Math. Nachr. **283,** No. 3, 392 – 442 (2010) / **DOI** 10.1002/mana.200910078

Weighted anisotropic product Hardy spaces and boundedness of sublinear operators

Marcin Bownik[*1], **Baode Li**[**2], **Dachun Yang**[***2], and **Yuan Zhou**[†2]

[1] Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, USA
[2] School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China

Received 27 March 2009, accepted 24 October 2009
Published online 26 February 2010

Key words Product space, expansive dilation, weight, Calderón reproducing formula, dyadic rectangle, atom, grand maximal function, Hardy space, quasi-Banach space, sublinear operator
MSC (2000) Primary: 42B30; Secondary: 42B20, 42B25, 42B35

Dedicated to the memory of Erhard Schmidt

Let A_1 and A_2 be expansive dilations, respectively, on $\mathbb{R}^n$ and $\mathbb{R}^m$. Let $\vec{A} \equiv (A_1,\, A_2)$ and $\mathcal{A}_p(\vec{A})$ be the class of product Muckenhoupt weights on $\mathbb{R}^n \times \mathbb{R}^m$ for $p \in (1,\, \infty]$. When $p \in (1,\, \infty)$ and $w \in \mathcal{A}_p(\vec{A})$, the authors characterize the weighted Lebesgue space $L^p_w(\mathbb{R}^n \times \mathbb{R}^m)$ via the anisotropic Lusin-area function associated with $\vec{A}$. When $p \in (0,\, 1]$, $w \in \mathcal{A}_\infty(\vec{A})$, the authors introduce the weighted anisotropic product Hardy space $H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A})$ via the anisotropic Lusin-area function and establish its atomic decomposition. Moreover, the authors prove that finite atomic norm on a dense subspace of $H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A})$ is equivalent with the standard infinite atomic decomposition norm. As an application, the authors prove that if T is a sublinear operator and maps all atoms into uniformly bounded elements of a quasi-Banach space $\mathcal{B}$, then T uniquely extends to a bounded sublinear operator from $H^p_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ to $\mathcal{B}$. The results of this paper improve the existing results for weighted product Hardy spaces and are new even in the unweighted anisotropic setting.

1 Introduction

The theory of Hardy spaces plays an important role in various fields of analysis and partial differential equations; see, for example, [17, 22, 32, 43, 54–56]. One of the most important applications of Hardy spaces is that they are good substitutes of Lebesgue spaces when $p \in (0,\, 1]$. For example, when $p \in (0,\, 1]$, it is well-known that Riesz transforms are not bounded on $L^p(\mathbb{R}^n)$, however, they are bounded on Hardy spaces $H^p(\mathbb{R}^n)$. There were several efforts of extending classical function spaces and related operators arising in harmonic analysis from Euclidean spaces to other domains and anisotropic settings; see [3, 9, 10, 28, 52, 59–61]. Fabes and Rivière [20, 21, 47] initiated the study of singular integrals with mixed homogeneity, and Calderón and Torchinsky [8–10] the study of Hardy spaces associated with anisotropic dilations. Recently, a theory of anisotropic Hardy spaces and their weighted theory were developed in [3, 6]. Another direction is the development of the theory of Hardy spaces on product domains initiated by Gundy and Stein [35]. In particular, Chang and Fefferman [12, 13] characterized the classical product Hardy spaces via atoms. Fefferman [26], Krug [38] and Zhu [66] established the weighted theory of the classical product Hardy spaces, and Sato [49, 50] established parabolic Hardy spaces on product domains. It was also proved that the classical product Hardy spaces are good substitutes of product Lebesgue spaces when $p \in (0, 1]$; see, for example, [23, 25, 26, 50, 53]. Recently, the boundedness of singular integrals on

* e-mail: mbownik@uoregon.edu, Phone: +1 5413465622, Fax: +1 5413460987
** e-mail: baodeli@mail.bnu.edu.cn, Phone: +86 1058805472, Fax: +86 1058805472
*** Corresponding author: e-mail: dcyang@bnu.edu.cn, Phone: +86 1058805472, Fax: +86 1058805472
† e-mail: yuanzhou@mail.bnu.edu.cn, Phone: +86 1058805472, Fax: +86 1058805472

product Lebesgue spaces was further proved to be useful in solving problems from the several complex variables by Nagel and Stein [44].

On the other hand, to establish the boundedness of operators on Hardy spaces, one usually appeals to the atomic decomposition characterization, see [8, 14, 16, 25, 28, 41, 58], which means that a function or distribution in Hardy spaces can be represented as a linear combination of functions of an elementary form, namely, atoms. Then, the boundedness of operators on Hardy spaces can be deduced from their behavior on atoms or molecules in principle. However, caution needs to be taken due to an example constructed in [4, Theorem 2]. There exists a linear functional defined on a dense subspace of $H^1(\mathbb{R}^n)$, which maps all $(1,\,\infty,\,0)$-atoms into bounded scalars, but yet it does not extend to a bounded linear functional on the whole $H^1(\mathbb{R}^n)$. This implies that the uniform boundedness of a linear operator T on atoms does not automatically guarantee the boundedness of T from $H^1(\mathbb{R}^n)$ to a Banach space $\mathcal{B}$.

Recently, there was a flurry of activity addressing the problem of boundedness of operators on $H^p(\mathbb{R}^n)$ via atomic decompositions in addition to older contributions; see [30, 41, 42, 58, 62] and the references therein. Let $p\in(0,1]$, $q\in[1,\,\infty]\cap(p,\,\infty]$ and s be an integer no less than $\lfloor n(1/p-1)\rfloor$, where and in what follows, $\lfloor\cdot\rfloor$ denotes the floor function. Using the Lusin-area function characterization of classical Hardy spaces, it was proved in [64] that if a sublinear operator T maps all smooth $(p,\,2,\,s)$-atoms into uniformly bounded elements of a quasi-Banach space $\mathcal{B}$, then T uniquely extends to a bounded sublinear operator from $H^p(\mathbb{R}^n)$ to $\mathcal{B}$. This result was generalized to the classical product Hardy spaces in [11]. At the same time, Meda, Sjögren, and Vallarino [40] independently obtained a related result using the grand maximal function characterization of $H^p(\mathbb{R}^n)$. Precisely, they proved that the norm of $H^p(\mathbb{R}^n)$ can be reached on some dense subspaces of $H^p(\mathbb{R}^n)$ via finite combinations of $(p,\,q,\,s)$-atoms when $q<\infty$ and continuous $(p,\,\infty,\,s)$-atoms. Their result immediately implies that if T is a linear operator and maps all $(p,\,q,\,s)$-atoms with $q<\infty$ or all continuous $(p,\,\infty,\,s)$-atoms into uniformly bounded elements of a Banach space $\mathcal{B}$, then T uniquely extends to a bounded linear operator from $H^p(\mathbb{R}^n)$ to $\mathcal{B}$. This result was further generalized to the weighted anisotropic Hardy spaces in [6] and the Hardy spaces on spaces of homogeneous type enjoying the reverse doubling property in [33] when $p\le 1$ and near to 1. Very recently, Ricci and Verdera [46] showed that if $p\in(0,1)$, then the uniform boundedness of a *linear* operator T on all $(p,\,\infty,\,s)$-atoms does guarantee the boundedness of T from $H^p(\mathbb{R}^n)$ to a Banach space $\mathcal{B}$.

In this paper, we always let A_1 and A_2 be expansive dilations, respectively, on $\mathbb{R}^n$ and $\mathbb{R}^m$. Let $\vec{A}\equiv(A_1,\,A_2)$ and $\mathcal{A}_p(\vec{A})$ be the class of product Muckenhoupt weights on $\mathbb{R}^n\times\mathbb{R}^m$ for $p\in(1,\,\infty]$. When $p\in(1,\,\infty)$ and $w\in\mathcal{A}_p(\vec{A})$, we characterize the anisotropic weighted Lebesgue space $L^p_w(\mathbb{R}^n\times\mathbb{R}^m)$ via the anisotropic Lusin-area function associated with expansive dilations. For $p\in(0,\,1]$ and $w\in\mathcal{A}_\infty(\vec{A})$ and admissible triplet $(p,\,q,\,\vec{s})_w$ (see Definition 4.2 below), we introduce the weighted anisotropic product Hardy space $H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\vec{A})$, the atomic one $H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n\times\mathbb{R}^m;\vec{A})$ and the finite atomic one $H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n\times\mathbb{R}^m;\vec{A})$, respectively, via the anisotropic Lusin-area function, $(p,\,q,\,\vec{s})_w$-atoms and finite linear combinations of $(p,\,q,\,\vec{s})^*_w$-atoms. We then prove that $H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\vec{A})$ coincides with $H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n\times\mathbb{R}^m;\vec{A})$, that $H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n\times\mathbb{R}^m;\vec{A})$ is dense in $H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\vec{A})$ and that both the quasi-norms $\|\cdot\|_{H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}$ and $\|\cdot\|_{H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}$ with $\vec{s}$ being sufficiently large are equivalent on $H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})$. As an application, we prove that if T is a sublinear operator and maps all $(p,\,q,\,\vec{s})^*_w$-atoms into uniformly bounded elements of a quasi-Banach space $\mathcal{B}$, then T uniquely extends to a bounded sublinear operator from $H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\vec{A})$ to $\mathcal{B}$.

We point out that the setting in this paper includes the classical isotropic product Hardy space theory of Gundy and Stein [35] and Chang and Fefferman [12, 13], the parabolic product Hardy space theory of Sato [49, 50] and the weighted product Hardy space theory of Fefferman [26], Krug [38] and Zhu [66]. Most results of this paper are new even in the unweighted setting. They also improve the corresponding results on the isotropic weighted product Hardy spaces in [26], [38] and [66]. The paper is organized as follows.

In Section 2, we recall some notation and definitions concerning expansive dilations, Muckenhoupt weights and maximal functions, whose basic properties are also presented. Moreover, we establish discrete Calderón reproducing formulae (see Proposition 2.16 below) associated to the product expansive dilations for distributions vanishing weakly at infinity, which were introduced by Folland and Stein [28] on homogeneous groups. These Calderón reproducing formulae are crucial tools for this paper. Another key tool used in this paper are the dyadic cubes of Christ [15], which substitute the role played by dilated balls and cubes in [3–5], and are used in deriving the atomic decomposition of product Hardy spaces via the Lusin-area function. Here we point out that a subtle relation between the dyadic cubes of Christ [15] and dilated balls associated to expansive dilations is established in Lemma 2.3(iv) according to the levels of dyadic cubes. This relation and the concept of the level of dyadic cubes play an important role in the whole paper, especially in the choice of dyadic rectangles of $\mathbb{R}^n \times \mathbb{R}^m$; see (4.1) and (5.4) below.

In Section 3, for $p \in (1, \infty)$ and $w \in \mathcal{A}_p(\vec{A})$ (resp. $w \in \mathcal{A}_p(A)$), with the aid of the theory of one-parameter vector-valued Calderón-Zygmund operators, we characterize the anisotropic weighted Lebesgue space $L^p_w(\mathbb{R}^n \times \mathbb{R}^m)$ (resp. $L^p_w(\mathbb{R}^n)$) via the anisotropic Lusin-area function associated with expansive dilations $\vec{A}$ (resp. A); see Theorem 3.2 and Theorem 3.4 below.

In Section 4, let $p \in (0, 1]$, $w \in \mathcal{A}_\infty(\vec{A})$ and $(p, q, \vec{s})_w$ be admissible. We introduce the Hardy space $H^p_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ and the atomic one $H^{p, q, \vec{s}}_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$, respectively, via the Lusin-area function and $(p, q, \vec{s})_w$-atoms. Using some ideas from [12, 13, 26, 66] and the Calderón reproducing formulae established in Proposition 2.16, we prove that $H^p_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ coincides with $H^{p, q, \vec{s}}_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$; see Theorem 4.5 below. We point out that since we are working on weighted anisotropic product Hardy spaces, when we decompose a distribution into a sum of atoms, the dual method for estimating norms of atoms in [12] does not work any more in the current setting. Instead, we invoke a method from Fefferman [26] with more subtle estimates involving rescaling techniques specific to the anisotropic setting. We also notice that a variant of Journé's covering lemma for expansive dilations established in Lemma 4.9 is crucial to the proof of the imbedding of $H^{p, q, \vec{s}}_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ into $H^p_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$. In fact, Lemma 4.9 plays an important role in obtaining the boundedness of operators on $H^p_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$. In particular, using Lemma 4.9, we obtain the boundedness of the anisotropic grand maximal function from $H^p_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ to $L^p_w(\mathbb{R}^n \times \mathbb{R}^m)$; see Proposition 4.11 below.

In Section 5, we introduce $H^{p, q, \vec{s}}_{w, \mathrm{fin}}(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ to be the set of all finite combinations of $(p, q, \vec{s})^*_w$-atoms. Via the Lusin-area function together with the Calderón reproducing formula and by using ideas from [40], we prove that $H^{p, q, \vec{s}}_{w, \mathrm{fin}}(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ is dense in $H^p_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ and that the quasi-norm $\| \cdot \|_{H^p_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})}$ is equivalent to $\| \cdot \|_{H^{p, q, \vec{s}}_{w, \mathrm{fin}}(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})}$ on $H^{p, q, \vec{s}}_{w, \mathrm{fin}}(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ with $\vec{s}$ being sufficiently large; see Theorem 5.2 below. In fact, by a careful choice of dyadic rectangles in $\mathbb{R}^n \times \mathbb{R}^m$ (see (5.4) below), we first construct some finite $(p, q, \vec{s})^*_w$-atoms and then by a subtle size estimate on the complement of the union of chosen rectangles, we prove that the difference between the original function and the linear combination of these finite $(p, q, \vec{s})^*_w$-atoms is still a $(p, q, \vec{s})^*_w$-atom multiplied by a small constant. We should point out that while the main idea comes from [40], Meda, Sjögren, and Vallarino used the grand maximal function characterization of the classical Hardy space $H^p(\mathbb{R}^n)$ to obtain the desired estimates instead. See also [6] for the weighted anisotropic Hardy space $H^p_w(\mathbb{R}^n; A)$. It is not clear if their approach [40] also works here, since so far, it is not known whether $H^p_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ can be characterized via the grand maximal function. Moreover, comparing with the non-product case (see [6, 33, 40]), our results require additional assumptions (5.1) and (5.2) on vanishing moments of atoms.

In Section 6, we present applications of Theorem 5.2. If T is a sublinear operator defined on $H^{p, q, \vec{s}}_{w, \mathrm{fin}}(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ and maps all $(p, q, \vec{s})^*_w$-atoms into uniformly bounded elements of a quasi-Banach space $\mathcal{B}$, then T uniquely extends to a bounded sublinear operator from $H^p_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ to $\mathcal{B}$; see Theorem 6.2 bellow. This result is an extension of [11, Theorem 1.1]. Using Theorem 6.2 and Journé's covering lemma, we establish a criteria on the boundedness of certain sublinear operators via their behavior on rectangular atoms, which extends and complements a result of Fefferman [25, Theorem 1].

We mention that there exist many predictable applications of our results in the study of boundedness of sublinear operators on the weighted product Hardy spaces. For example, in [39], we establish the boundedness on these weighted product Hardy spaces of singular integrals appearing in the work of Nagel and Stein [44].

We finally make some conventions. Throughout this paper, we always use C to denote a positive constant which is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts do not change through the whole paper. Denote by $\mathbb{N}$ the set $\{1, 2, \dots\}$ and by $\mathbb{Z}_+$ the set $\mathbb{N} \cup \{0\}$. We use $f \lesssim g$ or $g \gtrsim f$ to denote $f \le Cg$, and if $f \lesssim g \lesssim f$, we then write $f \sim g$. Denote by $M_n(\mathbb{R})$ the set of all real $n \times n$ matrices.

2 Preliminaries

We begin with the following notation and properties concerning expansive dilations.

Definition 2.1 $A \in M_n(\mathbb{R})$ is said to be an *expansive dilation*, shortly a *dilation*, if $\min_{\lambda\in\sigma(A)} |\lambda| > 1$, where $\sigma(A)$ is the set of all eigenvalues of A.

If A is diagonalizable over $\mathbb{C}$, we take $\lambda_- \equiv \min\{|\lambda|,\ \lambda \in \sigma(A)\}$ and $\lambda_+ \equiv \max\{|\lambda|,\ \lambda \in \sigma(A)\}$. Otherwise, let λ_- and λ_+ be two *positive numbers* such that

$$1 < \lambda_- < \min\{|\lambda|, \lambda \in \sigma(A)\} \le \max\{|\lambda|, \lambda \in \sigma(A)\} < \lambda_+.$$

Throughout the whole paper, for a fixed dilation A, we always let $b \equiv |\det A|$.

It was proved in [3, Lemma 2.2] that for a given dilation A, there exist an *open* and *symmetric convex* ellipsoid Δ and $r \in (1, \infty)$ such that $\Delta \subset r\Delta \subset A\Delta$, and one can additionally assume that $|\Delta| = 1$, where $|\Delta|$ denotes the n-dimensional Lebesgue measure of the set Δ. Throughout the whole paper, we set $B_k \equiv A^k\Delta$ for $k \in \mathbb{Z}$ and let σ be the *minimum integer* such that $2B_0 \subset A^\sigma B_0$. Then B_k is open, $B_k \subset rB_k \subset B_{k+1}$ and $|B_k| = b^k$. Obviously, $\sigma \ge 1$. For any subset E of $\mathbb{R}^n$, let $E^{\complement} \equiv \mathbb{R}^n \setminus E$. Then it is easy to prove (see [3, p. 8]) that for all $k,\ \ell \in \mathbb{Z}$, we have

$$B_k + B_\ell \subset B_{\max(k,\,\ell)+\sigma}, \tag{2.1}$$

$$B_k + (B_{k+\sigma})^{\complement} \subset (B_k)^{\complement}, \tag{2.2}$$

where $E + F$ denotes the algebraic sums $\{x + y : x \in E,\ y \in F\}$ of sets $E,\ F \subset \mathbb{R}^n$ (see [3, p. 8]).

Recall that the homogeneous quasi-norm associated with A was introduced in [3, Definition 2.3] as follows.

Definition 2.2 A *homogeneous quasi-norm* associated with an expansive dilation A is a measurable mapping $\rho : \mathbb{R}^n \to [0,\infty)$ satisfying that

(i) $\rho(x) = 0$ if and only if $x = 0$;

(ii) $\rho(Ax) = b\rho(x)$ for all $x \in \mathbb{R}^n$;

(iii) $\rho(x + y) \le H[\rho(x) + \rho(y)]$ for all $x,\ y \in \mathbb{R}^n$, where H is a constant no less than 1.

In the standard dyadic case $A = 2I_{n\times n}$, $\rho(x) = |x|^n$ is an example of homogeneous quasi-norms associated with A, where and in what follows, $I_{n\times n}$ always denotes the $n \times n$ *unit matrix* and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$.

Define the *step homogeneous quasi-norm* ρ associated with A and Δ by setting, for all $x \in \mathbb{R}^n$, $\rho(x) = b^k$ if $x \in B_{k+1} \setminus B_k$ or else 0 if $x = 0$. It was proved that all homogeneous quasi-norms associated with a given dilation A are equivalent (see [3, Lemma 2.4]). Therefore, for a given expansive dilation A, in what follows, for convenience, we always use the step homogeneous quasi-norm ρ.

For the step homogeneous quasi-norm ρ, from (2.1) and (2.2), it follows that for all $x,\ y \in \mathbb{R}^n$, $\rho(x + y) \le b^\sigma \max\{\rho(x),\ \rho(y)\} \le b^\sigma[\rho(x) + \rho(y)]$; see [3, p. 8].

