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Baggett's problem for frame wavelets

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1.1 Introduction

For a function $\psi \in L^2(\mathbb{R})$, we define its affine (or wavelet) system by

$$\mathcal{W}(\psi) = \{\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$$

If the system is an orthonormal basis of $L^2(\mathbb{R})$, then we call ψ a wavelet. In the more general case when the system forms a frame for $L^2(\mathbb{R})$, we call ψ a frame wavelet, or simply a framelet. If $\mathcal{W}(\psi)$ is a tight frame (with constant 1), i.e.,

$$||f||^2 = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 \qquad \text{for all } f \in L^2(\mathbb{R}),$$

then ψ is a tight framelet and also called a Parseval wavelet.

One of the fundamental problems in the theory of wavelets is a problem posed by Baggett in 1999. Baggett's problem asks whether every Parseval wavelet ψ must necessarily come from a generalized multiresolution analysis (GMRA). The precise meaning of this statement is explained later. Nonetheless, this problem can be reformulated in terms of the *space of negative dilates* of ψ defined as

$$V(\psi) = \overline{\text{span}}\{\psi_{j,k} : j < 0, k \in \mathbb{Z}\}. \tag{1.1}$$

Question 1 (Baggett, 1999). Let ψ be a Parseval wavelet with the space of negative dilates $V = V(\psi)$. Is it true that

$$\bigcap_{j\in\mathbb{Z}} D^j(V) = \{0\}\,?$$

Despite its simplicity Question 1 is a difficult open problem and only partial results are known. For example, Rzeszotnik and the author proved in [15] that if the dimension function (also called multiplicity function) of $V(\psi)$ is not identically ∞, then the answer to Question 1 is affirmative.

Question 1 is not only interesting for its own sake, but it also has several implications for other aspects of the wavelet theory. Rzeszotnik and the author [14] showed that a positive answer to Question 1 would imply that all compactly supported Parseval wavelets come from a MRA, thus generalizing the well-known result of Lemarié-Rieusset [1, 31] for compactly supported (orthonormal) wavelets. Furthermore, the answer to Question 1 would help in understanding the structure of the set of Parseval wavelets which was recently studied by Šikić, Speegle, and Weiss [37].

However, there is some evidence that the answer to Question 1 might be negative. This is because there exists a (non-tight) frame wavelet ψ with a very large space of negative dilates. The first example of such ψ was given by Rzeszotnik and the author in [14]. In fact, ψ has a dual frame wavelet and the space of negative dilates of ψ is the largest possible $V(\psi) = L^2(\mathbb{R})$. Here, we improve this result by showing that one can find such ψ with good smoothness and decay properties, e.g., ψ in the Schwartz class $\mathcal{S}(\mathbb{R})$.

1.2 Preliminaries

Despite the fact that all of our results are motivated by the classical case of dyadic dilations in $\mathbb{R}$ we will adopt a more general setting of an expansive integer-valued matrix, i.e., an $n \times n$ matrix whose eigenvalues have modulus greater than 1. That is, we shall assume that we are given an $n \times n$ expansive matrix A with integer entries, which plays the role of the usual dyadic dilation. The *dilation* operator D is given by $D\psi(x) = |\det A|^{1/2}\psi(Ax)$ and the *translation* operator T_k is given by $T_k f(x) = f(x-k)$, $k \in \mathbb{Z}^n$.

We say that a finite family $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$ is a *wavelet* if its associated affine system

$$\psi_{j,k} = D^j T_k \psi, \qquad j \in \mathbb{Z}, k \in \mathbb{Z}^n, \psi \in \Psi$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$. In the more general case, when the affine system is a frame or tight frame (with constant 1), we say that Ψ is a *frame wavelet* or a *Parseval wavelet*, resp. Moreover, a frame wavelet Ψ is called *semi-orthogonal* if

$$D^j W \perp D^{j'} W \qquad \text{for all } j \neq j' \in \mathbb{Z}.$$

where

$$W = W(\Psi) = \overline{\operatorname{span}}\{T_k\psi : k \in \mathbb{Z}^n, \psi \in \Psi\}. \tag{1.2}$$

The support of a function f defined on $\mathbb{R}^n$ is denoted by

$$\operatorname{supp} f = \{x \in \mathbb{R}^n : f(x) \neq 0\}.$$

Note that we are not taking the closure, since most of our functions are elements of $L^2(\mathbb{R}^n)$ and hence they are defined a.e. Given a Lebesgue measurable set $K \subset \mathbb{R}^n$, define the space

$$\check{L}^2(K) = \{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \hat{f} \subset K\}.$$

Here, the Fourier transform is defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle}\, dx.$$

1.2.1 GMRAs

Definition 1. *A sequence $\{D^j(V)\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$ is called a generalized multiresolution analysis (GMRA) if*

(M1) $T_k V = V$ *for all* $k \in \mathbb{Z}^n$*,*
(M2) $V \subset D(V)$*,*
(M3) $\overline{\bigcup_{j\in\mathbb{Z}} D^j(V)} = L^2(\mathbb{R}^n)$*,*
(M4) $\bigcap_{j\in\mathbb{Z}} D^j(V) = \{0\}$*.*
In addition, if (M5) holds,
(M5) $\exists\ \varphi \in V$ *such that* $\{T_k\varphi\}_{k\in\mathbb{Z}^n}$ *is an orthonormal basis of* V*,*

then $\{D^j(V)\}_{j\in\mathbb{Z}}$ *is a multiresolution analysis (MRA).*

A GMRA $\{D^j(V)\}_{j\in\mathbb{Z}}$ is customarily written as $\{V_j\}_{j\in\mathbb{Z}}$, where $V_j = D^j(V)$. The space V is called the *core space* of the GMRA. Condition (M1) means that V is shift-invariant (SI) and allows us to use the theory of shift-invariant spaces for understanding the connections between the GMRA structure and wavelets or framelets. This is a subject of an extensive study by several authors, e.g. [3, 4, 5, 7, 11, 13, 17, 29, 30].

For a family $\Psi \subset L^2(\mathbb{R}^n)$ we define its *space of negative dilates* by

$$V = V(\Psi) = \overline{\operatorname{span}}\{\psi_{j,k} : j < 0, k \in \mathbb{Z}^n, \psi \in \Psi\}. \tag{1.3}$$

We say that a frame wavelet Ψ is associated with a GMRA, or shortly comes from a GMRA, if its space $V = V(\Psi)$ satisfies (M1)–(M4). In addition, if V satisfies (M5), then V is associated with an MRA.

