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Characterization and perturbation of Gabor frame sequences with rational parameters

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Abstract

Let $A \subset L^2(\mathbb{R})$ be at most countable, and $p, q \in \mathbb{N}$. We characterize various frame-properties for Gabor systems of the form

$$G(1, p/q, A) = \{e^{2\pi imx} g(x - np/q) : m, n \in \mathbb{Z},\ \ g \in A\}$$

in terms of the corresponding frame properties for the row vectors in the Zibulski–Zeevi matrix. This extends work by [Ron and Shen, Weyl–Heisenberg systems and Riesz bases in $L_2(\mathbb{R}^d)$. Duke Math. J. 89 (1997) 237–282], who considered the case where A is finite. As a consequence of the results, we obtain results concerning stability of Gabor frames under perturbation of the generators. We also introduce the concept of rigid frame sequences, which have the property that all sufficiently small perturbations with a lower frame bound above some threshold value, automatically generate the same closed linear span. Finally, we characterize rigid Gabor frame sequences in terms of their Zibulski–Zeevi matrix.

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1. Introduction

Gabor systems are collections of functions

$$\mathcal{G}(a, b, g) = \{e^{2\pi i max} g(x - nb) : n, m \in \mathbb{Z}\} \tag{1.1}$$

which are built from a single function $g : \mathbb{R} \to \mathbb{C}$ by shifts in time and frequency determined by the parameters $a, b > 0$. Such systems, also called Weyl–Heisenberg systems, were introduced by Gabor [16] with the aim of constructing efficient, time–frequency localized expansions of signals as (infinite) linear combinations of elements in (1.1). A major development in the theory of Gabor systems is due to Daubechies et al. [11] who placed the problem of Gabor expansions in the framework of frames for a Hilbert space. We state the formal definition of frames and their main properties in Section 1.1. In particular, letting $g_{ma,nb}(x) = e^{2\pi i max} g(x - nb)$, if a Gabor system $\mathcal{G}(a, b, g)$ is a frame, the corresponding frame operator $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$Sf = \sum_{m,n \in \mathbb{Z}} \langle f, g_{ma,nb} \rangle g_{ma,nb}$$

is bounded and invertible. Letting $\tilde{g} = S^{-1} g$, this leads to the reconstruction formula

$$f = \sum_{m,n \in \mathbb{Z}} \langle f, g_{ma,nb} \rangle \tilde{g}_{ma,nb} = \sum_{m,n \in \mathbb{Z}} \langle f, \tilde{g}_{ma,nb} \rangle g_{ma,nb} \quad \text{for all } f \in L^2(\mathbb{R}), \tag{1.2}$$

where the series converges unconditionally in $L^2(\mathbb{R})$.

Since the appearance of [11], Gabor systems are a subject of intensive study, with research efforts directed at characterizing Gabor systems being frames, studying frame operators, and efficiently computing canonical duals and expansions (1.2), see for example [9,17,6,12,13]. The goal of this paper is to study the properties of Gabor expansions in the case when the product *ab* of shift and frequency parameters is rational. Applying a standard dilation argument one can assume that $a = 1$ and $b \in \mathbb{Q}$. The corresponding results involving Gabor systems with *ab* rational can be deduced from this special case.

We will, in fact, be more general and consider a multiple-generated Gabor system of the form

$$\mathcal{G}(1, p/q, \mathcal{A}) = \{e^{2\pi i mx} g(x - np/q) : m, n \in \mathbb{Z}, \ g \in \mathcal{A}\}.$$

Here, $\mathcal{A} \subset L^2(\mathbb{R})$ is at most countable set of generators, and $p, q \in \mathbb{N}$. We characterize various frame-properties for such systems in terms of the corresponding frame properties of the row vectors in the Zibulski–Zeevi matrix. Since these vectors have finite length, these equivalent conditions are considerably easier to verify than the frame conditions. This approach was taken already by Ron and Shen in [21], who considered the case where $\mathcal{A}$ is finite. We present an independent proof. At some points our approach is similar to the one used by Ron and Shen; in order for the exposition not being too long, we focus on the new parts, and only sketch the similar parts.

As a consequence of the results, we obtain results concerning stability of Gabor frames under perturbation of the generators. Under natural conditions, these results show that a small perturbation of a Gabor frame sequence has to be a frame sequence for the *same* subspace; a formalization of this observation leads to the definition of the concept of rigidity for a frame sequence. While rigid Gabor frame sequences exist, we prove that no finitely generated shift-invariant (SI) system has this property.

Note that the idea of characterizing frame-properties of a Gabor system in terms of certain matrix-valued functions originated in the work of Zibulski and Zeevi [23]. It was later used by

Gabardo and Han [14] to characterize Gabor frame sequences, also called subspace Gabor frames. The advantage of our approach is that it applies to multiple generators, and that the conditions are stated directly in terms of the rows of the Zibulski–Zeevi matrix; the corresponding condition in [14] is slightly more involved, see (2.7).

In the rest of this section we introduce some of the main tools, in particular frames, the Zibulski–Zeevi matrix and range functions, and state their main properties.

1.1. Frames

A (countable) sequence of elements $\{f_k\}$ in a separable Hilbert space $\mathbf{H}$ is a *frame* for $\mathbf{H}$ if

$$\exists A, B > 0: \ A\|f\|^2 \leqslant \sum |\langle f, f_k\rangle|^2 \leqslant B\|f\|^2 \quad \forall f \in \mathbf{H}. \tag{1.3}$$

The numbers A, B are called *frame bounds*. In case $\overline{\text{span}}\{f_k\}$ is just a subspace of $\mathbf{H}$ and (1.3) holds for $f \in \overline{\text{span}}\{f_k\}$, the sequence $\{f_k\}$ is a *frame sequence*. If at least the upper frame condition is satisfied, $\{f_k\}$ is a *Bessel sequence*. Finally, a *Riesz basis* (resp. Riesz sequence) is a frame (resp. frame sequence), which is at the same time a basis (resp. basis for the subspace $\overline{\text{span}}\{f_k\}$).