The following inequalities concerning A, ρ and the Euclidean norm $|\cdot|$ established in [3, Section 2] are used in the whole paper: There exists a positive constant C such that

$$C^{-1}[\rho(x)]^{\zeta_-} \le |x| \le C[\rho(x)]^{\zeta_+} \quad \text{for all} \quad \rho(x) \ge 1, \text{ and} \tag{2.3}$$

$$C^{-1}[\rho(x)]^{\zeta_+} \le |x| \le C[\rho(x)]^{\zeta_-} \quad \text{for all} \quad \rho(x) \le 1, \tag{2.4}$$

where and in what follows $\zeta_+ \equiv \ln(\lambda_+)/\ln b$ and $\zeta_- \equiv \ln(\lambda_-)/\ln b$, and that

$$C^{-1}b^{j\zeta_-}|x| \le |A^jx| \le Cb^{j\zeta_+}|x| \quad \text{for all} \quad j \ge 0, \text{ and} \tag{2.5}$$

$$C^{-1}b^{j\zeta_+}|x| \le |A^jx| \le Cb^{j\zeta_-}|x| \quad \text{for all} \quad j \le 0. \tag{2.6}$$

Moreover, $(\mathbb{R}^n,\ \rho,\ dx)$ is a space of homogeneous type in the sense of Coifman and Weiss [18], where dx is the n-dimensional Lebesgue measure. On such homogeneous spaces, Christ [15] provided an analogue of the grid of Euclidean dyadic cubes as follows.

Lemma 2.3 *Let A be a dilation. There exists a collection $\mathcal{Q} \equiv \{Q_\alpha^k \subset \mathbb{R}^n : k \in \mathbb{Z}, \alpha \in I_k\}$ of open subsets, where I_k is certain index set, such that*

(i) $\left|\mathbb{R}^n \setminus \cup_\alpha Q_\alpha^k\right| = 0$ for each fixed k and $Q_\alpha^k \cap Q_\beta^k = \emptyset$ if $\alpha \neq \beta$;

(ii) for any α, β, k, ℓ with $\ell \geq k$, either $Q_\alpha^k \cap Q_\beta^\ell = \emptyset$ or $Q_\alpha^\ell \subset Q_\beta^k$;

(iii) for each (ℓ, β) and each $k < \ell$ there exists a unique α such that $Q_\beta^\ell \subset Q_\alpha^k$;

(iv) there exist certain negative integer v and positive integer u such that for all Q_α^k with $k \in \mathbb{Z}$ and $\alpha \in I_k$, there exists $x_{Q_\alpha^k} \in Q_\alpha^k$ satisfying that for any $x \in Q_\alpha^k$, $x_{Q_\alpha^k} + B_{vk-u} \subset Q_\alpha^k \subset x + B_{vk+u}$.

In what follows, for convenience, we call k the *level* of the dyadic cube Q_α^k with $k \in \mathbb{Z}$ and $\alpha \in I_k$ and denote it by $\ell(Q_\alpha^k)$. Lemma 2.3 can be proved by a slight modification of the proof of [15, Theorem 11]. In fact, we only need to choose δ in the proof of [15, Theorem 11] to be b^v with v being negative integer. We omit the details. From now on, we call $\{Q_\alpha^k\}_{k\in\mathbb{Z},\, \alpha\in I_k}$ in Lemma 2.3 dyadic cubes.

For any locally integrable function f, the *Hardy-Littlewood maximal function* $\mathcal{M}(f)$ of f is defined by

$$\mathcal{M}(f)(x) \equiv \sup_{k\in\mathbb{Z}} \sup_{x\in y+B_k} \frac{1}{|B_k|} \int_{y+B_k} |f(z)|\, dz, \quad x \in \mathbb{R}^n.$$

It was proved in [3, Theorem 3.6] that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$ with $p \in (1, \infty]$ and bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$.

We now recall the weight class of Muckenhoupt associated with A introduced in [5].

Definition 2.4 Let $p \in [1, \infty)$, A a dilation and w a nonnegative measurable function on $\mathbb{R}^n$. The function w is said to belong to the *weight class of Muckenhoupt* $\mathcal{A}_p(A) \equiv \mathcal{A}_p(\mathbb{R}^n; A)$, if there exists a positive constant C such that when $p > 1$

$$\sup_{x\in\mathbb{R}^n} \sup_{k\in\mathbb{Z}} \left\{\frac{1}{|B_k|} \int_{x+B_k} w(y)\, dy\right\} \left\{\frac{1}{|B_k|} \int_{x+B_k} [w(y)]^{-1/(p-1)}\, dy\right\}^{p-1} \leq C,$$

and when $p = 1$

$$\sup_{x\in\mathbb{R}^n} \sup_{k\in\mathbb{Z}} \left\{\frac{1}{|B_k|} \int_{x+B_k} w(y)\, dy\right\} \left\{\operatorname*{esssup}_{y\in x+B_k} [w(y)]^{-1}\right\} \leq C;$$

and, the minimal constant C as above is denoted by $C_{p,\,A,\,n}(w)$.

Define $\mathcal{A}_\infty(A) \equiv \bigcup_{1\leq p<\infty} \mathcal{A}_p(A)$.

It is easy to see that if $1 \leq p \leq q \leq \infty$, then $\mathcal{A}_p(A) \subset \mathcal{A}_q(A)$.

In what follows, for any nonnegative local integrable function w and any Lebesgue measurable set E, let $w(E) \equiv \int_E w(x)\, dx$. For $p \in (0, \infty)$, denote by $L_w^p(\mathbb{R}^n)$ the *set of all measurable functions* f such that

$$\|f\|_{L_w^p(\mathbb{R}^n)} \equiv \left\{\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\right\}^{1/p} < \infty,$$

and $L_w^\infty(\mathbb{R}^n) \equiv L^\infty(\mathbb{R}^n)$. The space $L_w^{1,\infty}(\mathbb{R}^n)$ denotes the *set of all measurable functions* f such that

$$\|f\|_{L_w^{1,\infty}(\mathbb{R}^n)} \equiv \sup_{\lambda>0} \lambda w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) < \infty.$$

Moreover, we have the following conclusions.

Proposition 2.5 *(i) If $p \in [1, \infty)$ and $w \in \mathcal{A}_p(A)$, then there exists a positive constant C such that for all $x \in \mathbb{R}^n$ and $k, m \in \mathbb{Z}$ with $k \leq m$,*

$$C^{-1} b^{(m-k)/p} \leq \frac{w(x + B_m)}{w(x + B_k)} \leq C b^{(m-k)p};$$

(ii) If $p \in (1, \infty)$, then the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $L_w^p(\mathbb{R}^n)$ if and only if $w \in \mathcal{A}_p(A)$; if $p = 1$, then $\mathcal{M}$ is bounded from $L_w^1(\mathbb{R}^n)$ to $L_w^{1,\infty}(\mathbb{R}^n)$ if and only if $w \in \mathcal{A}_1(A)$.

Proposition 2.5(i) is just [6, Proposition 2.1(i)]. The proof of Proposition 2.5(ii) is also standard; see [30,32,57] for more details.

Let $\mathcal{S}(\mathbb{R}^n)$ be the *space of Schwartz functions on* $\mathbb{R}^n$ as in [3, p. 11], namely, the space of all smooth functions φ satisfying that for all $\alpha \in (\mathbb{Z}_+)^n$ and $m \in \mathbb{Z}_+$, $\|\varphi\|_{\alpha,\,m} \equiv \sup_{x\in\mathbb{R}^n}[\rho(x)]^m|\partial^\alpha\varphi(x)| < \infty$, where and in what follows, $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\partial^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1}\dots(\frac{\partial}{\partial x_n})^{\alpha_n}$. It is easy to see that $\mathcal{S}(\mathbb{R}^n)$ forms a locally convex complete metric space endowed with the seminorms $\{\|\cdot\|_{\alpha,\,m}\}_{\alpha\in(\mathbb{Z}_+)^n,\,m\in\mathbb{Z}_+}$. From (2.3) and (2.4), it follows that $\mathcal{S}(\mathbb{R}^n)$ coincides with the classical space of Schwartz functions; see [3, p. 11]. Moreover, we denote by $\mathcal{S}_s(\mathbb{R}^n)$ *the set of all* $\psi \in \mathcal{S}(\mathbb{R}^n)$ *satisfying that* $\int_{\mathbb{R}^n}\psi(x)x^\gamma\,dx = 0$ *for all* $\gamma \in (\mathbb{Z}_+)^n$ *with* $|\gamma| \le s$. Let $\mathcal{S}_\infty(\mathbb{R}^n) = \bigcap_{s\in\mathbb{N}}\mathcal{S}_s(\mathbb{R}^n)$.

The following lemma is a slight improvement of [6, Lemma 2.2]. We omit the details.

Lemma 2.6 *Let* $p \in [1, \infty]$ *and* $w \in \mathcal{A}_p(A)$. *Then*

(i) if $1/p + 1/p' = 1$, *then* $\mathcal{S}(\mathbb{R}^n) \subset L^{p'}_{w^{-1/(p-1)}}(\mathbb{R}^n)$;

(ii) $L^p_w(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ *and the inclusion is continuous.*

Lemma 2.7 $\mathcal{M}(\chi_{B_k})(x) \sim \frac{b^k}{b^k+\rho(x)}$ *for all* $k \in \mathbb{Z}$ *and* $x \in \mathbb{R}^n$.

P r o o f. Let σ be as in (2.1). If $x \in B_{k+\sigma}$, then

$$\mathcal{M}(\chi_{B_k})(x) \ge \frac{|B_k|}{|B_{k+\sigma}|} \gtrsim 1 \gtrsim \mathcal{M}(\chi_{B_k})(x),$$

which together with $\rho(x) \lesssim b^k$ yields the desired estimate in this case.

Assume now that $x \notin B_{k+\sigma}$. Then $\rho(x) \gtrsim b^k$. For any $y + B_\ell$ such that $x \in y + B_\ell$ and $(y + B_\ell) \cap B_k \ne \emptyset$, assume that $z_0 \in (y+B_\ell)\cap B_k$. By (2.1), we have $x \in z_0 + (y - z_0) + B_\ell \subset B_k + B_\ell + B_\ell \subset B_{\max(\ell+\sigma,\,k)+\sigma}$. From this and $x \notin B_{k+\sigma}$, it follows that $\ell + \sigma > k$ and further $x \in B_{\ell+2\sigma}$, which implies that $\rho(x) \lesssim b^\ell$. Moreover, by the definition of step homogeneous quasi-norm ρ, there exists $s \in \mathbb{Z}$ such that $x \in B_s \setminus B_{s-1}$, thus we obtain $B_s \subset B_{\ell+2\sigma}$ and $\rho(x) = |B_s|$. From this, $\rho(x) \lesssim b^\ell$ and $B_k \subset B_s$, it follows that

$$\mathcal{M}(\chi_{B_k})(x) = \sup_{\substack{y\in\mathbb{R}^n\\ \ell+\sigma>k}}\ \sup_{x\in y+B_\ell} b^{-\ell}\int_{y+B_\ell}\chi_{B_k}(z)\,dz \lesssim \frac{|B_k|}{\rho(x)} \sim \frac{|B_k|}{|B_s|} \lesssim \mathcal{M}(\chi_{B_k})(x),$$

which together with $\rho(x) \gtrsim b^k$ gives the desired estimate. This finishes the proof of the Lemma 2.7. □

Let $m,\,n \in \mathbb{N}$. In what follows, for convenience, we often let $n_1 \equiv n$ and $n_2 \equiv m$. For $i = 1,\,2$, let $A_i \in M_{n_i}(\mathbb{R})$ be a dilation and b_i, $B^{(i)}_{k_i}$, ρ_i, u_i and v_i associated with A_i as above.

For any locally integrable function f on $\mathbb{R}^n \times \mathbb{R}^m$, the *strong maximal function* $\mathcal{M}_s(f)$ is defined by setting, for all $x \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\mathcal{M}_s(f)(x) \equiv \sup_{k_1,\,k_2\in\mathbb{Z}}\ \sup_{x\in y+B^{(1)}_{k_1}\times B^{(2)}_{k_2}} \frac{1}{b_1^{k_1}b_2^{k_2}}\int_{y+B^{(1)}_{k_1}\times B^{(2)}_{k_2}}|f(z)|\,dz.$$

Obviously, $\mathcal{M}_s(f)(x) \le \mathcal{M}^{(1)}[\mathcal{M}^{(2)}(f)](x)$ for all $x \in \mathbb{R}^n \times \mathbb{R}^m$ and $\mathcal{M}_s$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $p \in (1, \infty]$, where $\mathcal{M}^{(i)}$ denotes the Hardy-Littlewood maximal operator on $\mathbb{R}^{n_i}$.

Remark 2.8 By a slight modification of the proof of Lemma 2.7, we also obtain that for all $k_1,\,k_2 \in \mathbb{Z}$ and $x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $\mathcal{M}_s\left(\chi_{B^{(1)}_{k_1}\times B^{(2)}_{k_2}}\right)(x) \sim \prod_{i=1}^2 \frac{b_i^{k_i}}{b_i^{k_i}+\rho_i(x_i)}$. We omit the details here.

Now we introduce the weight class of Muckenhoupt on $\mathbb{R}^n \times \mathbb{R}^m$ associated with A_1 and A_2, which coincides with the isotropic product weights as in [25] and [51] when $A_1 = 2I_{n\times n}$ and $A_2 = 2I_{m\times m}$. Among several equivalent ways of introducing product weights [30, Theorem VI.6.2] we adopt the following definition.

Definition 2.9 For $i = 1,\,2$, let A_i be a dilation on $\mathbb{R}^{n_i}$ and $\vec{A} = (A_1,\,A_2)$. Let $p \in (1, \infty)$ and w be a nonnegative measurable function on $\mathbb{R}^n \times \mathbb{R}^m$. The function w is said to be in the *weight class of Muckenhoupt* $\mathcal{A}_p(\vec{A}) \equiv \mathcal{A}_p(\mathbb{R}^n \times \mathbb{R}^m,\,\vec{A})$, if $w(x_1,\,\cdot) \in \mathcal{A}_p(A_2)$ for almost everywhere $x_1 \in \mathbb{R}^n$ and

$$\operatorname*{esssup}_{x_1\in\mathbb{R}^n} C_{p,\,A_2,\,m}(w(x_1,\,\cdot)) < \infty,$$

and $w(\cdot,\, x_2) \in \mathcal{A}_p(A_1)$ for almost everywhere $x_2 \in \mathbb{R}^m$ and $\operatorname{esssup}_{x_2\in\mathbb{R}^m} C_{p,\,A_1,\,n}(w(\cdot,\, x_2)) < \infty$. In what follows, let

$$C_{q,\,\vec{A},\,n,\,m}(w) \equiv \max\left\{\operatorname*{esssup}_{x_1\in\mathbb{R}^n} C_{p,\,A_2,\,m}(w(x_1,\,\cdot)),\ \operatorname*{esssup}_{x_2\in\mathbb{R}^m} C_{p,\,A_1,\,n}(w(\cdot,\,x_2))\right\}.$$

Define $\mathcal{A}_\infty(\vec{A}) \equiv \bigcup_{1<p<\infty} \mathcal{A}_p(\vec{A})$.

For any $w \in \mathcal{A}_\infty(\vec{A})$, define the critical index of w by

$$q_w \equiv \inf\{q \in (1,\,\infty):\ w \in \mathcal{A}_q(\vec{A})\}. \tag{2.7}$$

Obviously, $q_w \in [1,\,\infty)$. If $q_w \in (1,\,\infty)$, then $w \notin \mathcal{A}_{q_w}$, and if $q_w = 1$, Johnson and Neugebauer [36, p. 254] gave an example of $w \notin \mathcal{A}_1(2I_{n\times n})$ such that $q_w = 1$. It is easy to see that if $1 < p \le q \le \infty$, then $\mathcal{A}_p(\vec{A}) \subset \mathcal{A}_q(\vec{A})$. If $w \in \mathcal{A}_p(\vec{A})$ with $p \in (1,\,\infty)$, then there exists an $\epsilon \in (0,\,p-1]$ such that $w \in \mathcal{A}_{p-\epsilon}(\vec{A})$ by the reverse Hölder inequality.

Throughout the whole paper, for any measurable set $E \subset \mathbb{R}^n \times \mathbb{R}^m$ and $p \in \mathbb{R}$, we always set $w^p(E) \equiv \int_E [w(x)]^p\,dx$. Moreover, by the definition of $\mathcal{A}_p(\vec{A})$ and Proposition 2.5, we have the following proposition. We omit the details.

Proposition 2.10 *Let $\vec{A}$ be as in Definition* 2.9.

(i) If $p \in (1,\infty]$ and $w \in \mathcal{A}_p(\vec{A})$, there exists a positive constant C such that for all $x \in \mathbb{R}^n \times \mathbb{R}^m$ and $k_i,\ \ell_i \in \mathbb{Z}$ with $k_i \le \ell_i$,

$$C^{-1} b_1^{(\ell_1-k_1)/p} b_2^{(\ell_2-k_2)/p} \le \frac{w\big(x + B_{\ell_1}^{(1)} \times B_{\ell_2}^{(2)}\big)}{w\big(x + B_{k_1}^{(1)} \times B_{k_2}^{(2)}\big)} \le C b_1^{(\ell_1-k_1)p} b_2^{(\ell_2-k_2)p};$$

(ii) If $p \in (1,\,\infty)$, $w \in \mathcal{A}_p(\vec{A})$ and $q \in (1,\,\infty]$, then the strong maximal operator $\mathcal{M}_s$ is bounded on $L^p_w(\mathbb{R}^n \times \mathbb{R}^m)$ and moreover, there exists a positive constant C such that for all $\{f_j\}_{j\in\mathbb{N}} \subset L^p_w(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\left\|\left\{\sum_{j\in\mathbb{N}} [\mathcal{M}_s(f_j)]^q\right\}^{1/q}\right\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} \le C \left\|\left\{\sum_{j\in\mathbb{N}} |f_j|^q\right\}^{1/q}\right\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)}.$$

In fact, the vector-valued inequality (ii) can be obtained simply by iterating the corresponding vector-valued inequality for the Hardy-Littlewood maximal function in [1].

For $s_1,\, s_2 \in \mathbb{Z}_+$, let $\mathcal{S}_{s_1,\,s_2}(\mathbb{R}^n \times \mathbb{R}^m)$ be *the collection of all functions $\psi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ satisfying that $\int_{\mathbb{R}^n} \psi(x_1,\,x_2) x_1^\gamma\,dx_1 = 0$ for all $\gamma \in (\mathbb{Z}_+)^n$, $|\gamma| \le s_1$ and $x_2 \in \mathbb{R}^m$, and $\int_{\mathbb{R}^m} \psi(x_1,\,x_2) x_2^\beta\,dx_2 = 0$ for all $\beta \in (\mathbb{Z}_+)^m$, $|\beta| \le s_2$ and $x_1 \in \mathbb{R}^n$*. Let $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m) = \bigcap_{s_1,s_2\in\mathbb{N}} \mathcal{S}_{s_1,s_2}(\mathbb{R}^n \times \mathbb{R}^m)$.

Throughout the whole paper, for a dilation A, we always let A^* be its transpose. For functions φ on $\mathbb{R}^n$, ψ on $\mathbb{R}^n \times \mathbb{R}^m$ and $k,\ k_1,\ k_2 \in \mathbb{Z}$, let $\varphi_k(x) \equiv b^{-k}\varphi(A^{-k}x)$ for all $x \in \mathbb{R}^n$ and

$$\psi_{k_1,\,k_2}(x) \equiv b_1^{-k_1} b_2^{-k_2} \psi\big(A_1^{-k_1}x_1,\ A_2^{-k_2}x_2\big)$$

for all $x = (x_1,\,x_2) \in \mathbb{R}^n \times \mathbb{R}^m$.

Proposition 2.11 *(i) Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. For any $f \in \mathcal{S}(\mathbb{R}^n)$ (or $f \in \mathcal{S}'(\mathbb{R}^n)$), $f * \varphi_k \to f$ in $\mathcal{S}(\mathbb{R}^n)$ (or $\mathcal{S}'(\mathbb{R}^n)$) as $k \to -\infty$.*

(ii) Let $\varphi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ and $\int_{\mathbb{R}^n\times\mathbb{R}^m} \varphi(x)\,dx = 1$. For any $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ (or $f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$), $f * \varphi_{k_1,\,k_2} \to f$ in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ (or $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$) as $k_1,\,k_2 \to -\infty$.

In fact, Proposition 2.11(i) is just [3, Lemma 3.8]. The proof of Proposition 2.11(ii) is similar to that of (i). We omit the details.

