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Dimension Functions of Rationally Dilated GMRAs and Wavelets

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Abstract In this paper we study properties of generalized multiresolution analyses (GMRAs) and wavelets associated with rational dilations. We characterize the class of GMRAs associated with rationally dilated wavelets extending the result of Baggett, Medina, and Merrill. As a consequence, we introduce and derive the properties of the dimension function of rationally dilated wavelets. In particular, we show that any mildly regular wavelet must necessarily come from an MRA (possibly of higher multiplicity) extending Auscher's result from the setting of integer dilations to that of rational dilations. We also characterize all 3 interval wavelet sets for all positive dilation factors. Finally, we give an example of a rationally dilated wavelet dimension function for which the conventional algorithm for constructing integer dilated wavelet sets fails.

Keywords Wavelet · GMRA · Rational dilation · Wavelet dimension function

Mathematics Subject Classification (2000) Primary 42C40

1 Introduction

The wavelet dimension function is an important subject in the theory of wavelets. For a given orthonormal wavelet $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^N)$ associated with an

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integer expansive dilation $A \in M_N(\mathbb{Z})$, its dimension function is defined as

$$\mathfrak{D}_\Psi(\xi) = \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^N} |\widehat{\psi^\ell}((A^\mathsf{T})^j(\xi + k))|^2. \tag{1.1}$$

Initially, the wavelet dimension function was introduced in the one dimensional dyadic case by Lemarié-Rieusset [28, 29] to prove that all compactly supported wavelets are associated with a multiresolution analysis (MRA). After that Gripenberg [25] and Wang [35] used it to characterize all wavelets arising from an MRA. A further application of the wavelet dimension function is due to Auscher [2] who proved two fundamental results: (i) there are no regular wavelet bases for the Hardy space $H^2(\mathbb{R})$, and (ii) any mildly regular wavelet basis of $L^2(\mathbb{R}^N)$ must arise from an MRA (possibly of higher multiplicity). Furthermore, Auscher established that the wavelet dimension function $\mathfrak{D}_\Psi$ describes dimensions of certain subspaces of $\ell^2(\mathbb{Z}^N)$, and hence it is integer-valued.

A systematic study of properties of the wavelet dimension function (often called multiplicity function) was initiated by Baggett, Medina, and Merrill [3, 5] who introduced the concept of a generalized multi-resolution analysis (GMRA), and studied its relation to wavelets. In particular, they established the consistency equation for a multiplicity function

$$\sum_{\omega \in [(A^\mathsf{T})^{-1}\mathbb{Z}^N/\mathbb{Z}^N]} \mathfrak{D}(\xi + \omega) = L + \mathfrak{D}(A^\mathsf{T}\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^N, \tag{1.2}$$

where we use the convention that $[G/H]$ denotes a transversal (a set of representatives of distinct cosets) of a quotient group G/H. In a related development, Bownik, Rzeszotnik, and Speegle [22] characterized all possible integer-valued functions which are dimension functions of a wavelet. It turns out that every wavelet dimension function must satisfy an additional condition,

$$\sum_{k \in \mathbb{Z}^N} \mathbf{1}_\Delta(\xi + k) \ge \mathfrak{D}_\Psi(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^N, \tag{1.3}$$

where $\Delta = \{\xi \in \mathbb{R}^N : \mathfrak{D}_\Psi((A^\mathsf{T})^{-j}\xi) \ge 1 \text{ for all } j \in \mathbb{N} \cup \{0\}\}$. A similar result in the context of GMRAs was obtained by Baggett and Merrill [4], and then further generalized by Bownik and Rzeszotnik in [18]. In addition, the wavelet dimension function was extensively studied by a number of other authors including Behera [8], Paluszyński, Šikić, Weiss, and Xiao [30], Ron and Shen [33], and Weber [36]. As a result, the wavelet dimension function found a natural interpretation using the theory of shift-invariant spaces.

However, up to the present time, the wavelet dimension function has been studied exclusively for the class of integer dilations. The only exception is the work of Bownik and Speegle [21], who introduced an analogue of the wavelet dimension function for real dilations factors in one dimension. The goal of this paper is to initiate the study of the wavelet dimension functions in higher dimensions. We will concentrate on the class of rational expansive dilations $A \in M_N(\mathbb{Q})$, since most non-rational dilations admit only minimally supported frequency (MSF) wavelets due to results of Chui and Shi [24] and Bownik [13]. For every rational expansive dilation A there exists a plenty of non-MSF wavelets by a result of Bownik and Speegle [21]. Moreover, in one dimension Auscher [1] constructed rationally dilated wavelets with nice localization and smoothness properties. On the other hand, every MSF wavelet (or more generally combined MSF wavelet) has its space of negative dilates invariant under all translations. Hence, one can easily define the wavelet dimension function for such wavelets. However, this is rarely done since more information is carried by the wavelet set itself than by the wavelet dimension function.

There are some significant differences in the theory of dimension functions between the already well-understood case of integer dilations and that of rational dilations. The most prominent manifestation of that is a surprising gain of shift-invariance of GMRAs associated with wavelets. Namely, any GMRA associated with rationally dilated wavelet Ψ must be Γ-SI with $\Gamma = A\mathbb{Z}^N + \mathbb{Z}^N$. Note that in the classical case of $A \in M_N(\mathbb{Z})$, this self-improvement is non-existent. In addition, the corresponding multiplicity function must satisfy an analogue of the consistency equation (1.2). This leads to a characterization of GMRAs associated with rationally dilated wavelets extending the earlier result of Baggett, Medina, and Merrill [5]. As a consequence of this result we establish the formula for the wavelet dimension function as

$$\mathfrak{D}_\Psi(\xi) = \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} \sum_{k\in\Gamma^*} |\widehat{\psi^\ell}((A^\intercal)^j(\xi+k))|^2, \tag{1.4}$$

where $\Gamma^* = (A^\intercal)^{-1}\mathbb{Z}^N \cap \mathbb{Z}^N$ is a dual lattice to $\Gamma = A\mathbb{Z}^N + \mathbb{Z}^N$. It turns out the function $\mathfrak{D}_\Psi$ satisfies a collection of properties, including analogues of the consistency equation (1.2) and property (1.3), as in the case of integer dilations. While this collection of conditions characterizes all wavelet dimension functions in the integer case [22], it remains an open question whether the same is true in the rational case. Nevertheless, we show that the construction procedure for wavelet sets in [22], which is used in the sufficiency part of this characterization, fails for rational dilations already in one dimension. Finally, we also show that any mildly regular wavelet must necessarily come from an MRA (possibly of higher multiplicity). This extends Auscher's result [2] from the setting of integer dilations to that of rational dilations.

The paper is organized as follows. In Sect. 2 we recall the necessary facts about the spectral function of shift-invariant (SI) spaces, the concept of a GMRA, and quasi-affine systems for rational dilations. In the next section we establish an explicit formula for the spectral function of rationally dilated wavelets extending the result of Bownik and Rzeszotnik [18]. In Sect. 4 we generalize a theorem of Baggett, Medina, and Merrill [5] characterizing GMRAs which are associated with wavelets. We also give a sufficient condition on a dilation A, which guarantees that every wavelet associated with A comes from a GMRA. In Sect. 5 we derive properties of dimension functions of GMRAs. In the next section we introduce and derive the properties of dimension functions of rationally dilated wavelets. As an application we derive the extension of Auscher's result [2]. Section 7 is devoted to the characterization of all 3 interval wavelet sets for all positive dilation factors. Finally, in the last section we give an example of a wavelet dimension function for which the algorithm for constructing wavelet sets from [22] fails.

2 Preliminaries

In this section we recall some necessary results about shift invariant (SI) systems, the spectral function, the GMRA, and the quasi-affine systems for rational dilations. We start by establishing some necessary terminology. The translation operator by $y \in \mathbb{R}^N$ is $T_y f(x) = f(x-y)$; the dilation operator by the $N \times N$ invertible matrix $A \in M_N(\mathbb{R})$ is $D_A f(x) = \sqrt{|\det A|} f(Ax)$. For simplicity we shall write $Df = D_A f$. The Fourier transform of $f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ is

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i \langle x, \xi \rangle} dx \quad \text{for } \xi \in \mathbb{R}^N.$$

Given a measurable set $E \subset \mathbb{R}^N$, we define the translation-invariant space

$$\check{L}^2(E) = \{f \in L^2(\mathbb{R}^N) : \operatorname{supp} \hat{f} \subset E\}.$$

Definition 2.1 Suppose $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^N)$ and $A \in M_N(\mathbb{R})$ is an expansive matrix, i.e., all eigenvalues λ of A satisfy $|\lambda| > 1$. The *affine system* $\mathcal{A}(\Psi)$ associated with the dilation A is defined as

$$\mathcal{A}(\Psi) = \{\psi_{j,k} : j \in \mathbb{Z},\ k \in \mathbb{Z}^N,\ \psi \in \Psi\},$$

where $\psi_{j,k}(x) = |\det A|^{j/2} \psi(A^j x - k)$. We say that Ψ is a wavelet if $\mathcal{A}(\Psi)$ is an orthonormal basis of $L^2(\mathbb{R}^N)$. More generally, we say that Ψ is a *semi-orthogonal wavelet* if $\mathcal{A}(\Psi)$ is a Parseval frame for $L^2(\mathbb{R}^N)$, i.e.,

$$\|f\|^2 = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^N} |\langle f, \psi^\ell_{j,k} \rangle|^2 \quad \text{for all } f \in L^2(\mathbb{R}^N),$$

and different scales of $\mathcal{A}(\Psi)$ are mutually orthogonal, i.e.,

$$\langle \psi^\ell_{j,k}, \psi^{\ell'}_{j',k'} \rangle = 0 \quad \text{for all } \ell, \ell' = 1, \dots, L,\ j \neq j' \in \mathbb{Z},\ k, k' \in \mathbb{Z}^N.$$

2.1 Shift-invariant Spaces and the Spectral Function

The general properties of SI spaces were studied by a number of authors, see [11, 12, 31]. Here, we only list the results that will be used later on.

Definition 2.2 Suppose that Γ is a (full rank) lattice, i.e., $\Gamma = P\mathbb{Z}^N$, where $P \in M_N(\mathbb{R})$ is an $N \times N$ invertible matrix. We say that a closed subspace $V \subset L^2(\mathbb{R}^N)$ is *shift invariant* (SI) with respect to the lattice Γ, if

$$f \in V \Longrightarrow T_\gamma f \in V \quad \text{for all } \gamma \in \Gamma.$$

Given a countable family $\Phi \subset L^2(\mathbb{R}^N)$ and the lattice Γ we define the Γ-SI system $E^\Gamma(\Phi)$ by

$$E^\Gamma(\Phi) = \{T_\gamma \varphi : \varphi \in \Phi,\ \gamma \in \Gamma\}.$$

When $\Gamma = \mathbb{Z}^N$, we often drop the superscript Γ, and we simply say that V is SI. Likewise, $E(\Phi)$ means $E^{\mathbb{Z}^N}(\Phi)$.

The spectral function of SI spaces, which was introduced by Bownik and Rzeszotnik in [18], can be defined in several equivalent ways using range functions or dual Gramians. The following result, see [19, Lemma 2.5], can also serve as a definition.

Lemma 2.3 *If $V \subset L^2(\mathbb{R}^N)$ is Γ-SI and $\Phi \subset V$ is a countable family such that $E^\Gamma(\Phi)$ is a Parseval frame for V, then its spectral function is*

$$\sigma_V^\Gamma(\xi) = \frac{1}{|\mathbb{R}^N/\Gamma|} \sum_{\varphi\in\Phi} |\hat{\varphi}(\xi)|^2, \tag{2.1}$$

where $|\mathbb{R}^N/\Gamma|$ is the Lebesgue measure of the fundamental domain of $\mathbb{R}^N/\Gamma$, i.e., $|\mathbb{R}^N/\Gamma| = |\det P|$ if $\Gamma = P\mathbb{Z}^N$. In particular, (2.1) does not depend on the choice of Φ as long as $E^\Gamma(\Phi)$ is a Parseval frame for V.

Clearly, if $V \subset L^2(\mathbb{R}^N)$ is Γ-SI, then V is also Γ'-SI for any lattice $\Gamma' \subset \Gamma$. Hence, one can also talk about the spectral function $\sigma_V^{\Gamma'}$ with respect to a sparser lattice Γ'. Nevertheless, by a result of Bownik and Rzeszotnik [19, Corollary 2.7] both of these spectral functions coincide.

Theorem 2.4 *Suppose $\Gamma' \subset \Gamma$ are two lattices and $V \subset L^2(\mathbb{R}^N)$ is Γ-SI. Then,*

$$\sigma_V^\Gamma(\xi) = \sigma_V^{\Gamma'}(\xi) \quad \textit{for a.e. } \xi \in \mathbb{R}^N.$$

Consequently, we can drop the dependence of the spectral function of V on a lattice Γ and simply write σ_V instead of σ_V^Γ. This shows that the spectral function is a very fundamental notion of "size" of SI spaces which is independent of the underlying lattice Γ. In contrast, when working with the dimension function of a shift-invariant space, one needs to use the formula (2.2) when considering a sublattice $\Gamma' \subset \Gamma$, see [16, Lemma 2.4].

We recall that $\dim_V^\Gamma : \mathbb{R}^N \to \mathbb{N} \cup \{0, \infty\}$ is the multiplicity function of the projection-valued measure coming from the representation of Γ on V via translations by Stone's Theorem [3, 5]. Alternatively, one can define

$$\dim_V^\Gamma(\xi) = \dim \overline{\operatorname{span}}\{(\hat{\varphi}(\xi+k))_{k\in\Gamma^*} : \varphi \in \Phi\},$$

where $\Phi \subset V$ is a countable set of generators of V, i.e., $V = \overline{\operatorname{span}}\, E^\Gamma(\Phi)$.