It turns out that every semi-orthogonal frame wavelet Ψ comes from a GMRA. That is, the space $V = V(\Psi)$ satisfies the conditions (M1)–(M4) and, therefore, V is a core space of a GMRA. This is an easy consequence of the fact that the spaces V and W given by (1.2) and (1.3) satisfy

$$\bigoplus_{j\in\mathbb{Z}} D^j(W) = L^2(\mathbb{R}^n), \qquad V = \bigoplus_{j\le -1} D^j(W) = \Big(\bigoplus_{j\ge 0} D^j(W)\Big)^{\perp}. \tag{1.4}$$

Conversely, if we want to see when a GMRA gives rise to a wavelet, or a semi-orthogonal frame wavelet, then some knowledge of shift-invariant spaces is useful.

1.2.2 The spectral function of shift-invariant spaces

Every shift-invariant space $V \subset L^2(\mathbb{R}^n)$ has a *set of generators* Φ, that is, a countable family of functions whose integer shifts form a tight frame (with constant 1) for V, see [10, Theorem 3.3]. Although this family is not unique, the function

$$\sigma_V(\xi) = \sum_{\varphi\in\Phi} |\hat{\varphi}(\xi)|^2$$

does not depend (except on a set of null measure) on the choice of the family of generators. We call σ_V the *spectral function* of V. This notion was introduced by Rzeszotnik and the author in [13]. The basic property of σ is that it is additive on countable orthogonal sums of SI spaces and that $\sigma_{L^2(\mathbb{R}^n)} = 1$. The spectral function also behaves nicely under dilations since $\sigma_{D(V)}(\xi) = \sigma_V((A^T)^{-1}\xi)$. Moreover, if V is generated by a single function φ then

$$\sigma_V(\xi) = \begin{cases} |\hat{\varphi}(\xi)|^2(\sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(\xi+k)|^2)^{-1} & \text{for } \xi \in \operatorname{supp}\hat{\varphi}, \\ 0 & \text{otherwise.} \end{cases}$$

We also mention that there are several other equivalent ways of defining the spectral function among which we note the following formula

$$\sigma_V(\xi) = \lim_{\varepsilon\to 0} ||P_{\hat{V}}(\mathbf{1}_{(\xi-\varepsilon/2,\xi+\varepsilon/2)^n})||^2/\varepsilon^n \qquad \text{for a.e. } \xi \in \mathbb{R}^n,$$

where $P_{\hat{V}}$ denotes the orthogonal projection of $\mathcal{F}(V) = \hat{V}$ onto $L^2(\mathbb{R}^n)$.

The spectral function also allows us to define the *dimension function* of V

$$\dim_V(\xi) = \sum_{k\in\mathbb{Z}^n} \sigma_V(\xi+k).$$

The dimension function (also called the multiplicity function) takes values in $\mathbb{N}\cup\{0,\infty\}$. It is additive on countable orthogonal sums as the spectral function. Moreover, the minimal number of functions needed to generate V is equal to the L^∞ norm of $\dim_V$. In particular, V can be generated by a single function if and only if $\dim_V \le 1$. Moreover, condition (M5) is equivalent to the equation $\dim_V \equiv 1$. We refer the reader to [10, 13] for the proofs of all these facts.

1.2.3 Semi-orthogonal Parseval wavelets and GMRAs

The dimension function can be applied to connect GMRAs to semi-orthogonal Parseval wavelets. If V is a core space of a GMRA, then the space $W = D(V) \ominus V$ is shift-invariant and has a (possibly infinite) set of generators Ψ. From (M2), (M3), and (M4) it follows that

$$L^2(\mathbb{R}^n) = \bigoplus_{j\in\mathbb{Z}} D^j(W),$$

so we conclude that Ψ is a Parseval wavelet possibly of infinite order. That is, Ψ may have infinite number of generators and the affine system generated by the elements of Ψ forms a tight frame for $L^2(\mathbb{R}^n)$. Moreover, Ψ is clearly semi-orthogonal.

Conversely, if Ψ is a semi-orthogonal Parseval wavelet (possibly of infinite order), then the space V of its negative dilates satisfies conditions (M1)–(M4) due to (1.4). Therefore, there is a perfect duality between GMRA structures and semi-orthogonal Parseval wavelets (with possibly infinite number of generators).

Since we are interested in finitely generated frame wavelets, the following result provides the required connection.

Theorem 1. *Suppose that Ψ is a semi-orthogonal Parseval wavelet with L generators and V is the space of negative dilates of Ψ. Then, $\{D^j(V)\}_{j\in\mathbb{Z}}$ is a GMRA such that*

$$\dim_V(\xi) < \infty \qquad \text{for a.e. } \xi, \tag{1.5}$$

and

$$\sum_{d\in\mathcal{D}} \dim_V((A^*)^{-1}(\xi+d)) - \dim_V(\xi) \le L \qquad \text{for a.e. } \xi, \tag{1.6}$$

where $\mathcal{D}$ consists of representatives of distinct cosets of $\mathbb{Z}^n/(A^*\mathbb{Z}^n)$.

Conversely, if $\{D^j(V)\}_{j\in\mathbb{Z}}$ is a GMRA satisfying (1.5) *and* (1.6), *then there exists a a semi-orthogonal Parseval wavelet Ψ (with at most L generators) associated with this GMRA.*

Theorem 1 is a variant of the following well-known result of Baggett et al. [4]. For simplicity we state Theorem 2 in a shorter form. Its full form looks analogously as Theorem 1.

Theorem 2 (Baggett, Medina, Merrill, 1999). *A GMRA gives rise to a wavelet with L generators if and only if the dimension function of its core space V satisfies* (1.5) *and*

$$\sum_{d\in\mathcal{D}} \dim_V((A^*)^{-1}(\xi+d)) - \dim_V(\xi) = L \qquad \text{for a.e. } \xi. \tag{1.7}$$

Equation (1.7) is often referred as the *consistency equation* of Baggett. In order to establish Theorem 1 we recall the following fact shown in [13].

Lemma 1. *If Ψ is a semi–orthogonal Parseval wavelet and V is the space of negative dilates of Ψ, then*

$$\sigma_V(\xi) = \sum_{\psi\in\Psi}\sum_{j=1}^{\infty} |\hat{\psi}((A^*)^j\xi)|^2.$$

In particular,

$$\dim_V(\xi) = D_\Psi(\xi) \qquad \text{for a.e. } \xi,$$

where

$$D_\Psi(\xi) := \sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}^n}\sum_{j=1}^{\infty}|\hat\psi((A^*)^j(\xi+k))|^2. \tag{1.8}$$

The function D_Ψ is often referred to as the *wavelet dimension function* [1, 2, 16, 27, 35].