We recall from [28] that $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to *vanish weakly at infinity* if for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $f * \varphi_k \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $k \to \infty$. Denote by $\mathcal{S}'_{\infty,\,w}(\mathbb{R}^n)$ the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishing weakly at infinity. As pointed out in [28], if $f \in L^p(\mathbb{R}^n)$ with $p \in [1,\,\infty)$, then $f \in \mathcal{S}'_{\infty,\,w}(\mathbb{R}^n)$. Similarly, $f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ is said to *vanish weakly at infinity* if for any $\varphi^{(1)} \in \mathcal{S}(\mathbb{R}^m)$ and $\varphi^{(2)} \in \mathcal{S}(\mathbb{R}^n)$, $f * \varphi_{k_1, k_2} \to 0$ in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ as $k_1, k_2 \to \infty$, where $\varphi(x) \equiv \varphi^{(1)}(x_1)\varphi^{(2)}(x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m$. We also denote by $\mathcal{S}'_{\infty, w}(\mathbb{R}^n \times \mathbb{R}^m)$ *the set of all* $f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ *vanishing weakly at infinity*.

Now we establish the following Calderón reproducing formulae.

Lemma 2.12 *Let A be a dilation on $\mathbb{R}^n$ and A^* its transpose. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\widehat{\varphi}$ is compact and bounded away from the origin and for all $\xi \in \mathbb{R}^n \setminus \{0\}$,*

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi}\big((A^*)^j\xi\big) = 1. \tag{2.8}$$

Then for any $f \in L^2(\mathbb{R}^n)$, $f = \sum_{j\in\mathbb{Z}} f * \varphi_j$ in $L^2(\mathbb{R}^n)$. The same holds in $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$, respectively, for $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ or $f \in \mathcal{S}'_{\infty, w}(\mathbb{R}^n)$.

Proof. We first prove the lemma for $f \in L^2(\mathbb{R}^n)$. Define $F(\xi) \equiv \sum_{j\in\mathbb{Z}} |\widehat{\varphi}((A^*)^j\xi)|$ for all $\xi \in \mathbb{R}^n$. Obviously, $F(\xi) = F(A^*\xi)$ for all $\xi \in \mathbb{R}^n$, which implies that to show $F \in L^\infty(\mathbb{R}^n)$, it suffices to consider the values of F on $B_1^* \setminus B_0^*$, where B_0^* is the unit ball associated with the dilation A^*. Let ρ^* be the homogeneous quasi-norm associated with A^*. Since $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ and $\widehat{\varphi}(0) = 0$, we know that $|\widehat{\varphi}(\xi)| \lesssim \rho^*(\xi)^{-1}$ for all $\xi \in \mathbb{R}^n \setminus B_0^*$ and $|\widehat{\varphi}(\xi)| \lesssim |\xi|$ for $\xi \in B_1^*$. Thus by (2.6), $b > 1$ and $\zeta_- > 0$, for any $\xi \in B_1^* \setminus B_0^*$, we have

$$F(\xi) \lesssim \sum_{j\geq 0} \rho^*\big((A^*)^j\xi\big)^{-1} + \sum_{j<0} \big|(A^*)^j\xi\big| \lesssim \sum_{j\geq 0} b^{-j} + \sum_{j<0} b^{j\zeta_-} \lesssim 1. \tag{2.9}$$

Thus, $F \in L^\infty(\mathbb{R}^n)$. By this, the Lebesgue dominated convergence theorem and (2.8), for $f \in L^2(\mathbb{R}^n)$, we have $\widehat{f} = \sum_{j\in\mathbb{Z}} \widehat{\varphi}((A^*)^j\cdot)\widehat{f}$ in $L^2(\mathbb{R}^n)$, and thus $f = \sum_{j\in\mathbb{Z}} \varphi_j * f$ in $L^2(\mathbb{R}^n)$.

Now let us prove the lemma for $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ (or $f \in \mathcal{S}'_{\infty, w}(\mathbb{R}^n)$). Set $\phi \equiv \sum_{j=0}^\infty \varphi_j$. Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi_j(x) = b^{-j}\varphi(A^{-j}x)$, then ϕ is well-defined pointwise on $\mathbb{R}^n$. We claim that $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. Assuming the claim for the moment, by Proposition 2.11, we have $f * \phi_{-N} \to f$ in $\mathcal{S}(\mathbb{R}^n)$ (or $\mathcal{S}'(\mathbb{R}^n)$) as $N \to \infty$. On the other hand, by Hölder's inequality, for $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ (or by $f \in \mathcal{S}'_{\infty, w}(\mathbb{R}^n)$), we obtain that $f * \phi_N \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ (or $\mathcal{S}'(\mathbb{R}^n)$) as $N \to \infty$. Therefore, for $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ (or $f \in \mathcal{S}'_{\infty, w}(\mathbb{R}^n)$), we have $f * \phi_{-N} - f * \phi_N \to f$ in $\mathcal{S}(\mathbb{R}^n)$ (or $\mathcal{S}'(\mathbb{R}^n)$) as $N \to \infty$. Moreover, observing that $\phi_k = \sum_{j=0}^\infty (\varphi_j)_k = \sum_{j=k}^\infty \varphi_j$, and thus $\sum_{j=-N}^N \varphi_j = \phi_{-N} - \phi_{N+1}$, we obtain the lemma for $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ (or $f \in \mathcal{S}'_{\infty, w}(\mathbb{R}^n)$).

Let us now prove the above claim. Let $G(\xi) \equiv \sum_{j=0}^\infty \widehat{\varphi}((A^*)^j\xi)$ for all $\xi \in \mathbb{R}^n$. Then it suffices to prove that $G \in \mathcal{S}(\mathbb{R}^n)$, $\phi = \mathcal{F}^{-1}G$ and $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Since $\operatorname{supp}\widehat{\varphi}$ is compact, we may assume that $\operatorname{supp}\widehat{\varphi} \subset B_{k_0}^*$ for certain $k_0 \in \mathbb{Z}$. Then for any $j \in \mathbb{Z}_+$, we have $\operatorname{supp}\widehat{\varphi}((A^*)^j\cdot) \subset B_{k_0-j}^* \subset B_{k_0}^*$, which implies that $\operatorname{supp} G \subset B_{k_0}^*$. To prove $G \in \mathcal{C}^\infty(\mathbb{R}^n)$, for any $\alpha \in (\mathbb{Z}_+)^n$ and $\xi \in \mathbb{R}^n$, set $F_\alpha(\xi) \equiv \sum_{j\in\mathbb{Z}} |\partial^\alpha[\widehat{\varphi}((A^*)^j\xi)]|$. Let us now show $F_\alpha \in L^\infty(\mathbb{R}^n)$. Notice that for all $\xi \in \mathbb{R}^n$,

$$F_\alpha(A^*\xi) = \sum_{j\in\mathbb{Z}} \big|\partial^\alpha\big[\widehat{\varphi}\big((A^*)^{j+1}\xi\big)\big]\big| = \sum_{j\in\mathbb{Z}} |\partial^\alpha[\widehat{\varphi}((A^*)^j\xi)]| = F_\alpha(\xi),$$

which implies that to verify $F_\alpha \in L^\infty(\mathbb{R}^n)$, we only need to consider the values of F_α on $B_1^* \setminus B_0^*$. By (2.19) in [5], $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\rho^*(\xi) \sim 1$, we have

$$|\partial^\alpha\widehat{\varphi}((A^*)^j\xi)| \lesssim b^{j|\alpha|\zeta_+}|(\partial^\alpha\widehat{\varphi})((A^*)^j\xi)| \lesssim b^{j|\alpha|\zeta_+}\frac{1}{\rho^*((A^*)^j\xi)^{(1+|\alpha|\zeta_+)}} \lesssim b^{-j}$$

when $j > 0$, and $|\partial^\alpha\widehat{\varphi}((A^*)^j\xi)| \lesssim b^{j|\alpha|\zeta_-}$ when $j \leq 0$. From this, $b > 1$ and $\zeta_- > 0$, by (2.6), it follows that $F_\alpha(\xi) \lesssim \sum_{j\leq 0} b^{j|\alpha|\zeta_-} + \sum_{j>0} b^{-j} \lesssim 1$, and hence $F_\alpha \in L^\infty(\mathbb{R}^n)$. Notice that $\partial^\alpha G(\xi) = \sum_{j=0}^\infty \partial^\alpha[\widehat{\varphi}((A^*)^j\xi)]$ for all $\xi \in \mathbb{R}^n$. Thus, $G \in \mathcal{C}^\infty(\mathbb{R}^n)$. From this and $\operatorname{supp} G \subset B_{k_0}^*$, we deduce $G \in \mathcal{S}(\mathbb{R}^n)$.

Moreover, by the proof of $\operatorname{supp} G \subset B^*_{k_0}$, it is easy to see that $\operatorname{supp}\big(\sum_{j=0}^{\infty}|\widehat{\varphi}((A^*)^j\cdot)|\big) \subset B^*_{k_0}$, which together with Hölder's inequality and Minkowski's inequality implies that

$$\begin{aligned}\int_{\mathbb{R}^n}\sum_{j=0}^{\infty}|\widehat{\varphi}((A^*)^j\xi)|\,d\xi &\lesssim |B^*_{k_0}|^{1/2}\left\{\int_{\mathbb{R}^n}\left(\sum_{j=0}^{\infty}|\widehat{\varphi}((A^*)^j\xi)|\right)^2 d\xi\right\}^{1/2}\\ &\lesssim b^{k_0/2}\sum_{j=0}^{\infty}\left(\int_{\mathbb{R}^n}|\widehat{\varphi}((A^*)^j\xi)|^2\,d\xi\right)^{1/2}\\ &\lesssim b^{k_0/2}\sum_{j=0}^{\infty}b^{-j/2}\\ &\lesssim 1.\end{aligned}$$

Then by Fubini's theorem, we obtain $\mathcal{F}^{-1}G = \sum_{j\in\mathbb{Z}}\mathcal{F}^{-1}[\widehat{\varphi}((A^*)^j\cdot)] = \phi$ and hence, $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Let $e_1 \equiv (1,\,0,\,\ldots,\,0)$. Since $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$, by (2.8), we obtain

$$\int_{\mathbb{R}^n}\phi(x)\,dx = \widehat{\phi}(0) = \lim_{k\to-\infty}\widehat{\phi}\big((A^*)^k e_1\big) = \lim_{k\to-\infty}\sum_{j=0}^{\infty}\widehat{\varphi}\big((A^*)^{j+k}e_1\big) = \sum_{j\in\mathbb{Z}}\widehat{\varphi}\big((A^*)^j e_1\big) = 1,$$

which completes the proof of our claim and hence the proof of Lemma 2.12. □

Remark 2.13 From the proof of Lemma 2.12, it is easy to see that if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\widehat{\varphi}(0) = 0$, then $F(\xi) \equiv \sum_{j\in\mathbb{Z}}|\widehat{\varphi}((A^*)^j\xi)|$ for all $\xi \in \mathbb{R}^n$ is bounded on $\mathbb{R}^n$.

Using Lemma 2.12, we have the following Calderón reproducing formulae.

Proposition 2.14 *Let $s \in \mathbb{Z}_+$ and A be a dilation on $\mathbb{R}^n$. Then there exist $\theta,\,\psi \in \mathcal{S}(\mathbb{R}^n)$ such that*

(i) $\operatorname{supp}\theta \subset B_0$, $\int_{\mathbb{R}^n}x^\gamma\theta(x)\,dx = 0$ for all $\gamma \in (\mathbb{Z}_+)^n$ with $|\gamma| \le s$, $\widehat{\theta}(\xi) \ge C > 0$ for ξ in certain annulus, where C is a positive constant;

(ii) $\operatorname{supp}\widehat{\psi}$ is compact and bounded away from the origin;

(iii) $\sum_{j\in\mathbb{Z}}\widehat{\psi}((A^*)^j\xi)\widehat{\theta}((A^*)^j\xi) = 1$ for all $\xi \in \mathbb{R}^n\setminus\{0\}$.

Moreover, for all $f \in L^2(\mathbb{R}^n)$, $f = \sum_{j\in\mathbb{Z}} f*\psi_j*\theta_j$ in $L^2(\mathbb{R}^n)$. The same holds in $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$, respectively, for any $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ or $f \in \mathcal{S}'_{\infty,\,w}(\mathbb{R}^n)$.

We point out that the existences of such θ and ψ in Proposition 2.14 were proved in Theorem 5.8 of [5]. The conclusions of Proposition 2.14 then just follow from Lemma 2.12 by taking $\varphi = \theta*\psi$. Moreover, we also need the following variant on $\mathbb{R}^n\times\mathbb{R}^m$ of Lemma 2.12.

Lemma 2.15 *Let $i = 1,\,2$, A_i be a dilation on $\mathbb{R}^{n_i}$ and $\varphi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ such that $\operatorname{supp}\widehat{\varphi^{(i)}}$ is compact and bounded away from the origin and for all $\xi_i \in \mathbb{R}^{n_i}\setminus\{0\}$, (2.8) holds with A replaced by A_i, φ by $\varphi^{(i)}$ and ξ by ξ_i. Set $\varphi(x) \equiv \varphi^{(1)}(x_1)\varphi^{(2)}(x_2)$ for all $x = (x_1,\,x_2) \in \mathbb{R}^n\times\mathbb{R}^m$. Then for any $f \in L^2(\mathbb{R}^n\times\mathbb{R}^m)$, $f = \sum_{j_1,\,j_2\in\mathbb{Z}} f*\varphi_{j_1,\,j_2}$ in $L^2(\mathbb{R}^n\times\mathbb{R}^m)$. The same holds in $\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^m)$ or $\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^m)$, respectively, for any $f \in \mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$ or $f \in \mathcal{S}'_{\infty,\,w}(\mathbb{R}^n\times\mathbb{R}^m)$.*

Proof. We first prove the lemma for $f \in L^2(\mathbb{R}^n\times\mathbb{R}^m)$. For $\varphi = \varphi^{(1)}\varphi^{(2)}$, by (2.9), we obtain that for all $\xi = (\xi_1,\,\xi_2) \in \mathbb{R}^n\times\mathbb{R}^m$,

$$F(\xi) \equiv \sum_{j_1,\,j_2\in\mathbb{Z}}\widehat{\varphi}((A_1^*)^{j_1}\xi_1,\,(A_2^*)^{j_2}\xi_2) = \sum_{j_1\in\mathbb{Z}}\widehat{\varphi^{(1)}}((A_1^*)^{j_1}\xi_1)\sum_{j_2\in\mathbb{Z}}\widehat{\varphi^{(2)}}((A_2^*)^{j_2}\xi_2)$$

is bounded on $\mathbb{R}^n\times\mathbb{R}^m$. Then from this and the fact that $\sum_{j_1,\,j_2\in\mathbb{Z}}\widehat{\varphi}((A_1^*)^{j_1}\xi_1,\,(A_2^*)^{j_2}\xi_2) = 1$ for any $\xi \in (\mathbb{R}^n\times\mathbb{R}^m)\setminus\{(0,0)\}$, similarly to Lemma 2.12, we deduce the desired formula for $f \in L^2(\mathbb{R}^n\times\mathbb{R}^m)$.

For $f \in \mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$ or $f \in \mathcal{S}'_{\infty,\,w}(\mathbb{R}^n\times\mathbb{R}^m)$, observing that in the proof of Lemma 2.12, we have shown that $\phi^{(i)} \equiv \sum_{j_i=0}^{\infty}\varphi_j \in \mathcal{S}(\mathbb{R}^{n_i})$ and $\int_{\mathbb{R}^{n_i}}\phi(x_i)\,dx_i = 1$ for $i = 1,\,2$, which implies that $\phi \equiv \phi^{(1)}\phi^{(2)} \in \mathcal{S}(\mathbb{R}^n\times\mathbb{R}^m)$ and $\int_{\mathbb{R}^n\times\mathbb{R}^m}\phi(x)\,dx = 1$. Then, similarly to the proof of Lemma 2.12, we obtain the desired formulae, which completes the proof of Lemma 2.15. □

By Lemma 2.15, we have the following proposition.

Proposition 2.16 *Let $s_i \in \mathbb{Z}_+$ and A_i be a dilation on $\mathbb{R}^{n_i}$ for $i = 1, 2$. Suppose that $\theta^{(i)}, \psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ satisfy the conditions (i) through (iii) of Proposition* 2.14 *on $\mathbb{R}^{n_i}$. Set $\theta(\xi) \equiv \theta^{(1)}(\xi_1)\theta^{(2)}(\xi_2)$ and $\psi(\xi) \equiv \psi^{(1)}(\xi_1)\psi^{(2)}(\xi_2)$ for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^m$. Then for any $f \in L^2(\mathbb{R}^n \times \mathbb{R}^m)$,*

$$f = \sum_{j_1, j_2 \in \mathbb{Z}} f * \psi_{j_1, j_2} * \theta_{j_1, j_2}$$

in $L^2(\mathbb{R}^n \times \mathbb{R}^m)$. The same holds in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ or $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$, respectively, for any $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ or $f \in \mathcal{S}'_{\infty, w}(\mathbb{R}^n \times \mathbb{R}^m)$.

3 A weighted anisotropic Littlewood-Paley theory

We begin with the one parameter Lusin-area function.

Definition 3.1 Let A be a dilation on $\mathbb{R}^n$. Suppose $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\varphi}(0) = 0$. For all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define the *anisotropic Lusin-area function* of f by

$$S_\varphi(f)(x) \equiv \left\{ \sum_{k \in \mathbb{Z}} b^{-k} \int_{B_k} |f * \varphi_k(x - y)|^2 \, dy \right\}^{1/2}.$$

By the Plancherel formula and Remark 2.13, we have

$$\begin{aligned} \|S_\varphi(f)\|^2_{L^2(\mathbb{R}^n)} &= \sum_{k \in \mathbb{Z}} b^{-k} \int_{B_k} \int_{\mathbb{R}^n} |f * \varphi_k(x - y)|^2 \, dx \, dy \\ &= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right|^2 |\widehat{\varphi}_k(\xi)|^2 \, d\xi \\ &\lesssim \left\|\widehat{f}\right\|^2_{L^2(\mathbb{R}^n)} \\ &\lesssim \|f\|^2_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{3.1}$$

which implies that S_φ is bounded on $L^2(\mathbb{R}^n)$. Moreover, we have the following theorem.

Theorem 3.2 *Let A be a dilation on $\mathbb{R}^n$, $p \in (1, \infty)$, $w \in \mathcal{A}_p(A)$, and θ, ψ be as in Proposition* 2.14*. Suppose $\varphi \equiv \theta$ or ψ. Then $f \in L^p_w(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_{\infty, w}(\mathbb{R}^n)$ and $S_\varphi(f) \in L^p_w(\mathbb{R}^n)$. Moreover, for all $f \in L^p_w(\mathbb{R}^n)$, $\|f\|_{L^p_w(\mathbb{R}^n)} \sim \|S_\varphi(f)\|_{L^p_w(\mathbb{R}^n)}$.*

The proof of Theorem 3.2 will be given later. Similarly, we can introduce the product Lusin-area function as follows.

Definition 3.3 Let A_i be a dilation on $\mathbb{R}^{n_i}$ and $\varphi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ with $\widehat{\varphi^{(i)}}(0) = 0$ for $i = 1, 2$. Set $\varphi(x) \equiv \varphi^{(1)}(x_1)\varphi^{(2)}(x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m$. For all $f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ and $x \in \mathbb{R}^n \times \mathbb{R}^m$, define the *anisotropic product Lusin-area function* of f by

$$\vec{S}_\varphi(f)(x) \equiv \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} b_1^{-k_1} b_2^{-k_2} \int_{B^{(1)}_{k_1} \times B^{(2)}_{k_2}} |\varphi_{k_1, k_2} * f(x - y)|^2 \, dy \right\}^{1/2}.$$

Then by the Plancherel formula and Remark 2.13, similarly to (3.1), we know that $\vec{S}_\varphi$ is bounded on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$. Moreover, we have the following product version of Theorem 3.2 which will be proved later.