Lemma 2.5 *Suppose $\Gamma' \subset \Gamma$ are two lattices and $V \subset L^2(\mathbb{R}^N)$ is Γ-SI. Then,*

$$\dim_V^{\Gamma'}(\xi) = \sum_{\omega\in[(\Gamma')^*/\Gamma^*]} \dim_V^\Gamma(\xi+\omega) \quad \textit{for a.e. } \xi \in \mathbb{R}^N. \tag{2.2}$$

Here,

$$\Gamma^* = \{x \in \mathbb{R}^N : \langle x, k\rangle \in \mathbb{Z} \textit{ for all } k \in \Gamma\}$$

is the dual lattice of Γ, *and* $[(\Gamma')^*/\Gamma^*]$ *is a transversal of* $(\Gamma')^*/\Gamma^*$.

The following theorem summarizes properties of the spectral function.

Theorem 2.6 *Let* $\mathfrak{S}$ *be the collection of all possible SI subspaces of* $L^2(\mathbb{R}^N)$, *i.e.*, $V \in \mathfrak{S}$ *if and only if there exists a lattice* Γ *such that* V *is* Γ-*SI. The spectral function satisfies the following properties*: $(V, W \in \mathfrak{S})$

(a) $\sigma_V : \mathbb{R}^N \to [0,1]$ *is a measurable function*,
(b) $V = \bigoplus_{i=1}^N V_i$, *where* $V_i \in \mathfrak{S} \Longrightarrow \sigma_V(\xi) = \sum_{i=1}^N \sigma_{V_i}(\xi)$,
(c) $V = \bigoplus_{i\in\mathbb{N}} V_i$, *where* V_i *is* Γ-*SI for a fixed lattice* $\Gamma \Longrightarrow \sigma_V(\xi) = \sum_{i\in\mathbb{N}} \sigma_{V_i}(\xi)$,
(d) $V \subset W \Longrightarrow \sigma_V(\xi) \le \sigma_W(\xi)$,
(e) $V \subset W \Longrightarrow (V = W \iff \sigma_V(\xi) = \sigma_W(\xi))$,
(f) $\sigma_V(\xi) = \mathbf{1}_E(\xi) \iff V = \check{L}^2(E)$, *where* $E \subset \mathbb{R}^N$ *is a measurable set.*
(g) $V \subset \check{L}^2(E)$, *where* $E = \operatorname{supp}\sigma_V$,
(h) $\sigma_{D_A V}(\xi) = \sigma_V((A^\intercal)^{-1}\xi)$, *where* $A \in M_N(\mathbb{R})$ *is invertible*,
(i) *if* V *is* Γ-*SI then* $\dim_V^\Gamma(\xi) = \sum_{k\in\Gamma^*} \sigma_V(\xi + k)$.

Proof The proof of Theorem 2.6 can be found in [18, 19] with the exception of (g). To see (g), let Φ be the same as in Lemma 2.3. Take any $f \in V$. Then, $\hat{f}$ is an L^2 limit of functions of the form $\sum_{\varphi\in\Phi} r_\varphi(\xi)\hat{\varphi}(\xi)$, where only finitely many of the Γ^*-periodic trigonometric polynomials r_φ are non-zero. Consequently, $\hat{f}(\xi) = 0$ for all $\xi \notin \operatorname{supp}\sigma_V$, which proves (g). □

2.2 Generalized Multiresolution Analyses

The concept of a generalized multiresolution analysis (GMRA) was introduced by Baggett, Medina, and Merrill in their seminal work [5]. Its original formulation requires that the expansive dilation A preserves the underlying lattice Γ, i.e., $A\Gamma \subset \Gamma$. By a standard dilation argument this can be reduced to the case of the standard lattice $\mathbb{Z}^N$ and an integer dilation A.

In this work we are interested in a larger class of expansive dilations which do not necessarily preserve the lattice Γ, but nevertheless $A \cap A\Gamma$ is a (full rank) lattice. A reduction to the case of the standard lattice $\Gamma = \mathbb{Z}^N$ corresponds precisely to the class of rational dilations A. While the definition below does not mention this assumption explicitly, all of our results involve only rationally dilated GMRAs.

Definition 2.7 A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^N)$ is called a *generalized multiresolution analysis* (GMRA) if

(M1) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,
(M2) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
(M3) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^N)$,
(M4) $f \in V_0 \iff D^j f \in V_j$ for all $j \in \mathbb{Z}$,
M(5) V_0 is $\mathbb{Z}^N$-SI.

In addition, if (M6) holds,

(M6) $\exists \Phi = \{\varphi^1, \dots, \varphi^m\} \subset V_0$ such that $E(\Phi)$ is an orthonormal basis of V_0, then $\{V_j\}_{j\in\mathbb{Z}}$ is a *multiresolution analysis* (MRA) of multiplicity m. In the case when $m = 1$, we simply say that $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA.

The space V_0 is called the *core space* of the GMRA. Condition (M5) allows us to use the theory of shift-invariant spaces for understanding the connections between the GMRA structure and wavelets or framelets. This is a subject of an extensive study by several authors, e.g. [4–7, 10, 18, 36].

For a family $\Psi \subset L^2(\mathbb{R}^N)$ we define its *space of negative dilates* by

$$V(\Psi) = \overline{\operatorname{span}}\{\psi_{j,k} : j < 0, k \in \mathbb{Z}^N, \psi \in \Psi\}. \tag{2.3}$$

We say that a (semi-orthogonal) wavelet Ψ is associated with a GMRA $\{V_j\}_{j\in\mathbb{Z}}$, if the core space $V_0 = V(\Psi)$. In addition, if V_0 satisfies (M6), then we say that Ψ is associated with an MRA.

2.3 Quasi-Affine Systems

Quasi-affine systems are a variant of the usual wavelet (affine) systems that play an important role in the theory of wavelets. Originally quasi-affine systems have been introduced and investigated for integer expansive dilation matrices by Ron and Shen [32]. The importance of quasi-affine systems stems from the fact that the frame property carries over when moving from an affine system to its corresponding quasi-affine system, and vice versa. Furthermore, quasi-affine systems are shift-invariant and thus much easier to study than affine systems which are dilation invariant.

Bownik [14] extended the notion of quasi-affine frames to the class of rational expansive dilations in a manner which overlaps with the usual definition in the case of integer dilations. In order to introduce a quasi-affine system, we need to recall the concept of oversampling of shift-invariant systems which was introduced in [14].

Definition 2.8 Suppose that $\Phi \subset L^2(\mathbb{R}^N)$ is a countable collection, and the lattice Γ is rational. We define the *oversampled system* of $E^\Gamma(\Phi)$ by

$$O^\Gamma(\Phi) = E^{\mathbb{Z}^N+\Gamma}\left(\frac{1}{|\mathbb{Z}^N/(\mathbb{Z}^N\cap\Gamma)|^{1/2}}\Phi\right) = E\left(\bigcup_{\theta\in\Theta}\left\{\frac{1}{|\mathbb{Z}^N/(\mathbb{Z}^N\cap\Gamma)|^{1/2}}T_\theta\Phi\right\}\right),$$

where Θ is a transversal of $(\mathbb{Z}^N+\Gamma)/\mathbb{Z}^N$.

The main idea of Ron and Shen is to oversample negative scales of the affine system at a rate adapted to the scale in order for the resulting system to be shift invariant. In order to define quasi-affine systems for rational expansive dilations we need to oversample both negative and positive scales of the affine system (at a rate proportional to the scale) which results in a quasi-affine system that in general coincides with the affine system only at the scale zero.

Definition 2.9 Suppose that $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^N)$ and A is a rational dilation. We define the quasi-affine system associated with (Ψ, A) by

$$\mathcal{A}^q(\Psi) = \bigcup_{j\in\mathbb{Z}} O^{A^{-j}\mathbb{Z}^N}(D^j\Psi)$$

$$= E\Bigg(\bigcup_{j\in\mathbb{Z}}\bigcup_{\theta\in\Theta_j}\Bigg\{\frac{1}{|\mathbb{Z}^N/(\mathbb{Z}^N\cap A^{-j}\mathbb{Z}^N)|^{1/2}}T_\theta D^j\Psi\Bigg\}\Bigg), \tag{2.4}$$

where Θ_j is a transversal of $(\mathbb{Z}^N + A^{-j}\mathbb{Z}^N)/\mathbb{Z}^N$.

Even though the orthogonality of an affine system is not preserved by the corresponding quasi-affine system, it turns out the Parseval frame property still carries over between affine and quasi-affine systems. This result was first established in [14, Theorem 3.4]. A different proof of Theorem 2.10 was given in [27, Theorem 2.17].

Theorem 2.10 *Suppose that $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^N)$. The affine system $\mathcal{A}(\Psi)$ is a Parseval frame if and only if its quasi-affine counterpart $\mathcal{A}^q(\Psi)$ is a Parseval frame.*

We define the *negative part* of the quasi-affine system $\mathcal{A}^q(\Psi)$ as

$$\begin{aligned}\mathcal{A}^q_-(\Psi) &= \bigcup_{j<0} O^{A^{-j}\mathbb{Z}^N}(D^j\Psi)\\ &= E\Bigg(\bigcup_{j<0}\bigcup_{\theta\in\Theta_j}\Bigg\{\frac{1}{|\mathbb{Z}^N/(\mathbb{Z}^N\cap A^{-j}\mathbb{Z}^N)|^{1/2}}T_\theta D^j\Psi\Bigg\}\Bigg),\end{aligned} \tag{2.5}$$

where Θ_j is the same as in Definition 2.9. We will need the following result about $\mathcal{A}^q_-(\Psi)$.

Lemma 2.11 *Suppose that $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^N)$ is a rationally dilated semi-orthogonal wavelet such that its space of negative dilates*

$$V(\Psi) = \overline{\text{span}}\,\{\psi^\ell_{j,k} : j < 0,\ k \in \mathbb{Z}^N,\ \ell = 1, \dots, L\} \tag{2.6}$$

is $\mathbb{Z}^N$-shift invariant. Then the system $\mathcal{A}^q_-(\Psi)$ forms a Parseval frame for $V(\Psi)$.

Proof Define the lattice Γ_j for each $j \in \mathbb{Z}$ by $\Gamma_j = \mathbb{Z}^N + A^{-j}\mathbb{Z}^N$ and let $M_j \in M_N(\mathbb{Q})$ be such that $M_j\mathbb{Z}^N = \Gamma_j$. Then we can express the quasi-affine system as

$$\mathcal{A}^q(\Psi) = \{\tilde\psi^\ell_{j,k} : j \in \mathbb{Z},\ k \in \mathbb{Z}^N,\ \ell = 1, \dots, L\},$$

where for each $j \in \mathbb{Z}$, $k \in \mathbb{Z}^N$, and $\ell = 1, \dots, L$, we define

$$\tilde\psi^\ell_{j,k}(x) = \frac{|\det A|^{j/2}}{|\mathbb{Z}^N/(\mathbb{Z}^N\cap A^{-j}\mathbb{Z}^N)|^{1/2}}\psi^\ell(A^j(x + M_jk)).$$

In addition to $\mathcal{A}^q_-(\Psi)$, we consider the positive part of the quasi-affine system

$$\mathcal{A}^q_+(\Psi) = \{\tilde\psi^\ell_{j,k} : j \geq 0,\ k \in \mathbb{Z}^N,\ \ell = 1, \dots, L\}.$$

We claim that

$$\begin{aligned} V(\Psi) &= \overline{\operatorname{span}}\, \mathcal{A}^q_-(\Psi), \\ (V(\Psi))^\perp &= \overline{\operatorname{span}}\, \mathcal{A}^q_+(\Psi). \end{aligned} \tag{2.7}$$

Indeed, by the semi-orthogonality of Ψ we have

$$(V(\Psi))^\perp = \overline{\operatorname{span}}\{\psi^\ell_{j,k} : j \ge 0,\ k \in \mathbb{Z}^N,\ \ell = 1, \dots, L\}. \tag{2.8}$$

Given any $k_1 \in \mathbb{Z}^N$, we have $A^{-j}k_1 \in A^{-j}\mathbb{Z}^N \subset \mathbb{Z}^N + A^{-j}\mathbb{Z}^N = \Gamma_j$ and so there is some $k_2 \in \mathbb{Z}^N$ such that $A^j M_j k_2 = k_1$, implying that $\psi^\ell_{j,k_1} = |\mathbb{Z}^N/(\mathbb{Z}^N \cap A^{-j}\mathbb{Z}^N)|^{1/2}\tilde{\psi}^\ell_{j,k_2}$. Thus, we obtain the inclusions $\subset$ in (2.7). On the other hand, since $\mathbb{Z}^N \subset M_j^{-1}\Gamma_j = M_j^{-1}\mathbb{Z}^N + M_j^{-1}A^{-j}\mathbb{Z}^N$, then there are $\gamma, \beta \in \mathbb{Z}^N$ such that $M_j^{-1}\gamma + M_j^{-1}A^{-j}\beta = k_1$, implying that $\tilde{\psi}^\ell_{j,k_1} = \frac{1}{|\mathbb{Z}^N/(\mathbb{Z}^N \cap A^{-j}\mathbb{Z}^N)|^{1/2}} T_\gamma \psi^\ell_{j,\beta}$. Since $V(\Psi)$ is shift invariant, so is $(V(\Psi))^\perp$. This yields that $\tilde{\psi}^\ell_{j,k_1} \in V(\Psi)$ when $j < 0$, and $\tilde{\psi}^\ell_{j,k_1} \in (V(\Psi))^\perp$ when $j \ge 0$. Thus, we obtain the reverse inclusions $\supset$ in (2.7).