From the definition it is clear that orthonormal bases are special cases of frames. However, the frame conditions are considerably weaker than the conditions characterizing orthonormal bases; thus, it is in general much easier to design a frame with special properties than an orthonormal basis with the same properties. Also, the *Balian–Low Theorem* (see, e.g., [10,6] and the references therein) shows that good time–frequency localization is impossible for Gabor expansions based on an orthonormal basis; on the other hand, exponential decay simultaneous in time and frequency can be obtained via frame expansions (based on, e.g., the Gaussian). Also, the crucial expansion property for orthonormal bases has a counterpart for frames. In fact, if $\{f_k\}$ is a frame for $\mathbf{H}$, the *frame operator*

$$S : \mathbf{H} \to \mathbf{H}, \quad Sf = \sum \langle f, f_k\rangle f_k$$

is bounded and bijective, and each $f \in \mathbf{H}$ has the expansion

$$f = SS^{-1}f = \sum \langle f, S^{-1}f_k\rangle f_k.$$

For more detailed information on these concepts we refer to [6].

1.2. The Zibulski–Zeevi transform

The Zibulski–Zeevi transform is a map $G : L^2(\mathbb{R}) \to L^2([0, 1/q] \times [0, 1/p], M_{q,p}(\mathbb{C}))$; in fact, the image of $g \in L^2(\mathbb{R})$ is a matrix-valued function on $\mathbb{R}^2$, which we denote by G^g, and whose (l, r)th entry is

$$G^g_{l,r}(t, v) = \mathcal{Z}g(t - lp/q, v + r/p), \quad 0 \leqslant l \leqslant q - 1, \ 0 \leqslant r \leqslant p - 1, \ (t, v) \in \mathbb{R} \times \mathbb{R}.$$

Here $\mathcal{Z} : L^2(\mathbb{R}) \to L^2([0, 1]^2)$ is the Zak transform

$$\mathcal{Z}g(t, v) = \sum_{k\in\mathbb{Z}} g(t - k)e^{2\pi ikv} \quad \text{for a.e. } (t, v) \in \mathbb{R}^2.$$

The matrix G^g is called the *Zibulski–Zeevi matrix*. Often, we suppress the dependence on $g \in L^2(\mathbb{R})$ and write $G(t, v)$ instead of $G^g(t, v)$. The lth row of $G(t, v)$, $0 \leqslant l \leqslant q - 1$, is often denoted by $G_l(t, v)$.

It is convenient to equip the space of matrices $M_{q,p}(\mathbb{C})$ with the Hilbert–Schmidt norm $\|\cdot\|_{\mathrm{HS}}$. Consequently, the space $L^2([0,1]^2)$ can be identified with the space $L^2([0,1/q]\times[0,1/p], M_{q,p}(\mathbb{C}))$ by the map

$$F(t,v)\mapsto (F(t-lp/q, v+r/p))_{l=0,\ldots,q-1}^{r=0,\ldots,p-1}.$$

In addition, since $\mathcal{Z}$ is an isometric isomorphism we have the following observation.

Proposition 1.1. *The Zibulski–Zeevi transform* $G : L^2(\mathbb{R}) \to L^2([0,1/q]\times[0,1/p], M_{q,p}(\mathbb{C}))$ *is an isometric isomorphism.*

We now state a lemma, which characterizes the frame sequences $\mathcal{G}(1, p/q, \mathcal{A})$ which are Riesz bases, in terms of the Zibulski–Zeevi transform. It appears in different formulations in the literature, see [1,15]. In [15] the result is stated for the case of one generator, but the argument is valid for any finite collection of generators. Given a collection $\mathcal{A}=\{g_1,\ldots,g_n\}\subset L^2(\mathbb{R})$, let $G^{\mathcal{A}}$ denote the $nq\times p$ matrix with rows formed by the vectors

$$G_0^{g_1}, G_1^{g_1},\ldots, G_{q-1}^{g_1}, G_0^{g_2}, G_1^{g_2},\ldots, G_{q-1}^{g_2},\ldots, G_{q-1}^{g_n}. \tag{1.4}$$

Lemma 1.1. *Let* $\mathcal{A}=\{g_1,\ldots,g_n\}\subset L^2(\mathbb{R})$ *and assume that* $\mathcal{G}(1,p/q,\mathcal{A})$ *is a frame sequence. Then* $\mathcal{G}(1,p/q,\mathcal{A})$ *is a Riesz sequence if and only if* $\operatorname{rank} G^{\mathcal{A}} = nq$ *a.e.*

Since $G^{\mathcal{A}}$ is an $nq\times p$ matrix, this result implies that $\mathcal{G}(1,p/q,\mathcal{A})$ only can be a Riesz sequence if $nq\leqslant p$.

1.3. Range functions

A closed subspace $V\subset L^2(\mathbb{R})$ is shift-modulation invariant (SMI) if it is invariant under modulations and shifts

$$M_m T_{np/q} V = V \quad \text{for all } n,m\in\mathbb{Z}.$$

We will need a characterization of SMI spaces in terms of appropriate range functions. The analogous characterization of SI spaces dates back to Helson [18] and its proof can be found in [2, Proposition 1.5]. However, there are some significant differences between these two results. For example, the range function in the SMI setting takes values in subspaces of a finite-dimensional space instead of subspaces of $\ell^2(\mathbb{Z})$ as in the SI setting.

Definition 1.1. A *range function* is a map

$$J : [0,1/q]\times[0,1/p]\to\{E : E \text{ is a subspace of } \mathbb{C}^p\}.$$

Let $P(t,v)$ be the orthogonal projection of $\mathbb{C}^p$ onto $J(t,v)$. J is said to be *measurable* if the map $(t,v)\mapsto P(t,v)$ is operator measurable. In other words, we require that each entry of the matrix function corresponding to $P(t,v)$ is measurable.

The following result, Theorem 1.1, explains the relationship between SMI spaces and range functions. Its proof follows the line of the parallel result for SI systems in [2] and can be found in [3].

Theorem 1.1. *There is* $1-1$ *correspondence between SMI spaces* V *and measurable range functions* J: *given a measurable range function* J, *the associated SMI space is*

$$V = \{f \in L^2(\mathbb{R}) : G_l^f(t, v) \in J(t, v) \text{ for all } 0 \leqslant l \leqslant q-1, \text{ and for a.e. } (t, v)\}, \tag{1.5}$$

and if $V = \overline{\operatorname{span}}\, \mathcal{G}(1, p/q, \mathcal{A})$, *then the associated range function is*

$$J(t, v) = \operatorname{span}\{G_l^g(t, v) : 0 \leqslant l \leqslant q-1,\ g \in \mathcal{A}\}. \tag{1.6}$$

2. Paradigm of fiberization

The goal of this section is to prove a fiberization characterization of Gabor frames and Riesz sequences in terms of the Zibulski–Zeevi transform. This is reminiscent of an analogous characterization for SI systems by Ron and Shen [20] and its equivalent formulation by the first author [2, Theorem 2.3].