Theorem 3.4 *Let A_i be a dilation on $\mathbb{R}^{n_i}$ for $i = 1, 2$, $p \in (1, \infty)$, $w \in \mathcal{A}_p(\vec{A})$ and θ, ψ be as in Proposition* 2.16*. Suppose $\varphi \equiv \theta$ or ψ. Then $f \in L^p_w(\mathbb{R}^n \times \mathbb{R}^m)$ if and only if $f \in \mathcal{S}'_{\infty, w}(\mathbb{R}^n \times \mathbb{R}^m)$ and $\vec{S}_\varphi(f) \in L^p_w(\mathbb{R}^n \times \mathbb{R}^m)$. Moreover, for all $f \in L^p_w(\mathbb{R}^n \times \mathbb{R}^m)$, $\|f\|_{L^p_w(\mathbb{R}^n \times \mathbb{R}^m)} \sim \|\vec{S}_\varphi(f)\|_{L^p_w(\mathbb{R}^n \times \mathbb{R}^m)}$.*

Remark 3.5 For convenience, we can also rewrite $\vec{S}_\varphi(f)$ as

$$\vec{S}_\varphi(f)(x) = \left\{ \iint_{\Gamma(x)} |\varphi_{t_1,\,t_2} * f(y)|^2 \, dy \, \frac{d\sigma(t_1)\, d\sigma(t_2)}{b_1^{t_1} b_2^{t_2}} \right\}^{1/2},$$

where $\Gamma(x) \equiv \left\{(y,\ t) :\ y \in x + B_{t_1}^{(1)} \times B_{t_2}^{(2)},\ t = (t_1,\ t_2) \in \mathbb{R}^2\right\}$ and σ is the integer counting measure on $\mathbb{R}$, i. e., for all $E \subset \mathbb{R}$, $\sigma(E)$ is the number of integers contained in E.

Theorems 3.2 and 3.4 will be proved by viewing the Lusin-area function as the vector-valued Calderón-Zygmund operator and applying a duality argument. In fact, we will verify that the kernel of Lusin-area function satisfies the standard conditions of vector-valued Calderón-Zygmund operators, and then we will apply a well-known result on the boundedness of vector-valued Calderón-Zygmund operators in $L^p_w(\mathbb{R}^n)$ with $p \in (1,\ \infty)$; see Proposition 3.6.

To this end, we first recall the theory of vector-valued Calderón-Zygmund operators. Let $\mathcal{B}$ be a complex Banach space with norm $\|\cdot\|_{\mathcal{B}}$ and $\mathcal{B}^*$ its dual space with norm $\|\cdot\|_{\mathcal{B}^*}$. A function $f : \mathbb{R}^n \to \mathcal{B}$ is called $\mathcal{B}$-measurable, if there exists a measurable subset Ω of $\mathbb{R}^n$ such that $|\mathbb{R}^n \setminus \Omega| = 0$, the values of f on Ω are contained in some separable subspace $\mathcal{B}_0$ of $\mathcal{B}$, and for every $u^* \in \mathcal{B}^*$, the complex valued map $x \to \langle u^*,\ f(x)\rangle$ is measurable. From this definition and theorem in [65, p.131], it follows that the function $x \to \|f(x)\|_{\mathcal{B}}$ on $\mathbb{R}^n$ is measurable. For Banach spaces $\mathcal{B}_1$, $\mathcal{B}_2$, denote by $L(\mathcal{B}_1,\ \mathcal{B}_2)$ the space of all the bounded linear operators from $\mathcal{B}_1$ to $\mathcal{B}_2$.

For all $p \in (0,\ \infty]$, denote by $L^p(\mathbb{R}^n,\ \mathcal{B})$ the space of all $\mathcal{B}$-measurable functions f on $\mathbb{R}^n$ satisfying

$$\|f\|_{L^p(\mathbb{R}^n,\,\mathcal{B})} \equiv \left\{ \int_{\mathbb{R}^n} \|f(x)\|_{\mathcal{B}}^p \, dx \right\}^{1/p} < \infty$$

with a usual modification made when $p = \infty$. Denote by $L^\infty_c(\mathbb{R}^n,\ \mathcal{B})$ the space of $f \in L^\infty(\mathbb{R}^n,\ \mathcal{B})$ with compact support.

The proof of the following proposition is presented in Appendix at the end of this paper.

Proposition 3.6 *Let A be a dilation on $\mathbb{R}^n$, and $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces. Assume that $\mathcal{T}$ is a linear operator bounded from $L^2(\mathbb{R}^n,\ \mathcal{B}_1)$ to $L^2(\mathbb{R}^n,\ \mathcal{B}_2)$. Moreover, assume that there exists a continuous vector-valued function $\mathcal{K}$: $\mathbb{R}^n \setminus \{0\} \to L(\mathcal{B}_1,\ \mathcal{B}_2)$ such that for all $f \in L^\infty_c(\mathbb{R}^n,\ \mathcal{B}_1)$ and $x \notin \operatorname{supp} f$,*

$$\mathcal{T}(f)(x) = \int_{\mathbb{R}^n} \mathcal{K}(x-y) f(y)\, dy.$$

If there exist positive constants C and ϵ such that for all $y \in \mathbb{R}^n \setminus \{0\}$,

$$\|\mathcal{K}(y)\|_{L(\mathcal{B}_1,\,\mathcal{B}_2)} \le \frac{C}{\rho(y)}, \tag{3.2}$$

and for all $x,\ y \in \mathbb{R}^n \setminus \{0\}$ with $\rho(x-y) \le b^{-2\sigma}\rho(y)$,

$$\|\mathcal{K}(y) - \mathcal{K}(x)\|_{L(\mathcal{B}_1,\,\mathcal{B}_2)} \le C \frac{\rho(x-y)^\epsilon}{\rho(y)^{1+\epsilon}}, \tag{3.3}$$

then for all $p \in (1,\ \infty)$ and $w \in \mathcal{A}_p(A)$, $\mathcal{T}$ is bounded from $L^p_w(\mathbb{R}^n,\ \mathcal{B}_1)$ to $L^p_w(\mathbb{R}^n,\ \mathcal{B}_2)$.

Now we turn to the proofs of Theorems 3.2 and 3.4.

Proof of Theorem 3.2. Let $f \in L^p_w(\mathbb{R}^n)$. By Lemma 2.6, $f \in \mathcal{S}'(\mathbb{R}^n)$. To show that f vanishes weakly at infinity, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $k \in \mathbb{Z}_+$, by Hölder's inequality, we obtain

$$|\langle f,\ \varphi_k\rangle| \le \|f\|_{L^p_w(\mathbb{R}^n)} \|\varphi_k\|_{L^{p'}_{w^{-1/(p-1)}}(\mathbb{R}^n)}.$$

Moreover, by the definition of $\mathcal{A}_p(A)$ and Proposition 2.5(i), we have that for $j \in \mathbb{Z}_+$,

$$w^{-1/(p-1)}(B_j) = \int_{B_j} [w(x)]^{-1/(p-1)}\, dx \lesssim [w(B_j)]^{-1/(p-1)} |B_j|^{p'} \lesssim b^{j[p' - 1/(p(p-1))]}.$$

From this and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\varphi_k(x)|^{p'} [w(x)]^{-1/(p-1)}\, dx \\
&\quad \lesssim b^{-kp'} w^{-1/(p-1)}(B_k) + b^{-kp'} \sum_{j=k}^{\infty} \int_{B_{j+1}\setminus B_j} [b^{-k}\rho(x)]^{-p'} [w(x)]^{-1/(p-1)}\, dx \\
&\quad \lesssim \sum_{j=k}^{\infty} b^{-jp'} w^{-1/(p-1)}(B_j) \\
&\quad \lesssim \sum_{j=k}^{\infty} b^{-j/[p(p-1)]} \\
&\quad \lesssim b^{-k/[p(p-1)]},
\end{aligned}$$

which implies that f vanishes weakly at infinity and hence, $f \in \mathcal{S}'_{\infty,\,w}(\mathbb{R}^n)$.

We now prove the boundedness of S_φ on $L^p_w(\mathbb{R}^n)$ with $p \in (1, \infty)$. Let

$$\mathcal{H} \equiv \{F = \{f_k\}_{k\in\mathbb{Z}} :\ f_k \text{ is a measurable function on } B_k \text{ for any } k \in \mathbb{Z} \text{ and } \|F\|_{\mathcal{H}} < \infty\},$$

where $\|F\|_{\mathcal{H}} \equiv \left\{\sum_{k\in\mathbb{Z}} b^{-k} \int_{B_k} |f_k(y)|^2\, dy\right\}^{1/2}$. Obviously, $\mathcal{H}$ is a Hilbert space. For all $x \in \mathbb{R}^n \setminus \{0\}$, set $\mathcal{K}(x) \equiv \{\varphi_k(x-z) :\ k \in \mathbb{Z},\, z \in B_k\} \in L(\mathbb{C}, \mathcal{H})$, and for all $f \in L^\infty_c(\mathbb{R}^n)$ and $x \notin \operatorname{supp} f$, define $\mathcal{T} : L^\infty_c(\mathbb{R}^n) \to \mathcal{H}$ by

$$\mathcal{T}(f)(x) \equiv \int_{\mathbb{R}^n} \mathcal{K}(x-y) f(y)\, dy = \{\varphi_k * f(x-z) :\ z \in B_k,\, k \in \mathbb{Z}\}.$$

Then $\|\mathcal{T}(f)(x)\|_{\mathcal{H}} = S_\varphi(f)(x)$ for all $x \in \mathbb{R}^n$. From this and (3.1), it follows that $\mathcal{T}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n, \mathcal{H})$. To obtain the boundedness of S_φ on $L^p_w(\mathbb{R}^n)$, it suffices to prove $\mathcal{K}$ satisfies (3.2) and (3.3).

To see (3.2), for $z \in B_k$ and $y \in \mathbb{R}^n \setminus \{0\}$, let $j_0 \in \mathbb{Z}$ such that $\rho(y) = b^{j_0}$. By Definition 2.2(iii), $\rho(y) \le b^\sigma[\rho(z) + \rho(y-z)] \le b^\sigma[b^k + \rho(y-z)]$, which implies that $b^{j_0-k} \lesssim 1 + b^{-k}\rho(y-z)$. Then for all $y \in \mathbb{R}^n \setminus \{0\}$, we obtain

$$\begin{aligned}
\|\mathcal{K}(y)\|^2_{L(\mathbb{C},\,\mathcal{H})} &= \|\{\varphi_k(y-\cdot)\}_{k\in\mathbb{Z}}\|^2_{\mathcal{H}} \\
&= \sum_{k\in\mathbb{Z}} b^{-k} \int_{B_k} |\varphi_k(y-z)|^2\, dz \\
&\lesssim \sum_{k\in\mathbb{Z}} b^{-k} \int_{B_k} \frac{b^{-2k}}{[1 + b^{-k}\rho(y-z)]^4}\, dz \\
&\lesssim \sum_{k\le j_0} b^{-2k} b^{-4(j_0-k)} + \sum_{k>j_0} b^{-2k} \\
&\lesssim b^{-2j_0} \\
&\sim [\rho(y)]^{-2},
\end{aligned}$$

which gives (3.2).

To show (3.3), let $y, x \in \mathbb{R}^n$ with $y \neq 0$ and $\rho(x-y) \le b^{-2\sigma}\rho(y)$. Without loss of generality, we may assume that $\rho(x-y) = b^{j_0}$ and $\rho(y) = b^{j_0+j_1+2\sigma}$ for certain $j_0 \in \mathbb{Z}$ and $j_1 \in \mathbb{Z}_+$. Write

$$\begin{aligned}
&\|\mathcal{K}(x) - \mathcal{K}(y)\|^2_{L(\mathbb{C},\,\mathcal{H})} \\
&\quad = \|\{\varphi_k(y-\cdot) - \varphi_k(x-\cdot)\}_{k\in\mathbb{Z}}\|^2_{\mathcal{H}} \\
&\quad = \sum_{k\in\mathbb{Z}} b^{-3k} \int_{B_k} \left|\varphi\big(A^{-k}(y-z)\big) - \varphi\big(A^{-k}(x-z)\big)\right|^2 dz \le
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{k\in\mathbb{Z}} b^{-3k}\int_{B_k}\sup_{\xi\in B_{j_0}}\left|\nabla\varphi\big(A^{-k}(y-z-\xi)\big)\right|^2\left|A^{-k}(x-y)\right|^2\,dz\\
&\lesssim \left[\sum_{k<j_0}+\sum_{j_0\le k\le j_0+j_1}+\sum_{j_0+j_1<k}\right] b^{-3k}\int_{B_k}\sup_{\xi\in B_{j_0}}\left[1+\rho\big(A^{-k}(y-z-\xi)\big)\right]^{-4}\left|A^{-k}(x-y)\right|^2\,dz\\
&\equiv \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.
\end{aligned}
$$

To estimate I_1, since $\rho(A^{-k}(x-y))=b^{j_0-k}>1$ for $k<j_0$, by (2.3), we obtain

$$
|A^{-k}(x-y)|\lesssim[\rho(A^{-k}(x-y))]^{\zeta_+}=b^{\zeta_+(j_0-k)}. \tag{3.4}
$$

Observe that for $y\in B^{\complement}_{j_0+j_1+2\sigma}$, $z\in B_k$, $j_1\ge 0$, $j_0>k$ and $\xi\in B_{j_0}$, by (2.1) and (2.2), we have $A^{-k}(y-z-\xi)\in B^{\complement}_{j_0-k+j_1+2\sigma}+B_0+B_{j_0-k}\subset B^{\complement}_{j_0-k+j_1+\sigma}$, which implies that $\rho(A^{-k}(y-z-\xi))\ge b^{j_0-k+j_1+\sigma}$. From this, (3.4), $\zeta_+<1$, $\rho(x-y)=b^{j_0}$ and $\rho(y)=b^{j_0+j_1+2\sigma}$, it follows that

$$
\mathrm{I}_1\lesssim\sum_{k<j_0} b^{-2k}b^{-4(j_0-k+j_1)}b^{2\zeta_+(j_0-k)}\lesssim b^{-2j_0-4j_1}\lesssim\frac{[\rho(y-x)]^2}{[\rho(y)]^4}.
$$

To estimate I_2, since $\rho(A^{-k}(y-x))=b^{j_0-k}\le 1$ for $k\ge j_0$, by (2.4), we obtain

$$
|A^{-k}(x-y)|\lesssim[\rho(A^{-k}(x-y))]^{\zeta_-}\sim b^{\zeta_-(j_0-k)}. \tag{3.5}
$$

Moreover, observe that for $j_0\le k\le j_0+j_1$, $y\in B^{\complement}_{j_0+j_1+2\sigma}$, $z\in B_k$, $\xi\in B_{j_0}$ and $j_1\ge 0$, by (2.1) and (2.2), we still have that $\rho(A^{-k}(y-z-\xi))\ge b^{j_0-k+j_1+\sigma}$. From this, (3.5), $\rho(x-y)=b^{j_0}$ and $\rho(y)=b^{j_0+j_1+2\sigma}$, it follows that

$$
\mathrm{I}_2\lesssim\sum_{j_0\le k\le j_0+j_1} b^{-2k}b^{-4(j_0-k+j_1)}b^{2\zeta_-(j_0-k)}\lesssim b^{-2(j_0+j_1)}b^{-2\zeta_- j_1}\lesssim\frac{[\rho(y-x)]^{2\zeta_-}}{[\rho(y)]^{2(1+\zeta_-)}}.
$$

To estimate I_3, by (3.5), $\rho(x-y)=b^{j_0}$, $\rho(y)=b^{j_0+j_1+2\sigma}$ and $j_1\ge 0$, we have

$$
\mathrm{I}_3\lesssim\sum_{k>j_0+j_1} b^{-2k}b^{2(j_0-k)\zeta_-}\lesssim b^{-2(j_0+j)}b^{-2j_1\zeta_-}\lesssim\frac{[\rho(y-x)]^{2\zeta_-}}{[\rho(y)]^{2(1+\zeta_-)}}.
$$

Combining the estimates of I_1, I_2 and I_3 finishes the proof of (3.3). Thus, by Proposition 3.6, we obtain the boundedness of S_φ on $L^p_w(\mathbb{R}^n)$ for $p\in(1,\infty)$.

Conversely, let $f\in\mathcal{S}'_{\infty,w}(\mathbb{R}^n)$ and $S_\psi(f)\in L^p_w(\mathbb{R}^n)$ with $p\in(1,\infty)$. Set $\widetilde{\theta}(x)\equiv\theta(-x)$ for all $x\in\mathbb{R}^n$. For any $h\in\mathcal{S}(\mathbb{R}^n)$ with $\|h\|_{L^{p'}_{w^{-p'/p}}(\mathbb{R}^n)}\le 1$, by Proposition 2.14, the boundedness of $S_{\widetilde{\theta}}$ on $L^p_w(\mathbb{R}^n)$ with $p\in(1,\infty)$ and Hölder's inequality, we have

$$
\begin{aligned}
|\langle f,h\rangle| &= \left|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n} f*\psi_k*\theta_k(x)h(x)\,dx\right|\\
&= \left|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n} f*\psi_k(x)h*\widetilde{\theta}_k(x)\,dx\right|\\
&= \left|\sum_{k\in\mathbb{Z}} b^{-k}\int_{\mathbb{R}^n}\int_{x+B_k} f*\psi_k(x)h*\widetilde{\theta}_k(x)\,dy\,dx\right|\\
&= \left|\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}} b^{-k}\int_{y+B_k} f*\psi_k(x)h*\widetilde{\theta}_k(x)\,dx\,dy\right|\\
&\le \int_{\mathbb{R}^n}\left\{\sum_{k\in\mathbb{Z}} b^{-k}\int_{y+B_k}|f*\psi_k(x)|^2\,dx\right\}^{1/2}\left\{\sum_{k\in\mathbb{Z}} b^{-k}\int_{y+B_k}|h*\widetilde{\theta}_k(x)|^2\,dx\right\}^{1/2}dy\le
\end{aligned}
$$

$$\begin{aligned}&\le \|S_\psi(f)\|_{L^p_w(\mathbb{R}^n)}\|S_{\widetilde{\theta}}(h)\|_{L^{p'}_{w^{-p'/p}}(\mathbb{R}^n)}\\&\lesssim \|S_\psi(f)\|_{L^p_w(\mathbb{R}^n)}\|h\|_{L^{p'}_{w^{-p'/p}}(\mathbb{R}^n)}\\&\lesssim \|S_\psi(f)\|_{L^p_w(\mathbb{R}^n)},\end{aligned}$$

which together with the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^{p'}_{w^{-p'/p}}(\mathbb{R}^n)$ and $\left(L^{p'}_{w^{-p'/p}}(\mathbb{R}^n)\right)^* = L^p_w(\mathbb{R}^n)$ implies that $f \in L^p_w(\mathbb{R}^n)$ and $\|f\|_{L^p_w(\mathbb{R}^n)} \lesssim \|S_\psi(f)\|_{L^p_w}$. Similarly, for $f \in \mathcal{S}'_{\infty,\,w}(\mathbb{R}^n)$ and $S_\theta(f) \in L^p_w(\mathbb{R}^n)$, we have $f \in L^p_w(\mathbb{R}^n)$ and $\|f\|_{L^p_w(\mathbb{R}^n)} \lesssim \|S_\theta(f)\|_{L^p_w}$. This finishes the proof of Theorem 3.2. □

Proof of Theorem 3.4. We shall only prove that $\vec{S}_\varphi$ is bounded on $L^p_w(\mathbb{R}^n\times\mathbb{R}^m)$. This is because the proofs of the other conclusions are similar to those of Theorem 3.2.