By our hypothesis, the affine system $\mathcal{A}(\Psi)$ is a Parseval frame for $L^2(\mathbb{R}^N)$. This implies by Theorem 2.10 that $\mathcal{A}^q(\Psi)$ is also a Parseval frame. Since $\mathcal{A}^q(\Psi) = \mathcal{A}^q_-(\Psi) \cup \mathcal{A}^q_+(\Psi)$, (2.7) implies that $\mathcal{A}^q_-(\Psi)$ and $\mathcal{A}^q_+(\Psi)$ are Parseval frames for $V(\Psi)$ and $(V(\Psi))^\perp$, respectively. □

3 Spectral Function of Wavelets

The goal of this section is to establish the formula for the spectral function of rationally dilated wavelets. Hence, we wish to show the following generalization of a result of Bownik and Rzeszotnik [18, Theorem 4.2].

Theorem 3.1 *Suppose that $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^N)$ is a rationally dilated semi-orthogonal wavelet such that its space of negative dilates $V(\Psi)$ is $\mathbb{Z}^N$-shift invariant. Then,*

$$\sigma_{V(\Psi)}(\xi) = \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} |\widehat{\psi^\ell}((A^\intercal)^j \xi)|^2. \tag{3.1}$$

In the proof of Theorem 3.1 we will need the following elementary result about rational lattices.

Lemma 3.2 *Suppose that Γ is a rational lattice, i.e., $\Gamma = P\mathbb{Z}^N$ for some invertible $P \in M_N(\mathbb{Q})$. Then, we have the isomorphism*

$$(\mathbb{Z}^N + \Gamma)/\mathbb{Z}^N \simeq \Gamma/(\mathbb{Z}^N \cap \Gamma). \tag{3.2}$$

Moreover,

$$|\mathbb{Z}^N/(\mathbb{Z}^N \cap \Gamma)| = |\det P| \cdot |\Gamma/(\mathbb{Z}^N \cap \Gamma)|. \tag{3.3}$$

Proof To see (3.2) one can use the second isomorphism theorem for groups, whereas (3.3) follows from the elementary properties of the volume $d(\Gamma) = |\det P|$ of a lattice $\Gamma = P\mathbb{Z}^N$. In particular, if $\Gamma' \subset \Gamma$ are two lattices, then the quotient group Γ/Γ' has the order $d(\Gamma')/d(\Gamma)$, see [23]. □

Proof of Theorem 3.1 Using Lemmas 2.3 and 2.11, we can compute the spectral function as

$$\sigma^{\mathbb{Z}^N}_{V(\Psi)}(\xi) = \frac{1}{|\mathbb{R}^N/\mathbb{Z}^N|} \sum_{\ell=1}^{L} \sum_{j<0} \sum_{\theta\in\Theta_j} \frac{1}{|\mathbb{Z}^N/(\mathbb{Z}^N \cap A^{-j}\mathbb{Z}^N)|} |(\widehat{T_\theta D^j \psi^\ell})(\xi)|^2, \tag{3.4}$$

where Θ_j is the same as in Definition 2.9. Lemma 3.2 yields

$$\begin{aligned}|(\mathbb{Z}^N + A^{-j}\mathbb{Z}^N)/\mathbb{Z}^N| &= |A^{-j}\mathbb{Z}^N/(\mathbb{Z}^N \cap A^{-j}\mathbb{Z}^N)|,\\ |\mathbb{Z}^N/(\mathbb{Z}^N \cap A^{-j}\mathbb{Z}^N)| &= |\det A|^{-j} |A^{-j}\mathbb{Z}^N/(\mathbb{Z}^N \cap A^{-j}\mathbb{Z}^N)|.\end{aligned}$$

Hence, we conclude that $|\Theta_j| = |\mathbb{Z}^N/(\mathbb{Z}^N \cap A^{-j}\mathbb{Z}^N)| \cdot |\det A|^j$. This, along with the fact that

$$|(\widehat{T_\theta D^j \psi^\ell})(\xi)| = |\det A|^{-j/2} |\widehat{\psi^\ell}((A^\mathsf{T})^{-j}\xi)|$$

allows for the simplification

$$\begin{aligned}&\frac{1}{|\mathbb{Z}^N/(\mathbb{Z}^N \cap A^{-j}\mathbb{Z}^N)|} \sum_{\theta\in\Theta_j} |(\widehat{T_\theta D^j \psi^\ell})(\xi)|^2\\ &\quad= \frac{|\Theta_j|}{|\mathbb{Z}^N/(\mathbb{Z}^N \cap A^{-j}\mathbb{Z}^N)||\det A|^j} |\widehat{\psi^\ell}((A^\mathsf{T})^{-j}\xi)|^2\\ &\quad= |\widehat{\psi^\ell}((A^\mathsf{T})^{-j}\xi)|^2.\end{aligned}$$

Therefore, (3.4) becomes

$$\sigma_{V(\Psi)}(\xi) = \sum_{\ell=1}^{L} \sum_{j<0} |\widehat{\psi^\ell}((A^\mathsf{T})^{-j}(\xi))|^2 = \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} |\widehat{\psi^\ell}((A^\mathsf{T})^{j}(\xi))|^2.$$ □

Theorem 3.1 and Lemma 2.3 could be used to extend the definition of the wavelet dimension function to the case of rational dilations. Namely, if Ψ is a semi-orthogonal wavelet as in Theorem 3.1, then we could use Theorem 2.6(i) to define the wavelet dimension function as the dimension function of $\mathbb{Z}^N$-SI space $V(\Psi)$,

$$\dim^{\mathbb{Z}^N}_{V(\Psi)}(\xi) = \sum_{k\in\mathbb{Z}^N} \sigma_{V(\Psi)}(\xi + k) = \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}^N} |\widehat{\psi^\ell}((A^\mathsf{T})^{j}(\xi + k))|^2. \tag{3.5}$$

Note that (3.5) is exactly the same formula (1.1) as for the well-studied wavelet dimension function for integer dilations. Hence, one might think that the rationally dilated dimension function will enjoy similar properties as its integer dilated counterpart. As we will see later, this is not the case. In fact, it turns out that (3.5) is not the right way of extending the wavelet dimension function to the rational case despite that it coincides nicely with the usual formula in the integer case. Indeed, in Section 6 we will define the wavelet dimension function in such a way that it is concurrently:

- an extension of the usual integer dilated dimension function, and
- an encompassment of more information about a wavelet Ψ than (3.5).

The reason behind its existence is a somewhat unexpected gain of the shift-invariance of GMRAs associated with rationally dilated wavelets which will be explored in the next section. Consequently, the rationally dilated wavelet dimension function will be defined in terms of a SI space dimension function using the greater shift-invariance.

4 GMRAs and Wavelets

The following result is quite surprising, since it shows the self-improving property of GMRAs associated with rationally dilated wavelets. Namely, the core space V_0 of such a GMRA enjoys more shift-invariance than the ordinary $\mathbb{Z}^N$-SI. We should mention here that the study of integer dilated wavelets with improved shift-invariance goes back to Weber [37], see also [9, 34]. Note that in the case when A is integer-valued, no such improvement exists. This might explain why this rather elementary phenomenon remained unnoticed until this work.

Lemma 4.1 *Suppose a GMRA $\{V_j\}_{j\in\mathbb{Z}}$ associated with an arbitrary real dilation A gives rise to a semi-orthogonal wavelet Ψ, i.e., V_0 is the space of negative dilates $V(\Psi)$ of Ψ. Then, V_0 is Γ-SI, where $\Gamma = A\mathbb{Z}^N + \mathbb{Z}^N$.*

One should note that for a general dilation $A \in M_N(\mathbb{R})$, $\Gamma = A\mathbb{Z}^N + \mathbb{Z}^N$ need not be a lattice, that is a discrete subgroup of $\mathbb{R}^N$. In this case, Lemma 4.1 says that V_0 is invariant under translations T_y by $y \in \Gamma$, and hence also by $y \in \overline{\Gamma}$.

Proof Since V_0 is $\mathbb{Z}^N$-SI, the space V_1 is $A^{-1}\mathbb{Z}^N$-SI. Since

$$V_1 = V_0 \oplus W_0, \quad \text{where } W_0 = \overline{\operatorname{span}}\, E(\Psi), \tag{4.1}$$

V_1 is $\mathbb{Z}^N$-SI as well. Hence, V_1 is $(A^{-1}\mathbb{Z}^N + \mathbb{Z}^N)$-SI. Consequently, V_0 is $(A\mathbb{Z}^N + \mathbb{Z}^N)$-SI. □

Lemma 4.1 enables us to show the following extension of a theorem due to Baggett, Medina, and Merrill [5] to the case of rational dilations.

Theorem 4.2 *Suppose that a dilation $A \in M_N(\mathbb{Q})$ and Ψ is a semi-orthogonal wavelet with L generators associated with a GMRA $\{V_j\}_{j\in\mathbb{Z}}$. Then,*

$$V_0 \textit{ is } \Gamma\textit{-SI, where } \Gamma = A\mathbb{Z}^N + \mathbb{Z}^N, \tag{4.2}$$

and its Γ-dimension function $\mathfrak{D}(\xi) = \dim^{\Gamma}_{V_0}(\xi)$ satisfies

$$\mathfrak{D}(\xi) < \infty \quad \textit{for a.e. } \xi, \tag{4.3}$$

and the consistency inequality

$$\sum_{\omega\in[(A^\intercal)^{-1}\mathbb{Z}^N/\Gamma^*]} \mathfrak{D}(\xi+\omega) \le L + \sum_{\omega'\in[\mathbb{Z}^N/\Gamma^*]} \mathfrak{D}(A^\intercal\xi+\omega') \quad \textit{for a.e. } \xi. \tag{4.4}$$

In addition, if Ψ *is a wavelet, then we have equality in* (4.4), *i.e.*,

$$\sum_{\omega\in[(A^\intercal)^{-1}\mathbb{Z}^N/\Gamma^*]} \mathfrak{D}(\xi+\omega) = L + \sum_{\omega'\in[\mathbb{Z}^N/\Gamma^*]} \mathfrak{D}(A^\intercal\xi+\omega') \quad \textit{for a.e. } \xi. \tag{4.5}$$

Conversely, if a GMRA $\{V_j\}_{j\in\mathbb{Z}}$ *satisfies* (4.2), (4.3), *and* (4.4), *then there exists a semi-orthogonal wavelet* Ψ (*with at most* L *generators*) *associated with this GMRA. In addition, if* (4.5) *holds, then* Ψ *is a wavelet with* L *generators.*

Proof Suppose that Ψ is semi-orthogonal wavelet with L generators which is associated with $\{V_j\}_{j\in\mathbb{Z}}$. Lemma 4.1 guarantees that V_0 is Γ-SI. On the other hand, Theorem 3.1 gives an explicit formula for the spectral function of V_0. Thus, Theorem 2.6(i) yields

$$\dim^\Gamma_{V_0}(\xi) = \sum_{k\in\Gamma^*} \sigma_{V_0}(\xi+k). \tag{4.6}$$

Consequently,

$$\int_{\mathbb{R}^N/\Gamma^*} \dim^\Gamma_{V_0}(\xi) = \int_{\mathbb{R}^N} \sigma_{V_0}(\xi)d\xi = \sum_{\ell=1}^{L}\sum_{j=1}^{\infty}\int_{\mathbb{R}^N} |\widehat{\psi^\ell}((A^\intercal)^j\xi)|^2 d\xi$$

$$= \sum_{\ell=1}^{L}\sum_{j=1}^{\infty} \|\psi^\ell\|^2 |\det A|^j \le L/(|\det A|-1) < \infty.$$

Hence, (4.3) holds.

By (4.1) and Theorem 2.6(h) we have

$$\sigma_{V_0}(\xi) + \sigma_{W_0}(\xi) = \sigma_{V_1}(\xi) = \sigma_{D(V_0)}(\xi) = \sigma_{V_0}((A^\intercal)^{-1}\xi).$$

In particular,

$$\sum_{k\in\mathbb{Z}^N} \sigma_{V_0}(\xi+k) + \sum_{k\in\mathbb{Z}^N} \sigma_{W_0}(\xi+k) = \sum_{k\in\mathbb{Z}^N} \sigma_{V_0}((A^\intercal)^{-1}(\xi+k)). \tag{4.7}$$

It remains to describe the quantities appearing in (4.7) in terms of $\mathfrak{D}(\xi) = \dim^\Gamma_{V_0}(\xi)$. By the isomorphism $\mathbb{Z}^N \simeq (\mathbb{Z}^N/\Gamma^*)\times\Gamma^*$,

$$\sum_{k\in\mathbb{Z}^N} \sigma_{V_0}(\xi+k) = \sum_{\omega'\in[\mathbb{Z}^N/\Gamma^*]}\sum_{\gamma^*\in\Gamma^*} \sigma_{V_0}(\xi+\gamma^*+\omega') = \sum_{\omega'\in[\mathbb{Z}^N/\Gamma^*]} \mathfrak{D}(\xi+\omega'). \tag{4.8}$$

Since $E^{\mathbb{Z}^N}(\Psi)$ forms a Parseval frame for W_0, and Ψ has L generators, we have

$$\sum_{k\in\mathbb{Z}^N} \sigma_{W_0}(\xi+k) = \dim_{W_0}^{\mathbb{Z}^N}(\xi) \le L. \tag{4.9}$$

Finally, the isomorphism $(A^\intercal)^{-1}\mathbb{Z}^N \simeq ((A^\intercal)^{-1}\mathbb{Z}^N/\Gamma^*)\times\Gamma^*$ yields

$$\begin{aligned}\sum_{k\in\mathbb{Z}^N} \sigma_{V_0}((A^\intercal)^{-1}(\xi+k)) &= \sum_{\omega\in[(A^\intercal)^{-1}\mathbb{Z}^N/\Gamma^*]}\ \sum_{\gamma^*\in\Gamma^*} \sigma_{V_0}((A^\intercal)^{-1}\xi+\gamma^*+\omega)\\ &= \sum_{\omega\in[(A^\intercal)^{-1}\mathbb{Z}^N/\Gamma^*]} \mathfrak{D}((A^\intercal)^{-1}\xi+\omega).\end{aligned} \tag{4.10}$$

Combining (4.7)–(4.10) yields

$$\sum_{\omega'\in[\mathbb{Z}^N/\Gamma^*]} \mathfrak{D}(\xi+\omega') + L \ge \sum_{\omega\in[(A^\intercal)^{-1}\mathbb{Z}^N/\Gamma^*]} \mathfrak{D}((A^\intercal)^{-1}\xi+\omega),$$

which is equivalent with (4.4). In addition, if Ψ is a wavelet, then $E(\Psi)$ is a orthonormal basis for W_0. Since Ψ has L generators, we have

$$\sum_{k\in\mathbb{Z}^N} \sigma_{W_0}(\xi+k) = \dim_{W_0}^{\mathbb{Z}^N}(\xi) = L. \tag{4.11}$$

Consequently, we obtain (4.5).

