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Duality and interpolation of anisotropic Triebel–Lizorkin spaces

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Abstract We study properties of anisotropic Triebel–Lizorkin spaces associated with general expansive dilations and doubling measures on $\mathbb{R}^n$ using wavelet transforms. This paper is a continuation of (Bownik in J Geom Anal 2007, to appear, Trans Am Math Soc 358:1469–1510, 2006), where we generalized the isotropic methods of dyadic φ-transforms of Frazier and Jawerth (J Funct Anal 93:34–170, 1990) to non-isotropic settings. By working at the level of sequence spaces, we identify the duals of anisotropic Triebel–Lizorkin spaces. We also obtain several real and complex interpolation results for these spaces.

Keywords Anisotropic Triebel–Lizorkin space · Expansive dilation · Doubling measure · φ-transform · Wavelet transform · Dual space · Real interpolation · Complex interpolation

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1 Introduction and statements of main results

Triebel–Lizorkin spaces form a unifying class of function spaces encompassing many well-studied classical function spaces such as Lebesgue spaces, Hardy spaces, the Lipschitz spaces, and the space BMO. While these spaces were originally introduced on Euclidean spaces, there were several efforts of extending them to other domains and non-isotropic settings.

The usual isotropic dilations can be replaced by more general non-isotropic groups of dilations. This modification produces as a result many non-isotropic variants of the classical function spaces. Among many others we mention here parabolic Hardy spaces [12,13], Besov

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and Triebel–Lizorkin spaces for diagonal dilations [2,15,41,43–46]. The other possible direction is the study of weighted Besov and Triebel–Lizorkin spaces associated with general Muckenhoupt A_∞ weights, see [8–10,39]. One should also add that a significant portion of the theory of function spaces can also be done on very general domains such as spaces of homogeneous type introduced by Coifman and Weiss [14]; for example, see [25–28]. However, this high level of generality imposes restrictions on possible values of the integrability exponent p, i.e., $p > 1 - \delta$ for some possibly small $\delta > 0$.

To strike a balance between a high level generality and often diminishing range of possible results, we shall consider the class of non-isotropic dilation structures associated with expansive dilations, which includes previously considered diagonal setting. In the context of Hardy spaces this goal was achieved by the author in [3], where it was demonstrated that significant portion of a real-variable isotropic H^p theory extends to such anisotropic setting. Analogous extensions to anisotropic Besov and Triebel–Lizorkin spaces with doubling measures were studied in [4–6]. These studies show that the isotropic methods of dyadic φ-transforms of Frazier and Jawerth [17,19] extend to non-isotropic setting associated with general expansive dilations. Among other things proved in [4–6], weighted anisotropic Triebel–Lizorkin and Besov spaces are characterized by their wavelet transform coefficients and smooth atomic and molecular decompositions of these spaces are established. In addition, the localized version of Triebel–Lizorkin spaces in the endpoint case of $p = \infty$ is developed in [5].

The aim of this paper is to investigate further properties of these spaces such as duality and interpolation. Following Frazier and Jawerth [19] we will be working at the level of sequence spaces. We concentrate solely on Triebel–Lizorkin spaces. Indeed, Triebel [45,46] observed that (unweighted) Besov sequence spaces do not depend on the choice of an anisotropy, or in our case expansive dilation A, up to some natural rescaling of the smoothness parameter α. Hence, duality and interpolation results about anisotropic Besov spaces can be transferred from the isotropic setting without much effort. In what follows we summarize our results.

Suppose that A is an expansive dilation, i.e., $n \times n$ real matrix all of whose eigenvalues λ satisfy $|\lambda| > 1$. The corresponding Littlewood-Paley decomposition asserts [6,19] that any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ can be decomposed as

$$f = \sum_{j\in\mathbb{Z}} \varphi_j * f, \quad \text{where } \varphi_j(x) = |\det A|^j \varphi(A^j x),$$

with the convergence in $\mathcal{S}'/\mathcal{P}$ (modulo polynomials). Here, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a test function such that supp $\hat{\varphi}$ is compact and bounded away from origin, and $\sum_{j\in\mathbb{Z}} \hat{\varphi}((A^*)^j\xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Given a smoothness parameter $\alpha \in \mathbb{R}$, an integrability exponent $0 < p < \infty$, and a summability exponent $0 < q \le \infty$, we introduce the *anisotropic Triebel–Lizorkin space* $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ norm as

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha,q}} = \left\| \left(\sum_{j\in\mathbb{Z}} (|\det A|^{j\alpha} |f * \varphi_j|)^q \right)^{1/q} \right\|_{L^p(\mu)} < \infty. \tag{1.1}$$

Here, μ is a doubling measure respecting the action of A. That is,

$$\mu(B_{\rho_A}(x, 2r)) \le C\mu(B_{\rho_A}(x, r)) \quad \text{for all } x \in \mathbb{R}^n,\ r > 0,$$

where the balls $B_{\rho_A}(x, r)$ are defined with respect to a quasi-norm ρ_A associated with A. In the endpoint case of $p = \infty$, the naive definition of the space $\dot{\mathbf{F}}_p^{\alpha,q}$ using the norm (1.1) is unsatisfactory and we adopt the following definition. The space $\dot{\mathbf{F}}_\infty^{\alpha,q}(\mathbb{R}^n, A, \mu)$ is the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that,

$$\|f\|_{\dot{\mathbf{F}}_\infty^{\alpha,q}} = \sup_{P\in\mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \sum_{j=-\operatorname{scale}(P)}^{\infty} (|\det A|^{j\alpha} |f * \varphi_j(x)|)^q d\mu(x)\right)^{1/q} < \infty. \tag{1.2}$$

Here, $\mathcal{Q}$ is the collection of dilated cubes $Q = A^j([0,1]^n + k)$, with $\operatorname{scale}(Q) = j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$.

In Sect. 3 we show that the definition (1.2) is an extension of the case $p < \infty$. Indeed, we prove that there exists an operator $M^{\alpha,q}$ characterizing elements of $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ in terms of $L^p(\mu)$ norms for the entire range of $0 < p \le \infty$.

Theorem 1.1 *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is a doubling measure. Then,*

$$||M^{\alpha,q} f||_{L^p(\mu)} \asymp ||f||_{\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)} \quad \text{for all } f \in \mathcal{S}'/\mathcal{P}.$$

As a corollary of Theorem 1.1 we deduce that the space $\dot{\mathbf{F}}_\infty^{\alpha,q}$ is independent of the choice of a doubling measure μ. The following duality result is another manifestation of the fact that the endpoint space $\dot{\mathbf{F}}_\infty^{\alpha,q}$ is a continuous extension of $\dot{\mathbf{F}}_p^{\alpha,q}$, $p < \infty$. Theorem 1.2 is an extension of the well-known isotropic results of Triebel [43], Frazier and Jawerth [19], and Verbitsky [48] in the case of $(p,q) \in (1,\infty) \times (0,1)$. For convenience, we state it in the unweighted case where the duality pairing takes more natural form than in the weighted case.

Theorem 1.2 *Suppose $\alpha \in \mathbb{R}$, $0 < p, q < \infty$. Then, the dual space is*

$$(\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A))^* = \begin{cases} \dot{\mathbf{F}}_{p'}^{-\alpha,q'}(\mathbb{R}^n, A) & 1 \le p < \infty, \\ \dot{\mathbf{F}}_\infty^{-\alpha+(1/p-1),\infty}(\mathbb{R}^n, A) & 0 < p < 1. \end{cases}$$

In Sect. 5 we study real interpolation of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces. We prove that the expected isotropic results of Frazier and Jawerth [19] generalize to the non-isotropic setting.

Theorem 1.3 *Suppose $\alpha \in \mathbb{R}$, $0 < q \le \infty$, and $0 < p_0 < p < p_1 \le \infty$, and μ is a doubling measure. Then, the real interpolation space is*

$$(\dot{\mathbf{F}}_{p_0}^{\alpha,q}, \dot{\mathbf{F}}_{p_1}^{\alpha,q})_{\theta,p} = \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu), \quad \text{where } 1/p = (1-\theta)/p_0 + \theta/p_1.$$

Finally, in Sect. 6 we study complex interpolation of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces using Calderón products. We prove that the isotropic results of Triebel [43] and Frazier and Jawerth [19] extend to the non-isotropic setting. We also consider extensions of complex methods for quasi-Banach spaces [22,29,32]. We establish interpolation spaces for the entire range $0 < p, q \le \infty$, thus generalizing the results of Mendez and Mitrea [33].

Theorem 1.4 *Suppose $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, q_0 < \infty$, $0 < p_1, q_1 \le \infty$, and μ is a ρ_A-doubling measure. Then, for any $0 < \theta < 1$, the complex interpolation space is*

$$[\dot{\mathbf{F}}_{p_0}^{\alpha_0,q_0}, \dot{\mathbf{F}}_{p_1}^{\alpha_1,q_1}]_\theta = \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$, and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

2 Some background tools

We start by recalling basic definitions and properties of the Euclidean spaces associated with general expansive dilations.

2.1 Quasi-norms for expansive dilations

Definition 2.1 We say that a real $n \times n$ matrix is *expansive* if all of its eigenvalues satisfy $|\lambda| > 1$. A quasi-norm associated with an expansive matrix A is a Borel measurable mapping $\rho_A : \mathbb{R}^n \to [0, \infty)$ satisfying

$$\begin{aligned} &\rho_A(x) > 0, && \text{for } x \neq 0, \\ &\rho_A(Ax) = |\det A|\rho_A(x) && \text{for } x \in \mathbb{R}^n, \\ &\rho_A(x+y) \leq H(\rho_A(x) + \rho_A(y)) && \text{for } x, y \in \mathbb{R}^n, \end{aligned} \tag{2.1}$$

where $H \geq 1$ is a constant.

In the standard dyadic case $A = 2Id$, a quasi-norm ρ_A satisfies $\rho_A(2x) = 2^n \rho_A(x)$ instead of the usual scalar homogeneity. In particular, $\rho_A(x) = |x|^n$ is an example of a quasi-norm for $A = 2Id$, where $|\cdot|$ represent the Euclidean norm in $\mathbb{R}^n$. One can show that all quasi-norms associated to a fixed dilation A are equivalent, see [3, Lemma 2.4].

Definition 2.2 Let $\mathcal{B}$ be the collection of all ρ_A*-balls*

$$B_{\rho_A}(x, r) = \{y \in \mathbb{R}^n : \rho_A(x - y) < r\}, \quad x \in \mathbb{R}^n, \ r > 0.$$

Let $\mathcal{Q}$ be the collection of all *dilated cubes*

$$\mathcal{Q} = \{Q = A^j([0, 1]^n + k) : j \in \mathbb{Z}, \ k \in \mathbb{Z}^n\}$$

adapted to the action of a dilation A. Obviously, if $A = 2Id$ we obtain the usual collection of *dyadic cubes*.

The scale of a dilated cube $Q = A^j([0, 1]^n + k) \in \mathcal{Q}$ is defined as $\mathrm{scale}(Q) = j$. Alternatively, $\mathrm{scale}(Q) = \log_{|\det A|} |Q|$.

2.2 Doubling measures for expansive dilations

Definition 2.3 We say that a non-negative Borel measure μ on $\mathbb{R}^n$ is ρ_A*-doubling* if there exists $\beta = \beta(\mu) > 0$ such that

$$\mu(B_{\rho_A}(x, |\det A|r)) \leq |\det A|^\beta \mu(B_{\rho_A}(x, r)) \quad \text{for all } x \in \mathbb{R}^n, \ r > 0. \tag{2.2}$$

The smallest such β is called a doubling constant of μ.

For any Borel measurable function f define its Hardy-Littlewood maximal function $M_{\rho_A} f$ with respect to ρ_A-doubling measure μ by

$$M_{\rho_A} f(x) = \sup_{x \in B \in \mathcal{B}} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y).$$

It is easy to verify that we have the following fact. For rudimentary facts about spaces of homogeneous type we refer the reader to [14,21].

Proposition 2.1 *$(\mathbb{R}^n, \rho_A, \mu)$ is a space of homogeneous type, where ρ_A is a quasi-norm associated with an expansive dilation A, and μ is a ρ_A-doubling measure on $\mathbb{R}^n$.*

As a consequence the Fefferman-Stein vector-valued inequality [16] holds in our setting.

Theorem 2.2 *Suppose that $1 < p < \infty$, $1 < q \leq \infty$, and μ is a ρ_A-doubling measure. Then there exists a constant C such that*

$$\left\| \left(\sum_i |M_{\rho_A} f_i|^q \right)^{1/q} \right\|_{L^p(\mu)} \leq C \left\| \left(\sum_i |f_i|^q \right)^{1/q} \right\|_{L^p(\mu)}$$

holds for any $(f_i)_i \subset L^p(\mu)$.

We will also need several results about doubling measures and dilated cubes $\mathcal{Q}$. The following result can be found in [5].

Proposition 2.3 *Suppose that μ is ρ_A-doubling measure. Then:*

(a) For every $\eta > 0$ there exists a constant $c > 0$ such that

$$j \in \mathbb{Z},\ k_0, k_1 \in \mathbb{R}^n,\ |k_0 - k_1| < \eta \implies \mu(A^j([0,1]^n + k_0)) \leq c\mu(A^j([0,1]^n + k_1)).$$

(b) For fixed $x_0 \in \mathbb{R}^n$, let $P_j \in \mathcal{Q}$ be such that $\operatorname{scale}(P_j) = j$ *and $x_0 \in P_j$. Then*

$$\lim_{j\to\infty} \mu(P_j) = \infty.$$

Lemma 2.4 *Suppose that μ is ρ_A-doubling measure. Then, there exist constants $C, \varepsilon > 0$ such that for any $P, Q \in \mathcal{Q}$*

$$|Q| \leq |P|,\ Q \cap P \neq \emptyset \implies \mu(Q) \leq C \left(\frac{|Q|}{|P|} \right)^{\varepsilon} \mu(P). \tag{2.3}$$

Remark 2.1 One can show that the doubling condition (2.2) stipulates the estimate

$$|Q| \leq |P|,\ Q \cap P \neq \emptyset \implies \mu(Q) \geq C \left(\frac{|Q|}{|P|} \right)^{\beta} \mu(P).$$

Hence, Lemma 2.4 provides the converse decay estimate (2.3) of measures of small dilated cubes. In addition, it is clear that a similar estimate must hold for ρ_A-balls.

Proof Since A is expansive, there exists an integer $M > 0$ such that

$$A^{-M}([-1,1]^n) \subset y_0 + [0,1]^n \quad \text{for some } y_0 \in \mathbb{R}^n. \tag{2.4}$$

Let $P = A^{j_0}([0,1]^n + k_0) \in \mathcal{Q}$ be fixed. First, we prove that (2.3) holds under an additional assumption that $\operatorname{scale}(P) - \operatorname{scale}(Q)$ is divisible by M. Take any $Q = A^{j_0 - MN}([0,1]^n + k) \in \mathcal{Q}$, $N \in \mathbb{N}$, such that $Q \cap P \neq \emptyset$.

Define inductively the dilated cubes

$$Q_i = A^{j_0 - Mi}([0,1]^n + y_i), \quad \tilde{Q}_i = A^{j_0 - Mi}([-1,1]^n + y_i), \quad y_i \in \mathbb{R}^n,\ 0 \leq i \leq N,$$

as follows. Let $y_N = k$. Given $y_{i+1} \in \mathbb{R}^n$, $0 \leq i \leq N-1$, and hence $\tilde{Q}_{i+1}$ as above, we apply (2.4) to define $y_i \in \mathbb{R}^n$ so that

$$\tilde{Q}_{i+1} \subset A^{j_0 - Mi}([0,1]^n + y_i) = Q_i.$$

Define also the dilated cubes $Q'_i = A^{j_0 - Mi}([-1,0]^n + y_i)$. By Proposition 2.3 there exists a universal constant $c > 0$ such that $\mu(Q_i) \leq c\mu(Q'_i)$ for all $0 \leq i \leq N$. Hence,

$$\mu(Q_i) \geq \mu(\tilde{Q}_{i+1}) \geq \mu(Q_{i+1}) + \mu(Q'_{i+1}) \geq (1 + 1/c)\mu(Q_{i+1}).$$

Consequently,

$$\mu(Q_0) \geq (1+1/c)^N \mu(Q_N) = (1+1/c)^N \mu(Q).$$

Since $\operatorname{scale}(Q_0) = \operatorname{scale}(P) = j_0$ and $Q_0 \cap P \supset Q \cap P \neq \emptyset$, we have $c\mu(P) \geq \mu(Q_0)$ by Proposition 2.3. Combining the above yields

$$\mu(Q) \leq c(1+1/c)^{-N} \mu(P).$$

Since $|Q|/|P| = |\det A|^{-MN}$, (2.3) holds with $\varepsilon > 0$ so that $|\det A|^{M\varepsilon} = 1 + 1/c$.

Finally, we can remove the extra assumption on the scales of Q and P by increasing the constant C in (2.3). Indeed, take any $Q \in \mathcal{Q}$ with $Q \cap P \neq \emptyset$. Then, we can find $P' \in \mathcal{Q}$ such that $\operatorname{scale}(P') - \operatorname{scale}(Q)$ is divisible by M and $Q \cap P' \neq \emptyset$ and

$$\operatorname{scale}(P') > \operatorname{scale}(P) - M. \tag{2.5}$$

Consequently, we have

$$\mu(Q) \leq C\left(\frac{|Q|}{|P'|}\right)^{\varepsilon} \mu(P') \leq C|\det A|^M \left(\frac{|Q|}{|P|}\right)^{\varepsilon} \mu(P').$$

Note that $P \cap P' \neq \emptyset$ and the scales of P and P' are about the same by (2.5). Hence, an elementary argument using doubling of μ, e.g. [5, Lemma 2.9], shows that $\mu(P') \leq C\mu(P)$ for some constant C depending on M, but independent of $P, P' \in \mathcal{Q}$. This shows (2.3) in full generality. □

As a corollary of Lemma 2.4 we have

Lemma 2.5 *Suppose that μ is ρ_A-doubling measure and $\delta > 1$. Then, there exists $C = C(\delta) > 0$ such that*

$$\sum_{Q\in\mathcal{Q},\ |Q|\leq|P|,\ |Q\cap P|>0} (\mu(Q)/\mu(P))^{\delta} \leq C \quad \textit{for all } P \in \mathcal{Q}.$$

Proof Take any $P \in \mathcal{Q}$. Given $j \in \mathbb{Z}$, $j \leq \operatorname{scale}(P)$, define

$$P_j = \bigcup_{Q\in\mathcal{Q},\ \operatorname{scale}(Q)=j,\ |Q\cap P|>0} Q.$$

By Proposition 2.3 and [5, Lemma 6.5] we have $\mu(P_j) \leq C\mu(P)$. Hence, by Lemma 2.4,

$$\begin{aligned}
&\sum_{Q\in\mathcal{Q},\ |Q|\leq|P|,\ |Q\cap P|>0} (\mu(Q)/\mu(P))^{\delta} \\
&\quad\leq \sum_{j\leq\operatorname{scale}(P)} \frac{\mu(P_j)}{\mu(P)} \sup\left\{\left(\frac{\mu(Q)}{\mu(P)}\right)^{\delta-1} : Q \in \mathcal{Q},\ \operatorname{scale}(Q) = j,\ |Q\cap P| > 0\right\} \\
&\quad\leq C \sum_{j\leq\operatorname{scale}(P)} |\det A|^{(j-\operatorname{scale}(P))\varepsilon(\delta-1)} \leq C'.
\end{aligned}$$

□

2.3 Wavelet transforms for expansive dilations

Definition 2.4 We say that (φ, ψ) is an *admissible pair of dual frame wavelets* if φ, ψ are test functions in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ satisfying

$$\operatorname{supp}\hat{\varphi}, \operatorname{supp}\hat{\psi} \subset [-\pi, \pi]^n \setminus \{0\} \tag{2.6}$$

$$\sum_{j\in\mathbb{Z}} \overline{\hat{\varphi}((A^*)^j\xi)}\hat{\psi}((A^*)^j\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \tag{2.7}$$

where A^* is the adjoint (transpose) of A. Here,

$$\operatorname{supp}\hat{\varphi} = \overline{\{\xi \in \mathbb{R}^n : \hat{\varphi}(\xi) \neq 0\}},$$

and the Fourier transform of f is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i\langle x,\xi\rangle}dx.$$

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, define its *wavelet system* as

$$\{\varphi_Q(x) = |\det A|^{j/2}\varphi(A^jx - k) : Q = A^{-j}([0, 1]^n + k) \in \mathcal{Q}\}. \tag{2.8}$$

Definition 2.5 Suppose that (φ, ψ) is an admissible pair of dual frame wavelets. Define the φ-transform S_φ, also called the analysis transform, as the map taking $f \in \mathcal{S}'$ to the sequence $S_\varphi f = \{\langle f, \varphi_Q\rangle\}_{Q\in\mathcal{Q}}$, . Define the inverse φ-transform T_ψ, also called the synthesis transform, as the map taking $s = \{s_Q\}_Q$ to a distribution $T_\psi s = \sum_{Q\in\mathcal{Q}} s_Q\psi_Q$.