Let $\mathcal{H}_i$ be the space $\mathcal{H}$ as in the proof of Theorem 3.2 with B_k and b replaced, respectively, by $B^{(i)}_k$ and b_i with $i = 1, 2$. Let $\mathcal{H}_1\otimes\mathcal{H}_2$ be the set of all sequences $F = \{f_{k_1,\,k_2}\}_{k_1,\,k_2\in\mathbb{Z}}$ such that each f_{k_1,k_2} is measurable on $B^{(1)}_{k_1}\times B^{(2)}_{k_2}$ and

$$\begin{aligned}\|F\|_{\mathcal{H}_1\otimes\mathcal{H}_2} &\equiv \left\{\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} b_1^{-k_1}b_2^{-k_2}\int_{B^{(2)}_{k_2}}\int_{B^{(1)}_{k_1}}|f_{k_1,\,k_2}(y_1,\,y_2)|^2\,dy_1\,dy_2\right\}^{1/2}\\&= \left\{\sum_{k_2\in\mathbb{Z}} b_2^{-k_2}\int_{B^{(2)}_{k_2}}\|f_{\cdot,\,k_2}(\cdot,\,y_2)\|^2_{\mathcal{H}_1}\,dy_2\right\}^{1/2}.\end{aligned}$$

The last equation is the consequence of the fact that $\mathcal{H}_1\otimes\mathcal{H}_2$ can be thought of as a collection of measurable $\mathcal{H}_1$-valued functions $\{f_{\cdot,\,k_2}(\cdot,\,y_2)\}_{k_2\in\mathbb{Z}}$ defined almost everywhere for $y_2 \in B^{(2)}_{k_2}$. Clearly, $\mathcal{H}_1,\,\mathcal{H}_2,\,\mathcal{H}_1\otimes\mathcal{H}_2$ are Hilbert spaces. Here and in what follows, we always let

$$\varphi^{(1)}_{k_1} *_1 g(x_1,\,x_2) \equiv \int_{\mathbb{R}^{n_1}}\varphi^{(1)}_{k_1}(x_1-y_1)g(y_1,\,x_2)\,dy_1$$

and

$$\varphi^{(2)}_{k_2} *_2 g(x_1,x_2) \equiv \int_{\mathbb{R}^{n_2}}\varphi^{(2)}_{k_2}(x_2-y_2)g(x_1,\,y_2)\,dy_2.$$

For any $x_2\in\mathbb{R}^{n_2}\setminus\{0\}$, define $\mathcal{K}^{(2)}(x_2):\,\mathcal{H}_1\to\mathcal{H}_1\otimes\mathcal{H}_2$ by tensoring

$$\mathcal{K}^{(2)}(x_2) \equiv \left\{\varphi^{(2)}_{k_2}(x_2-z_2):\,k_2\in\mathbb{Z},\,z_2\in B^{(2)}_{k_2}\right\}.$$

As in the proof of Theorem 3.2, we know that $\mathcal{K}^{(2)}$ satisfies (3.2) and (3.3) with $\mathcal{B}_1 = \mathcal{H}_1$ and $\mathcal{B}_2 = \mathcal{H}_1\otimes\mathcal{H}_2$. Moreover, for any $F(\cdot) = \left\{F_{k_1}(y_1,\,\cdot):\,y_1\in B^{(1)}_{k_1}\right\}_{k_1\in\mathbb{Z}} \in L^\infty_c(\mathbb{R}^m,\,\mathcal{H}_1)$, define

$$\begin{aligned}\mathcal{T}(F)(x_2) &\equiv \mathcal{K}^{(2)} *_2 F(x_2)\\&\equiv \left\{\big(\varphi^{(2)}_{k_2} *_2 F\big)(x_2-y_2):\,y_2\in B^{(2)}_{k_2},\,k_2\in\mathbb{Z}\right\}\\&= \left\{\big(\varphi^{(2)}_{k_2} *_2 F_{k_1}\big)(y_1,\,x_2-y_2):\,y_1\in B^{(1)}_{k_1},\,y_2\in B^{(2)}_{k_2},\,k_1,\,k_2\in\mathbb{Z}\right\}.\end{aligned}$$

Denote by $\mathcal{F}_2$ the Fourier transform on the second variable. By the Plancherel formula and Remark 2.13, we have

$$\begin{aligned}&\|\mathcal{T}(F)\|^2_{L^2(\mathbb{R}^m,\,\mathcal{H}_1\otimes\mathcal{H}_2)}\\&\quad= \int_{\mathbb{R}^m}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} b_1^{-k_1}b_2^{-k_2}\int_{B^{(2)}_{k_2}}\int_{B^{(1)}_{k_1}}\left|\varphi^{(2)}_{k_2} *_2 F_{k_1}(y_1,\,x_2-y_2)\right|^2 dy_1\,dy_2\,dx_2 =\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k_1\in\mathbb{Z}} b_1^{-k_1} \int_{B^{(1)}_{k_1}} \int_{\mathbb{R}^m} \sum_{k_2\in\mathbb{Z}} \left|\widehat{\varphi^{(2)}_{k_2}}(\xi_2)\right|^2 |\mathcal{F}_2 F_{k_1}(y_1,\, \xi_2)|^2 d\,\xi_2\, dy_1\\
&\lesssim \int_{\mathbb{R}^m} \sum_{k_1\in\mathbb{Z}} b_1^{-k_1} \int_{B^{(1)}_{k_1}} \sum_{k_2\in\mathbb{Z}} |F_{k_1}(y_1,\, y_2)|^2\, dy_1\, dy_2\\
&\lesssim \|F\|^2_{L^2(\mathbb{R}^m,\, \mathcal{H}_1)}.
\end{aligned}
$$

Therefore by Proposition 3.6, for any $p \in (1, \infty)$ and $w \in \mathcal{A}_p(A_2)$, $\mathcal{T}$ is bounded from $L^p_w(\mathbb{R}^m, \mathcal{H}_1)$ to $L^p_w(\mathbb{R}^m, \mathcal{H}_1 \otimes \mathcal{H}_2)$.

Let $f \in L^\infty_c(\mathbb{R}^n \times \mathbb{R}^m)$. For any $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$, set

$$
F_{x_1}(x_2) \equiv \left\{(\varphi^{(1)}_{k_1} *_1 f)(x_1 - y_1,\, x_2) : y_1 \in B^{(1)}_{k_1},\, k_1 \in \mathbb{Z}\right\} \in \mathcal{H}_1.
$$

Then $F_{x_1} \in L^\infty_c(\mathbb{R}^m, \mathcal{H}_1)$ and we have

$$
\mathcal{T}(F_{x_1})(x_2) \equiv \left\{(\varphi_{k_1,\,k_2} * F)(x_1 - y_1,\, x_2 - y_2) : y_1 \in B^{(1)}_{k_1},\, k_1 \in \mathbb{Z},\, y_2 \in B^{(2)}_{k_2},\, k_2 \in \mathbb{Z}\right\},
$$

and $\vec{S}_\varphi(f)(x_1, x_2) = \|\mathcal{T}(F_{x_1})(x_2)\|_{\mathcal{H}_1\otimes\mathcal{H}_2}$. Recall that by Definition 2.9, for almost all x_1 (or x_2), $w(x_1, \cdot) \in \mathcal{A}_p(A_2)$ (or $w(\cdot, x_2) \in \mathcal{A}_p(A_1)$) and the weighted constants are uniformly bounded. Then, by Theorem 3.2 for $S_{\varphi^{(1)}}$, we have

$$
\begin{aligned}
\|\vec{S}_\varphi(f)\|^p_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} &= \int_{\mathbb{R}^n} \left\{\int_{\mathbb{R}^m} \|\mathcal{T}(F_{x_1})(x_2)\|^p_{\mathcal{H}_1\otimes\mathcal{H}_2} w(x_1,\, x_2)\, dx_2\right\} dx_1\\
&\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \|F_{x_1}(x_2)\|^p_{\mathcal{H}_1} w(x_1,\, x_2)\, dx_2\, dx_1\\
&\sim \int_{\mathbb{R}^m} \left\{\int_{\mathbb{R}^n} [S_{\varphi^{(1)}}(f(\cdot,\, x_2))(x_1)]^p w(x_1,\, x_2)\, dx_1\right\} dx_2\\
&\lesssim \|f\|^p_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)},
\end{aligned}
$$

which completes the proof of Theorem 3.4. □

4 Weighted anisotropic product Hardy spaces

We begin with the notion of weighted anisotropic product Hardy spaces.

Definition 4.1 Let $p \in (0, 1]$, $w \in \mathcal{A}_\infty(\vec{A})$ and q_w be as in (2.7), ψ be as in Proposition 2.16. Define the *weighted anisotropic product Hardy space* by

$$
H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A}) \equiv \left\{f \in \mathcal{S}'_{\infty,\,w}(\mathbb{R}^n \times \mathbb{R}^m) : \|f\|_{H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\, \vec{A})} \equiv \|\vec{S}_\psi(f)\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} < \infty\right\}.
$$

Notice that if $p \in (q_w, \infty)$, where q_w is as in (2.7), then by Theorem 3.4, we obtain $H^p_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A}) = L^p_w(\mathbb{R}^n \times \mathbb{R}^m)$ with equivalent norms. If $p \in (1, q_w]$, the element of $H^p_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ may be a distribution, and hence, $H^p_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A}) \neq L^p_w(\mathbb{R}^n \times \mathbb{R}^m)$; see [57, p. 86] for one parameter case. For applications considered in this paper, we concentrate only on $H^p_w(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ with $p \in (0, 1]$.

To define atomic Hardy spaces, we introduce the following notation and notions. Let A_i be a dilation on $\mathbb{R}^{n_i}$, and $\mathcal{Q}^{(i)}$, $\ell(Q_i)$, v_i, u_i be the same as in Lemma 2.3 corresponding to A_i for $i = 1, 2$. Let $\mathcal{R} \equiv \mathcal{Q}^{(1)} \times \mathcal{Q}^{(2)}$. For $R \in \mathcal{R}$, we always write $R = R_1 \times R_2$ with $R_i \in \mathcal{Q}^{(i)}$ and call R a dyadic rectangle. For $(k_1, k_2) \in \mathbb{Z} \times \mathbb{Z}$, define $\mathcal{R}_{k_1,k_2} \equiv \{R \in \mathcal{R} : \ell(R_1) = k_1,\, \ell(R_2) = k_2\}$. For $R \in \mathcal{R}$, let

$$
R_+ \equiv \{(y,\, t) : y \in R,\, t = (t_1,\, t_2) \in \mathbb{R}^2,\, t_i \sim v_i\ell(R_i) + u_i,\, i = 1,\, 2\}, \tag{4.1}
$$

where and in what follows, $t_i \sim v_i\ell(R_i) + u_i$ always means

$$
v_i\ell(R_i) + u_i + \sigma_i \le t_i < v_i(\ell(R_i) - 1) + u_i + \sigma_i, \tag{4.2}
$$

and σ_i is as in (2.1) and (2.2) associated with A_i for $i = 1,\ 2$. Note that the inequality (4.2) is seemingly reversed since the v_i's are negative.

Assume that Ω is an open set of $\mathbb{R}^n \times \mathbb{R}^m$. A dyadic rectangle $R \subset \Omega$ is said to be maximal in Ω if for any rectangle $S \subset \Omega$ satisfying that $R \subset S$, then $S = R$. Denote by $m(\Omega)$ the family of all maximal dyadic rectangles contained in Ω. We choose a positive integer $c_0 > 2$ such that $b_1^{-c_0 u_1} b_2^{-c_0 u_2} \le \big(b_1^{-2u_1} b_2^{-2u_2}/2\big)$ and set

$$\widetilde{\Omega} \equiv \big\{x \in \mathbb{R}^n \times \mathbb{R}^m,\ \mathcal{M}_s(\chi_\Omega)(x) > b_1^{-c_0 u_1} b_2^{-c_0 u_2}\big\}. \tag{4.3}$$

Definition 4.2 Let $w \in \mathcal{A}_\infty(\vec{A})$ and q_w be as in (2.7). The triplet $(p,\ q,\ \vec{s})_w$ is said to be *admissible* if $p \in (0,\ 1]$, $q \in [2,\ \infty) \cap (q_w,\ \infty)$ and $s_i \ge \lfloor (\frac{q_w}{p} - 1)\zeta_{i,-}^{-1} \rfloor$, where $\zeta_{i,-}$ is defined as in (2.3), $i = 1,\ 2$.

A function a is said to be a $(p,\ q,\ \vec{s})_w$*-atom* associated to an open set Ω of $\mathbb{R}^n \times \mathbb{R}^m$ with $w(\Omega) < \infty$ if

(I) a can be written as $a = \sum_{R \in m(\widetilde{\Omega})} a_R$ in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$, where a_R satisfies that

  (i) a_R is supported on $R'' = R_1'' \times R_2''$, where $R_i'' \equiv x_{R_i} + B^{(i)}_{v_i(\ell(R_i)-1)+u_i+3\sigma_i}$ for $i = 1,\ 2$.

  (ii) $\int_{\mathbb{R}^n} a_R(x_1,\ x_2) x_1^\alpha\, dx_1 = 0$ for all $|\alpha| \le s_1$ and almost all $x_2 \in \mathbb{R}^m$, and
  $\int_{\mathbb{R}^m} a_R(x_1,\ x_2) x_2^\beta\, dx_2 = 0$ for all $|\beta| \le s_2$ and almost all $x_1 \in \mathbb{R}^n$.

  Here a_R is called a particle associated with the rectangle R.

(II) $\|a\|_{L^q_w(\mathbb{R}^n \times \mathbb{R}^m)} \le [w(\Omega)]^{1/q-1/p}$ and $\sum_{R \in m(\widetilde{\Omega})} \|a_R\|^q_{L^q_w(\mathbb{R}^n \times \mathbb{R}^m)} \le [w(\Omega)]^{1-q/p}$.

Definition 4.3 Let $p \in (0,\ 1]$, $w \in \mathcal{A}_\infty(\vec{A})$ and q_w be as in (2.7) and $(p,\ q,\ \vec{s})_w$ be an admissible triplet. The *weighted atomic anisotropic product Hardy space* $H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A})$ is defined to be the collection of all $f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ of the form $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$, where $\sum_{j \in \mathbb{N}} |\lambda_j|^p < \infty$ and $\{a_j\}_{j \in \mathbb{N}}$ are $(p,\ q,\ \vec{s})_w$-atoms. For f in $H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A})$, the norm f on $H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A})$ is defined by

$$\|f\|_{H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A})} \equiv \inf \left\{ \left(\sum_{j \in \mathbb{N}} |\lambda_i|^p \right)^{1/p} \right\},$$

where the infimum is taken over all the above decompositions of f.

Remark 4.4 a) We remark here that the restriction $q \in [2, \infty)$ in Definition 4.2 seems reasonable, since we use the Lusin-area function to introduce $H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A})$. Moreover, from the known result on classical product Hardy spaces, we know that $\{s_i\}_{i=1,\,2}$ in Definition 4.2 are best possible.

b) Notice that if $(p,\ q,\ \vec{s})_w$ and $(p,\ r,\ \vec{t})_w$ are admissible, $q \le r$ and $s_i \le t_i$ for $i = 1,\ 2$, then a $(p,\ r,\ \vec{t})_w$-atom is a $(p,\ q,\ \vec{s})_w$-atom. Thus, the space $H^{p,\,r,\,\vec{t}}_w(\mathbb{R}^n \times \mathbb{R}^m) \subset H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A})$.

The main result of this section is as follows.

Theorem 4.5 *Let $w \in \mathcal{A}_\infty(\vec{A})$ and q_w be as in* (2.7)*. If $(p,\ q,\ \vec{s})_w$ is an admissible triplet, then*

$$H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A}) = H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A})$$

with equivalent norms.

From Theorem 4.5, we immediately deduce that the definition of the Hardy space $H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A})$ in Definition 4.1 is independent of the choice of ψ as in Proposition 2.16.

Since the proof of Theorem 4.5 is quite complicated, we will use several lemmas. Precisely, by choosing s_i such that $s_i \ge \lfloor (q_w/p - 1)\zeta_{i,-} \rfloor$ and $(s_i + 1)\zeta_{i,-} > 1$ for $i = 1,\ 2$, we first prove in Lemma 4.6 bellow that $H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A}) \subset H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A})$. Conversely, for all admissible $(p,\ q,\ \vec{s})_w$, in Lemma 4.8, we prove

$$[H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A}) \cap \mathcal{S}'_{\infty,\,w}(\mathbb{R}^n)] \subset H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A})$$

by using Journé covering lemma established in Lemma 4.9 below, and in Lemma 4.10, we further show that $H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A}) \subset \mathcal{S}'_{\infty,\,w}(\mathbb{R}^n)$. Combining Lemmas 4.6, 4.8, 4.9 and Remark 4.4 b) then finishes the proof of Theorem 4.5.

Lemma 4.6 *Let $w \in \mathcal{A}_\infty(\vec{A})$ and q_w be as in* (2.7)*. If $(p,\, q,\, \vec{s})_w$ is an admissible triplet and $(s_i+1)\zeta_{i,-} > 1$ for $i = 1,\, 2$, then there exists a positive constant C such that $\|f\|_{H_w^{p,\,q,\,\vec{s}}(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})} \le C\|f\|_{H_w^p(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}$ for all $f \in H_w^p(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A})$.*

P r o o f. To prove this lemma, we borrow some ideas from Fefferman [24, 26]. The whole proof is divided into 8 steps. In Step 1, we use the Calderón reproducing formula from Proposition 2.16 to decompose f into a sum of functions $\{e_R\}_R$ essentially supported in rectangles and recombine these functions (according to the size of the intersection between their corresponding rectangles and the level sets of the Lusin-area function) to obtain the particles $\{a_P\}_P$ and atoms $\{a_k\}_k$; see (4.6), (4.7) and (4.8). In Step 2 through Step 5, we show that $\{a_k\}_k$ are $(p,\, q,\, \vec{s})_w$-atoms. The crucial step is to estimate the size of these atoms in Step 3. Here we use the method from Fefferman [26] instead of the dual method used in [12] via a subtle inequality (4.10). Step 6 through Step 8 is devoted to proving the inequality (4.10), which when $n = m = 1$ was established in [13, 26]. To obtain (4.10) here, in Step 6, we conclude its proof to the proofs of the inequalities (4.17) and (4.18), which are given, respectively, in Step 7 and Step 8. To prove (4.17), a main technique used here is to scale the longer sides of considered rectangles to 1 via the anisotropic dilation invariance of the Lebesgue measure so that we can obtain a desired decreasing factor; see $|\ell(R_i) - \ell(P_i)|$ in (4.17).

We now start to prove Lemma 4.6 by letting ψ be as in Proposition 2.16 and $f \in H_w^p(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A})$.

Step 1. Decompose f by the Calderón reproducing formula.

For $k \in \mathbb{Z}$, set $\Omega_k \equiv \{x \in \mathbb{R}^n \times \mathbb{R}^m :\, \vec{S}_\psi(f)(x) > 2^k\}$ and

$$\mathcal{R}_k \equiv \{R \in \mathcal{R} :\, |R \cap \Omega_k| > |R|/2,\, |R \cap \Omega_{k+1}| \le |R|/2\}.$$

Then for each $R = R_1 \times R_2 \in \mathcal{R}$, there exists a unique $k \in \mathbb{Z}$ such that $R \in \mathcal{R}_k$. Thus,

$$\bigcup_{R\in\mathcal{R}} R = \bigcup_{k_1,\,k_2\in\mathbb{Z}} \bigcup_{R\in R_{k_1,\,k_2}} R = \bigcup_{k\in\mathbb{Z}} \bigcup_{R\in\mathcal{R}_k} R. \tag{4.4}$$

Moreover, for all $R \in \mathcal{R}_k$ and all $x \in R$, by Lemma 2.3(iv), we obtain

$$\begin{aligned}
\mathcal{M}_s(\chi_{\Omega_k})(x) &\ge \frac{1}{b_1^{v_1\ell(R_1)+u_1} b_2^{v_2\ell(R_2)+u_2}} \int_{x_R+B^{(1)}_{v_1\ell(R_1)+u_1}\times B^{(2)}_{v_2\ell(R_2)+u_2}} \chi_{\Omega_k}(y)\,dy\\
&\ge b_1^{-2u_1} b_2^{-2u_2} \frac{|\Omega_k \cap R|}{|R|}\\
&> b_1^{-c_0u_1} b_2^{-c_0u_2},
\end{aligned}$$

which implies that

$$\bigcup_{R\in\mathcal{R}_k} R \subset \widetilde{\Omega}_k, \tag{4.5}$$

where $\widetilde{\Omega}_k$ is as in (4.3).