Proof (Theorem 1). Suppose that Ψ is a semi-orthogonal Parseval wavelet with L generators and the spaces W and V are given by (1.2) and (1.3). We already know that $\{D^j(V)\}_{j\in\mathbb{Z}}$ is a GMRA. By Lemma 1,

$$\begin{aligned}\int_{[0,1]^n}\dim_V(\xi)d\xi &= \int_{\mathbb{R}^n}\sigma_V(\xi)d\xi = \sum_{\psi\in\Psi}\sum_{j=1}^{\infty}\int_{\mathbb{R}}|\hat\psi((A^*)^j\xi)|^2\\ &= \sum_{\psi\in\Psi}||\psi||^2/(|\det A|-1) \le L/(|\det A|-1) < \infty.\end{aligned} \tag{1.9}$$

Hence, (1.5) holds. Since $W\oplus V = D(V)$, we have

$$\sigma_W(\xi)+\sigma_V(\xi) = \sigma_{D(V)}(\xi) = \sigma_V((A^*)^{-1}\xi).$$

This implies that

$$\dim_W(\xi)+\dim_V(\xi) = \sum_{d\in\mathcal{D}}\dim_V((A^*)^{-1}(\xi+d)) \qquad \text{for a.e. } \xi, \tag{1.10}$$

where $\mathcal{D}$ consists of representatives of distinct cosets of $\mathbb{Z}^n/(A^*\mathbb{Z}^n)$. Since $\dim_W(\xi)\le L$, (1.6) holds.

Conversely, let $\{D^j(V)\}_{j\in\mathbb{Z}}$ be a GMRA satisfying (1.5) and (1.6). Let $W = D(V)\ominus V$. The consistency equation (1.10) and (1.6) yields

$$\dim_W(\xi)\le L \qquad \text{for a.e. } \xi.$$

By [10, Theorem 3.3] this implies that W has a set Ψ of $\le L$ generators. Since

$$V = \bigoplus_{j\le -1} D^j(W),$$

we infer that Ψ is a semi–orthogonal Parseval wavelet associated with the GMRA $\{D^j(V)\}_{j\in\mathbb{Z}}$.

1.3 Baggett's problem for Parseval wavelets

Baggett posed the following open problem during his talk at Washington University in 1999.

Question 2 (Baggett, 1999). Is every Parseval wavelet Ψ associated with a GMRA?

For the sake of historical accuracy, one should add that Baggett actually attempted to answer affirmatively Question 2 during his momentous lecture. This has sparked the interest of two listeners, Rzeszotnik and the author, who pointed out a missing argument in Baggett's approach. Despite several attempts in the next few years Question 2 remains unanswered as of now. Nonetheless, in his talk Baggett proved that Questions 1 and 2 are equivalent. Indeed, the following observation is due to Baggett.

Proposition 1 (Baggett, 1999). *If Ψ is a Parseval wavelet, then its space of negative dilates V is shift-invariant.*

Proof. It is enough to prove that the orthogonal complement $V^\perp$ of V is shift-invariant. It is clear that this complement is given by

$$V^\perp = \{\, f \in L^2(\mathbb{R}^n) : \|f\|_2^2 = \sum_{\psi\in\Psi}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} |\langle f, \psi_{j,k}\rangle|^2 \,\}$$

by the tight frame property. Thus, we can see immediately that the space $V^\perp$ is shift–invariant. □

We remark that the above result also holds if we assume that the framelet Ψ has a canonical dual framelet with the same number of generators, or equivalently, that Ψ has period one in the terminology of Daubechies and Han [23]. However, Proposition 1 in general is false for non-tight framelets and even for framelets which have a dual framelet. These facts were shown by Weber and the author in [17].

Proposition 1 proves that the space of negative dilates of a Parseval wavelet Ψ satisfies condition (M1). The other two conditions, (M2) and (M3), are clearly satisfied leaving only (M4). This crucial obstacle leads naturally to Question 1. Consequently, Questions 1 and 2 are equivalent.

In general, one might want to know what conditions on a shift-invariant space V guarantee that

$$\bigcap_{j\in\mathbb{Z}} D^j(V) = \{0\}. \tag{1.11}$$

A non-trivial result of this type was shown by Rzeszotnik in [36].

Proposition 2 (Rzeszotnik, 2001). *Let V be a shift-invariant space. If $\sigma_V \in L^1(\mathbb{R}^n)$, then condition (1.11) holds.*

In the case when V is a space of negative dilates we have a stronger result due to Rzeszotnik and the author [15].

Theorem 3 (Bownik, Rzeszotnik, 2006). *Let $\Psi \subset L^2(\mathbb{R}^n)$ be a Parseval wavelet and V be its space of negative dilates. If*

$$|\{\xi \in \mathbb{R}^n : \dim_V(\xi) < \infty\}| > 0, \tag{1.12}$$

then (1.11) holds and Ψ generates a GMRA.

While the complete proof of Theorem 3 can be found in [15], we present its outline containing the key idea of *semi-orthogonalization* appearing later in the proof of Theorem 4. This procedure constructs a semi-orthogonal wavelet which is associated to the same GMRA as a given Parseval wavelet. In practice, it may not even be known whether a Parseval wavelet Ψ, as in Theorem 3, is associated with a GMRA. Nevertheless, one can use the idea of semi-orthogonalization to eventually deduce this property.

Proof. Let $W = D(V) \ominus V$. Observe that W is a shift-invariant space generated by $\{\psi - P_V\psi\}_{\psi\in\Psi}$, where P_V is the orthogonal projection on V. Since Ψ is finite, W has a finite number of generators. That is, we have $\dim_W \le L$ for some $L \in \mathbb{N}$. The equation $D(V) = V \oplus W$ implies that

$$\sum_{d\in\mathcal{D}} m(B^{-1}\xi + d) = m(\xi) + \dim_W(\xi) \le m(\xi) + L, \tag{1.13}$$

where $m = \dim_V$ and $B = A^*$. To complete the proof we need the following result from [15].

Lemma 2. *Suppose that $m : \mathbb{R}^n \to [0,\infty)$ is $\mathbb{Z}^n$-periodic, measurable function such that*

$$\sum_{d\in D} m(\xi + d) \le m(B\xi) + L \qquad \textit{for a.e. } \xi \in \mathbb{T}^n, \tag{1.14}$$

for some $L \ge 0$. Then,

$$\int_{\mathbb{T}^n} m(\xi)d\xi \le L/(|\det A| - 1). \tag{1.15}$$

To apply Lemma 2 we need to show that m is finite a.e. This can be done using a simple ergodic argument.