It is not hard to show that the conditions (2.6), (2.7) imply that (φ, ψ) is a pair of dual frame wavelets in $L^2(\mathbb{R}^n)$. This means that $S_\varphi, S_\psi : L^2(\mathbb{R}^n) \to \ell^2(\mathcal{Q})$ are bounded, and hence, their adjoints $T_\varphi = (S_\varphi)^*, T_\psi = (S_\psi)^*$. Furthermore, $T_\psi \circ S_\varphi$ is the identity on $L^2(\mathbb{R})$, i.e.,

$$f = \sum_{Q\in\mathcal{Q}} \langle f, \varphi_Q\rangle\psi_Q \quad \text{for all } f \in L^2(\mathbb{R}^n), \tag{2.9}$$

where the above series converges unconditionally in L^2.

The above formula has a counterpart in the form of the reproducing identity (2.10) valid for tempered distributions modulo polynomials $\mathcal{S}'/\mathcal{P}$. Lemma 2.6 is an anisotropic generalization of its well-known dyadic analogue, see [17,19,20]. For the proof we refer the reader to [6].

Lemma 2.6 *If (φ, ψ) is an admissible pair of dual frame wavelets, then*

$$f = \sum_{Q\in\mathcal{Q}} \langle f, \varphi_Q\rangle\psi_Q, \quad \textit{for any } f \in \mathcal{S}'/\mathcal{P}, \tag{2.10}$$

where the convergence of the above series, as well as the equality, is in $\mathcal{S}'/\mathcal{P}$. More precisely, there exists a sequence of polynomials $\{P_k\}_{k=1}^\infty \subset \mathcal{P}$ and $P \in \mathcal{P}$ such that

$$f = \lim_{k\to\infty}\Bigg(\sum_{Q\in\mathcal{Q},\ |\det A|^{-k}\le|Q|\le|\det A|^k} \langle f, \varphi_Q\rangle\psi_Q + P_k\Bigg) + P,$$

with convergence in $\mathcal{S}'$.

2.4 Anisotropic $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces

We start by recalling the definition of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces in the range $0 < p < \infty$.

Definition 2.6 For $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, and a ρ_A-doubling measure μ, we define the *anisotropic Triebel–Lizorkin space* $\dot{\mathbf{F}}_p^{\alpha,q} = \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ as the collection of

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha,q}} = \left\| \left(\sum_{j\in\mathbb{Z}} (|\det A|^{j\alpha} |f * \varphi_j|)^q \right)^{1/q} \right\|_{L^p(\mu)} < \infty, \tag{2.11}$$

where $\varphi_j(x) = |\det A|^j \varphi(A^j x)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (2.12), (2.13)

$$\operatorname{supp} \hat{\varphi} := \overline{\{\xi \in \mathbb{R}^n : \hat{\varphi}(\xi) \neq 0\}} \subset [-\pi, \pi]^n \setminus \{0\}, \tag{2.12}$$

$$\sup_{j\in\mathbb{Z}} |\hat{\varphi}((A^*)^j \xi)| > 0 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.13}$$

In [5] we proved that this definition is independent of φ.

The discrete Triebel–Lizorkin sequence space $\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$ is defined as the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}} = \left\| \left(\sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q)^q \right)^{1/q} \right\|_{L^p(\mu)} < \infty, \tag{2.14}$$

where $\tilde{\chi}_Q = |Q|^{-1/2}\chi_Q$ is the L^2-normalized characteristic function of the dilated cube Q.

Since the naive definition of the space $\dot{\mathbf{F}}_p^{\alpha,q}$ using the norm (2.11) when $p = \infty$ is unsatisfactory, we adopt the following definition.

Definition 2.7 For $\alpha \in \mathbb{R}$, $0 < q \le \infty$, and a ρ_A-doubling measure μ, we define the *anisotropic Triebel–Lizorkin space* $\dot{\mathbf{F}}_\infty^{\alpha,q} = \dot{\mathbf{F}}_\infty^{\alpha,q}(\mathbb{R}^n, A, \mu)$ as the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that,

$$\|f\|_{\dot{\mathbf{F}}_\infty^{\alpha,q}(\mathbb{R}^n,A,\mu)} = \sup_{P\in\mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \sum_{j=-\operatorname{scale}(P)}^{\infty} (|\det A|^{j\alpha} |f * \varphi_j(x)|)^q d\mu(x) \right)^{1/q} < \infty, \tag{2.15}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (2.12) and (2.13). By [5] this definition is independent of φ. Moreover, in Sect. 3 we shall prove that this space independent of the choice of a doubling measure μ.

The sequence space, $\dot{\mathbf{f}}_\infty^{\alpha,q} = \dot{\mathbf{f}}_\infty^{\alpha,q}(A,\mu)$ is the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}(A,\mu)} = \sup_{P\in\mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \sum_{Q\in\mathcal{Q},\ \operatorname{scale}(Q)\le\operatorname{scale}(P)} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q(x))^q d\mu(x) \right)^{1/q} < \infty. \tag{2.16}$$

Naturally, if $q = \infty$, then (2.15) and (2.16) are interpreted as

$$\|f\|_{\dot{\mathbf{F}}_\infty^{\alpha,\infty}} = \sup_{j\in\mathbb{Z}} |\det A|^{j\alpha} \|f * \varphi_j\|_\infty < \infty, \quad \|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,\infty}} = \sup_{Q\in\mathcal{Q}} |Q|^{-\alpha-1/2} |s_Q| < \infty. \tag{2.17}$$

The following generalization of the fundamental result of Frazier and Jawerth [5,6,19] holds.

Theorem 2.7 *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and a μ is a doubling measure. Then, the φ-transform $S_\varphi : \dot{\mathbf{F}}_p^{\alpha,q} \to \dot{\mathbf{f}}_p^{\alpha,q}$ and the inverse φ-transform $T_\psi : \dot{\mathbf{f}}_p^{\alpha,q} \to \dot{\mathbf{F}}_p^{\alpha,q}$ are bounded, and $T_\psi \circ S_\varphi$ is the identity on $\dot{\mathbf{F}}_p^{\alpha,q}$.*

3 The local q-power function

The goal of this section is to show the existence of an operator $M^{\alpha,q}$ which characterizes $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ in the sense that $||M^{\alpha,q} f||_{L^p(\mu)} \asymp ||f||_{\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)}$ for the entire range of parameters $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$.

To do this we must obtain a better understanding of sequence spaces $\dot{\mathbf{f}}_p^{\alpha,q}(A, \mu)$ by following the approach of Frazier and Jawerth [19, Sect. 5]. In particular, we show that if μ is of the form $d\mu = w dx$ for some $w \in A_\infty$, then the space $\dot{\mathbf{f}}_\infty^{\alpha,q}(A, \mu)$ coincides with the unweighted space $\dot{\mathbf{f}}_p^{\alpha,q}(A)$ with equivalent norms. Consequently, we deduce that the Triebel–Lizorkin spaces $\dot{\mathbf{F}}_\infty^{\alpha,q}(\mathbb{R}^n, A, w)$ are independent of the choice of $w \in A_\infty$. At the same time we derive equivalent ways of computing $\dot{\mathbf{f}}_\infty^{\alpha,q}(A, \mu)$ norms. In particular, we derive the inclusion $\dot{\mathbf{F}}_\infty^{\alpha,q_1} \subset \dot{\mathbf{F}}_\infty^{\alpha,q_2}$ for $0 < q_1 < q_2 \le \infty$, which is a non-trivial consequence of the original localized definition (2.15).

Lemma 3.1 *Suppose that $\alpha \in \mathbb{R}$, $0 < q \le \infty$, and μ is a ρ_A-doubling measure. Fix $\varepsilon > 0$. Suppose that for each $Q \in \mathcal{Q}$, $E_Q \subset Q$ is a Borel set with $\mu(E_Q)/\mu(Q) > \varepsilon$. Then, for any $s = \{s_Q\}_Q$*

$$||s||_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)} \asymp \left\| \left(\sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha}|s_Q|\tilde{\chi}_{E_Q})^q \right)^{1/q} \right\|_{L^p(\mu)} \quad (0 < p < \infty) \tag{3.1}$$

$$\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}(A,\mu)} \asymp \sup_{P\in\mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \sum_{Q\in\mathcal{Q},\ |Q|\le|P|} (|Q|^{-\alpha}|s_Q|\tilde{\chi}_{E_Q}(x))^q d\mu(x) \right)^{1/q}, \tag{3.2}$$

where $\tilde{\chi}_{E_Q} = |Q|^{-1/2}\chi_{E_Q}$ is the normalized characteristic function of E_Q.

Proof Since $\tilde{\chi}_{E_Q} \le \tilde{\chi}_Q$, one direction of (3.1) is trivial. To prove the other direction, note that for any $r > 0$, $\chi_Q \le \varepsilon^{-1/r}(M(\chi_{E_Q}^r))^{1/r}$, where the maximal function M is defined over the dilated cubes $\mathcal{Q}$ rather than dilated balls $\mathcal{B}$. Clearly, M is pointwise dominated by the usual maximal function M_{ρ_A}. Choose $r < \min(p, q)$. Then, by Theorem 2.2,

$$\begin{aligned} ||s||_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)} &\le \varepsilon^{-1/r} \left\| \left(\sum_{Q\in\mathcal{Q}} (M(|Q|^{-\alpha}|s_Q|\tilde{\chi}_{E_Q})^r)^{q/r} \right)^{r/q} \right\|_{L^{p/r}(\mu)}^{1/r} \\ &\le C\varepsilon^{-1/r} \left\| \left(\sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha}|s_Q|\tilde{\chi}_{E_Q})^q \right)^{1/q} \right\|_{L^p(\mu)}, \end{aligned}$$

which proves (3.1).

Analogously, one direction of (3.2) is trivial. To prove the other direction, fix a dilated cube $P = A^{j_0}([0,1]^n + k_0) \in \mathcal{Q}$ and let $\bar{P}$ be the union of neighboring dilated cubes to P, i.e.,

$$\bar{P} = \sum_{k\in\mathbb{Z}^n,\ |k|<K} (P + A^{j_0}k). \tag{3.3}$$

Here, the constant K is chosen so that for any $Q \in \mathcal{Q}$ with $|Q \cap P| > 0$ and $|Q| \le |P|$, we have $Q \subset \bar{P}$. Then, by (3.1)

$$\begin{aligned}
&\frac{1}{\mu(P)} \int_P \sum_{Q \in \mathcal{Q},\ |Q| \le |P|} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q(x))^q d\mu(x) \\
&\le \frac{1}{\mu(P)} \int_{\mathbb{R}^n} \sum_{Q \in \mathcal{Q},\ Q \subset \bar{P},\ |Q| \le |P|} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q(x))^q d\mu(x) \\
&\le C \frac{1}{\mu(P)} \int_{\mathbb{R}^n} \sum_{Q \in \mathcal{Q},\ Q \subset \bar{P},\ |Q| \le |P|} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_{E_Q}(x))^q d\mu(x) \\
&\le C \sum_{k \in \mathbb{Z}^n,\ |k| < K} \frac{1}{\mu(P + A^{j_0} k)} \int_{P + A^{j_0} k} \sum_{Q \in \mathcal{Q},\ |Q| \le |P|} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_{E_Q}(x))^q d\mu(x),
\end{aligned}$$

where in the last step we used Proposition 2.3. This proves (3.2). □

In the case when the doubling measure μ is of the form $d\mu = w dx$ for some $w \in A_\infty$ one can reformulate Lemma 3.1

Lemma 3.2 *Suppose that $\alpha \in \mathbb{R}$, $0 < q \le \infty$, and $w \in A_\infty$. Fix $\varepsilon > 0$. Suppose that for each $Q \in \mathcal{Q}$, $E_Q \subset Q$ is a Lebesgue measurable set with $|E_Q|/|Q| > \varepsilon$. Then, for any $s = \{s_Q\}_Q$*

$$||s||_{\dot{\mathbf{f}}_p^{\alpha,q}(A,w)} \asymp \left\| \left(\sum_{Q \in \mathcal{Q}} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_{E_Q})^q \right)^{1/q} \right\|_{L^p(w)} \quad (0 < p < \infty)$$

$$\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}(A,w)} \asymp \sup_{P \in \mathcal{Q}} \left(\frac{1}{w(P)} \int_P \sum_{Q \in \mathcal{Q},\ |Q| \le |P|} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_{E_Q}(x))^q w(x) dx \right)^{1/q},$$

where $\tilde{\chi}_{E_Q} = |Q|^{-1/2} \chi_{E_Q}$ is the normalized characteristic function of E_Q.

Proof Since $w \in A_\infty$, there exists $0 < \delta < 1$ such that for any $Q \in \mathcal{Q}$ and measurable subset $E \subset Q$,

$$|E| \ge \varepsilon |Q| \implies w(E) \ge \delta w(Q).$$

This fact can be found in [40, Chapt. V]. Hence, it suffices to apply Lemma 3.1. Also we could redefine $\tilde{\chi}_{E_Q} = |E_Q|^{-1/2} \chi_{E_Q}$ as the L^2-normalized characteristic function of E_Q following [19] without affecting the validity of Lemma 3.2. □

We are now ready to define the concept of the local q-power function at the sequence space level.

Definition 3.1 For $s = \{s_P\}_P$ and $Q \in \mathcal{Q}$, we define a function $G^{\alpha,q}(s)$ and its localized version $G_Q^{\alpha,q}(s)$, which are used in computing $\dot{\mathbf{f}}_p^{\alpha,q}$ $(p < \infty)$ and $\dot{\mathbf{f}}_\infty^{\alpha,q}$ norms, respectively, by

$$G^{\alpha,q}(s) = \left(\sum_{P \in \mathcal{Q}} (|P|^{-\alpha} |s_P| \tilde{\chi}_P)^q \right)^{1/q}, \tag{3.4}$$

$$G_Q^{\alpha,q}(s) = \left(\sum_{P \in \mathcal{Q},\ |P| \le |Q|,\ |P \cap Q| > 0} (|P|^{-\alpha} |s_P| \tilde{\chi}_P)^q \right)^{1/q}. \tag{3.5}$$

For a fixed $0 < \varepsilon < 1$, the ε-median of $G_Q^{\alpha,q}(s)$ on Q is defined as

$$m_{\mu,Q}^{\alpha,q}(s) = \inf\{t : \mu(\{x \in Q : G_Q^{\alpha,q}(s)(x) > t\}) < \mu(Q)\varepsilon\}. \tag{3.6}$$

Finally, define the *local q-power function* of s as

$$m_\mu^{\alpha,q}(s)(x) = \sup_{Q \in \mathcal{Q}} m_Q^{\alpha,q}(s)\chi_Q(x). \tag{3.7}$$

Remark 3.1 Note that the definition of the local q-power function $m_\mu^{\alpha,q}$ depends not only on α, q and a measure μ, but also on the choice of $0 < \varepsilon < 1$. Since the exact value of ε is not important for most of the arguments, we shall specify this dependence only when necessary, e.g. in the proof of Theorem 5.3. Also, we shall often suppress the dependence on μ and simply write $m_Q^{\alpha,q}$ and $m^{\alpha,q}$ for functions (3.6) and (3.7), respectively.

Theorem 3.3 *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is a ρ_A-doubling measure. Then, for any sequence $s = \{s_Q\}_Q$*

$$||s||_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)} \asymp ||m_\mu^{\alpha,q}(s)||_{L^p(\mu)}. \tag{3.8}$$

In particular, when $p = \infty$ we have

$$||s||_{\dot{\mathbf{f}}_\infty^{\alpha,q}(A,\mu)} \asymp ||m_\mu^{\alpha,q}(s)||_{L^\infty}. \tag{3.9}$$

Proof Note that for any $t > 0$,

$$\{x : m^{\alpha,q}(s)(x) > t\} \subset \{x : M(\chi_E)(x) \ge \varepsilon\},$$

where $E = \{y : G^{\alpha,q}(s)(y) > t\}$. Here, the maximal function M is defined over the dilated cubes $\mathcal{Q}$ rather than dilated balls $\mathcal{B}$. Clearly, M is pointwise dominated by the usual maximal function M_{ρ_A}. Since M_{ρ_A} is of weak type (1, 1)

$$\begin{aligned}\mu(\{x : m^{\alpha,q}(s)(x) > t\}) &\le \mu(\{x : M(\chi_E)(x) \ge \varepsilon\}) \\ &\le \frac{C}{\varepsilon}||\chi_E||_{L^1(\mu)} = C'\mu(\{x : G^{\alpha,q}(s)(x) > t\}).\end{aligned}$$

Therefore, for $0 < p < \infty$,

$$||m^{\alpha,q}(s)||_{L^p(\mu)} \le C'||G^{\alpha,q}(s)||_{L^p(\mu)} = C'||s||_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)}.$$

If $p = \infty$, then we need a different argument. By Chebyshev's inequality

$$\begin{aligned}\mu(\{x \in Q : G_Q^{\alpha,q}(s)(x) > t\}) &\le t^{-q}\int_Q (G_Q^{\alpha,q}(s)(x))^q d\mu(x) \\ &\le t^{-q}\mu(Q)||s||^q_{\dot{\mathbf{f}}_\infty^{\alpha,q}(A,\mu)} < \varepsilon\mu(Q),\end{aligned}$$

for $t > \varepsilon^{-1/q}||s||_{\dot{\mathbf{f}}_\infty^{\alpha,q}(A,\mu)}$. Hence, $||m^{\alpha,q}(s)||_{L^\infty} \le \varepsilon^{-1/q}||s||_{\dot{\mathbf{f}}_\infty^{\alpha,q}(A,w)}$.