Let $\theta^{(i)}$ and $\psi^{(i)}$ be as in Proposition 2.14 such that each $\theta^{(i)}$ has the vanishing moments up to degree $s_3 \equiv 2\max(s_1, s_2) + 1$, where $s_i \ge \lfloor (q_w/p - 1)\zeta_{i,-}^{-1} \rfloor$ and $(s_i + 1)\zeta_{i,-} > 1$, $i = 1, 2$. Set $\theta \equiv \theta^{(1)}\theta^{(2)}$ and $\psi \equiv \psi^{(1)}\psi^{(2)}$. Then by Proposition 2.16, Lemma 2.3(i) and (4.4), for all $x \in \mathbb{R}^n \times \mathbb{R}^m$, we have

$$\begin{aligned}
f(x) &= \sum_{k_1,\,k_2\in\mathbb{Z}} \theta_{k_1,\,k_2} * \psi_{k_1,\,k_2} * f(x)\\
&= \sum_{k_1,\,k_2\in\mathbb{Z}} \sum_{\substack{m_1\sim v_1k_1+u_1\\ m_2\sim v_2k_2+u_2}} \int_{\mathbb{R}^n\times\mathbb{R}^m} \theta_{m_1,\,m_2}(x-y)\psi_{m_1,\,m_2} * f(y)\,dy\\
&= \sum_{k_1,\,k_2\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{k_1,\,k_2}} \sum_{\substack{m_1\sim v_1k_1+u_1\\ m_2\sim v_2k_2+u_2}} \int_R \theta_{m_1,\,m_2}(x-y)\psi_{m_1,\,m_2} * f(y)\,dy =
\end{aligned}$$

$$= \sum_{k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_k} \iint_{R_+} \theta_{t_1,\,t_2}(x-y)\psi_{t_1,\,t_2} * f(y)\,dy\,d\sigma(t_1)\,d\sigma(t_2)$$

in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$, where R_+ is as in (4.1) and σ is the counting measure on $\mathbb{R}$.

Set $\lambda_k \equiv 2^k[w(\Omega_k)]^{1/p}$ and $a_k \equiv \lambda_k^{-1} \sum_{R\in\mathcal{R}_k} e_R$, where for all $x \in \mathbb{R}^n \times \mathbb{R}^m$,

$$e_R(x) \equiv \iint_{R_+} \theta_{t_1,\,t_2}(x-y)\psi_{t_1,\,t_2} * f(y)\,dy\,d\sigma(t_1)\,d\sigma(t_2). \tag{4.6}$$

It is not hard to show that $e_R \in \mathcal{S}_{s_1,\,s_2}(\mathbb{R}^n \times \mathbb{R}^m)$. Let $m\big(\widetilde{\Omega}_k\big)$ be the set of all maximal dyadic rectangles contained in $\widetilde{\Omega}_k$. For each $R \in \mathcal{R}_k$, by (4.5), there exists at least one maximal dyadic rectangle in $m(\widetilde{\Omega}_k)$ containing R; if there exists only one such maximal dyadic rectangle, we then denote it by R^*; if there exist more than one such cubes, we denote the one which has the "longest" side in the $\mathbb{R}^n$ "direction" by R^*. We point out that R^* is unique by the choice. For each $P \in m\big(\widetilde{\Omega}_k\big)$, let

$$a_P \equiv \lambda_k^{-1} \sum_{R\in\mathcal{R}_k,\,R^*=P} e_R, \tag{4.7}$$

and then $a_k = \sum_{P\in m(\widetilde{\Omega}_k)} a_P$ in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$. Moreover, we rewrite f as

$$f = \sum_{k\in\mathbb{Z}} \lambda_k a_k = \sum_{k\in\mathbb{Z}} \lambda_k \sum_{P\in m(\widetilde{\Omega}_k)} a_P = \sum_{k\in\mathbb{Z}} \lambda_k \sum_{P\in m(\widetilde{\Omega}_k)} \sum_{R\in\mathcal{R}_k,\,R^*=P} \lambda_k^{-1} e_R \tag{4.8}$$

in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$.

Then we have

$$\sum_{k\in\mathbb{Z}} \lambda_k^p = \sum_{k\in\mathbb{Z}} 2^{pk} w(\Omega_k) \le \|\vec{S}_\psi(f)\|^p_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} = \|f\|^p_{H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}.$$

By this and (4.8), to conclude the proof of Lemma 4.6, we must show that each a_k is a fixed multiple of a $(p,\,q,\,\vec{s})_w$-atom associated with Ω_k.

Step 2. Show $\operatorname{supp} a_P \subset P'' \equiv P_1'' \times P_2''$.

If $x \in \operatorname{supp} a_P$, by (4.7), $a_P(x) \neq 0$ implies that there exists $R \in \mathcal{R}_k$ such that $R^* = P$ and $e_R(x) \neq 0$. Recall that for all $t_1,\,t_2 \in \mathbb{Z}$ and $(x_1,\,x_2) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\theta_{t_1,\,t_2}(x_1,\,x_2) = b_1^{-t_1} b_2^{-t_2} \theta^{(1)}_{t_1}(A_1^{-t_1}x_1)\theta^{(2)}_{t_2}(A_2^{-t_2}x_2)$$

and $\operatorname{supp}\theta^{(i)} \subset B_0^{(i)}$. If $e_R(x_1,\,x_2) \neq 0$, by (4.6), there exists $(y,\,(t_1,\,t_2)) \in R_+$ such that $A_i^{t_i}(x_i - y_i) \in B_0^{(i)}$. Moreover, by (4.2), we have $t_i < v_i[\ell(R_i) - 1] + u_i + \sigma_i$. Therefore, by Lemma 2.3(iv) and (2.1), we further have

$$x_i \in y_i + B^{(i)}_{t_i} \subset x_{R_i} + B^{(i)}_{v_i\ell(R_i)+u_i} + B^{(i)}_{v_i(\ell(R_i)-1)+u_i+\sigma_i} \subset x_{R_i} + B^{(i)}_{v_i(\ell(R_i)-1)+u_i+2\sigma_i} \equiv R_i'.$$

Thus,

$$\operatorname{supp} e_R \subset R' \equiv R_1' \times R_2'. \tag{4.9}$$

Since $R_i \subset P_i$, by Lemma 2.3(iv) and (2.1), we obtain

$$\begin{aligned}
R_i' = x_{R_i} + B^{(i)}_{v_i(\ell(R_i)-1)+u_i+2\sigma_i} &\subset x_{R_i} - x_{P_i} + x_{P_i} + B^{(i)}_{v_i(\ell(P_i)-1)+u_i+2\sigma_i}\\
&\subset x_{P_i} + B^{(i)}_{v_i\ell(P_i)+u_i} + B^{(i)}_{v_i(\ell(P_i)-1)+u_i+2\sigma_i}\\
&\subset x_{P_i} + B^{(i)}_{v_i(\ell(P_i)-1)+u_i+3\sigma_i}\\
&\equiv P_i''.
\end{aligned}$$

From this and (4.7), we obtain $\operatorname{supp} a_P \subset P'' \equiv P_1'' \times P_2''$.

Step 3. Prove $\|a_k\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)} \lesssim [w(\Omega_k)]^{1/q-1/p}$.
To this end, we need the following key lemma which will be shown in Steps 6-8 bellow.

Lemma 4.7 *Let θ, ψ be as in Proposition 2.16, $\mathcal{G}$ any set of dyadic rectangles in $\mathbb{R}^n\times\mathbb{R}^m$, and e_R as in (4.6) for any $R\in\mathcal{G}$. Then, there exists a positive constant C such that for all $x\in\mathbb{R}^n\times\mathbb{R}^m$,*

$$\left[\vec{S}_\theta\left(\sum_{R\in\mathcal{G}} e_R\right)(x)\right]^2 \le C\sum_{R\in\mathcal{G}}[\mathcal{M}_s(c_R\chi_R)(x)]^2, \tag{4.10}$$

where

$$c_R = \left\{\iint_{R_+} |\psi_{t_1,\,t_2} * f(y)|^2\,dy\,\frac{d\sigma(t_1)\,d\sigma(t_2)}{b_1^{t_1}b_2^{t_2}}\right\}^{1/2}.$$

Assuming Lemma 4.7 for the moment, since $q>q_w$, we have $w\in\mathcal{A}_q(\vec{A})$. By this, Theorem 3.4, Lemma 4.7 with $\mathcal{G}=\mathcal{R}_k$ and Proposition 2.10(ii), we have

$$\begin{aligned}\|a_k\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)} &\lesssim \lambda_k^{-1}\left\|\vec{S}_\theta\left(\sum_{R\in\mathcal{R}_k} e_R\right)\right\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}\\ &\lesssim \lambda_k^{-1}\left\|\left\{\sum_{R\in\mathcal{R}_k}[\mathcal{M}_s\,(c_R\chi_R)]^2\right\}^{1/2}\right\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}\\ &\lesssim \lambda_k^{-1}\left\|\left[\sum_{R\in\mathcal{R}_k} c_R^2\chi_R\right]^{1/2}\right\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}.\end{aligned}$$

Since for all $R\in\mathcal{R}_k$, $|R\cap\Omega_{k+1}|\le|R|/2$ and $R\subset\widetilde{\Omega}_k$ by Lemma 2.3(iv) and (4.5), then for all $x\in R$, we have

$$\mathcal{M}_s\Big(\chi_{R\cap(\widetilde{\Omega}_k\setminus\Omega_{k+1})}\Big)(x) \gtrsim \frac{1}{|R|}\int_R \chi_{R\cap(\widetilde{\Omega}_k\setminus\Omega_{k+1})}(y)\,dy \gtrsim \frac{|R|-|R|/2}{|R|}\gtrsim\chi_R(x).$$

From this and Proposition 2.10(ii), it follows that

$$\begin{aligned}\|a_k\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)} &\lesssim \lambda_k^{-1}\left\|\left\{\sum_{R\in\mathcal{R}_k}\Big[\mathcal{M}_s\Big(c_R\chi_{R\cap(\widetilde{\Omega}_k\setminus\Omega_{k+1})}\Big)\Big]^2\right\}^{1/2}\right\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}\\ &\lesssim \lambda_k^{-1}\left\|\left(\sum_{R\in\mathcal{R}_k} c_R^2\chi_{R\cap(\widetilde{\Omega}_k\setminus\Omega_{k+1})}\right)^{1/2}\right\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}.\end{aligned}\tag{4.11}$$

Moreover, fix $x\in\mathbb{R}^n\times\mathbb{R}^m$. If $R\in\mathcal{R}_k$ and $x\in R$, then for any $(y,\,t)\in R_+$, by Lemma 2.3(iv) and (2.1), $x_i-y_i\in B^{(i)}_{v_i\ell(R_i)+u_i+\sigma_i}\subset B^{(i)}_{t_i}$, which together with Remark 3.5 and the disjointness of R_+ implies that

$$\begin{aligned}&\sum_{R\in\mathcal{R}_k} c_R^2\chi_{R\cap(\widetilde{\Omega}_k\setminus\Omega_{k+1})}(x)\\ &\quad=\sum_{R\in\mathcal{R}_k}\iint_{R_+}|\psi_{t_1,\,t_2}*f(y)|^2\,dy\,\frac{d\sigma(t_1)\,d\sigma(t_2)}{b_1^{t_1}b_2^{t_2}}\chi_{R\cap(\widetilde{\Omega}_k\setminus\Omega_{k+1})}(x)\\ &\quad\lesssim\big[\vec{S}_\psi(f)(x)\big]^2\chi_{\widetilde{\Omega}_k\setminus\Omega_{k+1}}(x)\\ &\quad\lesssim 2^{2k}\chi_{\widetilde{\Omega}_k\setminus\Omega_{k+1}}(x).\end{aligned}\tag{4.12}$$

Notice that $w\big(\widetilde{\Omega}_k\big)\lesssim w(\Omega_k)$ by $w\in\mathcal{A}_q(\vec{A})$ and Proposition 2.10(ii). From these estimates, we deduce

$$\|a_k\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}\lesssim 2^{-k}[w(\Omega_k)]^{-1/p}2^k\big[w\big(\widetilde{\Omega}_k\big)\big]^{1/q}\lesssim[w(\Omega_k)]^{1/q-1/p}. \tag{4.13}$$

Step 4. Prove $\sum_{P\in m(\widetilde{\Omega}_k)} \|a_P\|^q_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)} \lesssim [w(\Omega_k)]^{1-q/p}$.

Similarly to the proof of (4.13), by Theorem 3.4, Lemma 4.7 with $\mathcal{G} = \{R \in \mathcal{R}_k : R^* = P\}$, the monotonicity of $\ell^{q/2}$ with $q \geq 2$, (4.12) and $w(\widetilde{\Omega}_k) \lesssim w(\Omega_k)$, we have

$$
\begin{aligned}
&\sum_{P\in m(\widetilde{\Omega}_k)} \|a_P\|^q_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}\\
&\lesssim \lambda_k^{-q} \sum_{P\in m(\widetilde{\Omega}_k)} \left\| \vec{S}_\theta\left(\sum_{R\in\mathcal{R}_k,\, R^*=P} e_R \right)\right\|^q_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}\\
&\lesssim \lambda_k^{-q} \sum_{P\in m(\widetilde{\Omega}_k)} \left\| \left\{ \sum_{R\in\mathcal{R}_k,\, R^*=P} c_R^2 \chi_{R\cap(\widetilde{\Omega}_k\setminus\Omega_{k+1})} \right\}^{1/2}\right\|^q_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}\\
&\lesssim \lambda_k^{-q} \left\| \vec{S}_\psi(f)\chi_{\widetilde{\Omega}_k\setminus\Omega_{k+1}} \right\|^q_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}\\
&\lesssim 2^{-qk}[w(\Omega_k)]^{-q/p} w(\widetilde{\Omega}_k) 2^{q(k+1)}\\
&\lesssim [w(\Omega_k)]^{1-q/p}.
\end{aligned}
\tag{4.14}
$$

Step 5. Show the vanishing moments of a_P.

By (4.12) and $w(\widetilde{\Omega}_k) \lesssim w(\Omega_k)$, we have

$$
\begin{aligned}
&\lambda_k^{-q} \int_{\mathbb{R}^n\times\mathbb{R}^m} \left\{ \iint_{\bigcup_{R\in\mathcal{R}_k} R_+} |f * \psi_{t_1,\,t_2}(y)|^2 \chi_{R\cap(\widetilde{\Omega}_k\setminus\Omega_k+1)}(x)\, dy \frac{d\sigma(t_1)\, d\sigma(t_2)}{b_1^{t_1} b_2^{t_2}} \right\}^{q/2} w(x)\, dx\\
&= \lambda_k^{-q} \left\| \left(\sum_{R\in\mathcal{R}_k} c_R^2 \chi_{R\cap(\widetilde{\Omega}_k\setminus\Omega_{k+1})} \right)^{1/2} \right\|^q_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}\\
&\lesssim [w(\Omega_k)]^{1-q/p} < \infty.
\end{aligned}
\tag{4.15}
$$

Take any $N \in \mathbb{N}$ and let $\mathcal{R}_{k,\,N} \equiv \{R \in \mathcal{R}_k : |\ell(R_i)| > N,\, i = 1,\, 2\}$. Replacing a_k by $\lambda_k^{-1} \sum_{R\in\mathcal{R}_{k,\,N}} e_R$, similarly to the estimate of (4.11), we obtain

$$
\begin{aligned}
&\left\| \lambda_k^{-1} \sum_{R\in\mathcal{R}_{k,\,N}} e_R \right\|^q_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}\\
&\lesssim \lambda_k^{-q} \left\| \left(\sum_{R\in\mathcal{R}_{k,\,N}} c_R^2 \chi_{R\cap(\widetilde{\Omega}_k\setminus\Omega_{k+1})} \right)^{1/2} \right\|^q_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}\\
&\sim \lambda_k^{-q} \int_{\mathbb{R}^n\times\mathbb{R}^m} \left\{ \iint_{\bigcup_{R\in\mathcal{R}_{k,\,N}} R_+} |f * \psi_{t_1,\,t_2}(y)|^2 \chi_{R\cap(\widetilde{\Omega}_k\setminus\Omega_k+1)}(x)\, dy \frac{d\sigma(t_1)\, d\sigma(t_2)}{b_1^{t_1} b_2^{t_2}} \right\}^{q/2} w(x)\, dx.
\end{aligned}
$$

Then by (4.15) and Lebesgue dominated convergence theorem, we have

$$
\left\| \lambda_k^{-1} \sum_{R\in\mathcal{R}_{k,\,N}} e_R \right\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)} \longrightarrow 0,
$$

as $N \to \infty$, which implies that $a_P = \lambda_k^{-1} \sum_{R\in\mathcal{R}_k,\, R^*=P} e_R$ converges in $L^q_w(\mathbb{R}^n\times\mathbb{R}^m)$, and thus for almost everywhere $x_2 \in \mathbb{R}^m$, $a_P(\cdot,\, x_2) \in L^q_{w(\cdot,\, x_2)}(\mathbb{R}^n)$. Moreover, recall that θ has vanishing moments $s_1 \geq \lfloor (q_w/p - 1)\zeta_{1,-}^{-1} \rfloor$ in the first variable and so is e_R. Let $h_1(x_1) \equiv x_1^\alpha \chi_{P_1''}(x_1)$ with $|\alpha| \leq s_1$ and $\widetilde{q} \in \mathbb{R}_+$

such that $q^{-1} + (\widetilde{q})^{-1} = 1$. Obviously, $h_1 \in L^{\widetilde{q}}_{w^{-\widetilde{q}/q}(\cdot,\, x_2)}(\mathbb{R}^n)$. Then by the fact that $\operatorname{supp} a_P(\cdot,\, x_2) \subset P_1''$, $\big(L^{\widetilde{q}}_{w^{-\widetilde{q}/q}(\cdot,\, x_2)}(\mathbb{R}^n)\big)^* = L^q_{w(\cdot,\, x_2)}(\mathbb{R}^n)$ and $\operatorname{supp} e_R(\cdot,\, x_2) \subset P_1''$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} a_P(x_1,\, x_2) x_1^{\alpha}\, dx_1 &= \langle a_P(\cdot,\, x_2),\, h_1\rangle \\ &= \sum_{R^*=P,\, R\in\mathcal{R}_k} \langle e_R(\cdot,\, x_2),\, h_1\rangle \\ &= \sum_{R^*=P,\, R\in\mathcal{R}_k} \int_{\mathbb{R}^n} e_R(x_1,\, x_2) x_1^{\alpha_1}\, dx_1 \\ &= 0.\end{aligned}$$

Thus, a_P has vanishing moments up to order s_1 in the first variable. By symmetry, a_P has vanishing moments up to order s_2 in the second variable.

Combining Steps 3 through 5 shows that a_k is a fixed multiple of a $(p,\, q,\, \vec{s})_w$-atom associated with Ω_k. To finish the proof of Lemma 4.6, we still need to show Lemma 4.7.

Step 6. Proof of Lemma 4.7.

