positively oriented contour that surrounds $\operatorname{sp}(T)$ in Ω. It is known that this definition does not depend on the choice of γ, see [29, Section 10.26].

Definition 6.4. Suppose that T is a normal operator acting on a Hilbert space $\mathcal{H}$, i.e., $TT^* = T^*T = I$. Let E be the spectral decomposition of T, see [29, Section 12.23]. Then, for any bounded Borel function h on $\operatorname{sp}(T)$, define

$$h(T) = \int_{\operatorname{sp}(T)} h(\lambda)\, dE(\lambda). \tag{6.39}$$

Theorem 6.7. *Suppose $V \subset L^2(\mathbb{R}^n)$ is an SMI space and $L : V \to V$ is a bounded SMI operator. Let $R(x,\xi)$ be its corresponding shift–modulation range operator as in Theorem* 6.1. *Assume that either*:

(i) *h is a holomorphic function on some neighborhood of* $\mathrm{sp}(L)$, *or*
(ii) *h is a bounded complex Borel function on* $\mathrm{sp}(L)$ *and L is normal.*

Then, $h(L)$ is also an SMI operator and its corresponding shift–modulation range operator is $(x,\xi) \mapsto h(R(x,\xi))$.

Proof. By Theorem 6.1, the range operator of $\lambda I - L$ is $(x,\xi) \mapsto \lambda I - R(x,\xi)$. Hence, by Theorem 6.2 we have $\mathrm{sp}(R(x,\xi)) \subset \mathrm{sp}(L)$ for a.e. (x,ξ). Likewise, if L is normal, then by Lemma 6.2 $R(x,\xi)$ is normal for a.e. (x,ξ). Therefore, it makes sense to speak of $h(R(x,\xi))$ in both cases. Consequently, if $\lambda \notin \mathrm{sp}(L)$, then the range operator of $(\lambda I - L)^{-1}$ is $(x,\xi) \mapsto (\lambda I - R(x,\xi))^{-1}$. Hence, by approximating the integral (6.38) by Riemann sums in the operator norm, we have the required conclusion in case (i).

To prove case (ii), observe that

$$p(T,T^*) = \int_{\mathrm{sp}(L)} p(\lambda,\bar{\lambda})\, dE(\lambda),$$

where p is any polynomial in two variables with complex coefficients. Clearly, the range operator of $p(T,T^*)$ is $(x,\xi) \mapsto p(R(x,\xi), R(x,\xi)^*)$. By Stone–Weierstrass theorem, polynomials $\lambda \mapsto p(\lambda,\bar{\lambda})$ are dense in $C(\mathrm{sp}(L))$. Hence, by [29, Theorem 12.24], the conclusion holds for continuous functions h. Finally, it suffices to use two basic facts. First, if $\{h_i\}$ is a uniformly bounded sequence of Borel functions converging pointwise to h on $\mathrm{sp}(L)$, then $\{h_i(T)\}$ converges to $h(T)$ in the strong operator topology. Second, the space of bounded Borel functions on a compact set $K \subset \mathbb{C}$ is the smallest space X containing $C(K)$ and closed under pointwise limits of uniformly bounded sequences in X. Consequently, the required conclusion holds also for bounded Borel functions. □

7. Duality of Gabor frames

As an illustration of our techniques we will prove several results about dual Gabor frames. Suppose that $\mathcal{A} = \{\varphi_m : m \in M\}$ is a family of generators in $L^2(\mathbb{R}^n)$, where M is at most countable, and the Gabor system $G(\mathcal{A},\Gamma)$ is a Bessel sequence. The *analysis operator* of this system $F : L^2(\mathbb{R}^n) \to \ell^2(\mathbb{Z}^n \times \Gamma \times M)$ is given by

$$Ff = \big(\langle f, M_k T_\gamma \varphi_m\rangle\big)_{(k,\gamma,m)\in\mathbb{Z}^n\times\Gamma\times M} \quad \text{for } f \in S(\mathcal{A},\Gamma). \tag{7.1}$$

The adjoint of F is called the *synthesis operator* $F^* : \ell^2(\mathbb{Z}^n \times \Gamma \times M) \to L^2(\mathbb{R}^n)$ and it is given by

$$F^*s = \sum_{(k,\gamma,m)\in\mathbb{Z}^n\times\Gamma\times M} s_{k,\gamma,m} M_k T_\gamma \varphi_m \quad \text{for } s = (s_{k,\gamma,m}) \in \ell^2\big(\mathbb{Z}^n \times \Gamma \times M\big). \tag{7.2}$$

Finally, the *frame operator* of $G(\mathcal{A},\Gamma)$ is the operator $L = F^*F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, which is self-adjoint and non-negative definite. Recall that $G(\mathcal{A},\Gamma)$ is a frame sequence with bounds $0 < c_0 \leqslant c_1 < \infty$ if and only if

$$c_0\|f\|^2 \leqslant \langle Lf, f\rangle \leqslant c_1\|f\|^2 \quad \text{for } f \in S(\mathcal{A},\Gamma).$$

This, in turn, is equivalent to $\operatorname{sp}(L) \subset \{0\} \cup [c_0, c_1]$.