The proof of the converse inequality is more delicate and it uses a stopping time argument. For each $x \in \mathbb{R}^n$ define its stopping scale $j(x)$ by

$$j(x) = \inf\left\{j \in \mathbb{Z} : \left(\sum_{\mathrm{scale}(P) \le j} (|P|^{-\alpha}|s_P|\tilde{\chi}_P(x))^q\right)^{1/q} \le m^{\alpha,q}(s)(x)\right\}.$$

Also define a Borel set $E_Q \subset Q \in \mathcal{Q}$ by

$$E_Q = \{x \in Q : j(x) \ge \mathrm{scale}(Q)\} = \{x \in Q : G_Q^{\alpha,q}(s)(x) \le m^{\alpha,q}(s)(x)\}.$$

By (3.6), $\mu(E_Q)/\mu(Q) \geq 1-\varepsilon$ and

$$\left(\sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha}|s_Q|\tilde{\chi}_{E_Q}(x))^q\right)^{1/q} \leq m^{\alpha,q}(s)(x) \quad \text{for all } x\in\mathbb{R}^n. \tag{3.10}$$

Therefore, by Lemma 3.1 we have $||s||_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)} \leq C||m^{\alpha,q}(s)||_{L^p(\mu)}$ for $0<p<\infty$. The same also holds for $p=\infty$ by Hölder's inequality and (3.2). □

Corollary 3.4 *Suppose that $\alpha\in\mathbb{R}$, $0<p,q\leq\infty$, and μ is a ρ_A-doubling measure. Fix $0<\varepsilon<1$. Then, for any $s=\{s_Q\}_Q$*

$$||s||_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)} \asymp \inf\left\{\left\|\left(\sum_{Q\in\mathcal{Q}}(|Q|^{-\alpha}|s_Q|\tilde{\chi}_{E_Q})^q\right)^{1/q}\right\|_{L^p(\mu)} : E_Q\subset Q,\ \mu(E_Q)/\mu(Q)>\varepsilon\right\},$$

where $E_Q\subset Q$ are Borel sets.

Proof Lemma 3.1 immediately implies the case $0<p<\infty$ and the upper bound for $||s||_{\dot{\mathbf{f}}_\infty^{\alpha,q}}$. To see the lower bound for $||s||_{\dot{\mathbf{f}}_\infty^{\alpha,q}}$, it suffices to choose E_Q's as in the proof of Theorem 3.3. □

One can reformulate the last result in the A_∞ setting as in Lemma 3.2.

Corollary 3.5 *Suppose that $\alpha\in\mathbb{R}$, $0<p,q\leq\infty$, and $w\in A_\infty$. Fix $0<\varepsilon<1$. Then, for any $s=\{s_Q\}_Q$*

$$||s||_{\dot{\mathbf{f}}_p^{\alpha,q}(A,w)} \asymp \inf\left\{\left\|\left(\sum_{Q\in\mathcal{Q}}(|Q|^{-\alpha}|s_Q|\tilde{\chi}_{E_Q})^q\right)^{1/q}\right\|_{L^p(w)} : E_Q\subset Q,\ |E_Q|/|Q|>\varepsilon\right\},$$

where $E_Q\subset Q$ are Lebesgue measurable sets.

In particular, when $p=\infty$, we have that the norm $||s||_{\dot{\mathbf{f}}_\infty^{\alpha,q}(A,w)}$ defined by (2.16) is independent (up to a multiplicative constant) of the choice of $w\in A_\infty$.

Question 3.1 Is it true that $\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}(A,\mu)}$ is equivalent to the unweighted norm $\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}}$ also for a general ρ_A-doubling measure μ, which is not necessarily in the class A_∞?

The next result shows that a big variety of possible $\dot{\mathbf{f}}_p^{\alpha,q}$ norms are all equivalent when $p=\infty$.

Theorem 3.6 *Suppose that $\alpha\in\mathbb{R}$ and $0<q\leq\infty$. Then, for each $0<r<\infty$ and ρ_A-doubling measure μ, we have the equivalence of norms*

$$\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}} \asymp \sup_{P\in\mathcal{Q}}\left(\frac{1}{\mu(P)}\int_P\left(\sum_{Q\in\mathcal{Q},\ \mathrm{scale}(Q)\leq\mathrm{scale}(P)}(|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q(x))^q\right)^{r/q}d\mu(x)\right)^{1/r}. \tag{3.11}$$

When $r=q$ and, then (3.11) coincides with the definition of $\dot{\mathbf{f}}_\infty^{\alpha,q}$ norm. In other situations, (3.11) gives an alternative way of computing $\dot{\mathbf{f}}_\infty^{\alpha,q}$ norm, which is sometimes more useful than the original definition.

Proof Suppose first that $r\geq q$. Then, the upper bound of $||s||_{\dot{\mathbf{f}}_\infty^{\alpha,q}}$ follows from Hölder's inequality. To show the lower bound, let E_Q be the same as in Corollary 3.4.

Fix a dilated cube $P = A^{j_0}([0,1]^n + k_0) \in \mathcal{Q}$ and let $\bar{P}$ be the union of neighboring dilated cubes to P given by (3.3). Then, by Lemma 3.1,

$$\begin{aligned}
&\frac{1}{\mu(P)} \int_P \Bigg(\sum_{Q \in \mathcal{Q},\ |Q| \le |P|} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q(x))^q \Bigg)^{r/q} d\mu(x) \\
&\le \frac{1}{\mu(P)} \int_{\mathbb{R}^n} \Bigg(\sum_{Q \in \mathcal{Q},\ Q \subset \bar{P},\ |Q| \le |P|} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q(x))^q \Bigg)^{r/q} d\mu(x) \\
&\le C \frac{1}{\mu(P)} \int_{\mathbb{R}^n} \Bigg(\sum_{Q \in \mathcal{Q},\ Q \subset \bar{P},\ |Q| \le |P|} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_{E_Q}(x))^q \Bigg)^{r/q} d\mu(x) \\
&\le C \Bigg\| \Bigg(\sum_{Q \in \mathcal{Q},\ Q \subset \bar{P},\ |Q| \le |P|} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_{E_Q})^q \Bigg)^{1/q} \Bigg\|_{L^\infty}^r \le C ||s||^r_{\dot{\mathbf{f}}^{\alpha,q}_\infty},
\end{aligned}$$

where the last inequality is a consequence of Corollary 3.4.

Next, suppose that $r < q$. This time the lower bound of $||s||_{\dot{\mathbf{f}}^{\alpha,q}_\infty}$ follows from Hölder's inequality. To show the upper bound, let E_Q be the same as in Corollary 3.4. Fix $P \in \mathcal{Q}$, and let

$$F(x) = \Bigg(\sum_{Q \in \mathcal{Q},\ Q \subset \bar{P},\ |Q| \le |P|} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_{E_Q}(x))^q \Bigg)^{1/q},$$

where $\bar{P}$ is as before. By Corollary 3.4

$$F(x)^q \le C F(x)^r ||s||^{q-r}_{\dot{\mathbf{f}}^{\alpha,q}_\infty}.$$

Hence, if $s \in \dot{\mathbf{f}}^{\alpha,q}_\infty$ then $F(x) < \infty$ a.e. and by Lemma 3.1,

$$\begin{aligned}
&\frac{1}{\mu(P)} \int_P \Bigg(\sum_{Q \in \mathcal{Q},\ |Q| \le |P|} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q(x))^q \Bigg)^{r/q} d\mu(x) \\
&\le C \frac{1}{\mu(P)} \Bigg(\int_{\mathbb{R}^n} F(x)^r d\mu(x) \Bigg)^{1/r} \\
&\le C \sum_{k \in \mathbb{Z}^n,\ |k| < K} \frac{1}{\mu(P + A^{j_0} k)} \Bigg(\int_{P + A^{j_0} k} F(x)^r d\mu(x) \Bigg)^{1/r} \le C ||s||_{\dot{\mathbf{f}}^{\alpha,q}_\infty},
\end{aligned}$$

where in the penultimate step we used Proposition 2.3. Finally, the case of arbitrary $s = \{s_Q\}_Q$ follows from the monotone convergence theorem. □

As an immediate corollary of Theorem 3.6 we have the following embedding result

Corollary 3.7 *Suppose that $\alpha \in \mathbb{R}$, $0 < p \le \infty$, and $0 < q_1 \le q_2 \le \infty$, and μ is a ρ_A-doubling measure. Then, the inclusion maps $i_1 \colon \dot{\mathbf{F}}^{\alpha,q_1}_p(\mathbb{R}^n, A, \mu) \hookrightarrow \dot{\mathbf{F}}^{\alpha,q_2}_p(\mathbb{R}^n, A, \mu)$ and $i_2 \colon \dot{\mathbf{f}}^{\alpha,q_1}_p(A, \mu) \hookrightarrow \dot{\mathbf{f}}^{\alpha,q_2}_p(A, \mu)$ are bounded.*

Proof The case $p < \infty$ is trivial by the continuity of the inclusion map $\ell^{q_1} \hookrightarrow \ell^{q_2}$ and the definition of respective spaces. However, the case $p = \infty$ is very far from being obvious due to the special way of defining our spaces. Nevertheless, Theorem 3.6 with $r = 1$ shows that the inclusion map i_2 is bounded also when $p = \infty$. By Theorem 2.7, i_1 is bounded as well. $\square$

Definition 3.2 Let $L_+(\mathbb{R}^n)$ be the vector space of all Borel functions on $\mathbb{R}^n$ with values in $[0, \infty]$. Define the operator $M^{\alpha,q} = M^{\alpha,q}_\mu \colon \mathcal{S}'/\mathcal{P} \to L_+(\mathbb{R}^n)$ by

$$M^{\alpha,q}(f) = m^{\alpha,q}_\mu(S_\varphi f) \quad \text{for } f \in \mathcal{S}'/\mathcal{P}.$$

Hence, $M^{\alpha,q}_\mu$ is a composition of the analysis transform S_φ with the sub-linear operator $m^{\alpha,q}_\mu$.

As a corollary of our results we have

Corollary 3.8 *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is a ρ_A-doubling measure. Then, we have*

$$||M^{\alpha,q} f||_{L^p(\mu)} \asymp ||f||_{\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n,A,\mu)} \quad \textit{for all } f \in \mathcal{S}'/\mathcal{P}.$$

Proof By Theorems 2.7 and 3.3,

$$||M^{\alpha,q} f||_{L^p(\mu)} = ||m^{\alpha,q}(S_\varphi f)||_{L^p(\mu)} \asymp ||S_\varphi f||_{\dot{\mathbf{f}}^{\alpha,q}_p(A,\mu)} \asymp ||f||_{\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n,A,\mu)}.$$

$\square$

4 Duality of $\dot{\mathbf{F}}^{\alpha,q}_p$ spaces

The goal of this section is to identify the duals of anisotropic $\dot{\mathbf{F}}^{\alpha,q}_p$ spaces for $0 < p, q < \infty$. In the classical setting this was done in [19,43] except the case $p > 1$ and $0 < q < 1$. This case was established by Verbitsky [48], see Remark 4.1.

Following Frazier and Jawerth [19] we identify the duals of $\dot{\mathbf{F}}^{\alpha,q}_p$ spaces by working on the sequence space level $\dot{\mathbf{f}}^{\alpha,q}_p$. We will use the convention that the conjugate exponent q' satisfies $1/q + 1/q' = 1$ if $1 < q \le \infty$ and $q' = \infty$ if $0 < q \le 1$. First, we will deal with the most interesting case when $p = 1$. Indeed, the case $1 < p < \infty$ is a consequence of standard duality results, whereas the case $0 < p < 1$ follows by certain imbedding arguments. The only exception in this scheme is the case $(p, q) \in (1, \infty) \times (0, 1)$, which requires special considerations.

Theorem 4.1 *Suppose $\alpha \in \mathbb{R}$, $0 < q < \infty$, and μ is a ρ_A-doubling measure. Then*

$$(\dot{\mathbf{f}}^{\alpha,q}_1(A, \mu))^* \approx \dot{\mathbf{f}}^{-\alpha,q'}_\infty(A).$$

More precisely, if $t = \{t_Q\}_Q \in \dot{\mathbf{f}}^{-\alpha,q'}_\infty$, then the map l

$$s = \{s_Q\}_Q \mapsto \langle s, t\rangle_\mu := \sum_{Q\in\mathcal{Q}} s_Q \overline{t_Q} \frac{\mu(Q)}{|Q|} \tag{4.1}$$

defines a continuous linear functional on $\dot{\mathbf{f}}^{\alpha,q}_1$ with $||l||_{(\dot{\mathbf{f}}^{\alpha,q}_1)^*} \asymp ||t||_{\dot{\mathbf{f}}^{-\alpha,q'}_\infty}$. Conversely, every $l \in (\dot{\mathbf{f}}^{\alpha,q}_1)^*$ is of the form (4.1) for some $t \in \dot{\mathbf{f}}^{-\alpha,q'}_\infty$.

Proof First, suppose that $1 \le q < \infty$ and $t \in \dot{\mathbf{f}}_\infty^{-\alpha,q'}$. For each $Q \in \mathcal{Q}$, let E_Q be the same as in the proof of Theorem 3.3, i.e.,

$$E_Q = \{x \in Q : G_Q^{-\alpha,q'}(t)(x) \le m^{-\alpha,q'}(t)(x)\}.$$

Note that $E_Q \subset Q$ and $\mu(E_Q)/\mu(Q) \ge 1-\varepsilon$, where $0 < \varepsilon < 1$ is the constant in Definition 3.1 of the local q-power function $m^{\alpha,q}$. Hence,

$$\begin{aligned}\left|\sum_{Q\in\mathcal{Q}} s_Q\overline{t_Q}\frac{\mu(Q)}{|Q|}\right| &\le \frac{1}{1-\varepsilon}\int_{\mathbb{R}^n}\sum_Q |Q|^{-\alpha}|s_Q|\tilde{\chi}_Q(x)|Q|^{\alpha}|t_Q|\tilde{\chi}_{E_Q}(x)d\mu(x)\\ &\le C\int_{\mathbb{R}^n} G^{\alpha,q}(s)(x)\left(\sum_Q(|Q|^{\alpha}|t_Q|\tilde{\chi}_{E_Q}(x))^{q'}\right)^{1/q'}d\mu(x)\\ &\le C\|s\|_{\dot{\mathbf{f}}_1^{\alpha,q}}\|m^{-\alpha,q'}(t)\|_{L^\infty} \le C\|s\|_{\dot{\mathbf{f}}_1^{\alpha,q}}\|t\|_{\dot{\mathbf{f}}_\infty^{-\alpha,q'}},\end{aligned}$$

where in the last two steps we used (3.10) and Theorem 3.3. This shows that functional l given by (4.1) satisfies $\|l\|_{(\dot{\mathbf{f}}_1^{\alpha,q})^*} \le C\|t\|_{\dot{\mathbf{f}}_\infty^{-\alpha,q'}}$ if $1 \le q < \infty$. The same is true for $0 < q < 1$ because of the trivial imbedding $\dot{\mathbf{f}}_1^{\alpha,q} \hookrightarrow \dot{\mathbf{f}}_1^{\alpha,1}$.

Conversely, take any $l \in (\dot{\mathbf{f}}_1^{\alpha,q})^*$. Then, l must be of the form (4.1) for some $t = \{t_Q\}_Q$, since sequences with finite support are dense in $\dot{\mathbf{f}}_1^{\alpha,q}$. For fixed $P \in \mathcal{Q}$ define a measure ν on $\mathcal{Q}$ by setting

$$\nu(\{Q\}) = \begin{cases}\mu(Q)/\mu(P) & \text{if } |Q\cap P| > 0,\ |Q| \le |P|,\\ 0 & \text{otherwise.}\end{cases}$$

First, suppose that $1 \le q < \infty$. By [5, Remark 3.5] and Theorem 3.6 with $r = q'$, there exists $P \in \mathcal{Q}$ such that

$$\begin{aligned}\|t\|_{\dot{\mathbf{f}}_\infty^{-\alpha,q'}} &\le C\left(\frac{1}{\mu(P)}\sum_{Q\in\mathcal{Q},\ |Q\cap P|>0,\ |Q|\le|P|}(|Q|^{\alpha-1/2}|t_Q|)^{q'}\mu(Q)\right)^{1/q'}\\ &= C\|\{|Q|^{\alpha-1/2}t_Q\}_Q\|_{\ell^{q'}(\mathcal{Q},d\nu)}\\ &\le C\sup_{\|s\|_{\ell^q(\mathcal{Q},d\nu)}\le 1}\left|\frac{1}{\mu(P)}\sum_{Q\in\mathcal{Q},\ |Q\cap P|>0,\ |Q|\le|P|} s_Q\overline{t_Q}|Q|^{\alpha-1/2}\mu(Q)\right|\\ &\le C\|l\|_{(\dot{\mathbf{f}}_1^{\alpha,q})^*}\frac{1}{\mu(P)}\sup_{\|s\|_{\ell^q(\mathcal{Q},d\nu)}\le 1}\|\{s_Q|Q|^{\alpha+1/2}\}_Q\|_{\dot{\mathbf{f}}_1^{\alpha,q}}.\end{aligned}$$

Let $\bar{P}$ be the union of neighboring cubes to P as in (3.3). Then, by Hölder's inequality

$$\begin{aligned}\|\{s_Q|Q|^{\alpha+1/2}\}_Q\|_{\dot{\mathbf{f}}_1^{\alpha,q}} &= \int_{\bar{P}}\left(\sum_{Q\in\mathcal{Q},\ |Q\cap P|>0,\ |Q|\le|P|}(|s_Q|\chi_Q(x))^q\right)^{1/q}d\mu(x)\\ &\le \mu(\bar{P})\left(\frac{1}{\mu(\bar{P})}\int_{\bar{P}}\sum_{Q\in\mathcal{Q},\ |Q\cap P|>0,\ |Q|\le|P|}(|s_Q|\chi_Q(x))^q d\mu(x)\right)^{1/q}\\ &\le \mu(\bar{P})\|s\|_{\ell^q(\mathcal{Q},d\nu)} \le \mu(\bar{P}).\end{aligned}$$

Since $\mu(\bar{P}) \le C\mu(P)$ for some $C > 0$ independent of P, we have

$$\|t\|_{\dot{\mathbf{f}}_\infty^{-\alpha,q'}} \le C\|l\|_{(\dot{\mathbf{f}}_1^{\alpha,q})^*}. \tag{4.2}$$

To show (4.2) when $0 < q < 1$, note that $q' = \infty$ and in the above argument it suffices to take extremal sequences s with only one non-zero element. More precisely, for $R \in \mathcal{Q}$ with $\nu(\{R\}) > 0$ define

$$(s^R)_Q = \begin{cases} |Q|^{\alpha+1/2}/\mu(Q) & \text{for } Q = R, \\ 0 & \text{otherwise.} \end{cases}$$

Since $||s^R||_{\dot{\mathbf{f}}_1^{\alpha,q}} = 1$ we have

$$\begin{aligned} ||t||_{\dot{\mathbf{f}}_\infty^{-\alpha,\infty}} &= \sup_{Q\in\mathcal{Q}} |Q|^{\alpha-1/2}|t_Q| = \sup_{R\in\mathcal{Q}} \left| \sum_{Q\in\mathcal{Q}} \delta_{Q,R}\overline{t_Q}|Q|^{\alpha-1/2} \right| = \sup_{R\in\mathcal{Q}} \left| \sum_{Q\in\mathcal{Q}} (s^R)_Q \overline{t_Q} \frac{\mu(Q)}{|Q|} \right| \\ &= \sup_{R\in\mathcal{Q}} |\langle s^R, t\rangle_\mu| \le ||l||_{(\dot{\mathbf{f}}_1^{\alpha,q})^*} \sup_{R\in\mathcal{Q}} ||s^R||_{\dot{\mathbf{f}}_1^{\alpha,q}} = ||l||_{(\dot{\mathbf{f}}_1^{\alpha,q})^*}, \end{aligned}$$

which proves (4.2). □

Next, we establish the general case.