Theorem 2.1. *Let* $0 < a \leqslant b < \infty$, *and* $\mathcal{A}$ *be at most countable. Then the following holds*:

(i) $\mathcal{G}(1, p/q, \mathcal{A})$ *is a Bessel sequence with bound* b *if and only if the system*

$$\{G_l^g(t, v) : 0 \leqslant l \leqslant q-1,\ g \in \mathcal{A}\} \tag{2.1}$$

is a Bessel sequence with bound pb *for a.e.* $(t, v) \in [0, 1/q] \times [0, 1/p]$.

(ii) $\mathcal{G}(1, p/q, \mathcal{A})$ *is a frame* (*resp. frame sequence*) *with bounds* a, b *if and only if* (2.1) *is a frame* (*resp. frame sequence*) *with bounds* pa, pb *for a.e.* $(t, v) \in [0, 1/q] \times [0, 1/p]$.

Proof. Standard arguments show that if system (2.1) satisfies either of the properties stated in Theorem 2.1 for a.e. $(t, v) \in [0, 1/q] \times [0, 1/p]$, then it satisfies the same property for a.e. $(t, v) \in \mathbb{R} \times \mathbb{R}$. Also, in [15, Lemma 3.2] it is proved that

$$\langle M_m T_{(nq+l)p/q} g, f \rangle = \int_0^1 \int_0^{1/p} e^{2\pi i m t} e^{-2\pi i n p v} \sum_{s=0}^{p-1} G_{l,s}^g(t, v) \overline{\mathcal{Z} f(t, v + s/p)}\, \mathrm{d}v\, \mathrm{d}t,$$

where $n \in \mathbb{Z}, l = 0, \ldots, q-1$. Using that $\{\sqrt{p} e^{2\pi i m t} e^{-2\pi i n p v} : m, n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2([0, 1] \times [0, 1/p])$ and some standard manipulations, this leads to

$$\sum_{g \in \mathcal{A}} \sum_{m,n \in \mathbb{Z}} |\langle M_m T_{np/q} g, f \rangle|^2 = \frac{1}{p} \sum_{g \in \mathcal{A}} \sum_{l,l'=0}^{q-1} \int_{[0,1/q]\times[0,1/p]} |\langle G_l^g(t, v), G_{l'}^f(t, v) \rangle|^2\, \mathrm{d}t\, \mathrm{d}v. \tag{2.2}$$

In order to prove the theorem, we first suppose that system (2.1) is a frame sequence with bounds pa, pb for a.e. $(t, v) \in [0, 1/q] \times [0, 1/p]$. Then

$$pa\|v\|^2 \leqslant \sum_{g \in \mathcal{A}} \sum_{l=0}^{q-1} |\langle G_l^g(t, v), v \rangle|^2 \leqslant pb\|v\|^2 \quad \text{for all } v \in J(t, v), \text{ and a.e. } (t, v), \tag{2.3}$$

where $J(t, v)$ is given by (1.6). By Theorem 1.1, if $f \in \overline{\operatorname{span}}\, \mathcal{G}(1, p/q, \mathcal{A})$, then $v = G_{l'}^f(t, v) \in J(t, v)$ for all $0 \leqslant l' \leqslant q-1$ and a.e. (t, v). Integrating (2.3) for $v = G_{l'}^f(t, v)$ over $[0, 1/q] \times$

$[0, 1/p]$, summing over $0 \leqslant l' \leqslant q-1$, and combining with (2.2) shows that $\mathcal{G}(1, p/q, \mathcal{A})$ is a frame sequence with bounds a, b.

For the converse, suppose first that $\mathcal{G}(1, p/q, \mathcal{A})$ is a Bessel sequence with bound b. Let $D \subset \mathbb{C}^p$ be a countable dense subset. To prove (2.3), it suffices to show that for any $v \in D$

$$\sum_{g\in\mathcal{A}} \sum_{l=0}^{q-1} |\langle G_l^g(t,\nu), v\rangle|^2 \leqslant pb\|P(t,\nu)v\|^2 \quad \text{a.e. } (t,\nu) \in [0,1/q]\times[0,1/p], \tag{2.4}$$

where $P(t, \nu)$ is the orthogonal projection of $\mathbb{C}^p$ onto $J(t, \nu)$. Assume on the contrary that (2.4) fails. Since D is countable, there exists a measurable set $E \subset [0, 1/q] \times [0, 1/p]$, with $|E| > 0$, $v_0 \in D$, and $\varepsilon > 0$, such that

$$\sum_{g\in\mathcal{A}} \sum_{l=0}^{q-1} |\langle G_l^g(t,\nu), P(t,\nu)v_0\rangle|^2 \geqslant (pb+\varepsilon)\|P(t,\nu)v_0\|^2 \quad \text{a.e.}(t,\nu) \in E, \tag{2.5}$$

Define $M \in L^2([0, 1/q] \times [0, 1/p], M_{q,p}(\mathbb{C}))$ by specifying its rows $M_{l'}(t, \nu)$, $0 \leqslant l' \leqslant q-1$, by

$$M_{l'}(t,\nu) = \begin{cases} P(t,\nu)v_0 & \text{for } l'=0 \text{ and } (t,\nu)\in E,\\ 0 & \text{otherwise.}\end{cases}$$