For $P \in \mathcal{R}$, let P_+ be as in (4.1). For all $x \in \mathbb{R}^n \times \mathbb{R}^m$, by Remark 3.5, we have

$$\begin{aligned}&\left[\vec{S}_\theta\left(\sum_{R\in\mathcal{G}} e_R\right)(x)\right]^2 \\ &\quad = \iint_{\Gamma(x)} \left|\theta_{t_1,\, t_2} * \left(\sum_{R\in\mathcal{G}} e_R\right)(y)\right|^2 dy\, \frac{d\sigma(t_1)\, d\sigma(t_2)}{b_1^{t_1} b_2^{t_2}} \qquad (4.16)\\ &\quad \le \sum_{P\in\mathcal{R},\ P_+\cap\Gamma(x)\neq\emptyset} \iint_{P_+} \left[\sum_{R\in\mathcal{G}} |e_R * \theta_{t_1,\, t_2}(y)|\right]^2 dy\, \frac{d\sigma(t_1)\, d\sigma(t_2)}{b_1^{t_1} b_2^{t_2}}.\end{aligned}$$

For any $(y,\, t) \in P_+$ with $P_+ \cap \Gamma(x) \neq \emptyset$, we will prove in Step 7 that if $P' \cap R' = \emptyset$, $e_R * \varphi_{t_1,\, t_2}(y) \equiv 0$, or else,

$$|e_R * \theta_{t_1,\, t_2}(y)| \lesssim c_R \mathcal{M}_s(\chi_R)(x) \prod_{i=1}^2 b_i^{(s_i+1)v_i|\ell(R_i)-\ell(P_i)|\zeta_{i,-}}. \tag{4.17}$$

For any $P \in \mathcal{R}$, we will show in Step 8 that

$$\sum_{R\in\mathcal{R},\ R'\cap P'\neq\emptyset} \prod_{i=1}^2 b_i^{(s_i+1)v_i|\ell(R_i)-\ell(P_i)|\zeta_{i,-}} \lesssim 1. \tag{4.18}$$

Assuming that (4.17) and (4.18) for the moment, for any $(y,\, t) \in P_+$ and $P_+ \cap \Gamma(x) \neq \emptyset$, by (4.17), the Cauchy-Schwarz inequality and (4.18), we obtain

$$\begin{aligned}\left(\sum_{R\in\mathcal{G}} |e_R * \theta_{t_1,\, t_2}(y)|\right)^2 &\lesssim \left\{\sum_{R\in\mathcal{G},\, R'\cap P'\neq\emptyset} c_R \mathcal{M}_s(\chi_R)(x) \prod_{i=1}^2 b_i^{(s_i+1)v_i|\ell(R_i)-\ell(P_i)|\zeta_{i,-}}\right\}^2 \\ &\lesssim \sum_{R\in\mathcal{G},\, R'\cap P'\neq\emptyset} c_R^2 [\mathcal{M}_s(\chi_R)(x)]^2 \prod_{i=1}^2 b_i^{(s_i+1)v_i|\ell(R_i)-\ell(P_i)|\zeta_{i,-}}.\end{aligned}$$

From this, (4.16) and (4.18), it follows that

$$\begin{aligned}
&\left[\vec{S}_\theta\left(\sum_{R\in\mathcal{G}} e_R\right)(x)\right]^2\\
&\quad\lesssim \sum_{P_+\cap\Gamma(x)\neq\emptyset}\iint_{P_+}\sum_{\substack{R\in\mathcal{G}\\ R'\cap P'\neq\emptyset}} c_R^2[\mathcal{M}_s(\chi_R)(x)]^2\prod_{i=1}^2 b_i^{(s_i+1)v_i|\ell(R_i)-\ell(P_i)|\zeta_{i,-}}\,dy\,\frac{d\sigma(t_1)\,d\sigma(t_2)}{b_1^{t_1}b_2^{t_2}}\\
&\quad\lesssim \sum_{R\in\mathcal{G}} c_R^2[\mathcal{M}_s(\chi_R)(x)]^2\left\{\sum_{\substack{P\in\mathcal{R}\\ R'\cap P'\neq\emptyset}}\prod_{i=1}^2 b_i^{(s_i+1)v_i|\ell(R_i)-\ell(P_i)|\zeta_{i,-}}\right\}\\
&\quad\lesssim \sum_{R\in\mathcal{G}} c_R^2[\mathcal{M}_s(\chi_R)(x)]^2,
\end{aligned}$$

which yields (4.10). To finish the proof of Lemma 4.7, we still need to show (4.17) and (4.18).

Step 7. Show (4.17).

Consider first the trivial case when $R'\cap P'=\emptyset$. In this case we claim that for $(y,\,(t_1,\,t_2))\in P_+$, we have $e_R*\theta_{t_1,\,t_2}(y)=0$. By (4.9), we have

$$e_R*\theta_{t_1,\,t_2}(y_1,\,y_2)=\iint_{R'} e_R(z_1,\,z_2)\theta_{t_1,\,t_2}(y_1-z_1,\,y_2-z_2)\,dz_1\,dz_2.$$

Recall that

$$\theta_{t_1,\,t_2}(y_1-z_1,\,y_2-z_2)=b_1^{-t_1}b_2^{-t_2}\theta^{(1)}\big(A_1^{-t_1}(y_1-z_1)\big)\theta^{(2)}\big(A_2^{-t_2}(y_2-z_2)\big),$$

and $\operatorname{supp}\theta^{(i)}\subset B_0^{(i)}$ for $i=1,\,2$. Moreover, since $(y,\,(t_1,\,t_2))\in P_+$, by (4.1), (4.2) and Lemma 2.3(iv), we obtain $y_i\in P_i\subset x_{P_i}+B^{(i)}_{v_i\ell(P_i)+u_i}$ and $t_i<v_i(\ell(P_i)-1)+u_i+\sigma_i$ for $i=1,\,2$. Therefore, if $(y,\,(t_1,\,t_2))\in P_+$ and $\theta_{t_1,\,t_2}(y_1-z_1,\,y_2-z_2)\neq 0$, then by (2.1), we have

$$z_i\in y_i+B^{(i)}_{t_i}\subset x_{P_i}+B^{(i)}_{v_i\ell(P_i)+u_i}+B^{(i)}_{v_i(\ell(P_i)-1)+u_i+\sigma_i}\subset x_{P_i}+B^{(i)}_{v_i(\ell(P_i)-1)+u_i+2\sigma_i}=P_i'.$$

Thus, for all $(y,\,(t_1,\,t_2))\in P_+$, we have

$$e_R*\theta_{t_1,\,t_2}(y)=\int_{R'\cap P'} e_R(z)\theta_{t_1,\,t_2}(y-z)\,dz, \tag{4.19}$$

and if $P'\cap R'=\emptyset$, we obtain $e_R*\theta_{t_1,\,t_2}(y)=0$.

We now consider the non-trivial case $R'\cap P'\neq\emptyset$. We shall establish (4.17) by considering the following four subcases.

Case I. $\ell(R_1)\geq\ell(P_1)$ and $\ell(R_2)\geq\ell(P_2)$. Let $i=1,\,2$. We first observe that for any $(y,\,(t_1,\,t_2))\in P_+$ and $z_i\in R_i'\equiv x_{R_i}+B^{(i)}_{v_i[\ell(R_i)-1]+u_i+2\sigma_i}$, by $t_i\geq v_i\ell(P_i)+u_i+\sigma_i$, we have

$$z_i'\equiv A_i^{-t_i}z_i\in A_i^{-t_i}x_{R_i}+B^{(i)}_{v_i[\ell(R_i)-1]+u_i+2\sigma_i-t_i}\subset A_i^{-t_i}x_{R_i}+B^{(i)}_{v_i[\ell(R_i)-1-\ell(P_i)]+\sigma_i}\equiv\bar{R}_i.$$

Let $\bar{R}\equiv\bar{R}_1\times\bar{R}_2$. Then for any $z_i'\in\bar{R}_i$, since $-v_i,\,\sigma_i>0$ and $v_i[\ell(R_i)-\ell(P_i)]\leq 0$, by (2.5) and (2.4), we obtain

$$|z_i'-A_i^{-t_i}x_{R_i}|=|A_i^{-v_i+\sigma_i}[A_i^{v_i-\sigma_i}(z_i'-A_i^{-t_i}x_{P_i})]|\lesssim b_i^{v_i[\ell(R_i)-\ell(P_i)]\zeta_{i,\,-}}. \tag{4.20}$$

On the other hand, by the Cauchy-Schwarz inequality, $\theta\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^m)$ and Lemma 2.3(iv), we have

$$\begin{aligned}
|e_R(x)|^2&\leq c_R^2\iint_{R_+}|\theta_{t_1,\,t_2}(x_1-y_1,\,x_2-y_2)|^2 b_1^{t_1}b_2^{t_2}\,dy_1\,dy_2\,d\sigma(t_1)\,d\sigma(t_2)\\
&\lesssim c_R^2\sum_{t_1\sim v_1\ell(R_1)+u_1}\;\sum_{t_2\sim v_2\ell(R_2)+u_2}|R|b_1^{-t_1}b_2^{-t_2}\\
&\lesssim c_R^2.
\end{aligned}\tag{4.21}$$

Let

$$\mathcal{P}_{w_i}^{(i)}(z_i) = \sum_{|\alpha_i| \le s_i} \frac{1}{\alpha_i!} \partial^{\alpha_i} \theta^{(i)}(w_i)(z_i - w_i)^{\alpha_i}$$

be the Taylor polynomial of $\theta^{(i)}$ about $w_i \in \mathbb{R}^{n_i}$ of degree s_i. For any $(y, (t_1, t_2)) \in P_+$, since $e_R \in \mathcal{S}_{s_1, s_2}(\mathbb{R}^n \times \mathbb{R}^m)$, $\theta = \theta^{(1)}\theta^{(2)}$ and $\theta^{(i)} \in \mathcal{S}_{s_i}(\mathbb{R}^{n_i})$ for $i = 1, 2$, by (4.9), Taylor's remainder theorem, (4.20) and (4.21), we obtain

$$\begin{aligned} &|e_R * \theta_{t_1, t_2}(y)| \\ &= \left| \int_{\bar{R}} e_R\big(A_1^{t_1} z_1,\, A_2^{t_2} z_2\big) \prod_{i=1}^{2} \theta^{(i)}\left(A_i^{-t_i} y_i - z_i\right)\, dz \right| \\ &= \left| \int_{\bar{R}} e_R\big(A_1^{t_1} z_1,\, A_2^{t_2} z_2\big) \prod_{i=1}^{2} \left(\theta^{(i)}\big(A_i^{-t_i} y_i - z_i\big) - \mathcal{P}^{(i)}_{A_i^{-t_i} y_i - A_i^{-t_i} x_{R_i}}\left(A_i^{-t_i} y_i - z_i\right) \right) dz \right| \\ &\le \int_{\bar{R}_1 \times \bar{R}_2} \big| e_R\big(A_1^{t_1} z_1,\, A_2^{t_2} z_2\big)\big| \prod_{i=1}^{2} \big| A_i^{-t_i} x_{R_i} - z_i \big|^{s_i+1}\, dz_1 dz_2 \\ &\lesssim c_R \prod_{i=1}^{2} b_i^{v_i[\ell(R_i) - \ell(P_i)]} b_i^{(s_i+1)v_i[\ell(R_i)-\ell(P_i)]\zeta_{i,-}}. \end{aligned} \tag{4.22}$$

Observing that since $\ell(P_i) \le \ell(R_i)$ and $P_i' \cap R_i' \ne \emptyset$ for $i = 1, 2$, by (2.1) and Lemma 2.3(iv), it is easy to see

$$R_i' \subset P_i''' \equiv x_{P_i} + B^{(i)}_{v_i(\ell(P_i)-1)+u_i+4\sigma_i}, \tag{4.23}$$

and hence $R' \subset P'''$. Moreover, for any $x \in \mathbb{R}^n \times \mathbb{R}^m$ and $\Gamma(x) \cap P_+ \ne \emptyset$, by (2.1) and Lemma 2.3(iv), we obtain

$$x \in P'. \tag{4.24}$$

By Lemma 2.3(iv), we have that $b_i^{v_i \ell(R_i)} \sim |R_i'|$ and $b_i^{v_i \ell(P_i)} \sim |P_i'''|$. By this, (4.23), (4.24), Lemma 2.3(iv) and Remark 2.8, we have that for any $x \in \mathbb{R}^n \times \mathbb{R}^m$ and $\Gamma(x) \cap P_+ \ne \emptyset$,

$$\prod_{i=1}^{2} b_i^{v_i[\ell(R_i)-\ell(P_i)]} \lesssim \mathcal{M}_s(\chi_{R'})(x) \lesssim \mathcal{M}_s(\chi_R)(x). \tag{4.25}$$

Combining this and (4.22) yields (4.17).

Case II. $\ell(R_1) < \ell(P_1)$ and $\ell(R_2) < \ell(P_2)$. In this case, for any $z_i \in P_i' = x_{P_i} + B^{(i)}_{v_i[\ell(P_i)-1]+u_i+2\sigma_i}$, we have

$$z_i' \equiv A_i^{-v_i \ell(R_i) - u_i} z_i \in A_i^{-v_i \ell(R_i) - u_i} x_{P_i} + B^{(i)}_{v_i[\ell(P_i)-1-\ell(R_i)]+2\sigma_i} \equiv \widetilde{P}_i. \tag{4.26}$$

Let $\widetilde{P} \equiv \widetilde{P}_1 \times \widetilde{P}_2$. For any $z_i' \in \widetilde{P}_i$, since $\ell(P_i) > \ell(R_i)$ and $-v_i, \sigma_i > 0$, by (2.5) and (2.4), similarly to the estimate of (4.20), we obtain

$$\left| z_i' - A_i^{-v_i \ell(R_i) - u_i} x_{P_i} \right| \lesssim b_i^{v_i[\ell(P_i)-\ell(R_i)]\zeta_{i,-}}. \tag{4.27}$$

Let $\widetilde{e}_R(z) \equiv e_R\left(A_1^{v_1 \ell(R_1)+u_1} z_1,\, A_2^{v_2 \ell(R_2)+u_2} z_2\right)$. For any $z \in \mathbb{R}^n \times \mathbb{R}^m$ and $(\alpha_1, \alpha_2) \in (\mathbb{Z}_+)^{n_1} \times (\mathbb{Z}_+)^{n_2}$, we have

$$|\partial_1^{\alpha_1} \partial_2^{\alpha_2} \widetilde{e}_R(z)| \lesssim c_R. \tag{4.28}$$

Indeed, for any $\gamma_i \geq v_i\ell(R_i) + u_i + \sigma_i$ and $z_i \in \mathbb{R}^{n_i}$, an application of chain rule yields

$$\left\|\partial^{\alpha_i}\left[\theta^{(i)}\left(A_i^{v_i\ell(R_i)+u_i-\gamma_i}\cdot\right)\right]\right\|_\infty \lesssim 1.$$

Hence (4.28) follows by the Cauchy-Schwarz inequality, (4.2) and Lemma 2.3(iv), similarly to the estimate of (4.21),

$$\begin{aligned}|\partial_1^{\alpha_1}\partial_2^{\alpha_2}\widetilde{e}_R(z)|^2 &= \bigg|\iint_{R_+}\partial_{z_1}^{\alpha_1}\partial_{z_2}^{\alpha_2}\left[\theta_{\gamma_1,\,\gamma_2}\left(A_1^{v_1\ell(R_1)+u_1}\cdot - y_1,\, A_2^{v_2\ell(R_2)+u_2}\cdot - y_2\right)\right](z_1,z_2)\\ &\qquad\qquad \times(\psi_{\gamma_1,\,\gamma_2} * f)(y)\,dy\,d\sigma(\gamma_1)\,d\sigma(\gamma_2)\bigg|^2\\ &\lesssim c_R^2\iint_{R_+} b_1^{-\gamma_1}b_2^{-\gamma_2}\,dy_1\,dy_2\,d\sigma(\gamma_1)\,d\sigma(\gamma_2)\\ &\lesssim c_R^2.\end{aligned}$$

Without loss of generality we can assume that

$$b_1^{(s_1+1)v_1(\ell(P_1)-\ell(R_1))\zeta_{1,\,-}} \leq b_2^{(s_2+1)v_2(\ell(P_2)-\ell(R_2))\zeta_{2,\,-}}, \tag{4.29}$$

since the other case is dealt in the same way. Let

$$\mathcal{P}_{w_1}(z_1,z_2) = \sum_{|\alpha_1|\leq s_3}\frac{\partial_{z_1}^{\alpha_1}\tilde{e}_R(w_1,z_2)}{\alpha_1!}(z_1-w_1)^{\alpha_1}$$

be the Taylor polynomial of $\tilde{e}_R(\cdot, z_2)$ in the first variable about $w_1 \in \mathbb{R}^{n_1}$ and degree s_3. For any $(y,\,(t_1,\,t_2)) \in \Gamma(x)\cap P_+$, by (4.19) and (4.26), the change of variables, and our hypothesis that each $\theta^{(i)}$ has vanishing moments up to degree $s_3 = 2\max(s_1, s_2) + 1$, we have

$$\begin{aligned}|e_R * \theta_{t_1,\,t_2}(y)| &= \bigg|\int_{\widetilde{P}}\left\{\widetilde{e}_R(z_1,\,z_2) - \mathcal{P}_{A_1^{-v_1\ell(R_1)-u_1}x_{P_1}}(z_1,z_2)\right\}\\ &\qquad\qquad\times\prod_{i=1}^2\theta_{t_i}^{(i)}\left(y_i - A_i^{v_i\ell(R_i)+u_i}z_i\right)b_i^{v_i\ell(R_i)+u_i}\,dz\bigg|\\ &\lesssim c_R\int_{\widetilde{P}}\left|z_1 - A_1^{-v_1\ell(R_1)-u_1}x_{P_1}\right|^{s_3+1}\prod_{i=1}^2\left|\theta_{t_i}^{(i)}\left(y_i - A_i^{v_i\ell(R_i)+u_i}z_i\right)\right|b_i^{v_i\ell(R_i)}\,dz\\ &\lesssim c_R b_1^{(s_3+1)v_1[\ell(P_1)-\ell(R_1)]\zeta_{1,\,-}}\prod_{i=1}^2 b_i^{v_i[\ell(P_i)-\ell(R_i)]}b_i^{-t_i+v_i\ell(R_i)}\\ &\lesssim c_R\prod_{i=1}^2 b_i^{(s_i+1)v_i(\ell(P_i)-\ell(R_i))\zeta_{i,\,-}}.\end{aligned}$$

Indeed, the first estimate is a consequence of Taylor's remainder theorem and (4.28), the second follows from (4.27), and the last follows from (4.29) and $b_i^{t_i} \sim b_i^{v_i\ell(P_i)}$ for $i = 1,\,2$.

Since $\ell(R_1) < \ell(P_1)$ and $\ell(R_2) < \ell(P_2)$, by (4.23) and symmetry, we obtain $P' \subset R'''$. From this, (4.24), Remark 2.8 and Lemma 2.3(iv), it follows that for $x \in P'$, $1 = \mathcal{M}_s(\chi_{R'''})(x) \lesssim \mathcal{M}_s(\chi_R)(x)$; see also (4.25). Then, combining this and (4.29) yields (4.17).

Case III. $\ell(R_1) \geq \ell(P_1)$ and $\ell(R_2) < \ell(P_2)$. In this case define

$$e_R^{(2)}(z_1,\,z_2) \equiv e_R\left(z_1,\,A_2^{v_2\ell(R_2)+u_2}z_2\right).$$

For any $z \in \mathbb{R}^n \times \mathbb{R}^m$ and $\alpha_2 \in (\mathbb{Z}_+)^{n_2}$, similarly to the estimate of (4.28), we obtain

$$\left|\partial_2^{\alpha_2} e_R^{(2)}(z_1, z_2)\right| \lesssim c_R. \tag{4.30}$$

Let $\bar{R}_1 \equiv A_1^{-t_1} x_{R_1} + B^{(1)}_{v_1[\ell(R_1)-1]+u_1+2\sigma_1-t_1}$ and $\widetilde{P}_2 \equiv A_2^{-v_2\ell(R_2)-u_2} x_{P_2} + B^{(2)}_{v_2[\ell(P_2)-1-\ell(R_2)]+2\sigma_2}$. Let $\mathcal{P}^{(1)}_{w_1}$ be the Taylor polynomial of $\theta^{(1)}$ about $w_1 \in \mathbb{R}^{n_1}$ of degree s_1, and let

$$\mathcal{P}_{w_2}(z_1, z_2) = \sum_{|\alpha_2| \le s_2} \frac{\partial_{z_2}^{\alpha_2} e_R^{(2)}(z_1, w_2)}{\alpha_2!}(z_2 - w_2)^{\alpha_2}$$

be the Taylor polynomial of $e_R^{(2)}(z_1, \cdot)$ in the second variable about $w_2 \in \mathbb{R}^{n_2}$ of degree s_2. For any $(y,\, (t_1,\, t_2)) \in \Gamma(x) \cap P_+$, by (4.19), the change of variables, and vanishing moment conditions, we have

$$\begin{aligned}
&e_R * \theta_{t_1,\, t_2}(y)\\
&\quad= \int_{\bar{R}_1 \times \widetilde{P}_2} e_R^{(2)}\big(A_1^{t_1} z_1,\, z_2\big)\theta^{(1)}\big(A_1^{-t_1} y_1 - z_1\big)\theta^{(2)}_{t_2}\left(y_2 - A_2^{v_2\ell(R_2)+u_2} z_2\right) b_2^{v_2\ell(R_2)+u_2}\, dz\\
&\quad= b_2^{v_2\ell(R_2)+u_2} \int_{\bar{R}_1 \times \widetilde{P}_2} \left(e_R^{(2)}\big(A_1^{t_1} z_1,\, z_2\big) - \mathcal{P}_{A_2^{-v_2\ell(R_2)-u_2} x_{P_2}}\big(A_1^{t_1} z_1, z_2\big)\right)\\
&\qquad\qquad \times \left(\theta^{(1)}\big(A_1^{-t_1} y_1 - z_1\big) - \mathcal{P}^{(1)}_{A_1^{-t_1} y_1 - A_1^{-t_1} x_{R_1}}\big(A_1^{-t_1} y_1 - z_1\big)\right)\\
&\qquad\qquad \times \theta^{(2)}_{t_2}\left(y_2 - A_2^{v_2\ell(R_2)+u_2} z_2\right) dz_1\, dz_2.
\end{aligned}$$

The last equation is a consequence of Fubini's theorem with the inside integration over the z_2 variable. Consequently, Taylor's remainder theorem, (4.20) for $i = 1$, and (4.27) for $i = 2$ yields

$$\begin{aligned}
|e_R * \theta_{t_1,\, t_2}(y)| &\lesssim c_R b_1^{v_1(\ell(R_1)-\ell(P_1))} b_2^{v_2[\ell(P_2)-\ell(R_2)]} b_2^{-t_2+v_2\ell(R_2)} \prod_{i=1}^{2} b_i^{(s_i+1)v_i|\ell(P_i)-\ell(R_i)|\zeta_{i,\,-}}\\
&\lesssim c_R b_1^{v_1[\ell(R_1)-\ell(P_1)]} \prod_{i=1}^{2} b_i^{(s_i+1)v_i|\ell(P_i)-\ell(R_i)|\zeta_{i,\,-}}.
\end{aligned} \tag{4.31}$$

Moreover, observing that $\ell(R_1) \ge \ell(P_1)$ and $\ell(R_2) < \ell(P_2)$, by (4.23) and symmetry, we obtain that $R_1' \subset P_1'''$ and $P_2' \subset R_2'''$. From this, $b_i^{v_i\ell(R_i)} \sim |R_i'|$, $b_i^{v_i\ell(P_i)} \sim |P_i'''|$, (4.24), Remark 2.8 and Lemma 2.3(iv), it follows that

$$\begin{aligned}
b_1^{v_1[\ell(R_1)-\ell(P_1)]} &\sim \frac{|R_1'|}{|P_1'''|} \sim \frac{|R_1' \cap P_1'''|}{|P_1'''|}\frac{|R_2''' \cap P_2'|}{|P_2'|} \lesssim \frac{|R_1''' \cap P_1'''|}{|P_1'''|}\frac{|R_2''' \cap P_2'''|}{|P_2'''|}\\
&\sim \frac{|R''' \cap P'''|}{|P'''|} \lesssim \mathcal{M}_s(\chi_{R'''})(x) \lesssim \mathcal{M}_s(\chi_R)(x);
\end{aligned}$$

see also (4.25). Combining this and (4.31) yields (4.17).