Theorem 4.2 *Suppose $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, and μ is a ρ_A-doubling measure. Then,*

$$(\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu))^* \approx \begin{cases} \dot{\mathbf{f}}_{p'}^{-\alpha,q'}(A,\mu) & 1 \le p < \infty, \\ \dot{\mathbf{f}}_\infty^{-\alpha,\infty}(A) & 0 < p < 1. \end{cases} \tag{4.3}$$

More precisely, a linear functional l is bounded on $\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$ if and only if l is of the form

$$l(s) = \langle s, t\rangle_{\mu,p} := \sum_{Q\in\mathcal{Q}} s_Q \overline{t_Q} \frac{\mu(Q)^{\max(1,1/p)}}{|Q|}, \quad \text{where } s = \{s_Q\}_Q, \tag{4.4}$$

for some sequence $t = \{t_Q\}_Q$, and we have

$$||l||_{(\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu))^*} \asymp \begin{cases} ||t||_{\dot{\mathbf{f}}_{p'}^{-\alpha,q'}(A,\mu)} & 1 \le p < \infty, \\ ||t||_{\dot{\mathbf{f}}_\infty^{-\alpha,\infty}(A)} & 0 < p < 1. \end{cases} \tag{4.5}$$

We shall split the proof of Theorem 4.2 into 3 remaining cases depending on the values of $(p,q) \in (1,\infty)\times[1,\infty)$, $(1,\infty)\times(0,1)$, or $(0,1)\times(0,\infty)$.

Proof of the case $(p,q) \in (1,\infty)\times[1,\infty)$ This case is a direct adaptation of the arguments in [19]. Assume initially that $1 < p < \infty$ and $0 < q < \infty$. Then, for any $s = \{s_Q\}$ and $t = \{t_Q\}$ we have

$$\begin{aligned} |\langle s, t\rangle_\mu| &\le \left| \sum_{Q\in\mathcal{Q}} s_Q\overline{t_Q}\frac{\mu(Q)}{|Q|} \right| \le \int_{\mathbb{R}^n} \sum_Q |Q|^{-\alpha}|s_Q|\tilde{\chi}_Q|Q|^\alpha|t_Q|\tilde{\chi}_Q d\mu(x) \\ &\le \int_{\mathbb{R}^n} \left(\sum_Q (|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q)^q\right)^{1/q} \left(\sum_Q (|Q|^\alpha|t_Q|\tilde{\chi}_Q)^{q'}\right)^{1/q'} d\mu(x) \\ &\le ||s||_{\dot{\mathbf{f}}_p^{\alpha,q}} ||t||_{\dot{\mathbf{f}}_{p'}^{-\alpha,q'}}. \end{aligned} \tag{4.6}$$

In the penultimate step we used Hölder's inequality if $q \ge 1$ and the imbedding $\ell^q \hookrightarrow \ell^1$ if $0 < q < 1$. The last step follows from Hölder's inequality.

To prove the converse we will use the following fact, see [43, Proposition 2.11.1].

Proposition 4.3 *Let (X, ν) be a measure space. For $1 < p < \infty$, $0 < q < \infty$, let $L^p(\ell^q)$ consists of sequences $f = \{f_j\}_{j\in\mathbb{Z}}$ of ν-measurable functions on X with the norm*

$$||f||_{L^p(\ell^q)} = \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{1/q} \right\|_{L^p(X,\nu)} < \infty.$$

Then, $(L^p(\ell^q))^* = L^{p'}(\ell^{q'})$ with the pairing

$$\langle f, g\rangle := \int_X \sum_{j\in\mathbb{Z}} f_j(x)\overline{g_j(x)}d\nu(x) \quad \textit{for } f \in L^p(\ell^q),\ g \in L^{p'}(\ell^{q'}). \tag{4.7}$$

We will apply Proposition 4.3 when $X = \mathbb{R}^n$ and the measure $\nu = \mu$. Let $I : \dot{\mathbf{f}}_p^{\alpha,q} \to L^p(l^q)$ be given by

$$I(s) = \{f_j(s)\}_{j\in\mathbb{Z}}, \quad \text{where } f_j(s) = \sum_{Q\in\mathcal{Q},\ \text{scale}(Q)=j} |Q|^{-\alpha} s_Q \tilde{\chi}_Q.$$

Clearly, I is an isometry. Let $P : L^{p'}(\ell^{q'}) \to \dot{\mathbf{f}}_{p'}^{-\alpha,q'}$ be given by

$$P(\{g_j\}_{j\in\mathbb{Z}}) = \{t_Q\}_{Q\in\mathcal{Q}}, \quad \text{where } t_Q = \frac{|Q|^{-\alpha+1/2}}{\mu(Q)} \int_Q g_j(x)d\mu(x),\ \text{scale}(Q) = j.$$

Note that $|t_Q| \le C|Q|^{-\alpha+1/2} M_{\rho_A} g_j(x)$ for $x \in Q$. Since $1 < p' < \infty$, Theorem 2.2 yields

$$||P(g)||_{\dot{\mathbf{f}}_{p'}^{-\alpha,q'}} \le C||\{M_{\rho_A} g_j\}_{j\in\mathbb{Z}}||_{L^{p'}(\ell^{q'})} \le C||\{g_j\}_{j\in\mathbb{Z}}||_{L^{p'}(\ell^{q'})}.$$

Thus, P is bounded.

Take any $l \in (\dot{\mathbf{f}}_p^{\alpha,q})^*$. By the Hahn-Banach Theorem, there exists $\tilde{l} \in (L^p(\ell^q))^*$ such that $\tilde{l} \circ I = l$ and $||\tilde{l}|| = ||l||$. Note that this step requires that $q \ge 1$. By Proposition 4.3, $\tilde{l}(f) = \langle f, g\rangle$ for some $g \in L^{p'}(\ell^{q'})$, where the pairing is given by (4.7). Define $t \in \dot{\mathbf{f}}_{p'}^{-\alpha,q'}$ by $t = P(g)$. Then,

$$\begin{aligned} l(s) = \tilde{l}(I(s)) = \langle I(s), g\rangle &= \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \left(\sum_{Q\in\mathcal{Q},\ \text{scale}(Q)=j} |Q|^{-\alpha} s_Q \tilde{\chi}_Q(x) \right) \overline{g_j(x)} d\mu(x) \\ &= \sum_{Q\in\mathcal{Q}} s_Q \overline{t_Q} \mu(Q)/|Q| = \langle s, t\rangle_\mu. \end{aligned}$$

Moreover, $||l||_{(\dot{\mathbf{f}}_p^{\alpha,q})^*} = ||g||_{L^{p'}(\ell^{q'})} \ge C^{-1}||t||_{\dot{\mathbf{f}}_{p'}^{-\alpha,q'}}$, which proves (4.5). □

Remark 4.1 The estimate (4.6) shows that for $p > 1$ and $0 < q < 1$, we have a continuous imbedding

$$\dot{\mathbf{f}}_{p'}^{-\alpha,\infty}(A, \mu) \hookrightarrow (\dot{\mathbf{f}}_p^{\alpha,q}(A, \mu))^*. \tag{4.8}$$

Frazier and Jawerth claim [19, Remark 5.14] that the above imbedding is an isomorphism identifying the dual of $\dot{\mathbf{f}}_p^{\alpha,q}$. Unfortunately, their argument uses the Hahn-Banach Theorem for $L^p(\ell^q)$, which is not a normed space when $p > 1$ and $0 < q < 1$. Hence, we must deal with this case separately.

In order to deal with the special case of $(p, q) \in (1, \infty) \times (0, 1)$, we will need the following extension of a result due to Verbitsky [47,48].

Theorem 4.4 *Suppose that μ is a ρ_A-doubling measure on $\mathbb{R}^n$ and $\{c_Q\}_{Q\in\mathcal{Q}'}$ is any positive sequence, where $\mathcal{Q}'\subset\mathcal{Q}$ is any subfamily of dilated cubes.*

(i) *Suppose* $0<p<r\le q\le\infty$. *Then, the inequality*

$$\left\| \left(\sum_{Q\in\mathcal{Q}'} |s_Q|^q (c_Q)^q \chi_Q \right)^{1/q} \right\|_{L^p(\mu)} \le C||s||_{\ell^r} \tag{4.9}$$

holds for all scalar sequences $s=\{s_Q\}_{Q\in\mathcal{Q}'}$ if and only if

$$\int_{\mathbb{R}^n} \sup_{Q\in\mathcal{Q}',\ x\in Q} ((c_Q)^r \mu(Q))^{p/(r-p)} d\mu(x) < \infty. \tag{4.10}$$

(ii) *Suppose* $0<q\le r<p<\infty$. *Then, the inequality*

$$\left\| \left(\sum_{Q\in\mathcal{Q}'} |s_Q|^q (c_Q)^q \chi_Q \right)^{1/q} \right\|_{L^p(\mu)} \ge C||s||_{\ell^r} \tag{4.11}$$

holds for all scalar sequences $s=\{s_Q\}_{Q\in\mathcal{Q}'}$ if and only if (4.10) *holds.*

Remark 4.2 Theorem 4.4 provides a characterization of imbeddings of weighted ℓ^r spaces into $\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$ and imbeddings of $\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$ into weighted ℓ^r spaces in parts (i) and (ii), resp. This result was established by Verbitsky [48, Theorems 3 and 4] in the dyadic case $A=2Id$, where μ is a locally finite measure on $\mathbb{R}^n$ (and hence not necessarily doubling). Moreover, Verbitsky's characterization works for the entire range of exponents $0<p,r<\infty$, $0<q\le\infty$, which splits into four fundamental subcases each for forward and reverse imbeddings of $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces into ℓ^r spaces. However, for our purposes we will need only the two special cases listed in Theorem 4.4.

Unlike [48], note the presence of the subfamily $\mathcal{Q}'\subset\mathcal{Q}$. While in part (i) we could dispense with $\mathcal{Q}'$ by allowing c_Q to be non-negative, this subfamily is essential in part (ii) since the exponent $p/(r-p)$ in (4.10) is negative. Finally, we remark that (ii) is actually equivalent to the duality of $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces. This can be seen in the proof of Theorem 4.2 in the case of $(p,q)\in(1,\infty)\times(0,1)$.

To establish Theorem 4.4 we will follow the original approach of Verbitsky. Thus, we will need the following theorem of Pisier [38, Theorem 1.1] about factorization through weak-L^r spaces.

Theorem 4.5 *Suppose that μ is a positive measure on X. Let $0<p<r<\infty$, I be any index set, and let $\{\phi\}_{i\in I}$ be a family in $L^p(\mu)$. Then, the inequality*

$$\left\| \sup_{i\in I}(|s_i|\phi_i) \right\|_{L^p(\mu)} \le C||s||_{\ell^r} \tag{4.12}$$

holds for all scalar sequences $s=\{s_i\}_{i\in I}\in\ell^r$ if and only if there exists a non-negative measurable function $F\ge 0$ with $\int_X f\,d\mu\le 1$, such that

$$\sup_{i\in I} ||F^{-1/p}\phi_i||_{L^{r,\infty}(\mu_0)} < \infty, \tag{4.13}$$

where $L^{r,\infty}(\mu_0)$ is a weak-L^r space with respect to the measure $d\mu_0=F\,d\mu$.

Recall that a measurable function f is in $L^{r,\infty}(\mu_0)$ if

$$||f||_{L^{r,\infty}(\mu_0)} = \left(\sup_{t>0} t^r \mu_0(\{x \in X : |f(x)| > t\})\right)^{1/r} < \infty.$$

We shall apply Theorem 4.5 in a special setting, where μ is a ρ_A-doubling measure on $\mathbb{R}^n$ and the family of functions $\{\phi_Q\}_{Q\in\mathcal{Q}'}$ is of the form $\phi_Q = c_Q\chi_Q$, where $c_Q > 0$.

Proof of Theorem 4.4(i) The implication (4.10) $\implies$ (4.9) is a direct consequence of a theorem due to Verbitsky [48, Theorem 1(i)], which holds in a general setting of measure spaces as Theorem 4.5. We note here that this direction will not be utilized further in the paper. To show the converse implication we shall reproduce Verbitsky's argument.

Suppose that (4.9) holds for $p < r$. By Theorem 4.5 and imbedding $\ell^q \hookrightarrow \ell^\infty$, there exists $F \geq 0$ with $\int F d\mu \leq 1$ such that

$$\sup_{Q\in\mathcal{Q}'} c_Q||F^{-1/p}\chi_Q||_{L^{r,\infty}(\mu_0)} < \infty, \tag{4.14}$$

where the measure μ_0 is given by $d\mu_0 = F d\mu$. Let $f = F^{-1/p}\chi_Q$. Note that $||f||_{L^p(\mu_0)} = \mu(Q)^{1/p}$. Suppose that $p < s < r$, and $1/s = t/p + (1-t)/r$, where $0 < t < 1$. By the elementary interpolation inequality, see [23, Proposition 1.1.14]

$$||f||_{L^s(\mu_0)} \leq C||f||^t_{L^{p,\infty}(\mu_0)}||f||^{1-t}_{L^{r,\infty}(\mu_0)}.$$

Hence, for any $Q \in \mathcal{Q}'$

$$\left(\int_Q F^{-s/p+1} d\mu\right)^{1/s} \leq C\mu(Q)^{t/p}||F^{-1/p}\chi_Q||^{1-t}_{L^{r,\infty}(\mu_0)}. \tag{4.15}$$

Letting $\delta = s/p - 1 > 0$ and combining (4.14) and (4.15), we have

$$\left(c_Q\mu(Q)^{1/r}\right)^{pr/(r-p)}\left(\frac{1}{\mu(Q)}\int_Q F^{-\delta} d\mu\right)^{1/\delta} \leq C < \infty.$$

On the other hand, Hölder's inequality yields

$$\left(\frac{1}{\mu(Q)}\int_Q F^{-\delta} d\mu\right)^{1/\delta}\left(\frac{1}{\mu(Q)}\int_Q F^{\varepsilon} d\mu\right)^{1/\varepsilon} \geq 1$$

for all $\delta, \varepsilon > 0$. Therefore,

$$\left(c_Q\mu(Q)^{1/r}\right)^{pr/(r-p)} \leq C\left(\frac{1}{\mu(Q)}\int_Q F^{\varepsilon} d\mu\right)^{1/\varepsilon} \leq C(M(F^\varepsilon)(x))^{1/\varepsilon} \quad \text{for } x \in Q,$$

where M denotes the maximal operator defined over $\mathcal{Q}$ rather than $\mathcal{B}$. Since the maximal operator M is bounded on $L^{1/\varepsilon}(\mu)$ for $0 < \varepsilon < 1$, we have

$$\int_{\mathbb{R}^n} \sup_{x\in Q}((c_Q)^r\mu(Q))^{p/(r-p)} d\mu(x) \leq C||M(F^\varepsilon)||^{1/\varepsilon}_{L^{1/\varepsilon}(\mu)} \leq C||F^\varepsilon||^{1/\varepsilon}_{L^{1/\varepsilon}(\mu)} \leq C < \infty.$$

This shows part (i) of Theorem 4.4. □

Proof of Theorem 4.4(ii) Suppose that (4.11) holds and $0 < q \le r < p < \infty$. We shall modify the original argument of Verbitsky [48] by taking advantage of the already established duality of $\dot{\mathbf{f}}_p^{0,1}$ spaces, where $p > 1$. The usual duality of $\ell^{r/q}$ yields

$$||s||_{\ell^r} = \sup_{t=\{t_Q\}} \frac{(\sum_Q |s_Q|^q |t_Q|^q)^{1/q}}{||t||_{\ell^{rq/(r-q)}}}.$$

Hence, (4.11) is equivalent to the inequality

$$\left(\sum_Q |s_Q|^q |t_Q|^q\right)^{1/q} \le C \left\| \left(\sum_{Q\in\mathcal{Q}} |s_Q|^q (c_Q)^q \chi_Q\right)^{1/q} \right\|_{L^p(\mu)} ||t||_{\ell^{rq/(r-q)}}. \tag{4.16}$$

On the other hand, by the already established duality $(\dot{\mathbf{f}}_{p/q}^{-1/2,1})^* = \dot{\mathbf{f}}_{p/(p-q)}^{1/2,\infty}$, where $1 < p/q < \infty$, we have that

$$\sup_{u=\{u_Q\}} \frac{\left|\sum_Q u_Q \overline{v_Q} \frac{\mu(Q)}{|Q|}\right|}{\left\|\left(\int \sum_Q |u_Q|^q \chi_Q\right)^{1/q}\right\|_{L^{p/q}(\mu)}}
= \sup_{u=\{u_Q\}} \frac{|\langle u, v\rangle_\mu|}{||u||_{\dot{\mathbf{f}}_{p/q}^{-1/2,1}}} \asymp ||v||_{\dot{\mathbf{f}}_{p/(p-q)}^{1/2,\infty}} = \left\| \sup_{x\in Q} |v_Q| \right\|_{L^{p/(p-q)}(\mu)}.$$

Letting

$$v_Q = \begin{cases} (t_Q)^q (c_Q)^{-q} \mu(Q)^{-1} & Q \in \mathcal{Q}', \\ 0 & Q \in \mathcal{Q} \setminus \mathcal{Q}', \end{cases}$$

and considering $u = \{u_Q\}$ of the form $u_Q = |s_Q|^q (c_Q)^q$ if $Q \in \mathcal{Q}'$ and anything if $Q \notin \mathcal{Q}'$, we have that

$$\sup_{s=\{s_Q\}} \frac{\left(\sum_Q |s_Q|^q |t_Q|^q\right)^{1/q}}{\left\|\left(\sum_{Q\in\mathcal{Q}'} |s_Q|^q (c_Q)^q \chi_Q\right)^{1/q}\right\|_{L^{p/q}(\mu)}}
\asymp \left(\int \sup_{Q\in\mathcal{Q}',\, x\in Q} ((t_Q)^q (c_Q)^{-q} \mu(Q)^{-1})^{p/(p-q)} d\mu(x)\right)^{pq/(p-q)}. \tag{4.17}$$

Let $p_1 = pq/(p-q)$, $r_1 = rq/(r-q)$, and $\tilde{c}_Q = (c_Q)^{-1}\mu(Q)^{-1/q}$. Combining (4.16) with (4.17) yields

$$\left(\int_{\mathbb{R}^n} \sup_{x\in Q} (\tilde{c}_Q t_Q)^{p_1} d\mu(x)\right)^{1/p_1} \le C||t||_{\ell^{r_1}}.$$

Finally, it suffices to apply Theorem 4.4(i) for $0 < p_1 < r_1 < q_1 = \infty$. Using $p_1 r_1/(r_1 - p_1) = pr/(p-r)$, the condition (4.10) (with p_1 and r_1 in place of p and r) yields

$$\int_{\mathbb{R}^n} \sup_{x\in Q} ((\tilde{c}_Q)^{r_1} \mu(Q))^{p_1/(r_1-p_1)} d\mu(x) = \int_{\mathbb{R}^n} \sup_{x\in Q} ((c_Q)^r \mu(Q))^{p/(r-p)} d\mu(x) < \infty.$$