By Proposition 1.1, there exists a unique $f \in L^2(\mathbb{R}^n)$ such that $G^f(t, \nu) = M(t, \nu)$ for a.e. $(t, \nu) \in [0, 1/q] \times [0, 1/p]$. By Theorem 1.1, $f \in \mathcal{G}(1, p/q, \mathcal{A})$. Hence, by (2.2) and (2.5)

$$\begin{aligned}\sum_{g\in\mathcal{A}}\sum_{m,n\in\mathbb{Z}} |\langle M_m T_{np/q} g, f\rangle|^2 &= \frac{1}{p}\sum_{g\in\mathcal{A}}\sum_{l,l'=0}^{q-1}\int_{[0,1/q]\times[0,1/p]} |\langle G_l^g(t,\nu), G_{l'}^f(t,\nu)\rangle|^2\,\mathrm{d}t\,\mathrm{d}\nu\\ &= \frac{1}{p}\sum_{g\in\mathcal{A}}\sum_{l=0}^{q-1}\int_E |\langle G_l^g(t,\nu), P(t,\nu)v_0\rangle|^2\,\mathrm{d}t\,\mathrm{d}\nu \geqslant (b+\varepsilon/p)\\ &\quad\times\int_E \|P(t,\nu)v_0\|^2\,\mathrm{d}t\,\mathrm{d}\nu\\ &= (b+\varepsilon/p)\int_{[0,1/q]\times[0,1/p]} \|M(t,\nu)\|_{\mathrm{HS}}^2\,\mathrm{d}t\,\mathrm{d}\nu\\ &= (b+\varepsilon/p)\|f\|^2,\end{aligned}$$

which is a contradiction with b being an upper bound of $\mathcal{G}(1, p/q, \mathcal{A})$. Thus (2.4) holds as desired, i.e., the system $\{G_l^g(t, \nu) : 0 \leqslant l \leqslant q-1,\ g \in \mathcal{A}\}$ is a Bessel sequence with bound pb. This concludes the proof of (i).

For the proof of the frame sequence case in (ii), we assume that $\mathcal{G}(1, p/q, \mathcal{A})$, in addition to being a Bessel sequence, also satisfies the lower frame condition. We have to prove that

$$pa\|P(t,\nu)v\|^2 \leqslant \sum_{g\in\mathcal{A}} \sum_{l=0}^{q-1} |\langle G_l^g(t,\nu), v\rangle|^2 \quad \text{a.e. } (t,\nu) \in [0,1/q]\times[0,1/p]$$

for any v in a countable dense subset of $\mathbb{C}^p$; however, assuming the opposite leads to a contradiction via the same arguments as above. This completes the proof for frame sequences.

The case of frames is an immediate consequence of the case of frame sequences and Theorem 1.1. Indeed, by Theorem 1.1, $\mathcal{G}(1, p/q, \mathcal{A})$ is complete if and only if its range function $J(t, \nu) =$

$\mathbb{C}^p$ for a.e. (t, v), or equivalently, system (2.1) is complete in $\mathbb{C}^p$ for a.e. (t, v). Hence, the frame sequence case shows that $\mathcal{G}(1, p/q, \mathcal{A})$ is a frame with bounds a, b if and only if system (2.1) is a frame in $\mathbb{C}^p$ with bounds pa, pb for a.e. (t, v). □

In the case of a finite set of generators, Theorem 2.1 has a counterpart, valid for Riesz bases:

Theorem 2.2. *If* $\mathcal{A} = \{g_1, \ldots, g_n\} \subset L^2(\mathbb{R})$, *then Theorem* 2.1 *also holds when the* frame *property is replaced by Riesz basis* (*resp. Riesz sequence*) *property*.

Proof. A Riesz sequence is just a special case of a frame sequence. Note that $\{G_l(t, v) : 0 \leqslant l \leqslant q-1,\ g \in \mathcal{A}\}$ being a Riesz sequence is equivalent to these nq vectors being linearly independent (this holds because we are dealing with a finite collection of vectors); and this is equivalent to rank $G^{\mathcal{A}} = nq$. Invoking Lemma 1.1 completes the proof. □

As a conclusion on this section, we note that the idea of stating frame properties of $\mathcal{G}(1, p/q, \mathcal{A})$ in terms of matrix-valued functions is due to Zibulski and Zeevi [22,23]. In the case when $\mathcal{A}$ is finite, Zibulski and Zeevi define $p \times p$ matrix-valued function

$$S(t, v) = S^{\mathcal{A}}(t, v) := \frac{1}{p} \sum_{g \in \mathcal{A}} G^g(t, v)^* G^g(t, v), \tag{2.6}$$

and they prove that $\mathcal{G}(1, p/q, \mathcal{A})$ is a frame with bounds a, b if and only if all eigenvalues of $S(t, v)$ lie in the interval $[a, b]$ for a.e. (t, v). The case of frame sequences was considered by Gabardo and Han [14]. Corollary 6.5.2 in [14] states that a single-generated system $\{e^{2\pi i m x} g(x - np/q)\}_{m,n \in \mathbb{Z}}$ is a frame sequence with bounds a, b if and only if

$$\begin{aligned} paG^*(t, v)G(t, v) &\leqslant [G^*(t, v)G(t, v)]^2 \\ &\leqslant pbG^*(t, v)G(t, v) \quad \text{a.e. } (t, v) \in [0, 1] \times [0, 1/p]. \end{aligned} \tag{2.7}$$

Then, it is not difficult to see that (2.7) is equivalent to the property that all *non-zero* eigenvalues of $S(t, v)$ lie in the interval $[a, b]$ for a.e. (t, v). The advantage of Theorem 2.1 is that the frame properties are characterized directly in terms of the rows of the Zibulski–Zeevi matrices.

3. Perturbation of Gabor frames sequences

The goal of this section is to prove perturbation results for Gabor frame sequences in terms of rank conditions on the Zibulski–Zeevi matrix. The analogous result for SI systems was shown by Kim, Kim, Lim, and the second author in [8].

Our approach is based on the following result:

Lemma 3.1. *Let* $\{f_k\}$ *and* $\{g_k\}$ *be finite sequences in a Hilbert space* $\mathbf{H}$, *and let* $W = \operatorname{span}\{f_k\}$, $\{V\} := \operatorname{span}\{g_k\}$. *Denote the bounds for* $\{f_k\}$ (*as frame for* W) *by* A, B. *Assume that there exists a constant* $\mu > 0$ *such that*

$$\left\| \sum c_k (f_k - g_k) \right\| \leqslant \mu \left(\sum |c_k|^2 \right)^{1/2} \tag{3.1}$$

for all scalar sequences $\{c_k\}$. *Then the following holds*:

(a) $\{g_k\}$ *is a Bessel sequence with bound* $B(1+\mu/\sqrt{B})^2$;

(b) *If* $\mu < \sqrt{A}$ *and* $\dim W = \dim V$, *then* $\{g_k\}$ *is a frame for V with bounds*

$$A(1-\mu/\sqrt{A})^2,\ B(1+\mu/\sqrt{B})^2. \tag{3.2}$$

Note that a finite collection of vectors always form a frame for the linear span of the elements; the role of Lemma 3.1 is that it provides us with estimates for the frame bounds.