Since the matrix $B = A^* : \mathbb{R}^n \to \mathbb{R}^n$ preserves the lattice $\mathbb{Z}^n$, it induces a measure preserving endomorphism $\tilde{B} : \mathbb{T}^n \to \mathbb{T}^n$. Moreover, $\tilde{B}$ is ergodic by [38, Corollary 1.10.1] because B is expansive. Define the set

$$E = \{\xi \in \mathbb{T}^n : m(\xi) < \infty\}.$$

The condition (1.13) implies that $\tilde{B}^{-1}E \subset E$. Since $\tilde{B}$ is measure preserving we must have $\tilde{B}^{-1}E = E$ (modulo null sets). Finally, by the ergodicity of $\tilde{B}$, we have either $|E| = 0$ or $|E| = 1$. Combining this with our hypothesis $|E| > 0$, proves that $m(\xi) < \infty$ for a.e. $\xi \in \mathbb{R}^n$.

Since all the assumptions of Lemma 2 are satisfied for our m, we get that $m \in L^1(\mathbb{T}^n)$. Equivalently, we have $\sigma_V \in L^1(\mathbb{R}^n)$. By Proposition 2, (1.11) holds and Ψ generates a GMRA.

We end this section by mentioning an interesting variant of Baggett's problem for single-generated Parseval wavelets [37].

Question 3 (Šikić, Speegle, and Weiss, 2007). Let V be the space of negative dilates of a Parseval wavelet ψ. Is it true that

$$\psi \notin V. \tag{1.16}$$

Naturally, an affirmative answer to Question 1 implies a positive answer to Question 3. However, the converse implication is not known. Nonetheless, the following equivalent statements about a Parseval wavelet ψ can be easily shown [37]:

(i) $\psi \in V$,
(ii) $V = DV$,
(iii) $V = L^2(\mathbb{R})$.

Once we relax the assumption that ψ is a Parseval wavelet, then Questions 1 and 3 are distinct. In Theorem 7, we shall exhibit a frame wavelet ψ such that $\psi \notin V$, but (1.11) fails.

1.4 Ramifications of Baggett's problem

A positive answer to Baggett's problem influences many other problems involving Parseval wavelets. The reason behind it is a *semi-orthogonalization* procedure which was introduced by Rzeszotnik and the author in [14].

Theorem 4. *Suppose that Ψ is a Parseval wavelet with L generators and its space of negative dilates V satisfies* (1.11)*. Then, there exists a semi-orthogonal Parseval wavelet Φ with $\leq L$ generators such that its space of negative dilates is also V. In other words, both Ψ and Φ are associated with the same GMRA $\{D^j(V)\}_{j\in\mathbb{Z}}$.*

Proof. Let V be the space of negative dilates of Ψ. By the hypothesis (1.11), the sequence $\{D^j(V)\}_{j\in\mathbb{Z}}$ is a GMRA. Let $W = D(V) \ominus V$. Observe that W is generated by L functions, namely $\psi - P_V\psi$, $\psi \in \Psi$, where P_V is the orthogonal projection onto V. Therefore, we can find a set Φ of $\leq L$ generators for W. As in the proof of Theorem 1, we have

$$V = \bigoplus_{j\leq -1} D^j(W).$$

Hence, we can infer that that Φ is a semi-orthogonal Parseval wavelet and V is the space of negative dilates of Φ. Therefore, Φ is associated to the same GMRA as Ψ.

Remark 1. A more explicit semi-orthogonalization procedure for the subclass of MRA Parseval wavelets was introduced recently by Šikić et al. [37]. Suppose that $\psi \in L^2(\mathbb{R})$ is a dyadic Parseval wavelet associated with an MRA. Let m be its generalized low-pass filter [32, 33, 37]. Then, the authors of [37] proved that one can modify the filter m in some minimal way to obtain a new filter corresponding to a semi-orthogonal Parseval wavelet ϕ which is associated with the same MRA as ψ.

As a corollary of Theorems 1 and 4 we deduce that Parseval wavelets give rise to the same class of GMRAs as **semi-orthogonal** Parseval wavelets. A priori, this is only true for Parseval wavelets associated with a GMRA which may (or may not) encompass all Parseval wavelets depending on the answer to Question 2.

Corollary 1. *Suppose that Ψ is a Parseval wavelet with L generators. Then, either $\{D^j(V)\}_{j\in\mathbb{Z}}$ is a GMRA satisfying* (1.5) *and* (1.6)*, or* $\dim_V \equiv \infty$.

Proof. If $\dim_V$ is not identically ∞, then $\{D^j(V)\}_{j\in\mathbb{Z}}$ is a GMRA by Theorem 3. Hence, Theorems 1 and 4 imply that (1.5) and (1.6) hold.

Next, we deduce that an affirmative answer to Baggett's problem implies that a compactly supported Parseval wavelet comes from an MRA [14].

Theorem 5 (Bownik, Rzeszotnik, 2005). *Let Ψ be a Parseval wavelet with $L = |\det A| - 1$ generators such that its space of negative dilates V satisfies condition* (1.11)*. Then, Ψ is associated with an MRA if and only if*

$$D_\Psi(\xi) = \sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}}\sum_{j=1}^{\infty} |\hat{\psi}((A^*)^j(\xi+k))|^2 > 0 \quad a.e. \tag{1.17}$$

Remark 2. We recall that the restriction on the number of generators $L = |\det A| - 1$ in Theorem 5 is a necessary condition for (orthogonal) wavelet Ψ to be associated with an MRA due to Lemma 1. In the case of Parseval wavelets it is possible to have MRA constructions resulting with bigger number of generators, see [20, 21, 24, 26, 34]. However, Theorem 5 is false if we relax the assumption $L = |\det A| - 1$.

Remark 3. We must emphasize that for general Parseval wavelets D_Ψ is not equal to $\dim_V$. This is unlike the case of semi-orthogonal wavelets, where Lemma 1 yields

$$D_\Psi \equiv \dim_V\,. \tag{1.18}$$

Conversely, by the results of Paluszyński et al. [33] the identity (1.18) forces a Parseval wavelet Ψ to be semi-orthogonal, see also [37, Theorem 3.15]. For the sake of accuracy, we should add that this result was shown only for dyadic, single generated, 1-dimensional Parseval wavelets.