Conversely, let $\{V_j\}_{j\in\mathbb{Z}}$ be a GMRA satisfying (4.2), (4.3), and (4.4). Define the space $W_0 = V_1 \ominus V_0$. Since V_0 is Γ-SI, $V_1 = D(V_0)$ is $A^{-1}\mathbb{Z}^N$-SI, and hence both spaces are $\mathbb{Z}^N$-SI. Thus, W_0 must be $\mathbb{Z}^N$-SI as well. The fact that $V_1 = V_0 \oplus W_0$ together with (4.7), (4.8), and (4.10) implies that

$$\begin{aligned}\dim_{W_0}^{\mathbb{Z}^N}(\xi) &= \sum_{k\in\mathbb{Z}^N} \sigma_{W_0}(\xi+k)\\ &= \sum_{\omega\in[(A^\intercal)^{-1}\mathbb{Z}^N/\Gamma^*]} \mathfrak{D}((A^\intercal)^{-1}\xi+\omega)\\ &\quad - \sum_{\omega'\in[\mathbb{Z}^N/\Gamma^*]} \mathfrak{D}(\xi+\omega') \le L.\end{aligned} \tag{4.12}$$

The last inequality and the fact that we can take the difference above follow from (4.3) and the consistency inequality (4.4). By [12, Theorem 3.3], the space W_0 has a set Ψ of at most L generators, such that $E^{\mathbb{Z}^N}(\Psi)$ is a Parseval frame for W_0. Since

$$V_0 = \bigoplus_{j\le -1} D^j(W_0),$$

we infer that Ψ is a semi-orthogonal wavelet associated with the GMRA $\{V_j\}_{j\in\mathbb{Z}}$. In addition, if (4.5) holds, then we have equality in (4.12). Thus, $E^{\mathbb{Z}^N}(\Psi)$ is an orthonormal basis of W_0. Consequently, Ψ is a wavelet associated with $\{V_j\}_{j\in\mathbb{Z}}$. □

Note that Theorem 4.2 only applies to rationally dilated wavelets associated with a GMRA. Thus, the following fundamental question remains open.

Question 1 Is every rationally dilated wavelet Ψ necessarily associated with a GMRA?

In the case when Ψ is integer dilated, the classical theory says that the answer is indeed positive. In general, this problem remains open and we shall only give a partial answer to it.

One may think of Question 1 as an analogue of the Baggett's problem for Parseval wavelets. For the background and importance of this problem we refer the reader to [17, 20]. Here, we shall only say that this problem asks whether every (integer dilated) Parseval wavelet Ψ must necessarily come from a GMRA. This problem can be reformulated in terms of the space $V(\Psi)$ of negative dilates of Ψ. One can show that the spaces $V_j = D^j(V(\Psi))$ satisfy all properties of a GMRA with a hypothetical exception of the intersection property (M2).

In our setting of rationally dilated wavelets, we face an analogous problem as for integer dilated Parseval wavelets. Indeed, if Ψ is a wavelet with respect to any real dilation A, then the spaces $V_j = D^j(V(\Psi))$ satisfy all properties of a GMRA with a possible exception of the SI property (M5). In the case A is not a rational dilation, the property (M5) may indeed fail. For an example of such wavelet associated with a dilation $\sqrt{2}$ in $L^2(\mathbb{R})$, see [20]. However, if A is a rational dilation, then no such counterexample is known to exist. In fact, we have only a positive evidence about validity of (M5). Finally, we should mention that if Ψ is a rationally dilated Parseval wavelet, then we encounter an accumulation of Baggett's problem together with Question 1. That is, both (M2) and (M5) are not certain to hold.

While Question 1 remains open, we will give a sufficient condition on a real dilation A, which guarantees that any wavelet (Ψ, A) is associated with a GMRA. To do this we need an elementary lemma.

Lemma 4.3 *Suppose $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^N)$ is a wavelet associated with a dilation $A \in M_N(\mathbb{R})$. Then, its space of negative dilates $V(\Psi)$ is shift invariant if and only if*

$$T_k \psi^{\ell}_{h,m} \perp \psi^{\ell'}_{j,n} \tag{4.13}$$

for all $h < 0$, $j \ge 0$, $k, m, n \in \mathbb{Z}^N$, $\ell, \ell' = 1, \dots, L$.

Proof Suppose that (4.13) holds. Then we can go one step further and say that for all $h < 0$, $j \ge 0$, $k_1, k_2, m, n \in \mathbb{Z}^N$, $\ell, \ell' = 1, \dots, L$, we have

$$\langle T_{k_1} \psi^{\ell}_{h,m}, T_{k_2} \psi^{\ell'}_{j,n} \rangle = \langle T_{k_1-k_2} \psi^{\ell}_{h,m}, \psi^{\ell'}_{j,n} \rangle = 0.$$

Hence, the SI spaces

$$W_1 = \overline{\operatorname{span}}\{T_k \psi^{\ell}_{j,n} : j < 0,\ k, n \in \mathbb{Z}^N,\ \ell = 1, \dots, L\},$$

$$W_2 = \overline{\operatorname{span}}\{T_k \psi^{\ell}_{j,n} : j \ge 0,\ k, n \in \mathbb{Z}^N,\ \ell = 1, \dots, L\}$$

are orthogonal $W_1 \perp W_2$.

Recall that $V(\Psi) = \overline{\operatorname{span}}\{\psi^\ell_{j,n} : j < 0,\ n \in \mathbb{Z}^N,\ \ell = 1,\dots,L\}$ and, hence, $(V(\Psi))^\perp = \overline{\operatorname{span}}\{\psi^\ell_{j,n} : j \geq 0,\ n \in \mathbb{Z}^N,\ \ell = 1,\dots,L\}$. Clearly, we have $V(\Psi) \subset W_1$ and $(V(\Psi))^\perp \subset W_2$. Furthermore, since $V(\Psi) \oplus (V(\Psi))^\perp = L^2(\mathbb{R}^N)$ and $W_1 \perp W_2$, we must have $W_1 = V(\Psi)$ and $W_2 = (V(\Psi))^\perp$. Since W_1 is, by definition, shift invariant, then so is $V(\Psi)$.

Conversely, if $V(\Psi)$ is SI, then so is $(V(\Psi))^\perp$. Thus, $W_1 = V(\Psi)$ and $W_2 = (V(\Psi))^\perp$, and $W_1 \perp W_2$. This implies (4.13). □

Theorem 4.4 *Suppose $\Psi = \{\psi^1,\dots,\psi^L\} \subset L^2(\mathbb{R}^N)$ is a wavelet associated with a dilation $A \in M_N(\mathbb{R})$. Then, $V(\Psi)$ is shift invariant if for all $h < 0$ and $j > 0$ we have*

$$\mathbb{Z}^N \subset A^h\mathbb{Z}^N + A^j\mathbb{Z}^N. \tag{4.14}$$

Proof Take any $h < 0$, $j \geq 0$, $k, m, n \in \mathbb{Z}^N$, $\ell, \ell' = 1,\dots,L$. If $j = 0$, then

$$\langle T_k\psi^\ell_{h,m}, \psi^{\ell'}_{j,n}\rangle = \langle T_k\psi^\ell_{h,m}, T_n\psi^{\ell'}\rangle = \langle \psi^\ell_{h,m}, T_{n-k}\psi^{\ell'}\rangle = \langle \psi^\ell_{h,m}, \psi^{\ell'}_{j,n-k}\rangle = 0.$$

If, on the other hand, $j > 0$ then we choose $\gamma, \beta \in \mathbb{Z}^N$ such that $k = A^{-h}\gamma + A^{-j}\beta$. We are guaranteed the existence of such $\gamma, \beta \in \mathbb{Z}^N$ by (4.14). Then,

$$\begin{aligned}\langle T_k\psi^\ell_{h,m}, \psi^{\ell'}_{j,n}\rangle &= \langle T_{A^{-h}\gamma + A^{-j}\beta} D^h T_m\psi^\ell, D^j T_n\psi^{\ell'}\rangle \\ &= \langle D^h T_{m+\gamma}\psi^\ell, D^j T_{n-\beta}\psi^{\ell'}\rangle = \langle \psi^\ell_{h,m+\gamma}, \psi^{\ell'}_{j,n-\beta}\rangle = 0.\end{aligned}$$

Therefore, $V(\Psi)$ is shift invariant by Lemma 4.3. □

As an illustration we demonstrate how Theorem 4.4 can provide for the shift invariance of wavelets associated with certain classes of dilations. For instance, it is well known that the space of negative dilates of any wavelet associated with an integer dilation is shift invariant. Theorem 4.4 provides a very quick proof of this fact by simply noting that if $A \in M_N(\mathbb{Z})$, then $\mathbb{Z}^N \subset A^h\mathbb{Z}^N$ for all $h < 0$. Slightly more interesting, however, is the case of diagonal rational dilations as we see below.

Proposition 4.5 *Suppose $\Psi = \{\psi^1,\dots,\psi^L\} \subset L^2(\mathbb{R}^N)$ is a wavelet associated with a dilation $A \in M_N(\mathbb{Q})$. If A is diagonal, then $V(\Psi)$ is shift invariant.*

Proof Suppose $A = \operatorname{diag}(\frac{p_1}{q_1},\dots,\frac{p_N}{q_N})$ where $p_i, q_i \in \mathbb{Z}$ with $\gcd(p_i, q_i) = 1$ for each $i = 1,\dots,N$. Given any $k = (k_1,\dots,k_N) \in \mathbb{Z}^N$ and any $h < 0$ and $j > 0$, choose $\omega = (\omega_1,\dots,\omega_N) \in \mathbb{Z}^N$ satisfying the two following conditions for each $i = 1,\dots,N$,

$$\omega_i \equiv -k_i \pmod{p_i^{-h}},$$

$$\omega_i \equiv 0 \pmod{q_i^j}.$$

Then for each $i = 1,\dots,N$, we have $(\frac{p_i}{q_i})^h(\omega_i + k_i) \in \mathbb{Z}$ and $(\frac{p_i}{q_i})^j\omega_i \in \mathbb{Z}$. Hence, $A^h(\omega + k) \in \mathbb{Z}^N$ and $A^j\omega \in \mathbb{Z}^N$. Thus, for each $m, n \in \mathbb{Z}^N$ and $\ell, \ell' = 1,\dots,L$ we have

$$\begin{aligned}\langle T_k\psi^{\ell}_{h,m}, \psi^{\ell'}_{j,n}\rangle &= \langle T_k D_h T_m \psi^{\ell}, D^j T_n \psi^{\ell'}\rangle = \langle T_{\omega+k} D_h T_m \psi^{\ell}, T_\omega D^j T_n \psi^{\ell'}\rangle \\ &= \langle D_h T_{A^h(\omega+k)+m}\psi^{\ell}, D^j T_{A^j\omega+n}\psi^{\ell'}\rangle = \langle \psi^{\ell}_{h,A^h(\omega+k)+m}, \psi^{\ell'}_{j,A^j\omega+n}\rangle \\ &= 0.\end{aligned}$$

Therefore, $V(\Psi)$ is shift invariant by Lemma 4.3. □

It is important to understand, however, that Theorem 4.4 does not guarantee the shift invariance of $V(\Psi)$ for every rationally dilated wavelet Ψ. The following simple example was communicated to us by Daniel Chan of the University of New South Wales, Australia.

Example 4.6 If $A = \begin{pmatrix} 1 & 2 \\ 3/2 & -1 \end{pmatrix}$, then A is an expansive matrix that does not satisfy (4.14) for $h = -1$ and $j = 1$.

Proof Note that A is expansive, since $A^2 = 4Id$. We claim that (4.14) fails for $h = -1$ and $j = 1$. Indeed, $A^{-1} = \frac{1}{4}A$ implies that

$$A^{-1}\mathbb{Z}^2 + A\mathbb{Z}^2 = \frac{1}{4}A\mathbb{Z}^2 + A\mathbb{Z}^2 = \frac{1}{4}A\mathbb{Z}^2.$$

Assuming that $\mathbb{Z}^2 \subset \frac{1}{4}A\mathbb{Z}^2$, we have $A\mathbb{Z}^2 \subset \frac{1}{4}A^2\mathbb{Z}^2 = \mathbb{Z}^2$, which is a contradiction. □

5 Dimension Function of GMRAs

The goal of this section is to derive the properties satisfied by a dimension function of any rationally dilated GMRA.