Likewise, one can define frame operator for an arbitrary Bessel sequence $(f_i)_{i\in I}$ in a Hilbert space $\mathcal{H}$. In particular, when $\{v_m\colon m \in M\}$ is a Bessel sequence in $\mathbb{C}^p$, then its frame operator L can be identified with $p \times p$ dual Gramian matrix given by (7.12).

Theorem 7.1. *Suppose that $G(\mathcal{A},\Gamma)$ is a Bessel sequence. The frame operator L of $G(\mathcal{A},\Gamma)$ is an SMI operator and its corresponding range operator R is such that each $R(x,\xi)$ is the frame operator of*

$$\frac{1}{\sqrt{p}}\mathcal{V}_{\mathcal{A}}(x,\xi) = \left\{\frac{1}{\sqrt{p}}\mathfrak{T}\varphi_m(x+l_j,\xi)\colon 1 \leqslant j \leqslant q,\ m \in M\right\} \subset \mathbb{C}^p \tag{7.3}$$

for a.e. (x,ξ). More explicitly,

$$\big\langle R(x,\xi)e_k, e_l\big\rangle = \frac{1}{p}\sum_{m\in M}\sum_{j=1}^{q} Z\varphi_m(x+l_j,\xi+d_l)\overline{Z\varphi_m(x+l_j,\xi+d_k)}, \tag{7.4}$$

where $\{e_1,\dots,e_p\}$ is the standard basis of $\mathbb{C}^p$, and $\{l_1,\dots,l_q\} \subset \Gamma$, $\{d_1,\dots,d_p\}$ are representatives of distinct cosets of $\Theta/\mathbb{Z}^n$, $\Xi^*/\mathbb{Z}^n$, respectively.

In particular, Theorem 7.1 implies that the frame operator L satisfies

$$\mathfrak{T}Lf(x,\xi) = R(x,\xi)\big(\mathfrak{T}f(x,\xi)\big) \quad \text{for a.e. } (x,\xi) \in \mathbb{R}^n \times \mathbb{T}^n, \text{ and } f \in L^2\big(\mathbb{R}^n\big), \tag{7.5}$$

where the linear map $R(x,\xi)$, which can be identified with $p \times p$ matrix, is given by (7.5). The formula (7.5) is often referred to as the Zibulski–Zeevi representation of the frame operator L. It was first proved in one-dimensional setting by Zibulski and Zeevi [31], see also [10, Chapter 1, Section 1.5] or [13,16,22–24,30]. However, its higher-dimensional analogue (7.5) appears to be new.

Before providing the proof of Theorem 7.1, which is a consequence of the more general Theorem 7.2, we can easily deduce the description of Gabor canonical dual frame sequences.

Definition 7.1. Suppose that $G(\mathcal{A},\Gamma)$ is a frame sequence and let L be its frame operator. The *canonical dual frame sequence* of $G(\mathcal{A},\Gamma)$ is the Gabor system $G(\mathcal{A}',\Gamma)$, where

$$\mathcal{A}' = \big\{\varphi'_m = L^{-1}\varphi_m\colon m \in M\big\}.$$

Here, L^{-1} is the generalized inverse of L. That is, $L^{-1} = (L|_{S(\mathcal{A},\Gamma)})^{-1}$, since $L|_{S(\mathcal{A},\Gamma)}$ is an isomorphism of $S(\mathcal{A},\Gamma)$.