Hence, (4.10) hold. The converse implication (4.10) $\implies$ (4.9) is a direct consequence of a theorem due to Verbitsky [48, Theorem 1(ii)], which holds in a general setting of measure spaces. Alternatively, one can show that this is a consequence of (4.8) using a reformulation argument as below. Since this direction will not be utilized further in the paper, we skip the details. This completes the proof of Theorem 4.4(ii). □

Proof of the case $(p, q) \in (1, \infty) \times (0, 1)$ Take any $l \in (\dot{\mathbf{f}}_p^{\alpha,q})^*$. Then, l must be of the form (4.4) for some $t = \{t_Q\}$ since sequences with finite support are dense in $\dot{\mathbf{f}}_p^{\alpha,q}$. Hence,

$$|l(s)| = |\langle s, t\rangle_\mu| = \left| \sum_{Q\in\mathcal{Q}} s_Q \overline{t_Q} \frac{\mu(Q)}{|Q|} \right| \le C||s||_{\dot{\mathbf{f}}_p^{\alpha,q}} \quad \text{for all } s = \{s_Q\}. \tag{4.18}$$

Define $\mathcal{Q}' = \{Q : t_Q \neq 0\}$, and let $u_Q = s_Q\overline{t_Q}\mu(Q)/|Q|$, $c_Q = |t_Q|^{-1}\mu(Q)^{-1}|Q|^{1/2-\alpha}$ for $Q \in \mathcal{Q}'$. Then, (4.18) can be rewritten as

$$||u||_{\ell^1} \le C \left\| \left(\sum_{Q\in\mathcal{Q}} |u_Q|^q (c_Q)^q \chi_Q \right)^{1/q} \right\|_{L^p(\mu)} \quad \text{for all } u = \{u_Q\}.$$

Theorem 4.4(ii) with $0 < q < r = 1 < p < \infty$, yields

$$\int_{\mathbb{R}^n} \sup_{Q\in\mathcal{Q}', x\in Q} (c_Q\mu(Q))^{p/(1-p)} d\mu(x)$$
$$= \int_{\mathbb{R}^n} \sup_{Q\in\mathcal{Q}, x\in Q} (|t_Q||Q|^{-1/2+\alpha})^{p/(p-1)} d\mu(x) = ||t||_{\dot{\mathbf{f}}_{p'}^{-\alpha,\infty}} < \infty,$$

where $1/p + 1/p' = 1$. This completes the proof of Theorem 4.2 in the case $1 < p < \infty$. □

Proof of the case $0 < p < 1$ Suppose that $t \in \dot{\mathbf{f}}_\infty^{-\alpha,\infty}$. Then, by Theorem 4.1 and Lemma 4.6,

$$|\langle s, t\rangle_{\mu,p}| = \left| \sum_{Q\in\mathcal{Q}} s_Q \overline{t_Q} \frac{\mu(Q)^{1/p}}{|Q|} \right| = \left| \sum_{Q\in\mathcal{Q}} s_Q \mu(Q)^{1/p-1} \overline{t_Q} \frac{\mu(Q)}{|Q|} \right|$$
$$\le C||\{s_Q\mu(Q)^{1/p-1}\}_Q||_{\dot{\mathbf{f}}_1^{\alpha,1}} ||t||_{\dot{\mathbf{f}}_\infty^{-\alpha,\infty}} \le C||s||_{\dot{\mathbf{f}}_p^{\alpha,q}} ||t||_{\dot{\mathbf{f}}_\infty^{-\alpha,\infty}}.$$

Lemma 4.6 *Suppose* $\alpha \in \mathbb{R}$, $0 < p < 1$, $0 < q < \infty$. *Define the sequence space* X_p^α *as the collection of* $s = \{s_Q\}_Q$ *such that*

$$||s||_{X_p^\alpha} =: ||\{s_Q\mu(Q)^{1/p-1}\}_Q||_{\dot{\mathbf{f}}_1^{\alpha,1}(A,\mu)} < \infty.$$

Then, we have continuous imbeddings

$$\dot{\mathbf{f}}_p^{\alpha,q}(A, \mu) \hookrightarrow \dot{\mathbf{f}}_p^{\alpha,\infty}(A, \mu) \hookrightarrow X_p^\alpha.$$

Proof The first imbedding is trivial since $||s||_{\dot{\mathbf{f}}_p^{\alpha,\infty}} \le ||s||_{\dot{\mathbf{f}}_p^{\alpha,q}}$. Observe also that

$$||s||_{X_p^\alpha} = \sum_{Q\in\mathcal{Q}} |Q|^{-\alpha-1/2}|s_Q|\mu(Q)^{1/p}.$$

By [5, Theorem 6.3] we can decompose any $s \in \dot{\mathbf{f}}_p^{\alpha,\infty}$ into ∞-atoms $\{r_j\}$ as

$$s = \sum_j \lambda_j r_j, \quad \text{where} \left(\sum_j |\lambda_j|^p \right)^{1/p} \le C||s||_{\dot{\mathbf{f}}_p^{\alpha,\infty}}.$$

Recall from [5, Sect. 6] that r is an ∞-atom for $\dot{\mathbf{f}}_p^{\alpha,\infty}$ supported near a cube $\bar{Q} \in \mathcal{Q}$, if

$$\begin{aligned} r_Q = 0 \quad &\text{if } \operatorname{scale}(Q) > \operatorname{scale}(\bar{Q}) \text{ or } |Q \cap \bar{Q}| = 0, \\ &||G^{\alpha,\infty}(r)||_{L^\infty} \le \mu(\bar{Q})^{-1/p}. \end{aligned}$$

Hence,

$$|(r_j)_Q| \le \begin{cases} |Q|^{\alpha+1/2}\mu(\bar{Q})^{-1/p} & \text{for } |Q| \le |\bar{Q}|,\ |Q \cap \bar{Q}| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$||r_j||_{X_p^\alpha} \le \sum_{Q \in \mathcal{Q},\ |Q| \le |\bar{Q}|,\ |Q \cap \bar{Q}| > 0} (\mu(Q)/\mu(\bar{Q}))^{1/p} \le C < \infty$$

by Lemma 2.5. Thus,

$$||s||_{X_p^\alpha} \le \sum_j |\lambda_j| ||r_j||_{X_p^\alpha} \le C\Big(\sum_j |\lambda_j|^p\Big)^{1/p} \le C||s||_{\dot{\mathbf{f}}_p^{\alpha,\infty}},$$

which completes the proof of Lemma 4.6. □

To show the converse direction in Theorem 4.2, take any $l \in (\dot{\mathbf{f}}_p^{\alpha,q})^*$. Then, l must be of the form (4.4) for some $t = \{t_Q\}_Q$, since sequences with finite support are dense in $\dot{\mathbf{f}}_p^{\alpha,q}$. For each $R \in \mathcal{Q}$ define the standard basis sequences s^R by $(s^R)_Q = 1$ if $Q = R$ and 0 otherwise. Then,

$$\begin{aligned} ||t||_{\dot{\mathbf{f}}_\infty^{-\alpha,\infty}} &= \sup_{Q \in \mathcal{Q}} |Q|^{\alpha-1/2}|t_Q| = \sup_{R \in \mathcal{Q}} \Big| \sum_{Q \in \mathcal{Q}} (s^R)_Q \overline{t_Q} |Q|^{\alpha-1/2} \Big| \\ &\le ||l||_{(\dot{\mathbf{f}}_p^{\alpha,q})^*} \sup_{R \in \mathcal{Q}} ||\{(s^R)_Q |Q|^{\alpha+1/2}\mu(Q)^{-1/p}\}_Q||_{\dot{\mathbf{f}}_p^{\alpha,q}} = ||l||_{(\dot{\mathbf{f}}_p^{\alpha,q})^*}, \end{aligned}$$

which completes the proof of Theorem 4.2. □

In the unweighted case, Theorem 4.2 takes a more familiar look for $0 < p < 1$ due to a simple rescaling of duality pairing (4.4) into the usual scalar product (4.20).

Corollary 4.7 *Suppose $\alpha \in \mathbb{R}$ and $0 < p, q < \infty$. Then*

$$(\dot{\mathbf{f}}_p^{\alpha,q})^* \approx \begin{cases} \dot{\mathbf{f}}_{p'}^{-\alpha,q'} & 1 \le p < \infty, \\ \dot{\mathbf{f}}_\infty^{-\alpha+(1/p-1),\infty} & 0 < p < 1. \end{cases} \tag{4.19}$$

More precisely, a linear functional l is bounded on $\dot{\mathbf{f}}_p^{\alpha,q}$ if and only if l is of the form

$$l(s) = \langle s, t\rangle := \sum_{Q \in \mathcal{Q}} s_Q \overline{t_Q}, \quad \text{where } s = \{s_Q\}_Q, \tag{4.20}$$

for some sequence $t = \{t_Q\}_Q$, and we have

$$||l||_{(\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu))^*} \asymp \begin{cases} ||t||_{\dot{\mathbf{f}}_{p'}^{-\alpha,q'}} & 1 \le p < \infty, \\ ||t||_{\dot{\mathbf{f}}_\infty^{-\alpha+(1/p-1),\infty}} & 0 < p < 1. \end{cases} \tag{4.21}$$

Corollary 4.7 enables us to prove the duality for unweighted $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces. Recall that

$$\mathcal{S}_0 = \mathcal{S}_0(\mathbb{R}^n) = \{\varphi \in \mathcal{S} : \int \varphi(x) x^\alpha dx = 0 \quad \text{for all multi-indices } \alpha\}.$$

If $0 < p, q < \infty$, then $\mathcal{S}_0$ is a dense subspace of $\dot{\mathbf{F}}_p^{\alpha,q}$ using Theorem 2.7 and the fact that finite sequences are dense in $\dot{\mathbf{f}}_p^{\alpha,q}$.

Theorem 4.8 *Suppose $\alpha \in \mathbb{R}$, $0 < p, q < \infty$. Then*

$$(\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A))^* \approx \begin{cases} \dot{\mathbf{F}}_{p'}^{-\alpha,q'}(\mathbb{R}^n, A) & 1 \le p < \infty, \\ \dot{\mathbf{F}}_\infty^{-\alpha+(1/p-1),\infty}(\mathbb{R}^n, A) & 0 < p < 1. \end{cases} \tag{4.22}$$

More precisely, l is a bounded linear functional on $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A)$ if and only if l is of the form

$$l(f) = \langle f, g\rangle := \overline{g}(f) \quad \text{for } f \in \mathcal{S}_0, \tag{4.23}$$

for some $g \in \dot{\mathbf{F}}_{p'}^{-\alpha,q'}$ if $1 \le p < \infty$ or $g \in \dot{\mathbf{F}}_\infty^{-\alpha+(1/p-1),\infty}$ if $0 < p < 1$. Moreover, we have

$$||l||_{(\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A))^*} \asymp \begin{cases} ||g||_{\dot{\mathbf{F}}_{p'}^{-\alpha,q'}(\mathbb{R}^n,A)} & 1 \le p < \infty, \\ ||g||_{\dot{\mathbf{F}}_\infty^{-\alpha+(1/p-1),\infty}(\mathbb{R}^n,A)} & 0 < p < 1. \end{cases} \tag{4.24}$$

The proof follows the original argument of Frazier and Jawerth [19, Theorem 5.13], which is included for completeness.

Proof Suppose that $\varphi = \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (2.6) and (2.7). Thus, it suffices to choose $\varphi = \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\operatorname{supp} \hat{\varphi} \subset [-\pi, \pi]^n \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} |\hat{\varphi}((A^*)^j \xi)|^2 = 1 \quad \text{for all } \xi \ne 0.$$

By Lemma 2.6 we have that for any $f \in \mathcal{S}_0$ and $g \in \mathcal{S}'/\mathcal{P}$,

$$\langle f, g\rangle = \left\langle f, \sum_{Q \in \mathcal{Q}} \langle g, \varphi_Q\rangle \varphi_Q \right\rangle = \sum_{Q \in \mathcal{Q}} \langle f, \varphi_Q\rangle\langle\varphi_Q, g\rangle = \langle S_\varphi f, S_\varphi g\rangle.$$

Suppose that $1 \le p < \infty$. Then, by Corollary 4.7 and Theorem 2.7

$$|\langle f, g\rangle| = |\langle S_\varphi f, S_\varphi g\rangle| \le C||S_\varphi f||_{\dot{\mathbf{f}}_p^{\alpha,q}} ||S_\varphi g||_{\dot{\mathbf{f}}_{p'}^{-\alpha,q'}} \le C||f||_{\dot{\mathbf{F}}_p^{\alpha,q}} ||g||_{\dot{\mathbf{F}}_{p'}^{-\alpha,q'}}.$$

Conversely, suppose that $l \in (\dot{\mathbf{F}}_p^{\alpha,q})^*$. Then, $l_1 = l \circ T_\varphi \in (\dot{\mathbf{f}}_p^{\alpha,q})^*$ and by Corollary 4.7, there exists $t \in \dot{\mathbf{f}}_{p'}^{-\alpha,q'}$ such that

$$l_1(s) = \langle s, t\rangle, \quad \text{for } s \in \dot{\mathbf{f}}_p^{\alpha,q}, \quad \text{and } ||l_1||_{(\dot{\mathbf{f}}_p^{\alpha,q})^*} \asymp ||t||_{\dot{\mathbf{f}}_{p'}^{-\alpha,q'}}.$$

Let $g = T_\varphi t = \sum_Q t_Q \varphi_Q$. By Theorem 2.7,

$$||g||_{\dot{\mathbf{f}}_{p'}^{-\alpha,q'}} \le C||t||_{\dot{\mathbf{f}}_{p'}^{-\alpha,q'}} \le C||l_1||_{(\dot{\mathbf{f}}_p^{\alpha,q})^*} \le ||l||_{(\dot{\mathbf{F}}_p^{\alpha,q})^*}.$$

Since $T_\varphi \circ S_\varphi$ is the identity on $\dot{\mathbf{F}}_p^{\alpha,q}$, hence

$$l(f) = l_1 \circ S_\varphi(f) = \langle S_\varphi f, t\rangle = \left\langle f, \sum_Q t_Q \varphi_Q \right\rangle = \langle f, g\rangle \quad \text{for } f \in \mathcal{S}_0.$$

Since the case $0 < p < 1$ is dealt in an identical way, this completes the proof of Theorem 4.8. $\square$

As a consequence of Theorem 4.8 we obtain the explicit pairing procedure between elements of $\dot{\mathbf{F}}_p^{\alpha,q}$ and its dual space.

Corollary 4.9 *Suppose that (φ, ψ) is an admissible pair of dual frame wavelets. Then, l is a bounded linear functional on $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A)$ if and only if*

$$l(f) = l_g(f) = \langle S_\psi f, S_\varphi g\rangle = \sum_{Q\in\mathcal{Q}} \langle f, \psi_Q\rangle\langle\varphi_Q, g\rangle \quad \text{for } f \in \dot{\mathbf{F}}_p^{\alpha,q}, \tag{4.25}$$

for some $g \in \dot{\mathbf{F}}_{p'}^{-\alpha,q'}$ if $1 \le p < \infty$ or $g \in \dot{\mathbf{F}}_\infty^{-\alpha+(1/p-1),\infty}$ if $0 < p < 1$. Moreover, the duality pairing $l_g(f)$ is independent of the choice of such (φ, ψ).

Proof By Theorem 4.8 we know that bounded linear functionals l on $\dot{\mathbf{F}}_p^{\alpha,q}$ are of the form

$$l(f) = \langle f, g\rangle = \left\langle f, \sum_{Q\in\mathcal{Q}} \langle g, \varphi_Q\rangle\psi_Q \right\rangle = \sum_{Q\in\mathcal{Q}} \langle f, \psi_Q\rangle\langle\varphi_Q, g\rangle = \langle S_\psi f, S_\varphi g\rangle \quad \text{for } f \in \mathcal{S}_0,$$

where g belongs to the appropriate dual space. Here, we used Lemma 2.6 which also implies that $l(f)$ is independent of the choice of (φ, ψ). By Corollary 4.7 and Theorem 2.7

$$|l(f)| = |\langle S_\psi f, S_\varphi g\rangle| \le C||S_\psi f||_{\dot{\mathbf{f}}_p^{\alpha,q}} ||S_\varphi g||_{\dot{\mathbf{f}}_{p'}^{-\alpha,q'}} \le C||f||_{\dot{\mathbf{F}}_p^{\alpha,q}} ||g||_{\dot{\mathbf{F}}_{p'}^{-\alpha,q'}}.$$

Hence, since $\mathcal{S}_0 \subset \dot{\mathbf{F}}_p^{\alpha,q}$ is dense, (4.25) must hold for all $f \in \dot{\mathbf{F}}_p^{\alpha,q}$. $\square$

The duality of weighted $\dot{\mathbf{F}}_p^{\alpha,q}$-spaces has a different form due to the form of pairing, which must involve the underlying measure μ. Example 4.1 shows that this is to be expected.

We say that $\Psi = \{\psi^1, \dots, \psi^L\} \subset \mathcal{S}(\mathbb{R}^n)$ is a Meyer-type orthonormal wavelet associated with dilation A, if for each $\psi \in \Psi$, supp $\hat{\psi}$ is compact and bounded away from the origin, and the wavelet system $\{\psi_Q : Q \in \mathcal{Q}, \ \psi \in \Psi\}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$, see [7,34].

Theorem 4.10 *Suppose $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, and μ is a ρ_A-doubling measure. Suppose that there exists a Meyer-type orthonormal wavelet Ψ for the dilation A. Then*

$$(\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu))^* \approx \begin{cases} \dot{\mathbf{F}}_{p'}^{-\alpha,q'}(\mathbb{R}^n, A, \mu) & 1 \le p < \infty, \\ \dot{\mathbf{F}}_\infty^{-\alpha,\infty}(\mathbb{R}^n, A) & 0 < p < 1. \end{cases} \tag{4.26}$$

More precisely, a linear functional l is bounded on $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ if and only if l is of the form

$$l(f) = \sum_{\psi\in\Psi} \sum_{Q\in\mathcal{Q}} \langle f, \psi_Q\rangle\langle\psi_Q, g\rangle \frac{\mu(Q)^{\max(1,1/p)}}{|Q|}, \tag{4.27}$$

for some $g \in \dot{\mathbf{F}}_{p'}^{-\alpha,q'}$ if $1 \le p < \infty$ or $g \in \dot{\mathbf{F}}_\infty^{-\alpha,\infty}$ if $0 < p < 1$, and we have

$$||l||_{(\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu))^*} \asymp \begin{cases} ||g||_{\dot{\mathbf{F}}_{p'}^{-\alpha,q'}(\mathbb{R}^n,A,\mu)} & 1 \le p < \infty, \\ ||g||_{\dot{\mathbf{F}}_\infty^{-\alpha,\infty}(\mathbb{R}^n,A)} & 0 < p < 1. \end{cases} \tag{4.28}$$

Proof By Theorem 2.7, the analysis transform $S_\Psi f = \{\langle f, \psi_Q\rangle\}_{Q\in\mathcal{Q},\ \psi\in\Psi}$ is bounded between $\dot{\mathbf{F}}_p^{\alpha,q}$ and the direct sum $\oplus_1^L \dot{\mathbf{f}}_p^{\alpha,q}$ of L copies of $\dot{\mathbf{f}}_p^{\alpha,q}$. Likewise, the synthesis transform $T_\Psi(s) = \sum_{\psi\in\Psi,\ Q\in\mathcal{Q}} s_Q^\psi \psi_Q$ is bounded between $\oplus_1^L \dot{\mathbf{f}}_p^{\alpha,q}$ and $\dot{\mathbf{F}}_p^{\alpha,q}$. Since Ψ is an orthonormal wavelet, $T_\Psi \circ S_\Psi$ is an identity on $\mathcal{S}_0 \subset L^2(\mathbb{R}^n)$ and it is also an identity on $\dot{\mathbf{F}}_p^{\alpha,q}$. Moreover, T_Ψ is also 1-to-1 due to orthogonality of the wavelet system generated by Ψ.