Theorem 5.1 *Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is a GMRA associated with the dilation $A \in M_N(\mathbb{Q})$. Then, its dimension function $\mathfrak{D}(\xi) = \dim^{\mathbb{Z}^N}_{V_0}(\xi)$ satisfies the following four conditions*:

(D1) $\mathfrak{D} : \mathbb{R}^N \to \mathbb{N}\cup\{0,\infty\}$ *is a measurable $\mathbb{Z}^N$-periodic function*,

(D2) $\mathfrak{D}$ *satisfies the consistency inequality*

$$\sum_{\omega\in[(A^{\mathsf{T}})^{-1}\tilde{\Gamma}/\mathbb{Z}^N]} \mathfrak{D}(\xi+\omega) \geq \sum_{\omega'\in[\tilde{\Gamma}/\mathbb{Z}^N]} \mathfrak{D}(A^{\mathsf{T}}\xi+\omega') \quad \textit{for a.e. } \xi\in\mathbb{R}^N \tag{5.1}$$

where $\tilde{\Gamma} = \mathbb{Z}^N + A^{\mathsf{T}}\mathbb{Z}^N$,

(D3) $\sum_{k\in\mathbb{Z}^N} \mathbf{1}_\Delta(\xi+k) \geq \mathfrak{D}(\xi)$ *for a.e. $\xi\in\mathbb{R}^N$, where*

$$\Delta = \big\{\xi\in\mathbb{R}^N : \mathfrak{D}\big((A^{\mathsf{T}})^{-j}\xi\big) \geq 1 \textit{ for all } j\in\mathbb{N}\cup\{0\}\big\}.$$

(D4) $\liminf_{n\to\infty} \mathfrak{D}((A^\intercal)^{-n}\xi) \ge 1$ *for a.e.* $\xi \in \mathbb{R}^N$.

Proof Clearly, the dimension function of any $\mathbb{Z}^N$-SI space V_0 must satisfy (D1). To show (D2) observe that the space V_1 is $A^{-1}\mathbb{Z}^N$-SI. Hence, the spaces V_0 and V_1 are SI with respect to the common lattice $\Gamma = \mathbb{Z}^N \cap (A^{-1}\mathbb{Z}^N)$. Note that $\Gamma^* = \mathbb{Z}^N + A^\intercal(\mathbb{Z}^N) = \tilde{\Gamma}$. The inclusion, $V_0 \subset V_1$ implies that

$$\dim^\Gamma_{V_1}(\xi) \ge \dim^\Gamma_{V_0}(\xi) \quad \text{for a.e. } \xi.$$

By Theorem 2.6 parts (h) and (i)

$$\begin{aligned}\dim^\Gamma_{V_1}(\xi) &= \sum_{\gamma\in\tilde{\Gamma}} \sigma_{V_1}(\xi+\gamma) = \sum_{\gamma\in\tilde{\Gamma}} \sigma_{V_0}((A^\intercal)^{-1}\xi + (A^\intercal)^{-1}\gamma) \\ &= \sum_{\omega\in[(A^\intercal)^{-1}\tilde{\Gamma}/\mathbb{Z}^N]} \sum_{k\in\mathbb{Z}^N} \sigma_{V_0}((A^\intercal)^{-1}\xi + k + \omega) \\ &= \sum_{\omega\in[(A^\intercal)^{-1}\tilde{\Gamma}/\mathbb{Z}^N]} \mathfrak{D}((A^\intercal)^{-1}\xi + \omega).\end{aligned}$$

On the other hand, Lemma 2.5 implies that

$$\dim^\Gamma_{V_0}(\xi) = \sum_{\omega'\in[\tilde{\Gamma}/\mathbb{Z}^N]} \mathfrak{D}(\xi + \omega').$$

Combining the last three results yields (D2).

To show the remaining two properties, note that $V_0 \subset V_1$ implies that

$$\sigma_{V_0}(\xi) \le \sigma_{V_1}(\xi) = \sigma_{V_0}((A^\intercal)^{-1}\xi), \quad \text{for a.e. } \xi \tag{5.2}$$

by Theorem 2.6(d) and (h). Thus, for a.e. ξ, $\sigma_{V_0}(\xi) \ne 0$ implies $\sigma_{V_0}((A^\intercal)^{-j}\xi) \ne 0$ for all $j \in \mathbb{N}\cup\{0\}$. Moreover, Theorem 2.6(i) yields

$$\sigma_{V_0}(\xi) \ne 0 \Longrightarrow \mathfrak{D}(\xi) = \dim_{V_0}(\xi) \ge 1.$$

Hence,

$$\operatorname{supp}\sigma_{V_0} = \left\{\xi \in \mathbb{R}^N : \sigma_{V_0}((A^\intercal)^{-j}\xi) \ne 0 \text{ for all } j \in \mathbb{N}\cup\{0\}\right\} \subset \Delta. \tag{5.3}$$

Therefore, by Theorem 2.6(a) and (i),

$$\sum_{k\in\mathbb{Z}^N} \mathbf{1}_\Delta(\xi+k) \ge \sum_{k\in\mathbb{Z}^N} \sigma_{V_0}(\xi+k) = \mathfrak{D}(\xi),$$

which shows (D3).

Finally, to see (D4) let

$$G = \left\{\xi \in \mathbb{R}^N : \liminf_{n\to\infty} \mathfrak{D}((A^\intercal)^{-n}\xi) \ge 1\right\}.$$

By (5.3) we have that $E = \operatorname{supp} \sigma_{V_0} \subset G$. Since $A^\intercal(G) = G$, we have $E_\infty = \bigcup_{j\in\mathbb{Z}}(A^\intercal)^j(E) \subset G$. On the other hand, Theorem 2.6(g) implies that $V_0 \subset \check{L}^2(E)$. Hence, each $V_j \subset \check{L}^2(E_\infty)$. By property (M3) of a GMRA this implies that $E_\infty = \mathbb{R}^N$ modulo null sets. This shows $G = \mathbb{R}^N$, and completes the proof of Theorem 5.1. □

In the case when the dilation $A \in M_N(\mathbb{Z})$, the consistency inequality (D2) reduces to

$$\sum_{\omega\in[(A^\intercal)^{-1}\mathbb{Z}^N/\mathbb{Z}^N]} \mathfrak{D}(\xi+\omega) \geq \mathfrak{D}(A^\intercal\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^N. \tag{5.4}$$

Hence, Theorem 5.1 is a generalization of a characterization of GMRA dimension functions due to Baggett and Merrill [4] and Bownik and Rzeszotnik [18, Theorem 3.6]. That is, a function $\mathfrak{D}$ is a dimension function of some integer dilated GMRA if and only if $\mathfrak{D}$ satisfies (D1)–(D4). However, Theorem 5.1 covers only the necessity of conditions (D1)–(D4). The sufficiency of (D1)–(D4) for rational dilations remains an open problem.

To establish the sufficiency of (D1)–(D4) for integer dilations, Bownik, Rzeszotnik, and Speegle [18, 22] have devised the following algorithm. A similar algorithm was given by Baggett and Merrill in [4]. Given any subset $E \subset \mathbb{R}^N$, let $E^P = \sum_{k\in\mathbb{Z}^N}(E+k)$. Let $\tau : \mathbb{R}^N \to [-1/2, 1/2)^N$ be the translation projection, $\tau(\xi) = \xi + k$, where $k \in \mathbb{Z}^N$ is the unique element such that $\xi + k \in [-1/2, 1/2)^N$. Finally, given any measurable subset $\tilde{E} \subset \mathbb{R}^N$, let $E \subset \tilde{E}$ be any measurable set such that $\tau(E) = \tau(\tilde{E})$ and $\tau|E$ is injective.

Algorithm 1 Assume that $\mathfrak{D}$ satisfies the conditions (D1)–(D4). For $m \in \mathbb{N}$, let

$$A_m = \{\xi \in [-1/2, 1/2)^N : \mathfrak{D}(\xi) \geq m\}.$$

(1) Let Q be any measurable subset of $\mathbb{R}^N$ satisfying the following four properties:
   (i) $Q \subset A^\intercal Q$,
   (ii) $\lim_{n\to\infty} \mathbf{1}_Q((A^\intercal)^{-n}\xi) = 1$ for a.e. $\xi \in \mathbb{R}^N$,
   (iii) $\tau|_Q$ is injective,
   (iv) $\mathfrak{D}(\xi) \geq 1$ for all $\xi \in Q$.
(2) Suppose that S_i's are already defined for all $i = 1, \dots, m-1$ and some $m \in \mathbb{N}$. In the case when $m = 1$ (meaning that none of S_i's were defined yet) let $\widetilde{F}_{m,1} = Q$. Otherwise, let $\widetilde{F}_{m,1} = (A^\intercal P_{m-1} \setminus P_{m-1}) \cap A_m^P$, where $P_{m-1} = \bigcup_{i=1}^{m-1} S_i$.
(3) For each $n \in \mathbb{N}$, define iteratively $\widetilde{F}_{m,n+1} = (A^\intercal F_{m,n} \setminus \bigcup_{i=1}^n F_{m,i}^P) \cap A_m^P$. Then, let $S_m = \bigcup_{n=1}^\infty F_{m,n}$.
(4) Finally, let $S = \bigcup_{m\in\mathbb{N}} S_m$.

Then, we have the following result due to [18, 22].

Theorem 5.2 *Assume that $A \in M_N(\mathbb{Z})$, and a function $\mathfrak{D}$, which is not ∞ constantly a.e., satisfies the conditions* (D1)–(D4). *Let S be the result of the above Algorithm. Define the spaces $V_j = \check{L}^2((A^\intercal)^j S)$ for $j \in \mathbb{Z}$. Then, $\{V_j\}_{j\in\mathbb{Z}}$ is a GMRA such that its dimension function $\dim_{V_0}^{\mathbb{Z}^N} \equiv \mathfrak{D}$.*

In particular, if $\mathfrak{D}$ satisfies the consistency equation (1.2), then the Algorithm produces a generalized scaling set S. Thus, defining W to be $A^{\mathsf{T}}S \setminus S$ gives a wavelet set associated with A. Nevertheless, it turns out that Theorem 5.2 is false for rational dilations A. For a counterexample see Sect. 8.

6 Dimension Function of Wavelets

The goal of this section is to establish the formula for the dimension function of rationally dilated wavelets. Moreover, we derive the necessary properties which every wavelet dimension function must satisfy. Theorem 4.2 suggests the following definition.

Definition 6.1 Suppose that $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^N)$ is a semi-orthogonal wavelet associated with a GMRA and the dilation $A \in M_N(\mathbb{Q})$. Define the wavelet dimension of Ψ as

$$\mathfrak{D}_\Psi(\xi) := \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} \sum_{k\in\Gamma^*} |\widehat{\psi^\ell}((A^{\mathsf{T}})^j(\xi + k))|^2, \tag{6.1}$$

where $\Gamma^* = (A^{\mathsf{T}})^{-1}\mathbb{Z}^N \cap \mathbb{Z}^N$ is a dual lattice to $\Gamma = A\mathbb{Z}^N + \mathbb{Z}^N$.

Theorem 6.2 *Suppose that* $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^N)$ *is a semi-orthogonal wavelet associated with a GMRA* $\{V_j\}_{j\in\mathbb{Z}}$. *Then, its dimension function* $\mathfrak{D}_\Psi$ *satisfies the following five conditions*:

(W1) $\mathfrak{D}_\Psi : \mathbb{R}^N \to \mathbb{N} \cup \{0\}$ *is a measurable* Γ^**-periodic function*,
(W2) $\mathfrak{D}_\Psi$ *satisfies the consistency inequality* (*equality if* Ψ *is a wavelet*)

$$\sum_{\omega\in[(A^{\mathsf{T}})^{-1}\mathbb{Z}^N/\Gamma^*]} \mathfrak{D}_\Psi(\xi + \omega) \le L + \sum_{\omega'\in[\mathbb{Z}^N/\Gamma^*]} \mathfrak{D}_\Psi(A^{\mathsf{T}}\xi + \omega') \quad \textit{for a.e. } \xi. \tag{6.2}$$

(W3) $\sum_{k\in\Gamma^*} \mathbf{1}_\Delta(\xi + k) \ge \mathfrak{D}_\Psi(\xi)$ for a.e. $\xi \in \mathbb{R}^N$, *where*

$$\Delta = \left\{\xi \in \mathbb{R}^N : \mathfrak{D}_\Psi\left((A^{\mathsf{T}})^{-j}\xi\right) \ge 1 \textit{ for all } j \in \mathbb{N} \cup \{0\}\right\}.$$

(W4) $\liminf_{n\to\infty} \mathfrak{D}_\Psi((A^{\mathsf{T}})^{-n}\xi) \ge 1$ *for a.e.* $\xi \in \mathbb{R}^N$,
(W5) $\int_{\mathbb{R}^N/\Gamma^*} \mathfrak{D}_\Psi(\xi)d\xi \le \frac{L}{|\det A|-1}$ (*with equality if* Ψ *is a wavelet*).