Since L is an SMI operator, so is L^{-1}, and we have the well-known identity

$$L^{-1}(M_k T_\gamma \varphi_m) = M_k T_\gamma \varphi'_m \quad \text{for all } (k,\gamma,m) \in \mathbb{Z}^n \times \Gamma \times M.$$

Moreover, $G(\mathcal{A}', \Gamma)$ is a also a frame sequence for $S(\mathcal{A}, \Gamma)$ with bounds $(c_1)^{-1}, (c_0)^{-1}$. As an immediate corollary of Theorem 7.1 and (7.5) we have

Corollary 7.1. *Suppose that $G(\mathcal{A}, \Gamma)$ is a frame sequence and J is the range function of $S(\mathcal{A}, \Gamma)$. Then its canonical dual frame sequence $G(\mathcal{A}', \Gamma)$ satisfies*

$$\mathfrak{T}\varphi'_m(x,\xi) = \big(R(x,\xi)\big)^{-1}\big(\mathfrak{T}\varphi_m(x,\xi)\big) \quad \text{for a.e. } (x,\xi), \tag{7.6}$$

where $(R(x,\xi))^{-1} = (R(x,\xi)|_{J(x,\xi)})^{-1}$ is the generalized inverse of $R(x,\xi)$. In particular, each system

$$\frac{1}{\sqrt{p}}\mathcal{V}_{\mathcal{A}'}(x,\xi) = \left\{ \frac{1}{\sqrt{p}}\mathfrak{T}\varphi'_m(x+l_j,\xi)\colon 1 \leqslant j \leqslant q,\ m \in M \right\} \subset \mathbb{C}^p \tag{7.7}$$

is the canonical dual of $\frac{1}{\sqrt{p}}\mathcal{V}_{\mathcal{A}}(x,\xi)$ for a.e. (x,ξ).

Theorem 7.1 is a special case of

Theorem 7.2. *Suppose that $\mathcal{A} = \{\varphi_m\colon m \in M\}$ and $\mathcal{A}' = \{\varphi'_m\colon m \in M\}$ are two families of generators in $L^2(\mathbb{R}^n)$, and their corresponding Gabor systems $G(\mathcal{A}, \Gamma)$ and $G(\mathcal{A}', \Gamma)$ are Bessel sequences. Let F and F' be their analysis operators. The mixed frame operator $L = F^*F'$ is an SMI operator and its corresponding range operator R is such that $R(x,\xi)$ is the mixed frame operator of $\frac{1}{\sqrt{p}}\mathcal{V}_{\mathcal{A}}(x,\xi)$ and $\frac{1}{\sqrt{p}}\mathcal{V}_{\mathcal{A}'}(x,\xi)$ for a.e. (x,ξ). More explicitly,*

$$\big\langle R(x,\xi)e_k, e_l\big\rangle = \frac{1}{p}\sum_{m\in M}\sum_{j=1}^{q} Z\varphi_m(x+l_j,\xi+d_l)\overline{Z\varphi'_m(x+l_j,\xi+d_k)}. \tag{7.8}$$

Proof. The fact that L is an SMI operator follows immediately from the well-known (almost) commutation relations between shifts T_γ and modulations M_k, and

$$Lf = \sum_{m\in M}\sum_{k\in\mathbb{Z}^n}\sum_{\gamma\in\mathbb{Z}^n}\big\langle f, M_k T_\gamma \varphi'_m\big\rangle M_k T_\gamma \varphi_m,$$

where the convergence is unconditional in L^2 for all $f \in L^2(\mathbb{R}^n)$.

Take any $f, g \in L^2(\mathbb{R})$ and $m \in M$. Mimicking the proof of Theorem 4.1, by Proposition 3.1 and (3.19)

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n}\sum_{\gamma\in\Gamma}\langle M_k T_\gamma \varphi_m, g\rangle\big\langle f, M_k T_\gamma \varphi'_m\big\rangle \\
&\quad = \sum_{k\in\mathbb{Z}^n}\sum_{\gamma\in\Gamma}\langle \mathfrak{T} M_k T_\gamma \varphi_m, \mathfrak{T} g\rangle\big\langle \mathfrak{T} f, \mathfrak{T} M_k T_\gamma \varphi'_m\big\rangle
\end{aligned}$$

$$= \sum_{k\in\mathbb{Z}^n}\sum_{\gamma\in\Gamma}\left(\int_{I_n\times I_{\Xi^*}} e^{2\pi i\langle k,x\rangle}\big\langle \mathfrak{T}\varphi_m(x+\gamma,\xi),\mathfrak{T}g(x,\xi)\big\rangle\,dx\,d\xi\right)$$

$$\times\left(\int_{I_n\times I_{\Xi^*}} e^{-2\pi i\langle k,x\rangle}\big\langle \mathfrak{T}f(x,\xi),\mathfrak{T}\varphi'_m(x+\gamma,\xi)\big\rangle\,dx\,d\xi\right)$$

$$= \sum_{k\in\mathbb{Z}^n}\sum_{j=1}^{q}\sum_{l\in\Xi}\left(\int_{I_n\times I_{\Xi^*}} e^{2\pi i\langle k,x\rangle}e^{2\pi i\langle l,\xi\rangle}\big\langle \mathfrak{T}\varphi_m(x+l_j,\xi),\mathfrak{T}g(x,\xi)\big\rangle\,dx\,d\xi\right)$$