Remark 3.1. Lemma 3.1 follows from a more general result stated in [8]. In fact, the result in [8] is valid for infinite-dimensional spaces. When W and V are infinite-dimensional, an extra condition on the angle between W and V is needed; in the finite-dimensional case [4, Corollary 2.9] shows that the angle condition is satisfied if $\dim W = \dim V$. Alternatively, see [19, Lemma 3.9].

For $\mathcal{A}' = \{h_1, \ldots, h_n\} \subset L^2(\mathbb{R})$, the matrix $G^{\mathcal{A}'}$ is defined similarly to the matrix $G^{\mathcal{A}}$, see (1.4).

Theorem 3.1. *Let* $\mathcal{A} = \{g_1, \ldots, g_n\} \subset L^2(\mathbb{R})$, *and assume that* $\mathcal{G}(1, p/q, \mathcal{A})$ *is a frame sequence with bounds* a, b. *Given another set of generators* $\mathcal{A}' = \{h_1, \ldots, h_n\} \subset L^2(\mathbb{R})$, *define*

$$\mu := \operatorname*{ess\,sup}_{(t,v)\in[0,1/q]\times[0,1/p]} \|G^{\mathcal{A}-\mathcal{A}'}(t,v)\| \quad \text{where } \mathcal{A}-\mathcal{A}' = \{g_i - h_i : 1 \leqslant i \leqslant n\}. \tag{3.3}$$

Then the following holds.

(i) *If* $\mu < \infty$, *then* $\mathcal{G}(1, p/q, \mathcal{A}')$ *is a Bessel sequence with bound* $b(1+\mu/\sqrt{pb})^2$.

(ii) *If* $\mu < \sqrt{pa}$ *and*

$$\operatorname{rank} G^{\mathcal{A}}(t,v) = \operatorname{rank} G^{\mathcal{A}'}(t,v) \quad a.e.\ (t,v), \tag{3.4}$$

then $\mathcal{G}(1, p/q, \mathcal{A}')$ *is also a frame sequence, with bounds*

$$a(1-\mu/\sqrt{pa})^2,\ b(1+\mu/\sqrt{pb})^2.$$

Proof. By Theorem 2.1, the system

$$\{G_l^g(t,v) : 0 \leqslant l \leqslant q-1,\ g \in \mathcal{A}\} \tag{3.5}$$

is a frame sequence for a.e. $(t,v) \in [0,1/q]\times[0,1/p]$, with bounds $A := pa$, $B := pb$. Note that (3.5) being a Bessel sequence with bound B for a.e. $(t,v) \in [0,1/q]\times[0,1/p]$ is equivalent with

$$\operatorname*{ess\,sup}_{(t,v)\in[0,1/q]\times[0,1/p]} \|G^{\mathcal{A}}(t,v)\| \leqslant \sqrt{B}$$

or

$$\operatorname*{ess\,sup}_{(t,v)\in[0,1/q]\times[0,1/p]} \left\| \left(G^{\mathcal{A}}\right)^*(t,v)\right\| \leqslant \sqrt{B}.$$

Since $G^{\mathcal{A}-\mathcal{A}'}(t,v) = G^{\mathcal{A}}(t,v) - G^{\mathcal{A}'}(t,v)$, the assumption in (i) implies that for a.e. $(t,v) \in [0,1/q]\times[0,1/p]$,

$$\|G^{\mathcal{A}'}(t,v)\| \leqslant \mu + \sqrt{B},$$

thus, for a.e. $(t, v) \in [0, 1/q] \times [0, 1/p]$, $\{G_l^g(t, v) : 0 \leqslant l \leqslant q-1,\ g \in \mathcal{A}'\}$ is a Bessel sequence with bound

$$\left(\mu + \sqrt{B}\right)^2 = B\left(1 + \frac{\mu}{\sqrt{B}}\right)^2 = pb\left(1 + \frac{\mu}{\sqrt{pb}}\right)^2.$$

Again by Theorem 2.1, it follows that $\mathcal{G}(1, p/q, \mathcal{A}')$ is a Bessel sequence with bound $b(1 + \mu/\sqrt{pb})^2$.

We now prove (ii). Let

$$W = \operatorname{span}\{G_l^g(t, v) : 0 \leqslant l \leqslant q-1,\ g \in \mathcal{A}\}, \tag{3.6}$$

$$V = \operatorname{span}\{G_l^g(t, v) : 0 \leqslant l \leqslant q-1,\ g \in \mathcal{A}'\}. \tag{3.7}$$

By Lemma 3.1, the assumptions that $\mu < \sqrt{A}$ and $\dim V = \dim W$ imply that for a.e. $(t, v) \in [0, 1/q] \times [0, 1/p]$, $\{G_l^g(t, v) : 0 \leqslant l \leqslant q-1,\ g \in \mathcal{A}'\}$ is a frame for V with lower bound

$$\left(\sqrt{A} - \mu\right)^2 = A\left(1 - \frac{\mu}{\sqrt{A}}\right)^2.$$

Again by Theorem 2.1, it follows that $\mathcal{G}(1, p/q, \mathcal{A}')$ is a frame sequence with bound $a(1 - \mu/\sqrt{pa})^2$. $\square$

Remark 3.2. In case $\mathcal{G}(1, p/q, \mathcal{A})$ is a Riesz sequence, condition (3.4) in Theorem 3.1 is superfluous, see Theorem 3.2 in [7]. By [5], it is also superfluous if $\mathcal{G}(1, p/q, \mathcal{A})$ is a frame for $L^2(\mathbb{R})$. However, in general, the condition is needed; see Example 4.1.

As a consequence of Theorem 3.1 we have the following corollary.

Corollary 3.1. *Assume that $\mathcal{A}$ and $\mathcal{A}'$ are the same as in Theorem 3.1 and that*

$$\operatorname{rank} G^{\mathcal{A}}(t, v) \in \{0, p\} \quad a.e.\ (t, v). \tag{3.8}$$

If $\mu < \sqrt{pa}$ and

$$\operatorname{supp} G^{\mathcal{A}'} \subset \operatorname{supp} G^{\mathcal{A}}, \tag{3.9}$$

then $\mathcal{G}(1, p/q, \mathcal{A}')$ is also a frame sequence, with bounds

$$a(1 - \mu/\sqrt{pa})^2,\ b(1 + \mu/\sqrt{pb})^2.$$

Furthermore, $\mathcal{G}(1, p/q, \mathcal{A})$ and $\mathcal{G}(1, p/q, \mathcal{A}')$ are frames for the same subspace.