Proof By Theorem 4.2, the core space V_0 is Γ-SI. Moreover, Theorem 3.1 and (4.6) yields

$$\dim^\Gamma_{V_0}(\xi) = \sum_{k\in\Gamma^*} \sigma^\Gamma_{V_0}(\xi + k) = \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} \sum_{k\in\Gamma^*} |\widehat{\psi^\ell}((A^{\mathsf{T}})^j(\xi + k))|^2 = \mathfrak{D}_\Psi(\xi). \tag{6.3}$$

Thus, (4.3) implies that $\mathfrak{D}_\Psi(\xi)$ takes values in $\mathbb{N} \cup \{0\}$ for a.e. ξ. Hence, (W1) holds.

The consistency inequality (W2) can be verified directly from the definition (6.1). However, it follows immediately from (4.4), (4.5), and (6.3).

To verify (W3), we simply repeat the proof of (D3) in Theorem 5.1. Indeed, (5.3) implies that $\sigma_{V_0}(\xi) \le \mathbf{1}_\Delta(\xi)$ for a.e. $\xi \in \mathbb{R}^N$. Therefore, by Theorem 2.6(i),

$$\sum_{k\in\Gamma^*} \mathbf{1}_\Delta(\xi+k) \ge \sum_{k\in\Gamma^*} \sigma^\Gamma_{V_0}(\xi+k) = \mathfrak{D}_\Psi(\xi),$$

which shows (W3).

Condition (W4) follows from the fact that for each $n \in \mathbb{N}$ and $\xi \in \mathbb{R}^N$ we have

$$\mathfrak{D}_\Psi\big((A^\top)^{-n}\xi\big) \ge \sum_{\ell=1}^{L}\sum_{j=1}^{\infty} \big|\widehat{\psi^\ell}\big((A^\top)^{j-n}\xi\big)\big|^2 = \sum_{\ell=1}^{L}\sum_{j=1-n}^{\infty} \big|\widehat{\psi^\ell}\big((A^\top)^{j}\xi\big)\big|^2,$$

and by the Calderón condition, see [14],

$$\liminf_{n\to\infty} \mathfrak{D}_\Psi\big((A^\top)^{-n}\xi\big) \ge \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} \big|\widehat{\psi^\ell}\big((A^\top)^{j}\xi\big)\big|^2 = 1 \quad \text{for a.e. } \xi.$$

Finally, (W5) is verified by the argument

$$\begin{aligned}
\int_{\mathbb{R}^N/\Gamma^*} \mathfrak{D}_\Psi(\xi)d\xi &= \sum_{\ell=1}^{L}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^N} \int_{\mathbb{R}^N/\Gamma^*} \big|\widehat{\psi^\ell}\big((A^\top)^{j}(\xi+k)\big)\big|^2 d\xi \\
&= \sum_{\ell=1}^{L}\sum_{j=1}^{\infty} \int_{\mathbb{R}^N} \big|\widehat{\psi^\ell}\big((A^\top)^{j}\xi\big)\big|^2 d\xi = \sum_{j=1}^{\infty} |\det A|^{-j} \cdot \sum_{\ell=1}^{L} \|\widehat{\psi^\ell}\|^2 \\
&\le \frac{L}{|\det A|-1}.
\end{aligned}$$

In the case when Ψ is a wavelet, the last step is an equality which proves (W5). □

As an application of Theorem 6.2, we will prove a generalization of Auscher's result on regular wavelets [2]. We say that a function $\psi \in L^2(\mathbb{R}^N)$ satisfies the regularity condition $(\mathfrak{R}^0)$, see [26, Sect. 7.6], if there exist C and $\delta > 0$, such that

$$\begin{aligned}
&|\hat\psi| \text{ is continuous on } \mathbb{R}^N \quad \text{and} \\
&|\hat\psi(\xi)| \le C|\xi|^{-N/2-\delta} \quad \text{for all } \xi \in \mathbb{R}^N.
\end{aligned} \tag{6.4}$$

Lemma 6.3 *Suppose that $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^N)$ satisfies $(\mathfrak{R}^0)$. Then $\mathfrak{D}_\Psi$ is a Γ^*-periodic function which is continuous on $\mathbb{R}^N \setminus \Gamma^*$.*

Proof The fact that A is expansive implies that there exists $\lambda > 1$ such that $|A^j x| \ge c\lambda^j |x|$ for all $j \in \mathbb{N}$. Using this and (6.4) one can show that $s(\xi) = \sum_{\ell=1}^{L}\sum_{j=1}^{\infty} |\widehat{\psi^\ell}((A^\top)^j\xi)|^2$ is continuous except possibly at $\xi = 0$ and $s(\xi) \le$

$C|\xi|^{-N-2\delta}$. Therefore, its Γ^*-periodization $\sum_{k\in\Gamma^*} s(\xi+k)$ is continuous on $\mathbb{R}^N$ except possibly on Γ^*. □

Theorem 6.4 *Suppose that* $\Psi=\{\psi^1,\dots,\psi^L\}\subset L^2(\mathbb{R}^N)$ *is a wavelet associated with a GMRA* $\{V_j\}_{j\in\mathbb{Z}}$. *If* Ψ *satisfies the regularity* $(\Re^0)$, *then* $\{V_j\}_{j\in\mathbb{Z}}$ *is an MRA* (*possibly of higher multiplicity*).

Proof By Lemma 6.3, the dimension function $\mathfrak{D}_\Psi$ is continuous except possibly at some discrete number of points. By Theorem 6.2, $\mathfrak{D}_\Psi$ is integer-valued, and hence $D_\Psi(\xi)=m$ for some $m\in\mathbb{N}$. By Lemma 2.5,

$$\dim_{V_0}^{\mathbb{Z}^N}(\xi)=\sum_{\omega\in[\mathbb{Z}^N/\Gamma^*]}\dim_{V_0}^{\Gamma}(\xi+\omega)=\sum_{\omega\in[\mathbb{Z}^N/\Gamma^*]}\mathfrak{D}_\Psi(\xi+\omega)=m|\mathbb{Z}^N/\Gamma^*|.$$

Hence, $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA of multiplicity $m|\mathbb{Z}^N/\Gamma^*|$. □

Note that Theorem 6.4 imposes a restriction on the number of generators that $(\Re^0)$-regular wavelet $\Psi=\{\psi^1,\dots,\psi^L\}$ can have. Indeed, by the property (W5),

$$\int_{\mathbb{R}^N/\Gamma^*}\mathfrak{D}_\Psi(\xi)d\xi=m|\mathbb{Z}^N/\Gamma^*|=\frac{L}{|\det A|-1}. \tag{6.5}$$

Thus, the number $L\in\mathbb{N}$ of generators of Ψ must be an integer multiple of $|\mathbb{Z}^N/\Gamma^*|(|\det A|-1)$. Hence, we obtain the following corollary which extends one dimensional result of Bownik and Speegle [21, Theorem 4.3].

Corollary 6.5 *Suppose that* A *is a diagonal dilation as in Proposition* 4.5. *Then*, *the minimal number of generators of* $(\Re^0)$-*regular wavelet* $\Psi=\{\psi^1,\dots,\psi^L\}$ *is*

$$L_{\min}=\prod_{j=1}^N|p_j|-\prod_{j=1}^N|q_j|.$$

Proof If A is as in Proposition 4.5, then $\Gamma^*=(A^\intercal)^{-1}\mathbb{Z}^N\cap\mathbb{Z}^N=q_1\mathbb{Z}\times\cdots\times q_N\mathbb{Z}$. Since $|\det A|=\prod_{j=1}^N|p_j/q_j|$, (6.5) immediately yields that L must be an integer multiple of $L_{\min}$. □

7 Characterization of 3 Interval Wavelet Sets

In this section we characterize all possible wavelet sets consisting of 3 intervals for all dilation factors $a>1$. While such characterization is of interest by itself, it also leads to a large class of examples of dimension functions. This direction will be explored in the next section.

7.1 Two Interval Wavelet Sets

As a preliminary, we first characterize 2 interval wavelet sets. We say that a measurable set $W \subset \mathbb{R}$ is a wavelet set associated with the dilation $a > 1$ if and only if

$$\sum_{k\in\mathbb{Z}} \mathbf{1}_W(\xi + k) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}, \tag{7.1}$$

$$\sum_{j\in\mathbb{Z}} \mathbf{1}_W(a^j \xi) = 1 \quad \text{for a.e.} \xi \in \mathbb{R}. \tag{7.2}$$

Notice that (7.2) implies that W cannot contain any intervals of positive length containing 0, even as an endpoint. Indeed, if $I \subset W$ is an interval with $0 \in I$, then for every $\xi > 0$, or $\xi < 0$, we would have $\xi \in a^j I$ for infinitely many $j \in \mathbb{Z}$ and, hence, $\sum_{j\in\mathbb{Z}} \mathbf{1}_W(a^j\xi) = \infty$. Furthermore, it also implies that W must have at least one negative component of positive measure and at least one positive component of positive measure. If, for instance, $|W \cap (-\infty, 0)| = 0$, then we would have $\sum_{j\in\mathbb{Z}} \mathbf{1}_W(a^j\xi) = 0$ for all $\xi < 0$. Thus, if we wish to construct a wavelet set W that is composed of exactly two intervals, we must have $W = [b, c] \cup [d, e]$ with $b < c < 0 < d < e$. Clearly, (7.2) is equivalent to $[b, c]$ and $[d, e]$ partitioning $(-\infty, 0)$ and $(0, \infty)$, respectively, by dilations (modulo null sets) as shown in Fig. 1. Note that in Figs. 1 & 2 we have $I_1 = [b, c]$ and $I_2 = [d, e]$.

On the other hand, (7.1) is equivalent to the fact that d is an integer shift of c and the lengths of the two intervals sum to 1, as shown in Fig. 2.

In other words, (7.2) is equivalent to $b = ac$ and $e = ad$, while (7.1) is equivalent to $(a - 1)(d - c) = 1$ and $d = c + n$ for some $n \in \mathbb{N}$ with $c + n > 0$. Solving for a yields $a = \frac{n+1}{n}$. Hence, we obtain the following theorem.

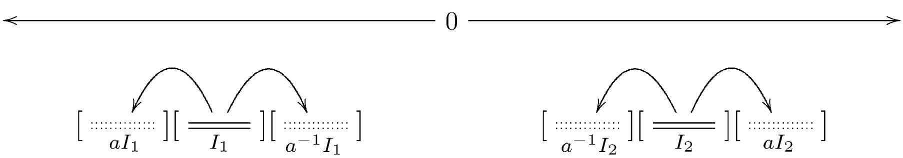

Fig. 1 Dilation partition condition for two intervals

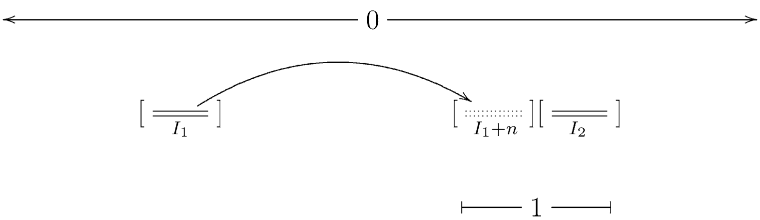

Fig. 2 Translation partition condition for two intervals

Theorem 7.1 *There exist two interval wavelet sets corresponding to the dilation a if and only if $a = \frac{n+1}{n}$ for some $n \in \mathbb{N}$. Furthermore, if $a = \frac{n+1}{n}$ for some $n \in \mathbb{N}$, then W is a two interval wavelet set corresponding to a if and only if:*

$$W = [ax, x] \cup \left[x + \frac{1}{a-1} \, , \, ax + \frac{a}{a-1} \right]$$

for some $x \in (\frac{-1}{a-1}, 0)$.

7.2 Construction of Three Interval Wavelet Sets

In [15] one can find an example of Speegle which provides a formula for a family of wavelet sets in $\mathbb{R}$ consisting of three intervals and depending on the dilation $a > 1$ (see [15], Remark 3). Our goal is to extend this example to a more general form, characterizing all wavelet sets in $\mathbb{R}$ consisting of three intervals.

We consider $W = [b, c] \cup [d, e] \cup [f, g]$ where $b < c < 0 < d < e < f < g$. This is sufficient since W satisfies (7.1) and (7.2) if and only if $-W$ does. Notice that in the case of three intervals (7.2) implies a slightly more complicated relationship. This condition is satisfied if and only if $[b, c]$ partitions $(-\infty, 0)$ by dilations (modulo null sets) and $[d, e]$ and $[f, g]$ partition $(0, \infty)$ by dilations (modulo null sets) in an interlacing pattern as shown in Fig. 3. Note that in Figs. 3–5 we have $I_1 = [b, c]$, $I_2 = [d, e]$, and $I_3 = [f, g]$. The number $p \in \mathbb{N}$ is called an *interlacing parameter*.