Consequently, S_Ψ and T_Ψ are isomorphisms and there is a 1-to-1 correspondence between linear functionals on $\dot{\mathbf{F}}_p^{\alpha,q}$ and $\dot{\mathbf{f}}_p^{\alpha,q}$. That is, for any $l \in (\dot{\mathbf{F}}_p^{\alpha,q})^*$ we associate $\tilde{l} = l \circ T_\Psi \in (\dot{\mathbf{f}}_p^{\alpha,q})^*$. Conversely, any $\tilde{l} \in (\dot{\mathbf{f}}_p^{\alpha,q})^*$ corresponds to $l = \tilde{l} \circ S_\Psi \in (\dot{\mathbf{F}}_p^{\alpha,q})^*$. Therefore, Corollary 4.7 immediately implies the duality (4.26) with the pairing (4.27). □

Note that Theorem 4.10 applies only if the dilation A admits Meyer-type wavelets. For the results about existence and non-existence of such wavelets we refer the reader to [3,7]. In the case when A does not admit Meyer-type wavelets, it is not clear whether the conclusion of Theorem 4.10 still holds.

Question 4.1 Let (φ, ψ) be an admissible pair of dual frame wavelets. Is any bounded linear functional l on $\dot{\mathbf{F}}_p^{\alpha,q}$ is of the form

$$l(f) = \sum_{Q\in\mathcal{Q}} \langle f, \phi_Q\rangle\langle\psi_Q, g\rangle \frac{\mu(Q)^{\max(1,1/p)}}{|Q|}, \tag{4.29}$$

for some for some g in the appropriate dual space such that (4.28) holds?

While this question remains open, as the last resort one can always describe the dual of $\dot{\mathbf{F}}_p^{\alpha,q}$ as a quotient of certain sequence spaces with the use of Theorem 4.2. Indeed, $\dot{\mathbf{F}}_p^{\alpha,q}$ is isomorphic with the subspace $S_\varphi(\dot{\mathbf{F}}_p^{\alpha,q}) \subset \dot{\mathbf{f}}_p^{\alpha,q}$ and any bounded linear functional on $S_\varphi(\dot{\mathbf{F}}_p^{\alpha,q})$ can be extended to $\dot{\mathbf{f}}_p^{\alpha,q}$ using the projection $S_\varphi T_\psi : \dot{\mathbf{f}}_p^{\alpha,q} \to \dot{\mathbf{f}}_p^{\alpha,q}$ onto $S_\varphi(\dot{\mathbf{F}}_p^{\alpha,q})$.

Example 4.1 Assume that our dilation matrix $A = 2Id$ and we are in the classical isotropic set-up. It is known that for any $1 < p < \infty$

$$\dot{\mathbf{F}}_p^{0,2}(\mathbb{R}^n, A, w) = L^p(w) \iff w \in A_p. \tag{4.30}$$

This is a consequence of the fact the square function is bounded on $L^p(w)$ if and only if $w \in A_p$, see [19, p. 125]. A basic functional analysis tells us that $(L^p(w))^* = L^{p'}(w)$, where $1/p + 1/p' = 1$. On the other hand, Theorem 4.10 implies that $(\dot{\mathbf{F}}_p^{0,2}(\mathbb{R}^n, A, w))^* \approx \dot{\mathbf{F}}_{p'}^{0,2}(\mathbb{R}^n, A, w)$. At the first sight these facts seem to lead to a contradiction with (4.30) in a situation when $w \in A_p$, but $w \notin A_{p'}$. Indeed, we have

$$L^{p'}(w) = (L^p(w))^* = (\dot{\mathbf{F}}_p^{0,2}(\mathbb{R}^n, A, w))^* \approx \dot{\mathbf{F}}_{p'}^{0,2}(\mathbb{R}^n, A, w) \neq L^{p'}(w).$$

Nevertheless, there is no paradox here, since the pairings in these two dualities are totally different. The pairing in $(L^p(w))^* = L^{p'}(w)$ is given by the integration against weighted Lebesgue measure, whereas the other pairing is given by a more complicated formula (4.27). Therefore, the only thing we can deduce from the above is that $\dot{\mathbf{F}}_p^{0,2}(\mathbb{R}^n, A, w)$ is isomorphic to $L^p(w)$ for all range of $1 < p < \infty$. This follows by a simple interpolation argument and the results of the next section.

5 Real interpolation of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces

In order to study real interpolation, it is useful to extend the class of Triebel–Lizorkin spaces to the limiting case $p = 0$ in addition to the already studied case $p = \infty$. Frazier and Jawerth [19] showed that this is possible at the sequence space level in the dyadic setting and the corresponding interpolation spaces turn out to be the usual $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces. Our goal is to prove that their techniques can be extended in a general anisotropic setting.

We recall some rudimentary facts about real interpolation spaces. Suppose that X_0, X_1 are two (compatible) quasi-normed vector spaces, or more generally, quasi-normed Abelian groups, see [1, Sect. 3.10]. Recall that a quasi-norm on an Abelian group satisfies exactly the same properties as a quasi-norm on a vector space with the exception that the homogeneity property $||\lambda x|| = |\lambda|||x||$ is replaced by $||-x|| = ||x||$. The development of the real interpolation method does not depend on the homogeneity property and hence we will allow our spaces to be quasi-normed Abelian groups.

The K-functional for a pair of spaces (X_0, X_1) is defined as

$$K(t) = K(t, x; X_0, X_1) = \inf_{x=x_0+x_1} (||x_0||_{X_0} + t||x_1||_{X_1}) \quad 0 < t < \infty.$$

It is also useful to consider an equivalent K_∞-functional

$$K_\infty(t) = K_\infty(t, x; X_0, X_1) = \inf_{x=x_0+x_1} \max(||x_0||_{X_0}, t||x_1||_{X_1}),$$

and the best-approximation E-functional

$$E(t) = E(t, x; X_0, X_1) = \inf_{||x_1||_{X_1} \le t} ||x - x_1||_{X_0}.$$

Definition 5.1 The *real interpolation space* $(X_0, X_1)_{\theta,q}$, where $0 < \theta < 1$ and $0 < q \le \infty$ is defined as the set of all $x \in X_0 + X_1$ such that

$$||x||_{\theta,q} = \left(\int_0^\infty (t^{-\theta} K(t, x; X_0, X_1))^q \frac{dt}{t} \right)^{1/q} < \infty.$$

For a space X and $\gamma > 0$, let X^γ denote the same space X with the norm $||\cdot||_{X^\gamma} = ||\cdot||_X^\gamma$. Then, we have the following fact about interpolation of L^p spaces on a general measure space due to Peetre and Sparr [37], see also [1, Theorem 7.2.2] or [19, Lemma 6.1],

$$(L^0, L^\infty)_{\theta,1/(1-\theta)}^{1/\theta} \approx L^p, \quad \text{where } 0 < \theta < 1, \ p = \theta/(1-\theta). \tag{5.1}$$

Definition 5.2 Define the space $\dot{\mathbf{f}}^0 = \dot{\mathbf{f}}^0(A, \mu)$ as the collection of all sequences $s = \{s_Q\}_{Q \in \mathcal{Q}}$ such that

$$||s||_{\dot{\mathbf{f}}^0(A,\mu)} = \mu\Bigg(\bigcup_{s_Q \ne 0} Q \Bigg).$$

Hence, $\dot{\mathbf{f}}^0$ is a natural extension of the scale of $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces since for any $\alpha \in \mathbb{R}$, $0 < q \le \infty$,

$$||s||_{\dot{\mathbf{f}}^0} = \left\| \left(\sum_{Q \in \mathcal{Q}} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q)^q \right)^{1/q} \right\|_{L^0(\mu)}.$$

Furthermore, Theorem 3.3 and Corollary 3.4 also hold for $\dot{\mathbf{f}}^0$.

Lemma 5.1 *Suppose that* $\alpha \in \mathbb{R}$, $0 < q \le \infty$, *and* $w \in A_\infty$. *Fix* $0 < \varepsilon < 1$. *Then, for any* $s = \{s_Q\}_Q$,

$$||s||_{\dot{\mathbf{f}}^0(A,\mu)} \asymp \inf \left\{ \mu\left(\bigcup_{s_Q \neq 0} E_Q \right) : E_Q \subset Q,\ \mu(E_Q)/\mu(Q) > \varepsilon \right\}, \tag{5.2}$$

where $E_Q \subset Q$ *are measurable sets. Moreover,*

$$||s||_{\dot{\mathbf{f}}^0(A,\mu)} \asymp ||m^{\alpha,q}(s)||_{L^0(\mu)}. \tag{5.3}$$

Proof Let M be the maximal function defined over dilated cubes $\mathcal{Q}$ rather than dilated balls $\mathcal{B}$. Since M is pointwise dominated by M_{ρ_A}, it is of weak type (1, 1) and

$$\mu\left(\bigcup_{s_Q \neq 0} Q \right) \le \mu(\{x : M(\chi_{\cup_{s_Q\neq 0} E_Q})(x) > \varepsilon\}) \le \frac{C}{\varepsilon} ||\chi_{\cup_{s_Q \neq 0} E_Q}||_{L^1(\mu)} = \frac{C}{\varepsilon} \mu\left(\bigcup_{s_Q \neq 0} E_Q \right)$$

Since the converse inequality is trivial, (5.2) is proved.

To prove (5.3), recall from the proof of Theorem 3.3 that

$$\mu(\{x : m^{\alpha,q}(s)(x) > 0\}) \le C\mu(\{x : G^{\alpha,q}(s)(x) > 0\}) = C\mu\left(\bigcup_{s_Q \neq 0} Q \right).$$

For the converse inequality, observe that if $s_Q \neq 0$, then $m_Q^{\alpha,q}(s) \neq 0$, and hence $m^{\alpha,q}(s)(x) \neq 0$ for all $x \in Q$. □

We need the following fact observed in [19, p. 82], which is stated here in a corrected form.

Proposition 5.2 *Suppose that* (X_0, X_1) *and* (Y_0, Y_1) *are two pairs of quasi-normed Abelian groups. Then, for any* $x \in X_0 + X_1$ *and* $y \in Y_0 + Y_1$, *the inequality*

$$K_\infty(t, x; X_0, X_1) \le cK_\infty(t, y; Y_0, Y_1) \quad \textit{for all } t > 0,$$

is equivalent to

$$E(cs, x; X_0, X_1) \le cE(s, y; Y_0, Y_1) \quad \textit{for all } s > 0,$$

where $c > 0$ *is a constant.*

Proof For brevity, let $K_\infty(t)$, $E(s)$ and $\tilde{K}_\infty(t)$, $\tilde{E}(s)$ represent K_∞, E-functionals computed for $x \in X_0 + X_1$ and $y \in Y_0 + Y_1$, respectively. It is easy to verify from the definitions that

$$K_\infty(t)/t > s \iff E(s)/s > t. \quad \text{for any } s, t > 0.$$

Furthermore, it is immediate that $K_\infty(t)/t$ and $E(s)/s$ are positive, non-increasing functions of $t > 0$, $s > 0$, respectively, and hence they are inverses of each other in the sense that:

$$K_\infty(t)/t = \sup\{s > 0 : E(s)/s > t\},$$
$$E(s)/s = \sup\{t > 0 : K_\infty(t)/t > s\}.$$

In particular, $K_\infty(t)/t, t > 0$, and $E(s)/s, s > 0$, are both right continuous.

Finally, if $K_\infty(t) \le c\tilde{K}_\infty(t)$ for all $t > 0$, then

$$\{t > 0 : K_\infty(t)/t > s\} \subset \{t > 0 : \tilde{K}_\infty(t)/t > s/c\} \quad s > 0.$$

Consequently, $E(s)/s \le \tilde{E}(s/c)/(s/c)$, $s > 0$, which proves one implication. The other implication follows in the same fashion. □

Lemma 5.1 and Proposition 5.2 imply the key equivalence of K-functionals as it was shown by Frazier and Jawerth in the dyadic setting [19, Theorem 6.4]. The proof given is a direct adaptation of this work.

Theorem 5.3 *Suppose that $\alpha \in \mathbb{R}$, $0 < q \le \infty$, and μ is a ρ_A-doubling measure. Then,*

$$K(t, s; \dot{\mathbf{f}}^0(A,\mu), \dot{\mathbf{f}}^{\alpha,q}_\infty(A,\mu)) \asymp K(t, m^{\alpha,q}(s); L^0(\mu), L^\infty),$$

with equivalence constants independent of $t > 0$ and $s = \{s_Q\}_{Q\in\mathcal{Q}}$.

Proof To prove

$$K(t, m^{\alpha,q}(s); L^0(\mu), L^\infty) \le C K(t, s; \dot{\mathbf{f}}^0(A,\mu), \dot{\mathbf{f}}^{\alpha,q}_\infty(A,\mu)) \quad \text{for } t > 0 \tag{5.4}$$

it is useful to emphasize the dependence on $0 < \varepsilon < 1$ of the definition of local q-power function despite the fact all such functions have equivalent norms, see Theorem 3.3. That is, let $m^{\alpha,q}_{Q,\varepsilon}(s)$ and $m^{\alpha,q}_{\varepsilon}(s)$ denote functions in (3.6) and (3.7), respectively.

Suppose that a sequence $s = s_0 + s_1$. It is easy to verify the subadditivity property

$$m^{\alpha,q}_{\varepsilon}(s_0+s_1) \le C_q(m^{\alpha,q}_{\varepsilon/2}(s_0) + m^{\alpha,q}_{\varepsilon/2}(s_1)).$$

Let $f_0 = \min(m^{\alpha,q}_{\varepsilon}(s), C_q m^{\alpha,q}_{\varepsilon/2}(s_0))$ and $f_1 = m^{\alpha,q}_{\varepsilon}(s) - f_0$. Then, $m^{\alpha,q}_{\varepsilon}(s) = f_0 + f_1$ and by Theorem 3.3 and Lemma 5.1

$$\begin{aligned} ||f_0||_{L^0(\mu)} &\le ||m^{\alpha,q}_{\varepsilon/2}(s_0)||_{L^0(\mu)} \le C||s_0||_{\dot{\mathbf{f}}^0(A,\mu)}, \\ ||f_1||_{L^\infty} &\le C_q||m^{\alpha,q}_{\varepsilon/2}(s_1)||_{L^\infty} \le C||s_1||_{\dot{\mathbf{f}}^{\alpha,q}_\infty(A,\mu)}, \end{aligned}$$

which proves (5.4).

To prove the converse inequality to (5.4), by Proposition 5.2, it suffices to prove that there exists $c > 0$ such that

$$E(ct, s; \dot{\mathbf{f}}^0(A,\mu), \dot{\mathbf{f}}^{\alpha,q}_\infty(A,\mu)) \le cE(t, m^{\alpha,q}(s); L^0(\mu), L^\infty) \quad \text{for } t > 0. \tag{5.5}$$

It is easy to verify that for any Borel function f we have

$$E(t, f; L^0(\mu), L^\infty) = \mu(\{x \in \mathbb{R}^n : |f(x)| > t\}). \tag{5.6}$$

Fix $t > 0$. Therefore, to prove (5.5) it suffices to find a splitting $s = s^0 + s^1$, such that

$$||s^0||_{\dot{\mathbf{f}}^0(A,\mu)} \le c\mu(\{x : m^{\alpha,q}(s)(x) > t\}) \tag{5.7}$$

$$||s^1||_{\dot{\mathbf{f}}^{\alpha,q}_\infty(A,\mu)} \le ct, \tag{5.8}$$

for some constant c independent of t. For each $Q \in \mathcal{Q}$, let

$$Q_+ = \{x \in Q : m^{\alpha,q}(s)(x) > t\}, \quad Q_- = Q \setminus Q_+,$$

and define partition $\mathcal{Q} = \mathcal{Q}_+ \dot{\cup} \mathcal{Q}_-$ by

$$\mathcal{Q}_0 = \{Q \in \mathcal{Q} : \mu(Q_+) > \mu(Q)/2\}, \quad \mathcal{Q}_1 = \{Q \in \mathcal{Q} : \mu(Q_-) \ge \mu(Q)/2\}.$$

Finally, define for each $Q \in \mathcal{Q}$,

$$(s^0)_Q = \begin{cases} s_Q & Q \in \mathcal{Q}_0, \\ 0 & Q \in \mathcal{Q}_1, \end{cases} \quad (s^1)_Q = s_Q - s^0_Q.$$

As in Theorem 3.3 define Borel set $E_Q \subset Q$ by

$$E_Q = \{x \in Q : G^{\alpha,q}_Q(s)(x) \le m^{\alpha,q}(s)(x)\}.$$

and recall that $\mu(E_Q)/\mu(Q) \geq 1-\varepsilon$, where $0<\varepsilon<1$ is the same as in the definition of local q-power function $m^{\alpha,q}(s)$. Now it convenient to specify that $\varepsilon = 1/4$ and define

$$\tilde{E}_Q = \begin{cases} E_Q \cap Q_+ & Q \in \mathcal{Q}_0, \\ E_Q \cap Q_- & Q \in \mathcal{Q}_1. \end{cases}$$

Since $\mu(\tilde{E}_Q)/\mu(Q) \geq 1/4$, by (5.2)

$$||s^0||_{\dot{\mathbf{f}}^0(A,\mu)} = \mu\Bigg(\bigcup_{s_Q \neq 0,\ Q \in \mathcal{Q}_0} Q \Bigg) \leq c\mu\Bigg(\bigcup_{s_Q \neq 0,\ Q \in \mathcal{Q}_0} \tilde{E}_Q \Bigg),$$

which shows (5.7), because $\tilde{E}_Q \subset Q_+$ for $Q \in \mathcal{Q}_0$.

By Corollary 3.4 and (3.10),

$$||s^1||_{\dot{\mathbf{f}}^{\alpha,q}_\infty(A,\mu)} \leq C \left\| \Bigg(\sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha} |(s^1)_Q| \tilde{\chi}_{\tilde{E}_Q})^q \Bigg)^{1/q} \right\|_{L^\infty} \leq C \sup_{x\in E} m^{\alpha,q}(s)(x) \leq Ct,$$

since $\tilde{E}_Q \subset E_Q$ and

$$E = \bigcup_{Q\in\mathcal{Q},\ (s^1)_Q \neq 0} \tilde{E}_Q \subset \bigcup_{Q\in\mathcal{Q}} Q_- \subset \{x : m^{\alpha,q}(s)(x) \leq t\}.$$

This shows (5.8) and completes the proof of Theorem 5.3. $\square$

By Proposition 5.2 we can restate Theorem 5.2 in the form of the estimate

$$\begin{aligned} \mu(\{x\in\mathbb{R}^n : m^{\alpha,q}(s)(x) > c_1 t\})/c_1 &\leq E(t,s;\dot{\mathbf{f}}^0(A,\mu), \dot{\mathbf{f}}^{\alpha,q}_\infty(A,\mu)) \\ &\leq \mu(\{x\in\mathbb{R}^n : m^{\alpha,q}(s)(x) > c_2 t\})/c_2 \quad \text{for all } t>0,\ s=\{s_Q\}, \end{aligned}$$

for some constants $0<c_2<c_1<\infty$. Hence, one can think of the best approximation functional $E(t,s;\dot{\mathbf{f}}^0,\dot{\mathbf{f}}^{\alpha,q}_\infty)$ (up to some universal constants) as the distribution function of local q-power function $m^{\alpha,q}(s)$ in the close analogy with the identification (5.6).