$$\times\left(\int_{I_n\times I_{\Xi^*}} e^{-2\pi i\langle k,x\rangle}e^{-2\pi i\langle l,\xi\rangle}\big\langle \mathfrak{T}f(x,\xi),\mathfrak{T}\varphi'_m(x+l_j,\xi)\big\rangle\,dx\,d\xi\right)$$

$$= \frac{1}{p}\sum_{j=1}^{q}\int_{I_n\times I_{\Xi^*}}\big\langle \mathfrak{T}\varphi_m(x+l_j,\xi),\mathfrak{T}g(x,\xi)\big\rangle\big\langle \mathfrak{T}f(x,\xi),\mathfrak{T}\varphi'_m(x+l_j,\xi)\big\rangle\,dx\,d\xi.$$

In the penultimate step we used (3.8) and the fact that every $\gamma\in\Gamma$ has a unique decomposition as $\gamma = l_j + l$ for some $1\leqslant j\leqslant q$ and $l\in\Xi$. In the last step we used the fact that $\{p^{1/2}e^{2\pi i(\langle k,x\rangle+\langle l,\xi\rangle)}\}_{k\in\mathbb{Z}^n,\ l\in\Xi}$ is an orthonormal basis of $L^2(I_n\times I_{\Xi^*})$. Summing the above formula over $m\in M$, we have

$$\langle Lf,g\rangle = \frac{1}{p}\sum_{m\in M}\sum_{j=1}^{q}\int_{I_n\times I_{\Xi^*}}\big\langle \mathfrak{T}\varphi_m(x+l_j,\xi),\mathfrak{T}g(x,\xi)\big\rangle\big\langle \mathfrak{T}f(x,\xi),\mathfrak{T}\varphi'_m(x+l_j,\xi)\big\rangle\,dx\,d\xi. \tag{7.9}$$

On the other hand, note that the mixed frame operator $R(x,\xi)$ of (7.3) and (7.7) is simply

$$R(x,\xi)a = \frac{1}{p}\sum_{m\in M}\sum_{j=1}^{q}\big\langle a,\mathfrak{T}\varphi'_m(x+l_j,\xi)\big\rangle\mathfrak{T}\varphi_m(x+l_j,\xi)\quad\text{for } a\in\mathbb{C}^p. \tag{7.10}$$

Consequently,

$$\langle Lf,g\rangle = \langle\mathfrak{T}Lf,\mathfrak{T}g\rangle = \int_{I_n\times I_{\Xi^*}}\big\langle R(x,\xi)\big(\mathfrak{T}f(x,\xi)\big),\mathfrak{T}g(x,\xi)\big\rangle\,dx\,d\xi. \tag{7.11}$$

Since $g\in L^2(\mathbb{R}^n)$ is arbitrary, we must have

$$\mathfrak{T}Lf(x,\xi) = R(x,\xi)\big(\mathfrak{T}f(x,\xi)\big)\quad\text{a.e. }(x,\xi),$$

which shows that $R = R(x,\xi)$ is the range operator corresponding to L. Finally, (7.8) is an immediate consequence of (3.4) and (7.10). □

Remark 7.1. The linear map $R(x,\xi)$ in (7.8), when thought of as a matrix, is often referred to as a *mixed dual Gramian* of (7.3) and (7.7), see for example [27,28]. When $\varphi_m = \varphi'_m$, then the matrix $R(x,\xi)$ is simply the *dual Gramian* of (7.3). Recall that the dual Gramian $\tilde{G} = (\tilde{G}_{i,j})_{i,j\in N}$ of a Bessel sequence $\{v_m\colon m \in M\}$ in the Hilbert space $\ell^2(N)$ is simply the matrix representation of the frame operator L of $\{v_m\colon m \in M\}$ with respect to the standard orthonormal basis $\{e_i\colon i \in N\}$ of $\ell^2(N)$. That is

$$\tilde{G}_{i,j} = \langle Le_j, e_i\rangle = \sum_{m\in M} \langle e_j, v_m\rangle\langle v_m, e_i\rangle. \tag{7.12}$$

Finally, we consider general Gabor dual frame sequences. We follow the terminology from [7,8].

Definition 7.2. Suppose that $G(\mathcal{A},\Gamma)$ is a frame sequence and $G(\mathcal{A}',\Gamma)$ is a Bessel sequence. We say that $G(\mathcal{A}',\Gamma)$ is a *generalized dual* of $G(\mathcal{A},\Gamma)$ if the mixed frame operator F^*F' is the identity on $S(\mathcal{A},\Gamma)$. In addition, if $G(\mathcal{A}',\Gamma)$ is a frame sequence, then $G(\mathcal{A}',\Gamma)$ is said to be an *oblique dual* frame sequence of $G(\mathcal{A},\Gamma)$.