On the other hand, the relationship implied by (7.1) is similar to the two interval case, with the exception that there are now two ways in which it can be satisfied. These are shown in Figs. 4 and 5. Initially we will concern ourselves only with three interval wavelet sets that satisfy the translation partition condition as shown in Fig. 4. Let us construct all such wavelet sets. Notice that each one represents a solution to

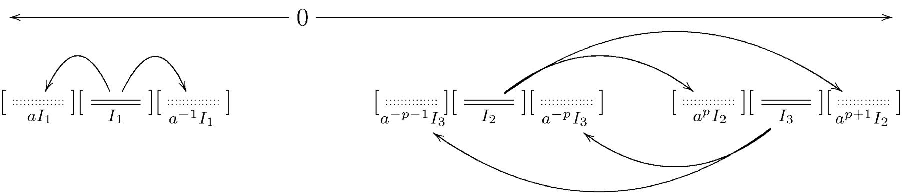

Fig. 3 Dilation partition condition for three intervals

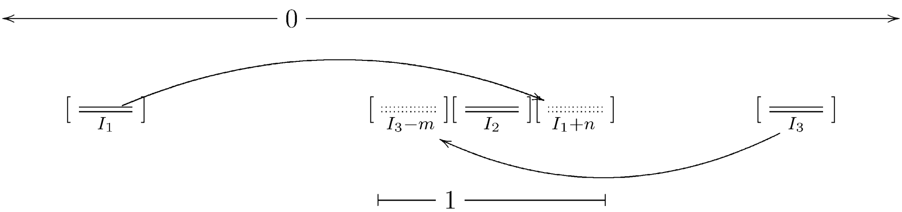

Fig. 4 Translation partition condition for three intervals (Option 1)

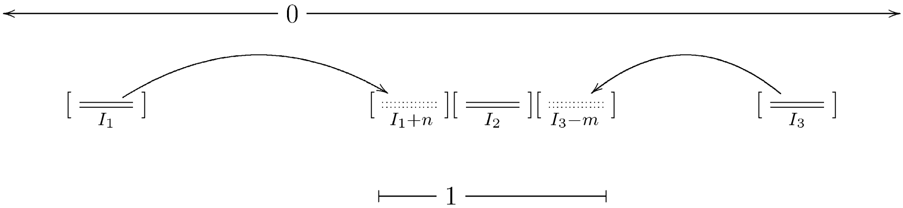

Fig. 5 Translation partition condition for three intervals (Option 2)

the following system of equations for some set of values $m, n, p \in \mathbb{N}$.

$$\begin{cases} m = g - d \\ n = e - b \\ 1 = c - b + e - d + g - f \\ 0 = -b + ac \\ 0 = a^p e - f \\ 0 = a^{p+1} d - g \end{cases} \tag{7.3}$$

For each fixed set of values $m, n, p \in \mathbb{N}$, (7.3) has a solution given by

$$\begin{aligned} [b, c] &= \left[\frac{a\big(m - n(a^p - 1) - 1\big)}{a^{p+1} - 1}, \frac{m - n(a^p - 1) - 1}{a^{p+1} - 1}\right], \\ [d, e] &= \left[\frac{m}{a^{p+1} - 1}, \frac{a(m-1) + n(a-1)}{a^{p+1} - 1}\right], \\ [f, g] &= \left[\frac{a^p\big(a(m-1) + n(a-1)\big)}{a^{p+1} - 1}, \frac{a^{p+1} m}{a^{p+1} - 1}\right]. \end{aligned} \tag{7.4}$$

Furthermore, the conditions $b < c < 0 < d < e < f < g$ impose the added conditions on m, n:

$$0 < n < \frac{a}{a-1}, \tag{7.5}$$

$$\frac{a}{a-1} - n < m < n(a^p - 1) + 1. \tag{7.6}$$

Next we will investigate three interval wavelet sets that satisfy the translation partition condition as shown in Fig. 5. Each of these sets represents a solution to the following system of equations for some set of values $m, n, p \in \mathbb{N}$.

$$\begin{cases} m = f - e \\ n = d - c \\ 1 = c - b + e - d + g - f \\ 0 = -b + ac \\ 0 = a^p e - f \\ 0 = a^{p+1} d - g \end{cases} \tag{7.7}$$

For each fixed set of values $m, n, p \in \mathbb{N}$, (7.7) has a solution given by

$$\begin{aligned}
[b,c] &= \left[\frac{m-n(a^{p+1}-1)+1}{a^p-1}, \frac{m-n(a^{p+1}-1)+1}{a(a^p-1)}\right], \\
[d,e] &= \left[\frac{m-n(a-1)+1}{a(a^p-1)}, \frac{m}{a^p-1}\right], \\
[f,g] &= \left[\frac{a^p m}{a^p-1}, \frac{a^p\big(m-n(a-1)+1\big)}{a^p-1}\right]
\end{aligned} \tag{7.8}$$

and the corresponding conditions on m, n are

$$0 < n < \frac{1}{a-1}, \tag{7.9}$$

$$\frac{1}{a-1} - n < m < n(a^{p+1}-1) - 1. \tag{7.10}$$

We are now ready to present our main result regarding three interval wavelet sets.

Theorem 7.2 *If $W \subset \mathbb{R}$ is comprised of three intervals, then W is a wavelet set if and only if W or $-W$ is as in (i) or (ii) below:*

(i) $[\frac{a(m-n(a^p-1)-1)}{a^{p+1}-1}, \frac{m-n(a^p-1)-1}{a^{p+1}-1}] \cup [\frac{m}{a^{p+1}-1}, \frac{a(m-1)+n(a-1)}{a^{p+1}-1}] \cup [\frac{a^p(a(m-1)+n(a-1))}{a^{p+1}-1}, \frac{a^{p+1}m}{a^{p+1}-1}]$ *for some $a > 1$, $p \in \mathbb{N}$, and $(n, m) \in \mathbb{Z}^2$ satisfying (7.5) and (7.6).*

(ii) $[\frac{m-n(a^{p+1}-1)+1}{a^p-1}, \frac{m-n(a^{p+1}-1)+1}{a(a^p-1)}] \cup [\frac{m-n(a-1)+1}{a^p-1}, \frac{m}{a^p-1}] \cup [\frac{a^p m}{a^p-1}, \frac{a^p(m-n(a-1)+1)}{a^p-1}]$ *for some $1 < a < 2$, $p \in \mathbb{N}$, and $(n, m) \in \mathbb{Z}^2$ satisfying (7.9) and (7.10).*

Proof This is a direct consequence of the fact that W is a wavelet set if and only if $-W$ is a wavelet set, and the fact that the matrices

$$\begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 1 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 1 & -1 & 1 & -1 & 1 \\
-1 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a^p & -1 & 0 \\
0 & 0 & a^{p+1} & 0 & 0 & -1
\end{bmatrix} \quad \text{and} \quad
\begin{bmatrix}
0 & 0 & 0 & -1 & 1 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
-1 & 1 & -1 & 1 & -1 & 1 \\
-1 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a^p & -1 & 0 \\
0 & 0 & a^{p+1} & 0 & 0 & -1
\end{bmatrix}$$

are non-singular for all $a > 1$, making the solutions to (7.3) and (7.7) unique. Moreover, the restriction $1 < a < 2$ in (ii) is a consequence of feasibility regions below. □

7.3 Analysis of the Feasible Regions

In (7.5) and (7.6) we are given a feasible region for (m, n) which depends only on the parameters a and p. We will denote this by $\mathcal{F}_1(a, p)$. One notices that $\mathcal{F}_1(a, p)$ is the interior of the triangle shown in Fig. 6.

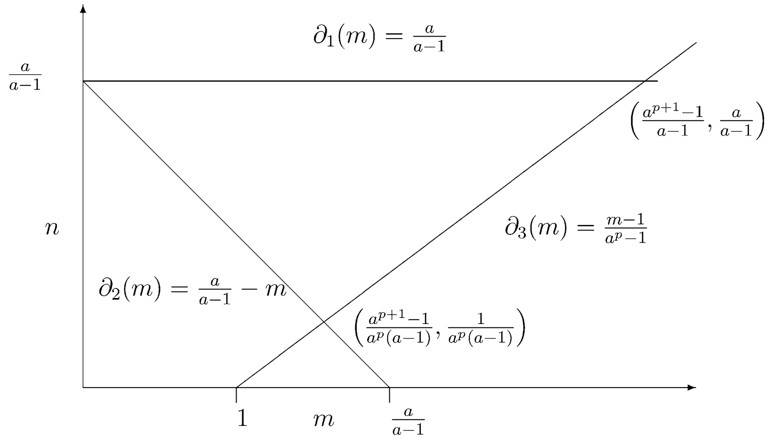

Fig. 6 $\mathcal{F}_1(a, p)$. Feasible region defined by (7.5) & (7.6)

Since (7.4) only gives us a wavelet set for integers n, m satisfying (7.5) and (7.6), it is of interest to know for what values of a, p is the intersection of $\mathcal{F}_1(a, b) \cap \mathbb{Z}^2$ nonempty.

First we consider the case when $\frac{a}{a-1} \notin \mathbb{Z}$. That is, $a \neq \frac{k+1}{k}$ for all $k \in \mathbb{N}$. In this case, it is easy to see that $\mathcal{F}_1(a, p) \cap \mathbb{Z}^2 \neq \varnothing$ for all $p \in \mathbb{N}$. Indeed, we clearly have $\lfloor \frac{a}{a-1} \rfloor < \frac{a}{a-1}$ and $\partial_2(1) = \frac{a}{a-1} - 1 < \lfloor \frac{a}{a-1} \rfloor$. Furthermore, $\partial_3(1) = 0 < \lfloor \frac{a}{a-1} \rfloor$. Therefore, $(1, \lfloor \frac{a}{a-1} \rfloor) \in \mathcal{F}_1(a, p)$ regardless of p. Note, however, that when $p = 1$ we have $\partial_1(2) - \partial_3(2) = 1$ in addition to $\partial_1(1) - \partial_2(1) = 1$. Thus, $(1, \lfloor \frac{a}{a-1} \rfloor)$ and $(2, \lfloor \frac{a}{a-1} \rfloor)$ are the only points in $\mathcal{F}_1(a, 1)$.

Suppose now that $a = \frac{k+1}{k}$ for some $k \in \mathbb{N}$ and, hence, $\frac{a}{a-1} \in \mathbb{Z}$. In this case, we have $\frac{a}{a-1} - 1 = \partial_2(1)$ and $\frac{a}{a-1} - 1 = \partial_3(2)$ since $\frac{a}{a-1} - 1 = \frac{1}{a-1}$. Hence, $(1, n) \notin \mathcal{F}_1(a, 1)$ and $(2, n) \notin \mathcal{F}_1(a, 1)$ for any $n \in \mathbb{Z}$. Furthermore, for all $m \in \mathbb{Z}$ with $3 \leq m \leq a + 1$ we have $\partial_1(m) - \partial_3(m) \leq \partial_1(2) - \partial_3(2) = 1$ and, hence, $(m, n) \notin \mathcal{F}_1(a, 1)$ for any $3 \leq m \leq a + 1$ and $n \in \mathbb{Z}$. Therefore, $\mathcal{F}_1(a, 1) \cap \mathbb{Z}^2 = \varnothing$. However, if $p > 1$ then we have $\partial_3(2) = \frac{1}{a^p-1} < \frac{1}{a-1} = \frac{a}{a-1} - 1$ and so in this case we have $(2, \frac{1}{a-1}) \in \mathcal{F}_1(a, p)$.

We can summarize this discussion in the following statement:

$$\mathcal{F}_1(a, p) \cap \mathbb{Z}^2 \neq \varnothing \iff \frac{1}{a-1} \notin \mathbb{N} \text{ or } p \geq 2. \tag{7.11}$$

On the other hand, conditions (7.9) and (7.10) give us a feasible region for (m, n) pictured as the interior of the triangle in Fig. 7. We wish to know for what values of a and p do we have $\mathcal{F}_2(a, p) \cap \mathbb{Z}^2 \neq \varnothing$.

Notice that the value $a = 2$ plays an even more critical role this time. Indeed, if $a \geq 2$, then $\frac{1}{a-1} \leq 1$ and so $\mathcal{F}_2(a, p) \cap \mathbb{Z}^2 = \varnothing$. Therefore, we can only have wavelet sets of this type for dilations $1 < a < 2$.

Let us now investigate the case $p = 1$ separately. Because $\frac{1}{a} < 1$ and the slope of ∂_3 is positive, it is clear that $\mathcal{F}_2(a, 1) \cap \mathbb{Z}^2 \neq \varnothing$ if and only if $(1, n) \in \mathcal{F}_2(a, 1)$ for some $n \in \mathbb{N}$. Since $\partial_1(1) - \partial_3(1) = \frac{1}{a-1} - \frac{2}{a^2-1} = \frac{a-1}{a^2-1} < 1$, this can only occur

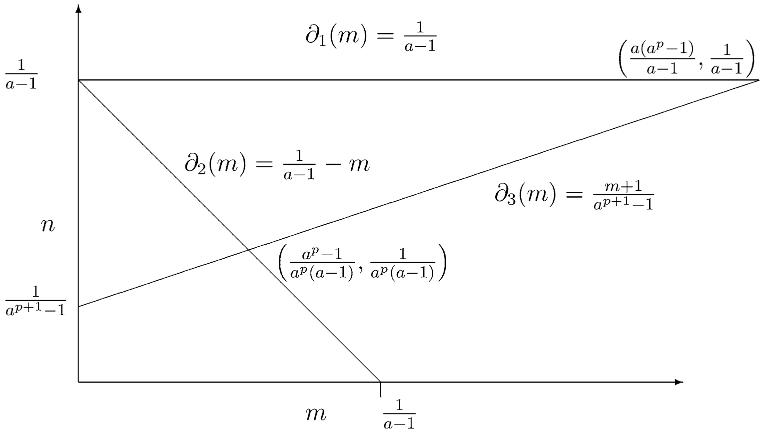

Fig. 7 $\mathcal{F}_2(a, p)$. Feasible region defined by (7.9) & (7.10)

for $n = \lfloor\frac{1}{a-1}\rfloor$. Thus, we wish to characterize the values of a for which we have $\frac{2}{a^2-1} < \lfloor\frac{1}{a-1}\rfloor < \frac{1}{a-1}$. With this in mind, let $n = \lfloor\frac{1}{a-1}\rfloor$. Notice that $\frac{2}{a^2-1} < n < \frac{1}{a-1}$ if and only if $\sqrt{1+\frac{2}{n}} < a < \frac{n+1}{n}$. Therefore, $(1, n) \in \mathcal{F}_2(a, 1)$ for some $n \in \mathbb{N}$ if and only if $\sqrt{1+\frac{2}{n}} < a < \frac{n+1}{n}$, in which case we have $n = \lfloor\frac{1}{a-1}\rfloor$. We conclude that $\mathcal{F}_2(a, 1) \cap \mathbb{Z}^2 \neq \varnothing$ if and only if $a \in \bigcup_{n\in\mathbb{N}}(\sqrt{1+\frac{2}{n}}, \frac{n+1}{n})$.