Proof. As in the proof of Theorem 3.1, for a fixed $(t, v) \in [0, 1/q] \times [0, 1/p]$, define the spaces W and V by (3.6) and (3.7). By (3.8) the space W is either a null space $\{0\}$ or $\mathbb{C}^p$ for a.e. (t, v). In the former case, the assumption (3.9) forces that $V = W = \{0\}$. In the latter case, Theorem 2.1 implies that the system

$$\{G_l^g(t, v) : 0 \leqslant l \leqslant q-1,\ g \in \mathcal{A}\}$$

is a frame for $\mathbb{C}^p$ with bounds $A = pa$ and $B = pb$. Hence, by a standard frame perturbation result (see [5,6, Theorem 15.1.1]) and the assumption $\mu < \sqrt{A}$, the system

$$\{G_l^g(t, v) : 0 \leqslant l \leqslant q-1,\ g \in \mathcal{A}'\}$$

is also a frame for $\mathbb{C}^p$ with bounds given by (3.2). Consequently, $V = W = \mathbb{C}^p$. Therefore, by Theorems 1.1 and 2.1, $\mathcal{G}(1, p/q, \mathcal{A}')$ is a frame sequence for the *same* subspace as $\mathcal{G}(1, p/q, \mathcal{A})$. □

Remark 3.3. In case of a single generator, $\mathcal{A} = \{g\}$, the rank condition (3.8) in Corollary 3.1 is

$$\operatorname{rank} G^g(t, v) \in \{0, p\} \quad \text{a.e. } (t, v). \tag{3.10}$$

By [15, Theorem 2.3], this is equivalent to the property that $\mathcal{G}(1, p/q, g)$ has a unique Gabor dual of type II in the terminology of Gabardo and Han [15].

In the next section we will prove that any such Gabor frame sequence can be perturbed only on the same subspace as the original system. That is, if we require that a perturbed Gabor frame sequence has a lower bound greater than some positive constant $\varepsilon > 0$, then it must generate the same subspace (as the original Gabor frame sequence) for sufficiently small perturbations. Hence, Gabor frame sequences satisfying (3.10) are in a sense very *rigid* under perturbations. And conversely, we will show that if a Gabor frame sequence is rigid in the above sense, then it must satisfy condition (3.10).

4. Optimality and rigidity of perturbations

There are two goals of this section. The first one is to prove that the rank condition (3.4) in Theorem 3.1 is optimal. More precisely, unless (3.4) is satisfied, in general, the perturbed Gabor system does not have to be a frame sequence regardless how small a perturbation (measured by the parameter μ) is. In particular, the support condition (3.9) in Corollary 3.1 is also optimal in the same sense. This leads naturally to the second goal of this section: a characterization of rigid Gabor frame sequences.

Rigid frame sequences are a special type of frame sequences which have the property that whenever they are perturbed by a sufficiently small perturbation then their closed linear span remains the same. We show that certain types of systems such as finitely generated SI frame sequences can never be rigid. Nevertheless, we prove the existence of rigid Gabor frame sequences using their characterization in terms of the rank of the Zibulski–Zeevi matrix.

The optimality of the rank condition (3.4) is shown by the following example.

Example 4.1. Let $\mathcal{A} = \{g_1, \ldots, g_n\} \subset L^2(\mathbb{R})$, and assume that $\mathcal{G}(1, p/q, \mathcal{A})$ is a frame sequence with bounds a, b. According to Remark 3.2 we assume that $\mathcal{G}(1, p/q, \mathcal{A})$ is neither a Riesz sequence nor a frame for all of $L^2(\mathbb{R})$.

Our aim is to construct another Gabor system $\mathcal{G}(1, p/q, \mathcal{A}')$, which is a very small perturbation of $\mathcal{G}(1, p/q, \mathcal{A})$, but yet it is not a frame sequence. To achieve this we consider, for any $1 \leqslant k_0 \leqslant n$, $0 \leqslant l_0 \leqslant q-1$, the set

$$\begin{aligned} E_{k_0,l_0} &= \{(t, v) \in [0, 1/q] \times [0, 1/p] : \operatorname{rank} G^{\mathcal{A}}(t, v) \\ &= \dim \operatorname{span}\{G_l^{g_k}(t, v) : (k, l) \neq (k_0, l_0)\}\}. \end{aligned}$$

In other words, $(t, v) \in E_{k_0,l_0}$ iff row $G_{l_0}^{g_{k_0}}(t, v)$ is a linear combination of the remaining $(nq-1)$ rows of $G^{\mathcal{A}}(t, v)$. We claim that there exist $1 \leqslant k_0 \leqslant n$, $0 \leqslant l_0 \leqslant q-1$ such that the set

$$E = E_{k_0,l_0} \cap \{(t, v) \in [0, 1/q] \times [0, 1/p] : \operatorname{rank} G^{\mathcal{A}}(t, v) \leqslant p-1\}. \tag{4.1}$$

has positive measure. Indeed, suppose on the contrary that (modulo null sets)

$$\begin{aligned}&\{(t,v):\operatorname{rank} G^{\mathcal{A}}(t,v)\leqslant nq-1\}\\ &\quad=\bigcup_{1\leqslant k\leqslant n,\ 0\leqslant l\leqslant q-1} E_{k,l}\subset\{(t,v):\operatorname{rank} G^{\mathcal{A}}(t,v)=p\}.\end{aligned}\tag{4.2}$$

Since $G^{\mathcal{A}}(t,v)$ is not a Riesz sequence, Lemma 1.1 shows that rank $G^{\mathcal{A}}(t,v)\leqslant nq-1$ on a set of positive measure. Consequently, (4.2) implies that $p\leqslant nq-1$. However, this also implies that

$$[0,1/q]\times[0,1/p]=\{(t,v):\operatorname{rank} G^{\mathcal{A}}(t,v)=p\},$$

which is a contradiction with Theorem 2.1 and our hypothesis that $\mathcal{G}(1,p/q,\mathcal{A})$ is not a frame; thus E has positive measure as claimed.