As a consequence of (5.1) and Theorems 3.3 and 5.3 we have the following corollary.

Corollary 5.4 *Suppose $\alpha\in\mathbb{R}$, $0<q\leq\infty$, $0<\theta<1$, and μ is ρ_A-doubling measure. Then,*

$$(\dot{\mathbf{f}}^0(A,\mu), \dot{\mathbf{f}}^{\alpha,q}_\infty(A,\mu))^{1/\theta}_{\theta,1/(1-\theta)} = \dot{\mathbf{f}}^{\alpha,q}_p(A,\mu), \quad \textit{where } p = \theta/(1-\theta).$$

Consequently, we obtain the following interpolation identities as Frazier and Jawerth [19].

Corollary 5.5 *Suppose $\alpha\in\mathbb{R}$, $0<q\leq\infty$, and $0<p_0<p<p_1\leq\infty$. Then*

$$(\dot{\mathbf{f}}^0, \dot{\mathbf{f}}^{\alpha,q}_{p_1})^{1/\theta}_{\theta,p/\theta} = \dot{\mathbf{f}}^{\alpha,q}_p \quad \textit{where } 1/\theta = 1 + 1/p - 1/p_1, \tag{5.9}$$

$$(\dot{\mathbf{f}}^{\alpha,q}_{p_0}, \dot{\mathbf{f}}^{\alpha,q}_{p_1})_{\theta,p} = \dot{\mathbf{f}}^{\alpha,q}_p \quad \textit{where } 1/p = (1-\theta)/p_0 + \theta/p_1. \tag{5.10}$$

Proof Recall two standard real interpolation identities:

$$(L^0, L^{p_1})^{1/\theta}_{\theta,p/\theta} = L^p \quad \text{where } 1/\theta = 1 + 1/p - 1/p_1, \tag{5.11}$$

$$(L^{p_0}, L^{p_1})_{\theta,p} = L^p \quad \text{where } 1/p = (1-\theta)/p_0 + \theta/p_1. \tag{5.12}$$

Indeed, (5.11) follows from the endpoint result in Proposition 5.2 and the reiteration and power theorems [1, Theorems 3.11.5–3.11.6]. For the proof of (5.12), see [1, Theorem 5.2.1].

There are two possible ways of proving Corollary 5.5. The first more direct approach establishes the equivalence formulas

$$K(t,s;\dot{\mathbf{f}}^0,\dot{\mathbf{f}}^{\alpha,q}_{p_1}) \asymp K(t,m^{\alpha,q}(s);L^0,L^{p_1}), \tag{5.13}$$

$$K(t,s;\dot{\mathbf{f}}^{\alpha,q}_{p_0},\dot{\mathbf{f}}^{\alpha,q}_{p_1}) \asymp K(t,m^{\alpha,q}(s);L^{p_0},L^{p_1}). \tag{5.14}$$

These can be shown by similar estimates as in Theorem 5.3, see also the proof of [18, Theorem 3.2]. Combining (5.11)–(5.14) yields Corollary 5.5.

Alternatively, we can deduce Corollary 5.5 as a consequence of the already shown endpoint interpolation result in Corollary 5.4 and a few standard facts about real interpolation spaces. Indeed, to prove (5.10) let $\theta_i = p_i/(1+p_i), i = 1, 2$, and $\rho = (1-\theta)/\theta_0+\theta/\theta_1 = (p+1)/p$. Then,

$$\begin{aligned}
(\dot{\mathbf{f}}^{\alpha,q}_{p_0},\dot{\mathbf{f}}^{\alpha,q}_{p_1})_{\theta,p} &= ((\dot{\mathbf{f}}^0,\dot{\mathbf{f}}^{\alpha,q}_{\infty})^{1/\theta_0}_{\theta_0,1/(1-\theta_0)},(\dot{\mathbf{f}}^0,\dot{\mathbf{f}}^{\alpha,q}_{\infty})^{1/\theta_1}_{\theta_1,1/(1-\theta_1)})_{\theta,p}\\
&= (((\dot{\mathbf{f}}^0,\dot{\mathbf{f}}^{\alpha,q}_{\infty})_{\theta_0,1/(1-\theta_0)},(\dot{\mathbf{f}}^0,\dot{\mathbf{f}}^{\alpha,q}_{\infty})_{\theta_1,1/(1-\theta_1)})_{\theta/(\theta_1\rho),\rho p})^{\rho}\\
&= ((\dot{\mathbf{f}}^0,\dot{\mathbf{f}}^{\alpha,q}_{\infty})_{1/\rho,\rho p})^{\rho} = \dot{\mathbf{f}}^{\alpha,q}_{p}.
\end{aligned}$$

In the first and the last steps we used Corollary 5.4, in the second step we used the power theorem [1, Theorem 3.11.6] and in the third step we applied the reiteration theorem [1, Theorem 3.11.5] for quasi-normed Abelian-groups together with the identity

$$\left(1-\frac{\theta}{\theta_1\rho}\right)\theta_0+\frac{\theta}{\theta_1\rho}\theta_1 = 1/\rho.$$

A similar argument proves (5.9) as well. □

The functorial property of interpolation implies the corresponding formulas for $\dot{\mathbf{F}}^{\alpha,q}_p$ spaces.

Corollary 5.6 *Suppose $\alpha \in \mathbb{R}$, $0 < q \le \infty$, and $0 < p_0 < p < p_1 \le \infty$, and μ is a ρ_A-doubling measure. Then,*

$$(\dot{\mathbf{F}}^{\alpha,q}_{p_0},\dot{\mathbf{F}}^{\alpha,q}_{p_1})_{\theta,p} = \dot{\mathbf{F}}^{\alpha,q}_{p}(\mathbb{R}^n,A,\mu), \quad \text{where } 1/p = (1-\theta)/p_0+\theta/p_1. \tag{5.15}$$

Proof Again, there are two ways of proving Corollary 5.6. The first more direct uses the equivalence formula

$$K(t,f;\dot{\mathbf{F}}^{\alpha,q}_{p_0},\dot{\mathbf{F}}^{\alpha,q}_{p_1}) \asymp K(t,S_\varphi f;\dot{\mathbf{f}}^{\alpha,q}_{p_0},\dot{\mathbf{f}}^{\alpha,q}_{p_1}) \asymp K(t,M^{\alpha,q}f;L^{p_0}(\mu),L^{p_1}(\mu)),$$

where $M^{\alpha,q}f$ is as in Definition 3.2. The first equivalence follows from Theorem 2.7 while the second follows from (5.14). Then, it suffices to use (5.12) together with Corollary 3.8.

Alternatively, we can use a standard retraction argument. That is, by Theorem 2.7 applied for an admissible pair of dual frame wavelets (φ,ψ), the functorial property of interpolation, and Corollary 5.5 we have retract diagrams

$$\begin{tikzcd}
 & \dot{\mathbf{f}}^{\alpha,q}_{p} \arrow[dr, "T_\psi"] & \\
\dot{\mathbf{F}}^{\alpha,q}_{p} \arrow[ur, "S_\varphi"] \arrow[rr, "\mathrm{Id}"] & & \dot{\mathbf{F}}^{\alpha,q}_{p}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \dot{\mathbf{f}}^{\alpha,q}_{p} \arrow[dr, "T_\psi"] & \\
(\dot{\mathbf{F}}^{\alpha,q}_{p_0},\dot{\mathbf{F}}^{\alpha,q}_{p_1})_{\theta,p} \arrow[ur, "S_\varphi"] \arrow[rr, "\mathrm{Id}"] & & (\dot{\mathbf{F}}^{\alpha,q}_{p_0},\dot{\mathbf{F}}^{\alpha,q}_{p_1})_{\theta,p}
\end{tikzcd} \tag{5.16}$$

Then, the identification (5.15) follows immediately from (5.16). □

Note that in Corollary 5.6 we assume that a pair of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces has fixed smoothness and summability parameters α and q, respectively. A similar result for a pair of (isotropic and inhomogeneous) $\mathbf{F}_p^{\alpha,q}$ spaces with fixed integrability exponent p is due to Triebel [43, Theorem 2.4.2]. It is remarkable that the corresponding interpolation space turns out to be an appropriate Besov space $\mathbf{B}_p^{\alpha,q}$. However, we will not pursue anisotropic analogues of this result here. Instead, we shall move to the complex interpolation method where no restrictions on the parameters α, p, and q are imposed.

6 Calderón Product and Complex Interpolation of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces

In this section we study complex interpolation of Triebel–Lizorkin spaces using Calderón products. We follow the approach of Frazier and Jawerth [19] which takes advantage of the fact that, unlike distribution spaces $\dot{\mathbf{F}}_p^{\alpha,q}$, the sequence spaces $\dot{\mathbf{f}}_p^{\alpha,q}$ are quasi-Banach lattices. Hence, by computing Calderón products of $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces we can deduce complex interpolation results for $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces. We start by defining the Calderón product of two quasi-Banach lattices.

Definition 6.1 Suppose that ν is a positive measure on Ω, and X is a quasi-Banach space of ν-measurable functions on Ω, which are identified if equal ν-a.e. We say that X is a *quasi-Banach lattice* on Ω if for any ν-measurable functions f and g

$$f \in X \text{ and } |g(x)| \le |f(x)| \ \nu\text{-a.e.} \implies g \in X \text{ and } ||g||_X \le ||f||_X.$$

Suppose that X_0 and X_1 are quasi-Banach lattices on Ω. Given $0 < \theta < 1$, define the *Calderón product* $X_0^{1-\theta} X_1^{\theta}$ as the collection of all ν-measurable functions u satisfying

$$||u||_{X_0^{1-\theta} X_1^{\theta}} = \inf\{M > 0 : |u(x)| \le M|v(x)|^{1-\theta}|w(x)|^{\theta} \quad \nu\text{-a.e.}$$
$$\text{for some } ||v||_{X_0} \le 1 \text{ and } ||w||_{X_1} \le 1\} < \infty.$$

Theorem 6.1 generalizes the result of Frazier and Jawerth [19, Theorem 8.2] on Calderón products of $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces.

Theorem 6.1 *Suppose $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1 \le \infty$, $0 < q_0, q_1 \le \infty$, and μ is a ρ_A-doubling measure. Then, for any $0 < \theta < 1$, we can identify the Calderón product as*

$$(\dot{\mathbf{f}}_{p_0}^{\alpha_0,q_0}(A,\mu))^{1-\theta}(\dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}(A,\mu))^{\theta} = \dot{\mathbf{f}}_p^{\alpha,q}(A,\mu), \tag{6.1}$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$, *and* $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

Proof For simplicity, let $X_0 = \dot{\mathbf{f}}_{p_0}^{\alpha_0,q_0}(A,\mu)$ and $X_1 = \dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}(A,\mu)$. Since $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces have a different definition when $p = \infty$, we shall split the proof into several cases. The inclusion $X_0^{1-\theta} X_1^{\theta} \subset \dot{\mathbf{f}}_p^{\alpha,q}$ can be proved in one stroke by following [19]. We recall this reasoning for the sake of completeness.

First, suppose that $s = \{s_Q\} \in X_0^{1-\theta} X_1^{\theta}$. Let $B = 2||s||_{X_0^{1-\theta} X_1^{\theta}}$. Then, there exist sequences $r = \{r_Q\}$ and $t = \{t_Q\}$ such that

$$||r||_{X_0} \le 1, \quad ||t||_{X_1} \le 1, \quad |s_Q| \le B|r_Q|^{1-\theta}|t_Q|^{\theta} \text{ for all } Q \in \mathcal{Q}.$$

According to Corollary 3.4, applied with $\varepsilon = 3/4$, there exist Borel sets $E^i_Q \subset Q$, $i = 0, 1$, for each dilated cube such that $\mu(E^i_Q)/\mu(Q) > 3/4$ and

$$||r||_{X_0} \asymp \left\| \left(\sum_{Q \in \mathcal{Q}} (|Q|^{-\alpha_0} |r_Q| \tilde{\chi}_{E^0_Q})^{q_0} \right)^{1/q_0} \right\|_{L^{p_0}(\mu)},$$

$$||t||_{X_1} \asymp \left\| \left(\sum_{Q \in \mathcal{Q}} (|Q|^{-\alpha_1} |t_Q| \tilde{\chi}_{E^1_Q})^{q_1} \right)^{1/q_1} \right\|_{L^{p_1}(\mu)}.$$

We remark that if $p_i < \infty$, $i = 0, 1$, then one can simply take $E^i_Q = Q$. Note that $E_Q := E^0_Q \cap E^1_Q \subset Q$ satisfies $\mu(E_Q)/\mu(Q) > 1/2$. Applying (discrete) Hölder's inequality with conjugate exponents $q_0/((1-\theta)q)$ and $q_1/(q\theta)$ yields

$$\begin{aligned} \left(\sum_Q (|Q|^{-\alpha} |s_Q| \tilde{\chi}_{E_Q})^q \right)^{1/q} &\le B \left(\sum_Q (|Q|^{-\alpha_0} |r_Q| \tilde{\chi}_{E_Q})^{q(1-\theta)} (|Q|^{-\alpha_1} |t_Q| \tilde{\chi}_{E_Q})^{q\theta} \right)^{1/q} \\ &\le B \left(\sum_Q (|Q|^{-\alpha_0} |r_Q| \tilde{\chi}_{E_Q})^{q_0} \right)^{(1-\theta)/q_0} \\ &\quad \times \left(\sum_Q (|Q|^{-\alpha_1} |t_Q| \tilde{\chi}_{E_Q})^{q_1} \right)^{\theta/q_1}. \end{aligned} \tag{6.2}$$

Computing $L^p(\mu)$ norms of (6.2) and applying again Hölder's inequality with conjugate exponents $p_0/((1-\theta)p)$ and $p_1/(p\theta)$ yields

$$||s||_{\dot{\mathbf{f}}^{\alpha,q}_p} \le C \left\| \left(\sum_Q (|Q|^{-\alpha} |s_Q| \tilde{\chi}_{E_Q})^q \right)^{1/q} \right\|_{L^p(\mu)} \le CB ||r||^{1-\theta}_{X_0} ||t||^{\theta}_{X_1} \le C ||s||_{X_0^{1-\theta} X_1^{\theta}}. \tag{6.3}$$

Here, the first inequality is a consequence of Corollary 3.4. Note that the above estimates are written as if all parameters p_i, q_i, $i = 0, 1$, were finite. Nevertheless, one can easily modify these estimates if any of them is infinite. Thus, (6.3) holds in full generality and it proves one direction of (6.1).

The proof of the opposite inclusion of (6.1) requires more work and it must be split into several cases.

Case 1 $p_0, p_1 < \infty$. Suppose that $s \in \dot{\mathbf{f}}^{\alpha,q}_p$. First, consider the subcase of $q_0, q_1 < \infty$. By the symmetry we may assume that $p_0/q_0 \le p_1/q_1$. For $k \in \mathbb{Z}$, define

$$\Omega_k = \left\{ x \in \mathbb{R}^n : \left(\sum_Q (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q(x))^q \right)^{1/q} > 2^k \right\}.$$

$$\mathcal{Q}_k = \{Q \in \mathcal{Q} : \mu(Q \cap \Omega_k) \ge \mu(Q)/2 \text{ and } \mu(Q \cap \Omega_{k+1}) < \mu(Q)/2\}.$$

Note that if $Q \notin \bigcup_{k \in \mathbb{Z}} \mathcal{Q}_k$, then $s_Q = 0$. In this case set $r_Q = t_Q = 0$. Otherwise, if $Q \in \mathcal{Q}_k$ for some $k \in \mathbb{Z}$, then we set

$$r_Q = (|s_Q|/A^0_Q)^{q/q_0} \quad \text{and} \quad t_Q = (|s_Q|/A^1_Q)^{q/q_1}, \tag{6.4}$$

where

$$A^i_Q = 2^{k\delta_i} |Q|^{u_i}, \quad \delta_i = 1 - pq_i/(qp_i), \; u_i = \alpha + 1/2 - (\alpha_i + 1/2) q_i/q, \; i = 0, 1.$$

A direct calculation shows that $|s_Q| = |r_Q|^{1-\theta}|t_Q|^{\theta}$. In addition, we claim that

$$||r||_{X_0} \leq C||s||_{\dot{\mathbf{f}}_p^{\alpha,q}}^{p/p_0} \tag{6.5}$$

$$||t||_{X_1} \leq C||s||_{\dot{\mathbf{f}}_p^{\alpha,q}}^{p/p_1}. \tag{6.6}$$

Assuming (6.5) and (6.6) for the moment, we can normalize r and t to get

$$|s_Q| \leq C||s||_{\dot{\mathbf{f}}_p^{\alpha,q}} \left(\frac{r_Q}{C||s||_{\dot{\mathbf{f}}_p^{\alpha,q}}^{p/p_0}}\right)^{1-\theta} \left(\frac{t_Q}{C||s||_{\dot{\mathbf{f}}_p^{\alpha,q}}^{p/p_1}}\right)^{\theta}.$$

Hence, we have $||s||_{X_0^{1-\theta}X_1^{\theta}} \leq C||s||_{\dot{\mathbf{f}}_p^{\alpha,q}}$.