The case $p \geq 2$ is most efficiently handled by splitting into two subcases: $1 < a \leq \frac{3}{2}$ and $\frac{3}{2} < a < 2$. We deal with the second of these cases first since it is by far the easiest. Notice that $\frac{3}{2} < a < 2$ implies that $\partial_2(1) < 1 < \partial_1(1)$. Furthermore, for such a we have that $p \geq 2 > \log_{\frac{3}{2}}(2)$ implies $\partial_3(1) = \frac{2}{a^{p+1}-1} < \frac{2}{(3/2)^{p+1}-1} < 1$. Thus, $(1, 1) \in \mathcal{F}_2(a, p)$. We conclude that $\mathcal{F}_2(a, p) \cap \mathbb{Z}^2 \neq \varnothing$ for all a and p satisfying $\frac{3}{2} < a < 2$ and $p \geq 2$.

The last remaining case is $p \geq 2$ and $1 < a < \frac{3}{2}$. First note that when $p \geq 2$ we have $1 < \frac{a^p-1}{a^p(a-1)}$. Thus, $(1, n) \in \mathcal{F}_2(a, p)$ if and only if $\partial_2(1) < n < \partial_1(1)$. Since $\partial_2(1) = \partial_1(1) - 1$, then this occurs if and only if $\frac{1}{a-1} \notin \mathbb{Z}$ (which is to say $a \neq \frac{k+2}{k+1}$ for all $k \in \mathbb{N}$). Suppose, then, that $a = \frac{k+2}{k+1}$ for some $k \in \mathbb{N}$. Then one wishes to know for what values of p we have $(2, \lfloor\frac{1}{a-1}\rfloor) \in \mathcal{F}_2(a, p)$. Indeed, $\partial_2(2) = \frac{1}{a-1} - 2 < \lfloor\frac{1}{a-1}\rfloor < \partial_1(2)$, thus $(2, \lfloor\frac{1}{a-1}\rfloor) \in \mathcal{F}_2(a, p)$ whenever, $\partial_3(2) < \frac{1}{a-1} - 1$. We see that this holds whenever $p > \log_a(\frac{2a-1}{2-a})$. But for all a satisfying $1 < a \leq \frac{3}{2}$ we have $\log_a(\frac{2a-1}{2-a}) \leq \log_{\frac{3}{2}}(4) < 3$. Therefore, we conclude that $\mathcal{F}_2(a, p) \cap \mathbb{Z}^2 \neq \varnothing$ for all $p \geq 3$ when $a = \frac{k+2}{k+1}$ for some $k \in \mathbb{N}$.

We summarize these results below which can be compared with the results for $\mathcal{F}_1(a, p)$.

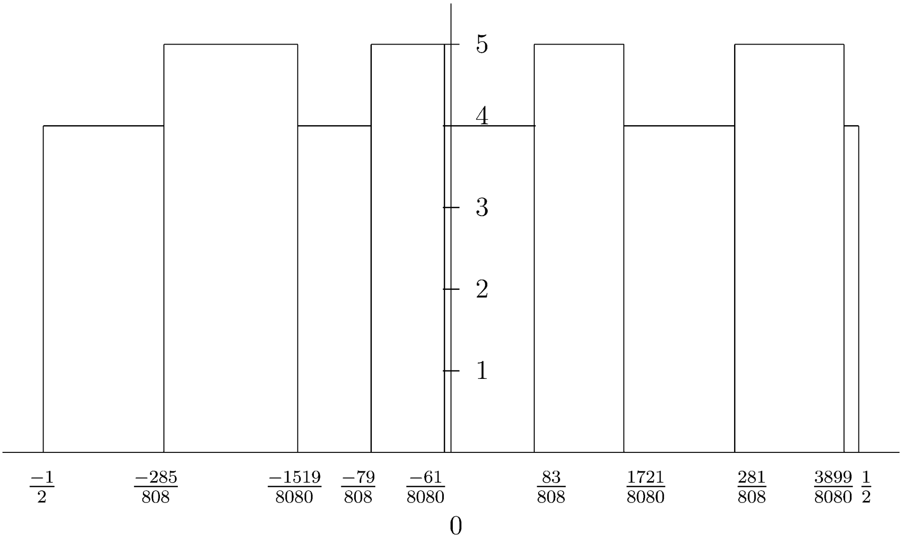

Fig. 8 The dimension function $\mathfrak{D}$

$$\mathcal{F}_2(a,p)\cap\mathbb{Z}^2\neq\varnothing \iff \begin{cases} a\in\bigcup_{n\in\mathbb{N}}\left(\sqrt{1+\frac{2}{n}},\frac{n+1}{n}\right) & p=1,\\ 1<a<2 \text{ and } \frac{1}{a-1}\notin\mathbb{N} & p=2,\\ 1<a<2 & p\geq 3.\end{cases} \tag{7.12}$$

8 An Example of Wavelet Dimension Function

In this section we provide an example of a dimension function of a rationally dilated wavelet for which Algorithm 1 fails to provide a wavelet set corresponding to that dimension function.

Consider the wavelet set W as given in Theorem 7.2(i) with the parameters $a=\frac{11}{9}, p=3, n=5, m=1$. That is, $W=[\frac{-3311}{808},\frac{-2709}{808}]\cup[\frac{6561}{8080},\frac{729}{808}]\cup[\frac{1331}{808},\frac{14641}{8080}]$. Define the wavelet $\psi\in L^2(\mathbb{R})$ by $\hat{\psi}=\mathbf{1}_W$. We wish to calculate the dimension function of ψ using (6.1). Since $\Gamma^*=9\mathbb{Z}$, we have $\mathfrak{D}_\psi(\xi)=\sum_{k\in 9\mathbb{Z}}\mathbf{1}_S(\xi+k)$, where $S=\bigcup_{j=1}^\infty a^{-j}W=[\frac{-2709}{808},\frac{6561}{8080}]\cup[\frac{729}{808},\frac{8019}{8080}]\cup[\frac{891}{808},\frac{9801}{8080}]\cup[\frac{1089}{808},\frac{11979}{8080}]$. Consequently, the wavelet ψ is associated with the GMRA $V_j=\check{L}^2(a^jS)$. In order to apply Algorithm 1 we must look at $\mathfrak{D}(\xi)=\dim_{V_0}^{\mathbb{Z}^N}(\xi)=\sum_{k\in\mathbb{Z}}\mathbf{1}_S(\xi+k)$. A calculation shows that

$$\mathfrak{D}(\xi)=\begin{cases}4 & \text{if } \xi+k\in X \text{ for some } k\in\mathbb{Z},\\ 5 & \text{otherwise,}\end{cases}$$

where $X=[\frac{-1}{2},\frac{-285}{808}]\cup[\frac{-1519}{8080},\frac{-79}{808}]\cup[\frac{-61}{8080},\frac{83}{808}]\cup[\frac{1721}{8080},\frac{281}{808}]\cup[\frac{3899}{8080},\frac{1}{2}]$. One period of $\mathfrak{D}$ is shown in Fig. 8. By Theorem 5.1, $\mathfrak{D}$ satisfies conditions (D1)–(D4).

We begin the algorithm by defining $Q = [-\varepsilon, 1-\varepsilon)$ for some $0 < \varepsilon < \frac{61}{8080}a^{-7}$. Clearly, Q satisfies conditions (i)–(iv) of Algorithm 1. Since $|F_{1,1}| = 1$, it follows that $E_1^P = \mathbb{R}$ and, hence, $F_{1,n} = \varnothing$ for all $n \geq 2$. Thus, $P_1 = S_1 = [-\varepsilon, 1-\varepsilon)$. Continuing with the algorithm, we have

$$F_{2,n} = \begin{cases} [-a^n\varepsilon, -a^{n-1}\varepsilon) \cup [a^{n-1}(1-\varepsilon), a^n(1-\varepsilon)) & \text{if } n \leq 3, \\ [a^3(1-\varepsilon), 2-a^3\varepsilon) & \text{if } n = 4, \\ \varnothing & \text{if } n \geq 5. \end{cases} \tag{8.1}$$

This gives $S_2 = [-(\frac{11}{9})^3\varepsilon, -\varepsilon) \cup [1-\varepsilon, 2-(\frac{11}{9})^3\varepsilon)$ and, hence, $P_2 = [-(\frac{11}{9})^3\varepsilon, 2-(\frac{11}{9})^3\varepsilon)$. Likewise, the next two iterations yield

$$\begin{aligned} S_3 &= [-a^5\varepsilon\ ,\ -a^3\varepsilon\big) \cup \big[2-a^3\varepsilon\ ,\ 3-a^5\varepsilon), \\ S_4 &= [-a^6\varepsilon\ , -a^5\varepsilon\big) \cup \big[3-a^5\varepsilon\ ,\ 4-a^6\varepsilon). \end{aligned} \tag{8.2}$$

Now, one must use caution when approaching the fifth iteration due to the fact that $A_5^P \neq \mathbb{R}$, unlike $A_2^P, \ldots, A_4^P$. Note that

$$\widetilde{F}_{5,1} = (A_5^P \cap [-a^7\varepsilon\ , -a^6\varepsilon)) \cup (A_5^P \cap [4-a^6\varepsilon\ , 4a-a^7\varepsilon)).$$

Since $\frac{-61}{8080} < -a^7\varepsilon$, the above reduces to

$$\widetilde{F}_{5,1} = A_5^P \cap [4-a^6\varepsilon\ , 4a-a^7\varepsilon).$$

Furthermore, a direct calculation shows that

$$5 - \frac{1519}{8080} < 4a - a^7\varepsilon < 5 - \frac{79}{808}.$$

Therefore, we are left with

$$F_{5,1} = \left[4+\frac{83}{808}\ ,\ 4+\frac{1721}{8080}\right] \cup \left[4+\frac{281}{808}\ ,\ 4+\frac{3899}{8080}\right] \cup \left[5-\frac{285}{808}\ ,\ 5-\frac{1519}{8080}\right].$$

Finally, we claim that $F_{5,2} = \varnothing$ and, hence, $F_{5,n} = \varnothing$ for all $n \geq 2$. This is true because $5 - \frac{61}{8080} < \frac{11}{9}(4+\frac{83}{808})$ and $\frac{11}{9}(5+\frac{-1519}{8080}) < 6 - \frac{79}{808}$. Therefore, the algorithm stops, giving the output

$$S' = [-a^6\varepsilon, 4-a^6\varepsilon) \cup F_{5,1}.$$

Define the spaces $V_j' = \check{L}^2(a^j S')$ for $j \in \mathbb{Z}$. While $\{V_j'\}_{j\in\mathbb{Z}}$ is a GMRA, its dimension function $\mathfrak{D}'(\xi) = \dim_{V_0'}^{\mathbb{Z}^N}(\xi) = \sum_{k\in\mathbb{Z}} \mathbf{1}_{S'}(\xi+k)$ is different from $\mathfrak{D}$. Indeed, we have

$$\mathfrak{D}'(\xi) = \begin{cases} 4 & \text{if } \xi + k \in \bar{X} \text{ for some } k \in \mathbb{Z}, \\ 5 & \text{otherwise,} \end{cases}$$

where $\bar{X} = [\frac{-1}{2}, \frac{-285}{808}] \cup [\frac{-1519}{8080}, \frac{83}{808}] \cup [\frac{1721}{8080}, \frac{281}{808}] \cup [\frac{3899}{8080}, \frac{1}{2}]$. Thus, $\mathfrak{D}' \neq \mathfrak{D}$ and Algorithm 1 fails, see Fig. 9.

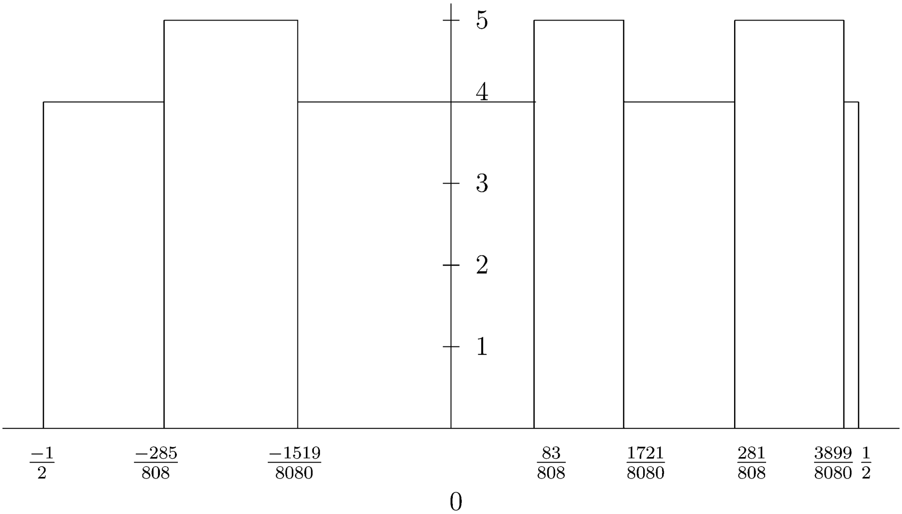

Fig. 9 The dimension function $\mathfrak{D}'$

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