Let J be the range function associated to the shift-modulation space $\overline{\operatorname{span}}\,\mathcal{G}(1,p/q,\mathcal{A})$, that is, $J(t,v)$ is given by (1.6). Define another range function J' by

$$J'(t,v)=\begin{cases}J(t,v)^{\perp} & (t,v)\in E,\\ \{0\} & \text{otherwise.}\end{cases}$$

Since J' is a non-zero measurable range function, there exists by Theorem 1.1 $0\neq f\in L^2(\mathbb{R})$ such that

$$G_l^f(t,v)\in J'(t,v)\quad\text{for all } 0\leqslant l\leqslant q-1,\ \text{and a.e. } (t,v).$$

By Proposition 1.1 we can find f as above such that all rows of $G^f(t,v)$, except the l_0th row $G^f_{l_0}(t,v)$, are zero. Let $\delta>0$ be any constant. By a simple rescaling argument we can also assume that

$$\|G^f_{l_0}(t,v)\|\leqslant\delta\quad\text{for a.e. } (t,v)\tag{4.3}$$

and for all $r>0$,

$$|\{(t,v):0<\|G^f_{l_0}(t,v)\|<r\}|>0.\tag{4.4}$$

Finally, define the perturbed system $\mathcal{A}'=\{h_1,\ldots,h_n\}$ by $h_{k_0}=g_{k_0}+f$ and $h_k=g_k$ for $k\neq k_0$. By (4.3) the perturbation parameter μ in (3.3) is at most δ and it can be made arbitrarily small. However, we claim that the Gabor system $\mathcal{G}(1,p/q,\mathcal{A}')$ is not a frame sequence. Indeed, by our construction $G^f_l(t,v)$ can only be non-zero if $l=l_0$ and $(t,v)\in E_{k_0,l_0}$. Consequently, for a.e. (t,v)

$$\begin{aligned}J''(t,v)&=\operatorname{span}\{G_l^{h_k}(t,v):1\leqslant k\leqslant n,\ 0\leqslant l\leqslant q-1\}\\ &=\operatorname{span}(\{G_l^{g_k}(t,v):(k,l)\neq(k_0,l_0)\},G_{l_0}^{g_{k_0}}(t,v)+G^f_{l_0}(t,v))\\ &=\operatorname{span}(J(t,v),G^f_{l_0}(t,v)).\end{aligned}$$

Likewise, using the orthogonality condition $G^f_{l_0}(t,v)\perp J(t,v)$ we have

$$\sum_{1\leqslant k\leqslant n,\ 0\leqslant l\leqslant q-1}|\langle G^f_{l_0}(t,v),G_l^{h_k}(t,v)\rangle|^2=|\langle G^f_{l_0}(t,v),G_{l_0}^{h_{k_0}}(t,v)\rangle|^2=\|G^f_{l_0}(t,v)\|^4.$$

Therefore, the lower bound of the frame sequence $\{G_l^{h_k}(t,v):1\leqslant k\leqslant n,\ 0\leqslant l\leqslant q-1\}$ is at most $\|G^f_{l_0}(t,v)\|^2$ for a.e. (t,v) such that $G^f_{l_0}(t,v)\neq 0$. By (4.4) this value can be made arbitrarily close to zero on sets of positive measure. By Theorem 3.1 our perturbation $\mathcal{G}(1, p/q, \mathcal{A}')$ is not a frame sequence.

Our next aim is to characterize rigid Gabor frame sequences. To state precisely our result we adopt the following definition of rigidity.

Definition 4.1. We say that $\mathcal{G}(1, p/q, \mathcal{A})$ is a *rigid* frame sequence if for every $\varepsilon > 0$, there exists $\delta > 0$, such that whenever $\mathcal{G}(1, p/q, \mathcal{A}')$ is a frame sequence with lower bound ε and the Bessel constant of $\mathcal{G}(1, p/q, \mathcal{A} - \mathcal{A}')$ is less than δ, then both frames sequences generate that same subspace.

Recall that the (optimal) Bessel bound of $\mathcal{G}(1, p/q, \mathcal{A} - \mathcal{A}')$ equals $\mu/\sqrt{p}$, where μ is given by (3.3). Hence, a Gabor frame sequence $\mathcal{G}(1, p/q, \mathcal{A})$ is rigid if and only if any sufficiently small perturbation $\mathcal{G}(1, p/q, \mathcal{A}')$, in the sense that $\mu < \delta$, generates the same subspace as $\mathcal{G}(1, p/q, \mathcal{A})$. Naturally, we are restricting to perturbations $\mathcal{G}(1, p/q, \mathcal{A}')$ with lower frame sequence bound above some threshold value of $\varepsilon > 0$.

Theorem 4.1. *Let $\mathcal{G}(1, p/q, \mathcal{A})$ be a Gabor frame sequence. Then $\mathcal{G}(1, p/q, \mathcal{A})$ is rigid if and only if* (3.8) *holds.*

Proof. Suppose that (3.8) fails, that is

$$E = \{(t, v) \in [0, 1/q] \times [0, 1/p] : 0 < \operatorname{rank} G^{\mathcal{A}}(t, v) < p\}$$

has positive measure. Let U be a $p \times p$ diagonal unitary matrix with diagonal entries $e^{i\alpha_1}, \ldots, e^{i\alpha_p}$. Given $\delta > 0$, we choose α_j's to be all different and $0 < |\alpha_j| < \delta$. Define the perturbed set of generators $\mathcal{A}' = \{h_1, \ldots, h_n\}$ by their Zibulski–Zeevi matrix

$$G_l^{h_k}(t, v) = U(G_l^{f_k}(t, v)) \quad \text{for all } 1 \leqslant k \leqslant n,\ 0 \leqslant l \leqslant q-1,\ (t, v) \in [0, 1/q] \times [0, 1/p]. \tag{4.5}$$

By Proposition 1.1 and Theorem 2.1, $\mathcal{G}(1, p/q, \mathcal{A}')$ is a Gabor frame sequence with the same bounds as $\mathcal{G}(1, p/q, \mathcal{A})$. Furthermore, these Gabor systems generate the same subspace if and only if

$$U(J(t, v)) = J(t, v) \quad \text{for a.e. } (t, v) \in [0, 1/q] \times [0, 1/p], \tag{4.6}$$

where $J = J(t, v)$ is the range function of $\overline{\operatorname{span}}\, \mathcal{G}(1, p/q, \mathcal{A})$. In addition, $\mathcal{G}(1, p/q, \mathcal{A}')$ is a small perturbation of $\mathcal{G}(1, p/q, \mathcal{A})$, since

$$(G^{\mathcal{A}-\mathcal{A}'})^*(t, v) = (I - U)((G^{\mathcal{A}})^*(t, v))$$

and $\|I - U\| = |1 - e^{i\delta}| < \delta$. Consequently, $\mathcal{G}(1, p/q, \mathcal{A} - \mathcal{A}')$ is a Bessel sequence with a bound less than the product of δ and the upper bound of $\mathcal{G}(1, p/q, \mathcal{A})$. Thus, if (4.6) fails then $\mathcal{G}(1, p/q, \mathcal{A})$ is not rigid.