To prove (6.5), we use Lemma 3.1 with sets $E_Q = Q \cap \Omega_k$, $Q \in \mathcal{Q}_k$,

$$\begin{aligned}
||r||_{X_0}^{p_0} &\leq C \int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_k} |Q|^{-(\alpha_0+1/2)q_0} (A_Q^0)^{-q} |s_Q|^q \chi_{E_Q}\right)^{p_0/q_0} d\mu \\
&\leq C \int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \chi_{\Omega_k} \sum_{Q\in\mathcal{Q}_k} 2^{-k\delta_0 q} (|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q)^q\right)^{p_0/q_0} d\mu \\
&\leq C \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q)^q\right)^{(1-\delta_0)p_0/q_0} d\mu = C||s||_{\dot{\mathbf{f}}_p^{\alpha,q}}^p.
\end{aligned}$$

Here, in the penultimate step we used that $\delta_0 \leq 0$ and

$$2^{-k\delta_0 q}\chi_{\Omega_k} \leq \left(\sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q)^q\right)^{-\delta_0}.$$

To prove (6.6), we use a similar argument as above by redefining $E_Q = Q \cap (\Omega_{k+1})^c$, $Q \in \mathcal{Q}_k$,

$$\begin{aligned}
||t||_{X_1}^{p_1} &\leq C \int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_k} |Q|^{-(\alpha_1+1/2)q_1} (A_Q^1)^{-q} |s_Q|^q \chi_{E_Q}\right)^{p_1/q_1} d\mu \\
&\leq C \int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \chi_{(\Omega_{k+1})^c} \sum_{Q\in\mathcal{Q}_k} 2^{-k\delta_1 q} (|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q)^q\right)^{p_1/q_1} d\mu \\
&\leq C \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q)^q\right)^{(1-\delta_1)p_1/q_1} d\mu = C||s||_{\dot{\mathbf{f}}_p^{\alpha,q}}^p.
\end{aligned}$$

This change is needed, since $\delta_1 \geq 0$ and consequently

$$2^{-(k+1)\delta_1 q}\chi_{(\Omega_{k+1})^c} \leq \left(\sum_{Q\in\mathcal{Q}_k} (|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q)^q\right)^{-\delta_1}.$$

To prove the subcase of $q_0 < \infty$ and $q_1 = \infty$ we define r_Q as in (6.4) and let $t_Q = (A_Q^0)^{1/\theta}$. Again it is not difficult to show that $|s_Q| = |r_Q|^{1-\theta}|t_Q|^{\theta}$ and (6.5) holds by adopting the above arguments and observing that $\delta_0 = \theta p/p_1 > 0$. Finally, (6.6) follows by Lemma 3.1 applied with $E_Q = Q \cap \Omega_k$,

$$||t||_{X_1}^{p_1} \le C \int_{\mathbb{R}^n} \left(\sup_{k\in\mathbb{Z}} \sup_{Q\in\mathcal{Q}_k} |Q|^{-\alpha_1} |t_Q| \tilde{\chi}_{E_Q} \right)^{p_1} d\mu \le C \int_{\mathbb{R}^n} \left(\sup_{k\in\mathbb{Z}} 2^{kp/p_1} \chi_{\Omega_k} \right)^{p_1} d\mu$$
$$\le C \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q)^q \right)^{p/q} d\mu = C ||s||^p_{\dot{\mathbf{f}}_p^{\alpha,q}}.$$

Likewise, the subcase of $q_0 = \infty$ and $q_1 < \infty$ follows by the symmetry. Finally, in the subcase of $q_0 = q_1 = \infty$ we select r_Q and t_Q so that

$$(|Q|^{-\alpha_0} |r_Q| \tilde{\chi}_Q)^{p_0} = (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q)^p = (|Q|^{-\alpha_1} |t_Q| \tilde{\chi}_Q)^{p_1}.$$

Hence, we have $|s_Q| = |r_Q|^{1-\theta} |t_Q|^\theta$ and equalities in (6.5) and (6.6). This proves Case 1.

Case 2 $p_0 < \infty$ and $p_1 = \infty$. Suppose that $s \in \dot{\mathbf{f}}_p^{\alpha,q}$. First, consider the subcase of $q_0, q_1 < \infty$. Define sequences r and t as in (6.4) with the understanding that $\delta_1 = 1$. The estimate (6.5) is shown the same way as before. Our goal is to prove (6.6) which is now understood as

$$||t||_{X_1} \le C. \tag{6.7}$$

This inequality follows by Corollary 3.4 with sets $E_Q = Q \cap (\Omega_{k+1})^c$,

$$\begin{aligned} ||t||_{X_1} &\le C \left\| \left(\sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_k} |Q|^{-(\alpha_1+1/2)q_1} (A_Q^1)^{-q} |s_Q|^q \chi_{E_Q} \right)^{1/q_1} \right\|_{L^\infty} \\ &\le C \left\| \left(\sum_{k\in\mathbb{Z}} \chi_{(\Omega_{k+1})^c} \sum_{Q\in\mathcal{Q}_k} 2^{-k\delta_1 q} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q)^q \right)^{1/q_1} \right\|_{L^\infty} \\ &\le C \left\| \left(\sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q)^q \right)^{(1-\delta_1)/q_1} \right\|_{L^\infty} = C. \end{aligned} \tag{6.8}$$

To prove the subcase of $q_0 \le \infty$ and $q_1 = \infty$, we define r_Q as in (6.4) and let $t_Q = (A_Q^0)^{1/\theta}$. Hence, to be explicit we set

$$r_Q = (|s_Q|/|Q|^{\theta(\alpha_1+1/2)})^{1/(1-\theta)}, \quad t_Q = |Q|^{\alpha_1+1/2}. \tag{6.9}$$

Again it is not difficult to show that $|s_Q| = |r_Q|^{1-\theta} |t_Q|^\theta$ and (6.5) and (6.7) hold by direct calculations.

Finally, in the subcase of $q_0 = \infty$, $q_1 < \infty$, we let $r_Q = (A_Q^1)^{1/(1-\theta)}$ and define t_Q as in (6.4) with the understanding that $\delta_1 = 1$. Then, (6.5) follows by Lemma 3.1 applied with $E_Q = Q \cap \Omega_k$,

$$||r||_{X_0}^{p_0} \le C \int_{\mathbb{R}^n} \left(\sup_{k\in\mathbb{Z}} \sup_{Q\in\mathcal{Q}_k} |Q|^{-\alpha_0} |r_Q| \tilde{\chi}_{E_Q} \right)^{p_0} d\mu \le C \int_{\mathbb{R}^n} \left(\sup_{k\in\mathbb{Z}} 2^{kp/p_0} \chi_{\Omega_k} \right)^{p_0} d\mu$$
$$\le C \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q)^q \right)^{p/q} d\mu = C ||s||^p_{\dot{\mathbf{f}}_p^{\alpha,q}}.$$

The estimate (6.7) follows along the lines of (6.8). This proves Case 2.

Case 3 $p_0 = p_1 = \infty$. Suppose that $s \in \dot{\mathbf{f}}_\infty^{\alpha,q}$. By Corollary 3.4, applied with $\varepsilon = 3/4$, there exist Borel sets $F_Q \subset Q$ such that $\mu(F_Q)/\mu(Q) > 3/4$ and

$$||s||_{\dot{\mathbf{f}}_\infty^{\alpha,q}} \asymp \left\| \left(\sum_{Q \in \mathcal{Q}} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_{F_Q})^q \right)^{1/q} \right\|_{L^\infty}.$$

First, consider the subcase of $q_0, q_1 < \infty$. For $k \in \mathbb{Z}$, we redefine

$$\Omega_k = \{x \in \mathbb{R}^n : \left(\sum_Q (|Q|^{-\alpha} |s_Q| \tilde{\chi}_{F_Q}(x))^q \right)^{1/q} > 2^k\}.$$

Define sequences r and t as in (6.4) by regarding that $p/p_0 = p/p_1 = 1$. Then, we have $|s_Q| = |r_Q|^{1-\theta} |t_Q|^\theta$. By setting $E_Q = F_Q \cap \Omega_k$, $Q \in \mathcal{Q}_k$, one can adapt the proof of (6.5) given in Case 1 with the use of Corollary 3.4, since $\mu(E_Q)/\mu(Q) > 3/4 - 1/2 = 1/4$. Likewise, by setting $E_Q = F_Q \cap (\Omega_{k+1})^c$, $Q \in \mathcal{Q}_k$, one can adapt the proof of (6.6).

In the final subcase of $q_0 \le \infty$ and $q_1 = \infty$, we define sequences r and t by (6.9). Again it is not difficult to show that $|s_Q| = |r_Q|^{1-\theta} |t_Q|^\theta$. By direct calculations one can verify that

$$||r||_{X_0} \le C ||s||_{\dot{\mathbf{f}}_\infty^{\alpha,q}}^{1/(1-\theta)}, \quad ||t||_{X_1} = 1.$$

Hence, we have $||s||_{X_0^{1-\theta} X_1^\theta} \le C ||s||_{\dot{\mathbf{f}}_\infty^{\alpha,q}}$. This completes Case 3 and the proof of Theorem 6.1. □

Once Theorem 6.1 is established we can obtain complex interpolation results exactly in the same way as Frazier and Jawerth did in [19]. Let $[X_0, X_1]_\theta$ be the Calderón's complex interpolation space, see Definition 6.2. Then, we have

Theorem 6.2 *Suppose* $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, q_0 < \infty$, $0 < p_1, q_1 \le \infty$, *and* μ *is a* ρ_A*-doubling measure. Then, for any* $0 < \theta < 1$, *we can identify the complex interpolation spaces*

$$[\dot{\mathbf{f}}_{p_0}^{\alpha_0,q_0}(A, \mu), \dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}(A, \mu)]_\theta = \dot{\mathbf{f}}_p^{\alpha,q}(A, \mu), \tag{6.10}$$

$$[\dot{\mathbf{F}}_{p_0}^{\alpha_0,q_0}(\mathbb{R}^n, A, \mu), \dot{\mathbf{F}}_{p_1}^{\alpha_1,q_1}(\mathbb{R}^n, A, \mu)]_\theta = \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu), \tag{6.11}$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$, *and* $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

Proof in the Banach space case. If $\min(p_0, q_0, p_1, q_1) \ge 1$, then $X_0 = \dot{\mathbf{f}}_{p_0}^{\alpha_0,q_0}(A, \mu)$ and $X_1 = \dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}(A, \mu)$ are Banach spaces. Then, it suffices to use Calderón's result [11, p. 125], which identifies the Calderón product $X = X_0^{1-\theta} X_1^\theta$ of Banach lattices X_0 and X_1, with the complex interpolation space $[X_0, X_1]_\theta$. This identification holds under the hypothesis that for any $f \in X$,

$$f_n(x) \to f(x) \ \nu\text{-a.e. and } |f_n(x)| \le |f(x)| \ \nu\text{-a.e.} \implies ||f_n||_X \to ||f||_X \text{ as } n \to \infty.$$

Since $p, q < \infty$, $X = \dot{\mathbf{f}}_p^{\alpha,q}$ satisfies the above property by the Lebesgue dominated convergence theorem. Hence, Theorem 6.1 yields (6.10). The functorial property of interpolation and Theorem 2.7 imply (6.11). □

Note that the above argument does not treat the case when $p_j < 1$ or $q_j < 1$, $j = 0, 1$, since the Calderón's complex method is restricted to Banach spaces. There are many ways of extending the complex interpolation to quasi-Banach spaces. Triebel [42], [43, Sect. 2.4.7] used the method of Calderón and Torchinsky [12,13] to obtain complex interpolation results in the usual isotropic setting for the full range of parameters $\alpha \in \mathbb{R}, 0 < p, q \leq \infty$. However, it is not known whether this method has a functorial interpolation property (for bounded linear operators), see [43, Sect. 2.4.8]. We will avoid this potential shortcoming by pursuing another complex method originally introduced by Janson and Jones [29]. A slight modification of this method, which is due to Kalton and Mitrea [32], is given below.

Definition 6.2 Let (X_0, X_1) be an interpolation couple of quasi-Banach spaces, i.e., X_j, $j = 0, 1$, are continuously imbedded in a larger Hausdorff topological vector space. Let $S = \{z \in \mathbb{C} : 0 < \Re z < 1\}$. Define $\mathcal{F}$ as the space of bounded analytic functions $f : S \to X_0 + X_1$, which extend continuously to the closure $\bar{S}$, and such that the traces $t \mapsto f(j+it)$ are bounded continuous functions into X_j, $j = 0, 1$. We endow $\mathcal{F}$ with the quasi-norm

$$||f||_{\mathcal{F}} = \max\left(\sup_t ||f(it)||_{X_0}, \sup_t ||f(1+it)||_{X_1}, \sup_{z\in S} ||f(z)||_{X_0+X_1}\right). \tag{6.12}$$

For $0 < \theta < 1$, define the complex interpolation space

$$[X_0, X_1]_\theta = \{x \in X_0 + X_1 : x = f(\theta) \quad \text{for some } f \in \mathcal{F}\},$$

with the quasi-norm

$$||x||_{[X_0,X_1]_\theta} = \inf\{||f||_{\mathcal{F}} : f \in \mathcal{F},\ f(\theta) = x\}.$$

Remark 6.1 In the case when X_0 and X_1 are Banach space, the last term in definition of $||\cdot||_{\mathcal{F}}$ is obsolete and we obtain the usual Calderón complex interpolation space, see [1, Chapt. 4] or [11]. However, in the case of quasi-Banach spaces this term is designed to ensure both the completeness of $\mathcal{F}$ and the continuity of evaluation functions $\mathcal{F} \ni f \mapsto f(z)$.

Another variant of the complex method is obtained by replacing $\mathcal{F}$ by its subspace $\mathcal{F}_0$ generated by functions with finite dimensional range, see [29]. Hence, $f \in \mathcal{F}_0$ if and only if $f(z) = \sum_{j=1}^m f_j(z)x_j$, $x_j \in X_0 \cap X_1$, and f_j's are bounded analytic scalar functions in S and continuous on $\bar{S}$. The advantage of this approach is that one can avoid discussing the meaning of analytic functions with values in quasi-Banach spaces [32].

Kalton [31] introduced and studied the following important subclass of quasi-Banach spaces.

Definition 6.3 We say that a quasi-Banach space is A-convex (analytically convex) if there exists a constant $C > 0$ such that for every polynomial $P : \mathbb{C} \to X$, we have

$$||P(0)||_X \leq C \max_{|z|=1} ||P(z)||_X.$$

The main advantage of this class is that it possesses some of the useful properties of Banach spaces such as the Maximum Modulus Principle [31]. In particular, if $X_0 + X_1$ is A-convex, then the last term in (6.12) can be dropped. Furthermore, Triebel–Lizorkin spaces are known to be A-convex [33].

Lemma 6.3 *Suppose $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, and μ is a ρ_A-doubling measure. Then, the spaces $\dot{\mathbf{f}}_p^{\alpha,q}(A, \mu)$ and $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ are A-convex.*

Proof We say that a quasi-Banach lattice $(X, ||\cdot||_X)$ is called *lattice r-convex* if there exists a constant $C > 0$ such that

$$\left\|\left(\sum_{j=1}^m |f_j|^r\right)^{1/r}\right\|_X \leq C\left(\sum_{j=1}^m ||f_j||_X^r\right)^{1/r} \tag{6.13}$$

for any finite family $\{f_j\}_{1\le j\le m}$ in X. By the results of Kalton [30, Theorem 2.2] and [31, Theorem 4.4.], see also [33, Theorem 4.4], for any quasi-Banach lattice X we have

$$X \text{ is A-convex} \iff X \text{ is lattice } r\text{-convex for some } r > 0.$$

Let $X = \dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$. Then, it is not difficult to verify that X is a quasi-Banach lattice satisfying (6.13) with $r = \min(p,q)$ and $C = 1$. Hence, by the above mentioned result of Kalton, X is A-convex. By Theorem 2.7, one can treat $\dot{\mathbf{F}}_p^{\alpha,q}$ as a closed subspace of $\dot{\mathbf{f}}_p^{\alpha,q}$, since we have the equivalence of norms $||f||_{\dot{\mathbf{F}}_p^{\alpha,q}} \asymp ||S_\varphi f||_{\dot{\mathbf{f}}_p^{\alpha,q}}$. Consequently, $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A,\mu)$ is A-convex as well. □

We will need the following result due to Kalton and Mitrea [32, Theorem 3.4] and its $\mathcal{F}_0$-variant due to Gomez and Milman [22].

Theorem 6.4 *Let Ω be a Polish space and ν a σ-finite Borel measure on Ω. Let X_0 and X_1 be a pair of quasi-Banach function spaces on (Ω,ν). Suppose that both X_0 and X_1 are A-convex and separable. Then, $X_0 + X_1$ is A-convex and*

$$X_\theta = [X_0, X_1]_\theta = X_0^{1-\theta} X_1^\theta,$$

in the sense of equivalence of norms.

We are now ready to prove Theorem 6.2.

Proof of Theorem 6.2 By Lemma 6.3, the spaces $X_0 = \dot{\mathbf{f}}_{p_0}^{\alpha_0,q_0}$ and $X_1 = \dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}$ are A-convex. Then, by Theorem 6.4 the complex interpolation space $[X_0, X_1]_\theta$ equals the Calderón product $X = X_0^{1-\theta} X_1^\theta$ which is identified with $\dot{\mathbf{f}}_p^{\alpha,q}$ by Theorem 6.1. Note that even if X_1 is not separable ($p_1 = \infty$ or $q_1 = \infty$), then Theorem 6.4 still applies by the remark following [33, Proposition 4.9]. Hence, we have (6.10). The functorial property of interpolation and Theorem 2.7 imply (6.11). □

Finally, we discuss two other interpolation methods which work very well for quasi-Banach spaces. Nilsson [35, Theorem 2.1] proved that in the case of quasi-Banach lattices X_0 and X_1, there are two methods which can be conveniently described in terms of Calderón product $X = X_0^{1-\theta} X_1^\theta$. These are:

1. Gagliardo's method of interpolation [35,36]
$$\langle X_0, X_1\rangle_\theta = X^0,$$
where X^0 is the closure of $X_0 \cap X_1$ in X,
2. $\pm$ method of interpolation of Gustavsson and Peetre [24]
$$X \subset \langle X_0, X_1, \theta\rangle \subset X^{\sim},$$
where $X^{\sim}$ is the Gagliardo closure of X in $X_0 + X_1$.

Nilsson's result holds under the assumption that some convexifications of X_0 and X_1 are Banach lattices, see [35, Definition 1.7], which is satisfied for $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces. Consequently, we have the following generalization of a result of Frazier and Jawerth [19, Sect. 8].

Corollary 6.5 *Suppose $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, q_0 \le \infty$, $0 < p_1, q_1 \le \infty$, and μ is a ρ_A-doubling measure. Then, for any $0 < \theta < 1$,*

$$\langle \dot{\mathbf{f}}_{p_0}^{\alpha_0,q_0}, \dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}\rangle_\theta = \dot{\mathbf{f}}_p^{\alpha,q}(A,\mu), \quad \langle \dot{\mathbf{F}}_{p_0}^{\alpha_0,q_0}, \dot{\mathbf{F}}_{p_1}^{\alpha_1,q_1}\rangle_\theta = \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A,\mu), \quad (p,q<\infty), \tag{6.14}$$

$$\langle \dot{\mathbf{f}}_{p_0}^{\alpha_0,q_0}, \dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}, \theta\rangle = \dot{\mathbf{f}}_p^{\alpha,q}(A,\mu), \quad \langle \dot{\mathbf{F}}_{p_0}^{\alpha_0,q_0}, \dot{\mathbf{F}}_{p_1}^{\alpha_1,q_1}, \theta\rangle = \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A,\mu), \tag{6.15}$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$, and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

Idea of the proof It suffices to prove only the first parts of (6.14) and (6.15) by the functorial property of interpolation and Theorem 2.7.

Let $X_0 = \dot{\mathbf{f}}_{p_0}^{\alpha_0,q_0}(A,\mu)$ and $X_1 = \dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}(A,\mu)$. By Theorem 6.1, $X = X_0^{1-\theta} X_1^{\theta} = \dot{\mathbf{f}}_p^{\alpha,q}$. Since finite sequences are dense in $\dot{\mathbf{f}}_p^{\alpha,q}$ for $p, q < \infty$, we have $\dot{\mathbf{f}}_p^{\alpha,q} = (\dot{\mathbf{f}}_p^{\alpha,q})^o$, which proves (6.14). Likewise, (6.15) is a consequence of

$$\dot{\mathbf{f}}_p^{\alpha,q} = (\dot{\mathbf{f}}_p^{\alpha,q})\tilde{}. \tag{6.16}$$

The proof of (6.16) is a straightforward modification of the argument in [19, Theorem 8.5], and hence it is skipped. □

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