Despite that (1.18) may fail we have that for any Parseval wavelet Ψ

$$\operatorname{supp} D_\Psi = \operatorname{supp} \dim_V, \tag{1.19}$$

see [14]. Indeed, by Proposition 1, V is a shift-invariant space generated by the functions

$$\{D^{-j}\psi : \psi \in \Psi, j = 1, 2, \ldots\}.$$

This, combined with an equivalent definition of the dimension function of shift-invariant spaces in terms of its range function, see [8, 10], yields

$$\dim_V(\xi) = \dim \operatorname{span}\{(\hat{\psi}((A^*)^j(\xi + k))_{k\in\mathbb{Z}^n} : \psi \in \Psi, j = 1, 2, \ldots\},$$

which shows (1.19).

Proof (Theorem 5). First, suppose that Ψ is associated with an MRA, i.e., its space of negative dilates satisfies $\dim_V \equiv 1$. By (1.18) we have that $\operatorname{supp} D_\Psi = \mathbb{R}^n$ and thus (1.17) holds.

Conversely, assume (1.17). We need to show that (M5) is satisfied, or equivalently that $\dim_V \equiv 1$. Let Φ be the semi-orthogonal Parseval wavelet obtained from Ψ by Theorem 4. By Lemma 1 and the estimate (1.9) with Φ taking place of Ψ, we have

$$\int_{[0,1]^n} \dim_V(\xi)\, d\xi = \sum_{\varphi\in\Phi} \|\hat{\varphi}\|^2/(|\det A| - 1) \le L/(|\det A| - 1) = 1.$$

On the other hand, (1.17) and (1.18) imply that $\dim_V(\xi) > 0$ for a.e. ξ. Since $\dim_V$ is integer–valued we have that $\dim_V \equiv 1$, which concludes the proof of Theorem 5. □

As a corollary of Theorem 5 we have the following extension of a result of Lemarié-Rieusset [31] to Parseval wavelets.

Corollary 2 (Bownik, Rzeszotnik, 2005). *Suppose that a Parseval wavelet Ψ satisfies the assumptions of Theorem 5 and at least one generator of Ψ is compactly supported. Then, Ψ is associated with an MRA.*

Combining Corollary 2 with Theorem 3, we have the following corollary.

Corollary 3. *Suppose that a Parseval wavelet Ψ has $L = |\det A| - 1$ generators and at least one of them is compactly supported. If the space V of negative dilates of Ψ satisfies* (1.12)*, then Ψ comes from an MRA.*

1.5 Frame wavelets with large spaces of negative dilates

In this section we prove that the assumption in Question 1 on ψ being a Parseval wavelet is necessary. This result is due to Rzeszotnik and the author [14] who constructed an example of a dyadic framelet $\psi \in L^2(\mathbb{R})$, such that its space of negative dilates V is the largest possible, i.e., $V = L^2(\mathbb{R})$. Furthermore, such a framelet can have frame bounds arbitrarily close to 1 and it has a dual framelet. Here, we shall improve the example in [14] by showing that such a framelet can also have good smoothness and decay properties.

Theorem 6. *For any $\delta > 0$, there exists a frame wavelet $\psi \in L^2(\mathbb{R})$ such that:*

(i) $\hat{\psi}$ *is C^∞ and all its derivatives have exponential decay,*
(ii) the frame bounds of $\mathcal{W}(\psi)$ are 1 and $1+\delta$,
(iii) the space of negative dilates of ψ is equal to $L^2(\mathbb{R})$,
(iv) ψ has a dual frame wavelet.

While the proof of Theorem 6 follows the general construction method of [14], there are also some significant changes due to the additional smoothness requirement on ψ. In the proof of Theorem 6 we will use the following two standard results. Lemma 3 gives a sufficient condition for an affine system to be a Bessel sequence. Its proof can be found in [28, Theorem 13.0.1]. Lemma 4 is a basic perturbation result for frames which can be found in [19, Corollary 15.1.5].

Lemma 3. *Suppose that $\psi \in L^2(\mathbb{R})$ is such that $\hat{\psi} \in L^\infty(\mathbb{R})$ and*

$$\hat{\psi}(\xi) = O(|\xi|^{\delta}) \qquad \text{as } \xi \to 0, \tag{1.20}$$
$$\hat{\psi}(\xi) = O(|\xi|^{-1/2-\delta}) \qquad \text{as } |\xi| \to \infty, \tag{1.21}$$
$$\tag{1.22}$$

for some $\delta > 0$. Then the affine system $\mathcal{W}(\psi)$ is a Bessel sequence.

Lemma 4. *Suppose that $\mathcal{H}$ is a Hilbert space, $\{f_j\} \subset \mathcal{H}$ is a frame with constants C_1 and C_2,*

$$C_1||f||^2 \le \sum_j |\langle f, f_j\rangle|^2 \le C_2||f||^2 \qquad \text{for all } f \in \mathcal{H},$$

and $\{g_j\} \subset \mathcal{H}$ is a Bessel sequence with constant C_0,

$$\sum_j |\langle f, g_j\rangle|^2 \le C_0||f||^2 \qquad \text{for all } f \in \mathcal{H}.$$

If $C_0 < C_1$, then $\{f_j + g_j\}$ is a frame with constants $((C_1)^{1/2} - (C_0)^{1/2})^2$ and $((C_2)^{1/2} + (C_0)^{1/2})^2$.

We will also need the following fact about the scale averaging of periodic functions. Lemma 5 can be considered as a special case of a result due to Bui and Laugesen [18, Lemma 9] which also holds for functions in $L^p_{loc}(\mathbb{R}^n)$ and fairly general dilation matrices. This result is very close in spirit to the classical results of Banach-Saks and Szlenk asserting that weak convergence in L^p implies norm convergence of arithmetic means. Since we impose weaker assumptions on Ψ than in [18], we present the proof of Lemma 5 for completeness.

Lemma 5. *Suppose $\Psi \in L^2(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. In other words, Ψ is a 1-periodic function in $L^2_{loc}(\mathbb{R})$. Let $\Psi_j(x) = \Psi(2^j x)$, and $c = \int_0^1 \Psi$. Then, for any strictly increasing sequence $(l_j)_{j\in\mathbb{N}} \subset \mathbb{N}$,*

$$\lim_{J\to\infty} \frac{1}{J} \sum_{j=1}^{J} \Psi_{l_j} = c \qquad \text{in } L^2_{loc}(\mathbb{R}). \tag{1.23}$$

Proof. Without loss of generality we can assume that $c = \int_0^1 \Psi = 0$. Otherwise, it suffices to apply (1.23) for a function $\Psi - c$. For the purpose of Lemma 5, let $||\Psi|| = (\int_0^1 |\Psi|^2)^{1/2}$ be the norm in $L^2(\mathbb{T})$ with the corresponding scalar product $\langle \cdot, \cdot \rangle$.