Likewise, suppose that (4.6) holds. By our choice of U, its only invariant subspaces are of the form

$$V = \operatorname{span}\{e_{i_1}, \ldots, e_{i_m}\}$$

for some $1 \leqslant i_1 < \cdots < i_m \leqslant p$. Hence, if (4.6) holds, then $J(t, v) = V$ on some set of positive measure $E' \subset E$. Let $1 \leqslant i_0 \leqslant p$ be such that $i_0 \neq i_1, \ldots, i_m$. Let U' be the unitary matrix which is the identity on $\operatorname{span}(e_{i_0}, e_{i_1})^\perp$ and

$$U'(e_{i_1}) = \cos(\delta)e_{i_1} + \sin(\delta)e_{i_0}, \quad U'(e_{i_0}) = \sin(\delta)e_{i_1} - \cos(\delta)e_{i_0}.$$

Define the perturbed set of generators $\mathcal{A}' = \{h_1, \ldots, h_n\}$ by (4.5) with U' in place of U. By the same argument as before we conclude that $\mathcal{G}(1, p/q, \mathcal{A})$ is not rigid as well.

Conversely, suppose that (3.8) holds. Let a, b be the frame sequence bounds of $\mathcal{G}(1, p/q, \mathcal{A})$. Take any $\varepsilon > 0$ and suppose that $\mathcal{G}(1, p/q, \mathcal{A}')$ is a frame sequence with lower bound ε and the Bessel constant of $\mathcal{G}(1, p/q, \mathcal{A} - \mathcal{A}')$ is less than $\varepsilon/2$. Without loss of generality we can assume that $0 < \varepsilon < a/2$. By Theorem 2.1, the rows of $G^{\mathcal{A}}(t, v)$ form a frame for $\mathbb{C}^p$ on the set

$$E_p = \{(t, v) \in [0, 1/q] \times [0, 1/p] : \operatorname{rank} G^{\mathcal{A}}(t, v) = p\}.$$

Consequently, the rows of $G^{\mathcal{A}'}(t, v)$ form a frame for $\mathbb{C}^p$ for a.e. $(t, v) \in E_p$ as well. On the other hand, if (t, v) belongs to the set

$$E_0 = \{(t, v) \in [0, 1/q] \times [0, 1/p] : \operatorname{rank} G^{\mathcal{A}}(t, v) = 0\},$$

then $G^{\mathcal{A}'}(t, v) = -G^{\mathcal{A}-\mathcal{A}'}(t, v)$. By Theorem 2.1, the rows of $G^{\mathcal{A}'}(t, v)$ form a frame sequence with the lower bound $p\varepsilon$ and at the same time a Bessel sequence with constant $p\varepsilon/2$. Consequently, $G^{\mathcal{A}'}(t, v) = 0$ for a.e. $(t, v) \in E_p$. This shows that the range functions corresponding to the shift-modulation spaces $\overline{\operatorname{span}}\, \mathcal{G}(1, p/q, \mathcal{A})$ and $\overline{\operatorname{span}}\, \mathcal{G}(1, p/q, \mathcal{A}')$ are identical. By Theorem 1.1, the Gabor frame sequence $\mathcal{G}(1, p/q, \mathcal{A})$ is rigid. $\square$

The mere existence of rigid Gabor frame sequences is far from being obvious. Indeed, one could contrast Theorem 4.1 with the SI setting.

Definition 4.2. Let $\mathcal{A} \subset L^2(\mathbb{R})$ be finite and suppose that the SI system

$$\mathcal{E}(\mathcal{A}) = \{g(x-k) : k \in \mathbb{Z},\ g \in \mathcal{A}\}$$

is a frame sequence. We say that $\mathcal{E}(\mathcal{A})$ is *rigid* if it satisfies Definition 4.1, where each appearance of a Gabor system $\mathcal{G}(1, p/q, \mathcal{A})$ is replaced by the corresponding SI system $\mathcal{E}(\mathcal{A})$.

Theorem 4.2. *No finitely generated SI frame sequence $\mathcal{E}(\mathcal{A})$ is rigid.*

Proof. By [2, Theorem 2.3], $\mathcal{E}(\mathcal{A})$ is a frame sequence if and only $\{\mathcal{T}g_1(\xi), \ldots, \mathcal{T}g_n(\xi)\} \subset \ell^2(\mathbb{Z})$ is a frame sequence for a.e. $\xi \in [0, 1]$. Here,

$$\mathcal{T}g(\xi) = (\hat{g}(\xi + k))_{k \in \mathbb{Z}}, \quad \hat{g}(\xi) = \int_{\mathbb{R}} g(x)e^{-2\pi i x\xi}\, dx.$$

Given any $\delta > 0$, let U be any unitary operator on $\ell^2(\mathbb{Z})$ such that $\|I - U\| < \delta$ and (4.7) fails

$$U(J(\xi)) = J(\xi) \quad \text{for a.e. } \xi \in [0, 1], \text{ where } J(\xi) = \operatorname{span}\{\mathcal{T}g_1(\xi), \ldots, \mathcal{T}g_n(\xi)\}. \tag{4.7}$$

It is not difficult to prove the existence of such U following the approach in the proof of Theorem 4.1 and using the fact $J(\xi)$ are finite-dimensional subspaces of $\ell^2(\mathbb{Z})$.

Define the perturbed set of generators $\mathcal{A}' = \{h_1, \ldots, h_n\}$ by $\mathcal{T}h_k(\xi) = U(\mathcal{T}g_k(\xi))$. Using [2, Theorem 2.3] one can conclude that $\mathcal{E}(\mathcal{A}')$ is a frame sequence with the same bounds as $\mathcal{E}(\mathcal{A})$, which is a small perturbation of $\mathcal{E}(\mathcal{A})$. However, [2, Proposition 1.5] shows that these two systems generate distinct SI spaces. Consequently, $\mathcal{E}(\mathcal{A})$ is not rigid. $\square$

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