Remark 7.2. In the special case when $S(\mathcal{A},\Gamma) = L^2(\mathbb{R}^n)$, then both of these notions are identical and yield the usual concept of a *dual frame*.

As an immediate consequence of Theorems 4.1 and 7.2 we have the following results.

Theorem 7.3. *Suppose that $G(\mathcal{A},\Gamma)$ is a frame sequence, $G(\mathcal{A}',\Gamma)$ is a Bessel sequence, and J is the range function of $S(\mathcal{A},\Gamma)$.*

- (i) *$G(\mathcal{A}',\Gamma)$ is a generalized dual of $G(\mathcal{A},\Gamma)$ if and only if the mixed dual Gramian $R(x,\xi)$ given by* (7.8) *is the identity on $J(x,\xi)$ for a.e. (x,ξ).*
- (ii) *Assume, in addition, that $G(\mathcal{A}',\Gamma)$ is a frame sequence. Then, $G(\mathcal{A}',\Gamma)$ is an oblique dual of $G(\mathcal{A},\Gamma)$ if and only if the mixed dual Gramian $R(x,\xi)$ given by* (7.8) *is the identity on $J(x,\xi)$ for a.e. (x,ξ).*

In the special case when $S(\mathcal{A},\Gamma) = L^2(\mathbb{R}^n)$, we can extend the one-dimensional result of Zibulski and Zeevi [31] to higher dimensions.

Corollary 7.2. *Suppose that $G(\mathcal{A},\Gamma)$ and $G(\mathcal{A}',\Gamma)$ are two Bessel sequences. Then $G(\mathcal{A}',\Gamma)$ is a dual frame of $G(\mathcal{A},\Gamma)$ if and only if the system* (7.7) *is a dual frame of* (7.3) *for a.e. (x,ξ). Equivalently, $R(x,\xi)$ given by* (7.8) *is the identity on $\mathbb{C}^p$ for a.e. (x,ξ).*

Finally, we illustrate how functional calculus can be used for computing canonical tight generators (also called windows) for Gabor frames.

Definition 7.3. Let $G(\mathcal{A},\Gamma)$ be a Gabor frame, $\mathcal{A} = \{\varphi_m\colon m \in M\}$, and L be its frame operator. Define the *canonical tight generators* by

$$L^{-1/2}\mathcal{A} = \{\psi_m\colon \psi_m = L^{-1/2}\varphi_m,\ m \in M\}. \tag{7.13}$$

Then, it is easy to see that $G(L^{-1/2}\mathcal{A}, \Gamma)$ is a tight frame, often called the *canonical tight Gabor frame*.

Canonical tight windows for Gabor frames were studied in 1-dimensional case by Janssen and Strohmer [24] with the use of functional calculus. The functional calculus is particularly useful for rational Gabor systems where the Zak transform methods, such as Zibulski–Zeevi matrices, can be used to replace operators of L^2 by finite size matrices, see [24, Section 1.1]. By Theorems 6.7 and 7.1, the same methods can be extended to higher dimensions.

Corollary 7.3. *Suppose that $G(\mathcal{A}, \Gamma)$ is a frame and L is its frame operator. Let R be the range operator corresponding to L, which is given by* (7.4). *Let h be a bounded Borel function on* $\mathrm{sp}(L)$. *Then, $h(L)$ is an SMI operator with the range operator $(x,\xi) \mapsto h(R(x,\xi))$. That is,*

$$\big[\mathfrak{T} \circ h(L)\big] f(x,\xi) = h\big(R(x,\xi)\big)\big(\mathfrak{T} f(x,\xi)\big) \quad \textit{for a.e. } (x,\xi) \in \mathbb{R}^n \times \mathbb{T}^n,\ f \in L^2\big(\mathbb{R}^n\big).$$

Consequently, the functional calculus of the frame operator L reduces to functional calculus of positive definite $p \times p$ matrices, which is useful for doing computations. In particular, if $h(\lambda) = \lambda^{-1/2}$, then the canonical tight generators (7.13) are given by

$$\mathfrak{T}\psi_m(x,\xi) = R(x,\xi)^{-1/2}\big(\mathfrak{T}\varphi_m(x,\xi)\big) \quad \text{for a.e. } (x,\xi),$$

where $R(x,\xi)$ is a positive definite $p \times p$ matrix given by (7.4).

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