We claim that the sequence (Ψ_j) converges to c weakly in $L^2(\mathbb{T})$. Indeed, let f be 1-periodic and continuous. Take any $\varepsilon > 0$ and choose $j \in \mathbb{N}$ such that $|x - y| \le 2^{-j} \implies |f(x) - f(y)| \le \varepsilon$. Since

$$\int_{k/2^j}^{(k+1)/2^j} \Psi(2^j x) dx = 0$$

we have

$$\left| \int_0^1 \Psi_j f \right| = \left| \sum_{k=0}^{2^j-1} \int_{k/2^j}^{(k+1)/2^j} \Psi(2^j x)(f(x) - f(k/2^j)) dx \right| \le \varepsilon \int_0^1 |\Psi|.$$

A standard approximation argument using Luzin's theorem and $||\Psi_j|| = ||\Psi||$, shows the claim. In particular, we have that

$$d_j := |\langle \Psi, \Psi_j \rangle| \to 0 \qquad \text{as } j \to \infty. \tag{1.24}$$

For any $j \le k \in \mathbb{N}$, the change of variables and 1-periodicity of Ψ yields

$$|\langle \Psi_j, \Psi_k \rangle| = |\langle \Psi, \Psi_{k-j} \rangle| = d_{k-j}.$$

Thus, we have the estimate

$$||\Psi_{l_1} + \ldots + \Psi_{l_J}||^2 = \left| \sum_{j=1}^{J} \sum_{k=1}^{J} \langle \Psi_{l_j}, \Psi_{l_k} \rangle \right| \le \sum_{j=1}^{J} \sum_{k=1}^{J} d_{|l_j - l_k|} \le 2J \sum_{j=0}^{J-1} d_j^*.$$

Here, we used $d_{|l_j - l_k|} \le d^*_{|j-k|}$, where $d_j^* = \sup\{d_k : k \ge j\}$ is a decreasing sequence dominating (d_j). Hence, by (1.24)

$$\left\| \frac{\Psi_{l_1} + \ldots + \Psi_{l_J}}{J} \right\|^2 \le \frac{2}{J} \sum_{j=0}^{J-1} d_j^* \to 0 \qquad \text{as } J \to \infty.$$

This shows (1.23) and completes the proof of Lemma 5.

Proof (Theorem 6). Define the sets Z_1, Z_2 by

$$Z_1 = \bigcup_{k\in\mathbb{Z}} (k + (-1/4, 1/4)),$$
$$Z_2 = \mathbb{R} \setminus Z_1.$$

Suppose that $\psi^0 = \psi^1 + \psi^2$, where $\psi^1 \in \check{L}^2(Z_1)$ and $\psi^2 \in \check{L}^2(Z_2)$. As usual, define

$$W_j^l = \overline{\text{span}}\{\psi_{j,k}^l : k \in \mathbb{Z}\} \qquad \text{for } l = 0, 1, 2.$$

Lemma 6.

$$W_j^0 = W_j^1 \oplus W_j^2 \qquad \textit{for } j \in \mathbb{Z}. \tag{1.25}$$

Proof. It suffices to show (1.25) for $j = 0$. Take any $f \in W_0^1$ and $g \in W_0^2$. By the results in [8, 10] we have

$$W_0^l = \{f \in L^2 : \hat{f}(\xi) = m(\xi)\hat{\psi}^l(\xi), \quad m \text{ is measurable and 1-periodic}\}. \tag{1.26}$$

Since $\operatorname{supp}\hat{f} \subset Z_1$, $\operatorname{supp}\hat{g} \subset Z_2$ we have $f \perp g$. Thus, $W_0^1 \perp W_0^2$. Finally, it suffices to prove $W_0^1 \oplus W_0^2 \subset W_0^0$, since the converse inclusion is trivial. Take any $f \in W_0^1 \oplus W_0^2$. By (1.26) there are 1-periodic measurable functions m_1 and m_2 such that

$$\hat{f}(\xi) = m_1(\xi)\hat{\psi}^1(\xi) + m_2(\xi)\hat{\psi}^2(\xi) = m_1(\xi)\mathbf{1}_{Z_1}(\xi)\hat{\psi}^0(\xi) + m_2(\xi)\mathbf{1}_{Z_2}(\xi)\hat{\psi}^0(\xi). \tag{1.27}$$

Since the sets Z_1 and Z_2 are invariant under integer shifts, $m = m_1\mathbf{1}_{Z_1} + m_2\mathbf{1}_{Z_2}$ is 1-periodic. Hence, by (1.26) and (1.27) $f \in W_0^0$, which shows $W_0^0 = W_0^1 \oplus W_0^2$.

It now remains to choose ψ^1 and ψ^2 appropriately. The idea is that negative dilates of ψ^1 will generate functions whose Fourier transform is supported near the origin, whereas the negative dilates of ψ^2 will exhaust all functions which are supported away from the origin (in the Fourier domain).

Let ψ^1 be a Parseval wavelet such that $\hat{\psi}^1$ is C^∞ and

$$\operatorname{supp}\hat{\psi}^1 = (-1/4, -1/16) \cup (1/16, 1/4).$$

Such a frame wavelet can be constructed by a standard method, for example see [12]. Indeed, it suffices to take the convolution of $\mathbf{1}_{(-3/16,-1/8)\cup(1/8,3/16)}$ with a non-negative smooth bump function supported on $(-1/16, 1/16)$ and normalize the result to obtain the Calderón condition

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}^1(2^j\xi)|^2 = 1 \qquad \text{for } \xi \in \mathbb{R} \setminus \{0\}.$$

Note that $\psi^1 \in \check{L}^2(Z_1)$ and by (1.26), $W_0^1 = \check{L}^2((-1/4, -1/16) \cup (1/16, 1/4))$. Hence,

$$W_j^1 = \check{L}^2((-2^{j-2}, -2^{j-4}) \cup (2^{j-4}, 2^{j-2})) \qquad \text{for any } j \in \mathbb{Z},$$

and therefore, the space of negative dilates of ψ^1 is

$$\begin{aligned} V^1 = \overline{\operatorname{span}} \bigcup_{j<0} W_j^1 &= \check{L}^2\Big(\bigcup_{j=-\infty}^{-1} (-2^{j-2}, -2^{j-4}) \cup (2^{j-4}, 2^{j-2}) \Big) \\ &= \check{L}^2(-1/8, 1/8). \end{aligned} \tag{1.28}$$

The function ψ^2 should be regarded as a perturbation term of $\psi^0 = \psi^1 + \psi^2$. We are now ready to describe the construction procedure of ψ^2.

Let $\{\varphi_m : m \in \mathbb{N}\}$ be some enumeration of the "truncated" Gabor system

$$\{\mathbf{1}_{(k,k+1)} e^{2\pi i j \xi} : j \in \mathbb{Z}, k \in \mathbb{Z}, k \neq -1, 0\}.$$

Clearly, $\{\varphi_m : m \in \mathbb{N}\}$ is an orthonormal basis of $L^2((-\infty, -1) \cup (1, \infty))$. For any $m \in \mathbb{N}$, let $k_m \in \mathbb{Z}$ denote the left endpoint of the support of φ_m, i.e., $\operatorname{supp} \varphi_m = (k_m, k_m + 1)$.

Let Ψ be a 1-periodic function such that

$$\Psi \in C^\infty, \qquad \operatorname{supp} \Psi \subset Z_2, \qquad \int_0^1 \Psi = 1. \tag{1.29}$$

Let $(m_p)_{p\in\mathbb{N}}$ be a sequence of natural numbers such that each natural number occurs infinitely many times. We construct by induction a sequence of functions $\{\phi_p : p \in \mathbb{N}\}$ and a sequence of natural numbers $(l_p)_{p\in\mathbb{N}}$.

Let $\phi_1 = D^{-l_1}(\varphi_{m_1})\Psi$ and $l_1 = 1$. Suppose we have constructed functions $\phi_1, \ldots, \phi_p$ and integers $l_1, \ldots, l_p$ up to some $p \in \mathbb{N}$. Define l_{p+1} to be the smallest integer such that

$$\operatorname{supp} \phi_1 \cup \ldots \cup \operatorname{supp} \phi_p \subset (-2^{l_{p+1}}, 2^{l_{p+1}}), \tag{1.30}$$

and

$$\phi_{p+1} = D^{-l_{p+1}}(\varphi_{m_{p+1}})\Psi. \tag{1.31}$$

It is easy to see that the sequence $(l_p)_{p\in\mathbb{N}}$ is increasing and the supports of ϕ_p's are included in pairwise disjoint open intervals. Finally, define $\psi^2 \in \check{L}^2(Z_2)$ by

$$\widehat{\psi^2}(\xi) = \sum_{p\in\mathbb{N}} c_p \phi_p(\xi) = \sum_{p\in\mathbb{N}} c_p D^{-l_p}(\varphi_{m_p})\Psi, \tag{1.32}$$

for some sufficiently fast decaying sequence $(c_p)_{p\in\mathbb{N}}$ of positive numbers. More precisely, we can choose c_p's such that $0 < c_{p+1} < c_p/(p+1)$ for all $p \in \mathbb{N}$ and all derivatives of $\widehat{\psi^2}$ have exponential decay. This will guarantee that $\psi^0 = \psi^1 + \psi^2$ satisfies property (i) of Theorem 6. In particular, by Lemma 3, the affine system generated by ψ^2 is a Bessel sequence.

Our next goal is to show the following key fact.

Lemma 7. *Suppose that ψ^2 given by (1.32) is constructed as above. Let V^2 be the space of negative dilates of ψ^2 and P be the orthogonal projection onto $\check{L}^2((-\infty,-1)\cup(1,\infty))$, i.e.,*

$$\widehat{(Pf)}(\xi) = \hat{f}(\xi)\mathbf{1}_{(-\infty,-1)\cup(1,\infty)} \qquad \textit{for } f \in L^2(\mathbb{R}).$$

Then, $P(V^2)$ is dense in $\check{L}^2((-\infty,-1)\cup(1,\infty))$.

Proof. Since

$$\tilde{V}^2 := \overline{\operatorname{span}}\{\psi^2_{-l_p,0} : p \in \mathbb{N}\} \subset V^2$$

it suffices to show that $P(\tilde{V}^2)$ is dense in $\check{L}^2((-\infty,-1)\cup(1,\infty))$. Hence, we need to show that each basis element φ_m, $m \in \mathbb{N}$, of $L^2((-\infty,-1)\cup(1,\infty))$ belongs to the closure of $\mathcal{F}(P(\tilde{V}^2))$. Given $r \in \mathbb{N}$,

$$\widehat{\psi^2_{-l_r,0}} = D^{l_r}(\widehat{\psi^2}) = \sum_{p\in\mathbb{N}} c_p D^{l_r}(\phi_p).$$

By (1.30) and (1.31), $\operatorname{supp} D^{l_r}(\phi_p) \subset (-1,1)$ for $p<r$, and we have

$$\begin{aligned}(P(\psi^2_{-l_r,0}))\hat{} &= \sum_{p\ge r} c_p D^{l_r}(\phi_p) = \sum_{p\ge r} c_p D^{l_r-l_p}(\varphi_{m_p})\Psi_{l_r} \\ &= c_r\Psi_{l_r}\left[\varphi_{m_r} + \sum_{p>r}\frac{c_p}{c_r}D^{l_r-l_p}(\varphi_{m_p})\right].\end{aligned}$$

Since $c_{r+1}/c_r < 1/(r+1)$,

$$\left|\left|\sum_{p>r}\frac{c_p}{c_r}D^{l_r-l_p}(\varphi_{m_p})\right|\right| \le \sum_{p>r}\frac{1}{(r+1)(r+2)\dots p}||D^{l_r-l_p}(\varphi_{m_p})|| < 2/r,$$

we conclude that $\Psi_{l_r}(\varphi_{m_r}+\eta_r)$ belongs to $\mathcal{F}(P(\tilde{V}^2))$ for some $\eta_r \in L^2$ with $||\eta_r|| < 2/r$.

For a fixed $m \in \mathbb{N}$, let $R = \{r \in \mathbb{N} : m_r = m\}$. By our construction $R = \{r_1, r_2, \dots\}$ is infinite. By Lemma 5

$$\frac{\Psi_{l_{r_1}} + \dots + \Psi_{l_{r_J}}}{J} \to 1 \quad \text{as } J \to \infty \qquad \text{in } L^2(k_m, k_m+1).$$

Hence, as $J \to \infty$

$$\begin{aligned}&\frac{\Psi_{l_{r_1}}(\varphi_{m_{r_1}}+\eta_{r_1}) + \dots + \Psi_{l_{r_J}}(\varphi_{m_{r_J}}+\eta_{r_J})}{J} \\ &= \varphi_m\frac{\Psi_{l_{r_1}} + \dots + \Psi_{l_{r_J}}}{J} + \frac{\Psi_{l_{r_1}}\eta_{r_1} + \dots + \Psi_{l_{r_J}}\eta_{r_J}}{J} \to \varphi_m \qquad \text{in } L^2(\mathbb{R}),\end{aligned}$$

since $||\Psi_l||_\infty = ||\Psi||_\infty < \infty$ and

$$\frac{\eta_{r_1} + \ldots + \eta_{r_J}}{J} \to 0 \qquad \text{in } L^2(\mathbb{R}) \text{ as } J \to \infty.$$

Therefore, φ_m belongs to the closure of $\mathcal{F}(P(\tilde{V}^2))$. Since $m \in \mathbb{N}$ is arbitrary and $\{\varphi_m : m \in \mathbb{N}\}$ is an orthonormal basis of $L^2((-\infty,-1) \cup (1,\infty))$, this completes the proof of Lemma 7.

Lemma 8. *Suppose that V^2 is the same as in Lemma 7. Let P_j be the orthogonal projection onto $\check{L}^2((-\infty,-2^j) \cup (2^j,\infty))$, i.e.,*

$$\widehat{(P_j f)}(\xi) = \hat{f}(\xi)\mathbf{1}_{(-\infty,-2^j)\cup(2^j,\infty)} \qquad \text{for } f \in L^2(\mathbb{R}).$$

Then, $P_j(V^2)$ is dense in $\check{L}^2((-\infty,-2^j) \cup (2^j,\infty))$ for any $j \in \mathbb{Z}$.

Proof. Since the case $j \geq 0$ follows immediately from Lemma 7, we may assume that $j < 0$. A straightforward calculation shows that $P_j = D^j P D^{-j}$. Take any $f \in \check{L}^2((-\infty,-2^j) \cup (2^j,\infty))$. Since $D^{-j} f \in \check{L}^2((-\infty,-1) \cup (1,\infty))$, by Lemma 7 there exists a sequence $\{f_k : k \in \mathbb{N}\} \subset V^2$ such that $P_0 f_k \to D^{-j} f$ as $k \to \infty$. Hence, $P_j D^j f_k \to f$ as $k \to \infty$. Since $D^j f_k \in V^2$ for $j \leq 0$, this shows Lemma 8.

We are now ready to conclude the proof of Theorem 6. Let V^0 be the space of negative dilates of ψ^0. By (1.25),

$$V^0 = \overline{\text{span}}\left(\bigcup_{j<0} W_j^0\right) = \overline{\text{span}}\left(\bigcup_{j<0} (W_j^1 \cup W_j^2)\right) = \overline{\text{span}}(V^1 \cup V^2).$$

Therefore, by (1.28) and by Lemma 8

$$\overline{P_{-3}(V^2)} = \check{L}^2((-\infty,-1/8) \cup (1/8,\infty)),$$

we have that V^0 is dense in $L^2(\mathbb{R})$. Since V^0 is closed it must be equal to $L^2(\mathbb{R})$. It remains to show that one can also find a framelet with this property.

Recall that $\psi^0 = \psi^1 + \psi^2$, where ψ^1 is a Parseval wavelet and ψ^2 generates a Bessel affine system. Therefore, by Lemma 4, there exists $\varepsilon > 0$ such that $\psi' = \psi^1 + \varepsilon\psi^2$ is a framelet with frame bounds $1-\delta/3$ and $1+\delta/3$. Moreover, since $\varepsilon\psi^2$ is also of the form (1.32), the space of negative dilates of ψ' is also $L^2(\mathbb{R})$. Therefore, $\psi = (1-\delta/3)^{-1/2}\psi'$ is a framelet with constants 1 and $1+\delta$ whose space of negative dilates is $L^2(\mathbb{R})$. In fact, a more delicate argument shows that the lower frame bound of ψ' is ≥ 1 and the last normalization step is not necessary.

Finally, to show that ψ has a dual frame wavelet we employ the well-known characterizing equations [9, 25, 27]. We recall that functions $\phi, \psi \in L^2(\mathbb{R})$ whose respective affine systems are Bessel sequences form a pair of dual framelets if and only if

$$\sum_{j\in\mathbb{Z}} \hat{\phi}(2^j\xi)\overline{\hat{\psi}(2^j\xi)} = 1 \qquad \text{a.e. } \xi,$$

$$\sum_{j=0}^{\infty} \hat{\phi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+q))} = 0 \qquad \text{a.e. } \xi \text{ and for odd } q.$$

Thus, using $\operatorname{supp}\hat{\psi}^i \subset Z_i$, $i = 1, 2$, one can show that $\phi = (1-\delta/3)^{1/2}\psi^1$ is a dual framelet to $\psi = (1-\delta/3)^{-1/2}(\psi^1 + \varepsilon\psi^2)$. This completes the proof of Theorem 6.

We finish this section by showing that the affirmative answer to Question 3 does not imply a positive answer to Question 1 for general frame wavelets ψ.

Theorem 7. *For any $\delta > 0$, there exists a frame wavelet $\psi \in L^2(\mathbb{R})$ such that:*

(i) $\hat{\psi}$ is C^∞ and all its derivatives have exponential decay,
(ii) the frame bounds of $\mathcal{W}(\psi)$ are 1 and $1+\delta$,
(iii) the space V of negative dilates of ψ satisfies $\bigcap_{j\in\mathbb{Z}} D^j(V) \neq \{0\}$,
(iv) $\psi \notin V$,
(v) ψ has a dual frame wavelet.

Proof. Let ψ_1 and ψ_2 be the same as in the proof of Theorem 6. Then, a frame wavelet constructed by Theorem 6 is of the form $\psi' = c_1\psi^1 + c_2\psi^2$ for some constants c_1, c_2.

Define a function $\psi = c_1\psi^1 + c_2\psi_+$, where ψ_+ is given by $\hat{\psi}_+ = \hat{\psi}\mathbf{1}_{(0,\infty)}$. We claim that ψ satisfies all properties of Theorem 7. Indeed, (i) is trivial. The property (ii) follows from the same perturbation argument as in Theorem 6. Likewise, the same argument as in Theorem 6 shows that the space of negative dilates V satisfies $\check{L}^2(0,\infty) \subset V$. This is mainly due to the decomposition

$$L^2(\mathbb{R}) = H_+(\mathbb{R}) \oplus H_-(\mathbb{R}), \quad \text{where } H^2_-(\mathbb{R}) = \check{L}^2(-\infty,0),\ H^2_+(\mathbb{R}) = \check{L}^2(0,\infty),$$

and the fact that Hardy spaces $H^2_-(\mathbb{R})$ and $H^2_+(\mathbb{R})$ are invariant under the action of D and T_k. On the other hand, it is clear that $V \neq L^2(\mathbb{R})$ and hence (iv) holds. Finally, (v) is shown exactly in the same way as in Theorem 6 with $\phi = (c_1)^{-1}\psi_1$ being a dual framelet to ψ.

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