Case IV. Finally, the case $\ell(R_1) < \ell(P_1)$ and $\ell(R_2) \ge \ell(P_2)$ follows from Case III by the symmetry. This completes the proof of the crucial estimate (4.17).

Step 8. Verify (4.18).

Let $\sharp E$ be the cardinality of the set E. For $i = 1,\, 2$, by (4.23) and Lemma 2.3, we have

$$\sharp\big\{R_i \in \mathcal{Q}^{(i)} :\, R_i' \cap P_i' \ne \emptyset,\, |R_i| \le |P_i|,\, \ell(R_i) = k_i\big\} \lesssim \frac{|P_i'''|}{|R_i|} \sim b_i^{-v_i|k_i-\ell(P_i)|}$$

and

$$\sharp\big\{R_i \in \mathcal{Q}^{(i)} :\, R_i' \cap P_i' \ne \emptyset,\, |R_i| \ge |P_i|,\, \ell(R_i) = k_i\big\} = 1.$$

Then by this and $(s_i+1)\zeta_{i,-}-1>0$, we obtain

$$
\begin{aligned}
\sum_{R'\cap P'\neq\emptyset}\prod_{i=1}^{2} b_i^{(s_i+1)v_i|\ell(R_i)-\ell(P_i)|\zeta_{i,-}} &\lesssim \prod_{i=1}^{2}\sum_{k_i\in\mathbb{Z}}\sum_{\substack{\ell(R_i)=k_i\\ R_i'\cap P_i'\neq\emptyset}} b_i^{(s_i+1)v_i|\ell(R_i)-\ell(P_i)|\zeta_{i,-}}\\
&\lesssim \prod_{i=1}^{2}\sum_{k_i\in\mathbb{Z}} b_i^{v_i[(s_i+1)\zeta_{i,-}-1]|k_i-\ell(P_i)|}\\
&\lesssim 1,
\end{aligned}
$$

which shows (4.18) and hence, completes the proof of Lemma 4.6. □

We now prove the converse of Lemma 4.6.

Lemma 4.8 *Let the assumptions be as in Theorem* 4.5. *Then there exists a positive constant C such that for all $f\in H_w^{p,\,q,\,\vec{s}}(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A}\,)\cap\mathcal{S}'_{\infty,\,w}(\mathbb{R}^n\times\mathbb{R}^m)$, $\|f\|_{H_w^{p,\,q,\,\vec{s}}(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A}\,)}\le C\|f\|_{H_w^p(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A}\,)}$.*

To prove Lemma 4.8, we need a variant of the Journé's covering lemma established in [37,45]; see also [7] for some different variants. We first recall some notation and definitions. Let $\Omega\subset\mathbb{R}^n\times\mathbb{R}^m$ be an open set. Denote by $m_i(\Omega)$ the family of all dyadic rectangles $R\subset\Omega$ which are maximal in the $\mathbb{R}^{n_i}$ "direction", where $i=1,\,2$. Recall that $n_1=n$ and $n_2=m$. Let $\eta_0\in(0,\,1)$. For $R=R_1\times R_2\in m_1(\Omega)$, let $\widehat{R}_2\equiv\widehat{R}_2(R_1)$ be the "longest" dyadic cube containing R_2 such that $\big|\big(R_1\times\widehat{R}_2\big)\cap\Omega\big|>\eta_0\big|R_1\times\widehat{R}_2\big|$; and for $R=R_1\times R_2\in m_2(\Omega)$, let $\widehat{R}_1\equiv\widehat{R}_1(R_2)$ be the "longest" dyadic cube containing R_1 such that

$$|(\widehat{R}_1\times R_2)\cap\Omega|>\eta_0|\widehat{R}_1\times R_2|. \tag{4.32}$$

For $R_i\in\mathcal{Q}^{(i)}$ and $j_i\in\mathbb{N}$, we denote by $(R_i)_{j_i}$ the unique dyadic cube in $\mathcal{Q}^{(i)}$ containing R_i with $\ell((R_i)_{j_i})=\ell(R_i)-j_i$. Obviously, $(R_i)_0=R_i$. Also, let $h:[0,\infty)\to[0,\infty)$ be an increasing function such that $\sum_{j=0}^{\infty}jh\big(C_0\delta_0^j\big)<\infty$, where $C_0\equiv\max\big\{b_1^{2u_1-1},\,b_2^{2u_2-1}\big\}$ and $\delta_0\equiv\max\{b_1^{v_1},\,b_2^{v_2}\}$.

The following result is a variant of the well-known Journé's covering lemma in [45] and is adapted to expansive dilations.

Lemma 4.9 *Let A_i be a dilation on $\mathbb{R}^{n_i}$ for $i=1,\,2$, $w\in\mathcal{A}_\infty(\vec{A}\,)$ and q_w be as in* (2.7). *Let $\eta_0\in(0,1)$. Then there exists a positive constant C, only depending on n, m, η_0 and $C_{q,\,\vec{A},\,n,\,m}(w)$ with $q\in(q_w,\infty)$, such that for all open sets $\Omega\subset\mathbb{R}^n\times\mathbb{R}^m$ with $w(\Omega)<\infty$,*

$$\sum_{R=R_1\times R_2\in m_1(\Omega)} w(R)h\bigg(\frac{|R_2|}{|\widehat{R}_2|}\bigg)\le Cw(\Omega) \tag{4.33}$$

and

$$\sum_{R=R_1\times R_2\in m_2(\Omega)} w(R)h\bigg(\frac{|R_1|}{|\widehat{R}_1|}\bigg)\le Cw(\Omega). \tag{4.34}$$

P r o o f. Since the proofs for (4.33) and (4.34) are similar, we only show (4.34).

Let $R_1\in\mathcal{Q}^{(1)}$ such that $R_1\times R_2\in m_2(\Omega)$ for certain R_2. Notice that for any given $R_1\in\mathcal{Q}^{(1)}$, there may exist more than one $P_2\in\mathcal{Q}^{(2)}$ such that $R_1\times P_2\in m_2(\Omega)$. Based on this, for any $j_1\in\mathbb{N}$, we define

$$A_{R_1,\,j_1}\equiv\big\{P_2\in\mathcal{Q}^{(2)}:\,R_1\times P_2\in m_2(\Omega),\ \widehat{R}_1\equiv\widehat{R}_1(P_2)=(R_1)_{j_1-1}\big\}. \tag{4.35}$$

If $A_{R_1,\,j_1}\neq\emptyset$, for each $R_2\in A_{R_1,\,j_1}$, then by Lemma 2.3(iv), we have

$$x_{R_1}+B^{(1)}_{v_1\ell(R_1)-u_1}\subset R_1\subset x_{R_1}+B^{(1)}_{v_1\ell(R_1)+u_1}$$

and $x_{\widehat{R}_1}+B^{(1)}_{v_1\ell(\widehat{R}_1)-u_1}\subset\widehat{R}_1\subset x_{\widehat{R}_1}+B^{(1)}_{v_1\ell(\widehat{R}_1)+u_1}$. From this, it follows that

$$b_1^{-2u_1}b_1^{v_1(j_1-1)}\le\frac{|R_1|}{|\widehat{R}_1|}\le b_1^{2u_1}b_1^{v_1(j_1-1)}. \tag{4.36}$$

Let $\widetilde{C} \equiv b_1^{2u_1-1}$. By (4.36) and the disjointness of $\{R_2 :\ R_1 \times R_2 \in m_2(\Omega)\}$, we have

$$\begin{aligned}
&\sum_{R=R_1\times R_2\in m_2(\Omega)} w(R)h\left(\frac{|R_1|}{|\widehat{R}_1|}\right)\\
&\quad=\sum_{\{R_1:\, R_1\times R_2\in m_2(\Omega)\}}\ \sum_{j_1\in\mathbb{N},\, A_{R_1,\,j_1}\neq\emptyset}\ \sum_{R_2\in A_{R_1,\,j_1}} w(R_1\times R_2)h\left(\frac{|R_1|}{|\widehat{R}_1|}\right)\\
&\quad\le \sum_{j_1\in\mathbb{N}} h\left(\widetilde{C}b_1^{v_1 j_1}\right) \sum_{\{R_1:\, R_1\times R_2\in m_2(\Omega),\, A_{R_1,\,j_1}\neq\emptyset\}} w\left(R_1\times\bigcup_{R_2\in A_{R_1,\,j_1}} R_2\right).
\end{aligned}$$

Set $E_{R_1} \equiv \bigcup_{R_1\times R_2\subset\Omega} R_2$. For any $j_1 \in \mathbb{N}$ and any given $R_1 \in \mathcal{Q}^{(1)}$ satisfying $A_{R_1,\,j_1} \neq \emptyset$, if $x_2 \in \bigcup_{R_2\in A_{R_1,\,j_1}} R_2$, then there exists a dyadic cube $R_2 \in \mathcal{Q}^{(2)}$ such that $R_1 \times R_2 \in m_2(\Omega)$, $x_2 \in R_2$ and $\widehat{R}_1 = (R_1)_{j_1-1}$ by (4.35). By (4.32) and the maximality of $\widehat{R}_1$, we have $\left|\left((R_1)_{j_1-1}\times R_2\right)\cap\Omega\right| > \eta_0\left|(R_1)_{j_1-1}\times R_2\right|$ and $\left|\left((R_1)_{j_1}\times R_2\right)\cap\Omega\right| \le \eta_0\left|(R_1)_{j_1}\times R_2\right|$, which implies that $\left|\left((R_1)_{j_1}\times R_2\right)\cap\left((R_1)_{j_1}\times E_{(R_1)_{j_1}}\right)\right| \le \eta_0\left|(R_1)_{j_1}\times R_2\right|$, namely, $\left|(R_1)_{j_1}\times\left(R_2\cap E_{(R_1)_{j_1}}\right)\right| \le \eta_0\left|(R_1)_{j_1}\times R_2\right|$. Therefore, $\left|R_2\cap E_{(R_1)_{j_1}}\right| \le \eta_0|R_2|$, and hence, $\left|R_2\cap\left(E_{(R_1)_{j_1}}\right)^{\complement}\right| > (1-\eta_0)|R_2|$, where $\left(E_{(R_1)_{j_1}}\right)^{\complement} \equiv \left(\mathbb{R}^m\setminus E_{(R_1)_{j_1}}\right)$. From this and $R_2 \subset E_{R_1}$, it follows that for $x_2 \in R_2$, $\mathcal{M}^{(2)}\left(\chi_{E_{R_1}\setminus E_{(R_1)_{j_1}}}\right)(x_2) > 1-\eta_0$, where $\mathcal{M}^{(2)}$ is the Hardy-Littlewood maximal operator with respect to the second variable, namely, on $\mathbb{R}^m$. Thus, for any $j_1 \in \mathbb{N}$, we obtain

$$\bigcup_{R_2\in A_{R_1,\,j_1}} R_2 \subset K \equiv \left\{x_2\in\mathbb{R}^m :\ \mathcal{M}_2\left(\chi_{E_{R_1}\setminus E_{(R_1)_{j_1}}}\right)(x_2) > 1-\eta_0\right\}. \tag{4.37}$$

Since $w \in \mathcal{A}_\infty(\vec{A})$ implies that there exists $q \in (1, \infty)$ such that $w \in \mathcal{A}_q(\vec{A})$. Then by Definition 2.9, for almost all $x_1 \in \mathbb{R}^n$, we obtain that $w(x_1, \cdot) \in \mathcal{A}_q(A_2)$ and the weighted constants are uniformly bounded. By this, (4.37) and Proposition 2.5(ii), we have

$$w\left(R_1\times\left(\bigcup_{R_2\in A_{R_1,\,j_1}} R_2\right)\right) \le w(R_1\times K) \lesssim w\left(R_1\times\left(E_{R_1}\setminus E_{(R_1)_{j_1}}\right)\right). \tag{4.38}$$

For $i = 1, \ldots, j_1$, by the disjointness of sets $\left\{(R_1)_{i-1}\times\left(E_{(R_1)_{i-1}}\setminus E_{(R_1)_i}\right)\subset\Omega :\ R_1\in\mathcal{Q}^{(1)}\right\}$, we have

$$\sum_{\{R_1:\, R_1\times R_2\in m_2(\Omega),\, A_{R_1,\,j_1}\neq\emptyset\}} w\left((R_1)_{i-1}\times\left(E_{(R_1)_{i-1}}\setminus E_{(R_1)_i}\right)\right) \le w(\Omega).$$

By this, $R_1 \subset (R_1)_{i-1}$ for $i \in \mathbb{N}$ and (4.38), we obtain

$$\begin{aligned}
&\sum_{R=R_1\times R_2\in m_2(\Omega)} w(R)h\left(\frac{|R_1|}{|\widehat{R}_1|}\right)\\
&\quad\lesssim \sum_{j_1\in\mathbb{N}} h\left(\widetilde{C}b_1^{v_1 j_1}\right) \sum_{\{R_1:\, R_1\times R_2\in m_2(\Omega),\, A_{R_1,\,j_1}\neq\emptyset\}} w\left(R_1\times\left(E_{R_1}\setminus E_{(R_1)_{j_1}}\right)\right)\\
&\quad\lesssim \sum_{j_1\in\mathbb{N}} h\left(\widetilde{C}b_1^{v_1 j_1}\right) \sum_{\{R_1:\, R_1\times R_2\in m_2(\Omega),\, A_{R_1,\,j_1}\neq\emptyset\}}\ \sum_{i=1}^{j_1} w\left((R_1)_{i-1}\times\left(E_{(R_1)_{i-1}}\setminus E_{(R_1)_i}\right)\right)\\
&\quad\lesssim w(\Omega)\sum_{j_1\in\mathbb{N}} j_1 h\left(\widetilde{C}b_1^{v_1 j_1}\right)\\
&\quad\lesssim w(\Omega),
\end{aligned}$$

which completes the proof of Lemma 4.9. □

Proof of Lemma 4.8. We prove Lemma 4.8 by the following 7 steps.

Step 1. Reduce to the uniform estimates on atoms.

Let ψ be as in Proposition 2.16. It suffices to prove that for all $(p,\, q,\, \vec{s})_w$-atoms a,

$$\|\vec{S}_\psi(a)\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} \lesssim 1. \tag{4.39}$$

In fact, for any $f \in H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n\times\mathbb{R}^m;\, \vec{A})$, there exist $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ and $(p,\, q,\, \vec{s})_w$-atoms $\{a_k\}_{k\in\mathbb{N}}$ such that $f = \sum_{k\in\mathbb{N}} \lambda_k a_k$ in $\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^m)$ and $\sum_{k\in\mathbb{N}} |\lambda_k|^p \lesssim \|f\|^p_{H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}$. By this, $\psi \in \mathcal{S}(\mathbb{R}^n\times\mathbb{R}^m)$, Minkowski's inequality, Fatou's lemma, and the monotonicity of the ℓ^p-norm with $p\in(0,1]$ and (4.39), we have

$$\|\vec{S}_\psi(f)\|^p_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} \le \sum_{k\in\mathbb{N}} \lambda_k^p \|\vec{S}_\psi(a)\|^p_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} \lesssim \|f\|^p_{H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}.$$

Let us now show (4.39) by Step 2 through Step 7.

Step 2. Estimate $\vec{S}_\psi(a)$ on a "finite" expansion of the support of a.

Assume that a is a $(p,\, q,\, \vec{s})_w$-atom associated with an open set Ω satisfying $w(\Omega)<\infty$ as in Definition 4.2. Let $\widetilde{\Omega}$ be as in (4.3) and $\eta_0 \equiv b_1^{v_1-5\sigma_1} b_2^{v_2-5\sigma_2}$. Obviously, $\eta_0\in(0,1)$. For each $R = R_1\times R_2 \in m(\widetilde{\Omega})$, let $\widehat{R}_1$ be the "longest" dyadic cube containing R_1 such that $\big|(\widehat{R}_1\times R_2)\cap\widetilde{\Omega}\big| > \eta_0\big|\widehat{R}_1\times R_2\big|$. For $\widetilde{\Omega}$, we define $\widetilde{\Omega}' \equiv \{x\in\mathbb{R}^n\times\mathbb{R}^m :\, \mathcal{M}_s(\chi_{\widetilde{\Omega}})(x) > b_1^{-2u_1}b_2^{-2u_2}\eta_0\}$. Similarly, we define $\widetilde{\Omega}''$ and $\widetilde{\Omega}'''$ by replacing $\widetilde{\Omega}$ in the definition of $\widetilde{\Omega}'$, respectively, by $\widetilde{\Omega}'$ and $\widetilde{\Omega}''$. Obviously, $(\widehat{R}_1\times R_2)\subset\widetilde{\Omega}'$. For any given $\widetilde{R}_1\times R_2\in m_1(\widetilde{\Omega}')$ and $\widetilde{R}_1\supset R_1$, let $\widehat{R}_2$ be the "longest" dyadic cube containing R_2 such that $\big|(\widetilde{R}_1\times\widehat{R}_2)\cap\widetilde{\Omega}'\big| > \eta_0\big|\widetilde{R}_1\times\widehat{R}_2\big|$. Set $\bar{R}^*\equiv\bar{R}_1^*\times\bar{R}_2^*\equiv\Big(x_{R_1}+B^{(1)}_{v_1(\ell(\widehat{R}_1)-1)+u_1+5\sigma_1}\Big)\times\Big(x_{R_2}+B^{(2)}_{v_2(\ell(\widehat{R}_2)-1)+u_2+5\sigma_2}\Big)$. Then we have

$$w\Bigg(\bigcup_{R\in m(\widetilde{\Omega})}\bar{R}^*\Bigg)\lesssim w(\Omega). \tag{4.40}$$

In fact, to prove (4.40), let $R^\sharp\equiv\Big(x_{\widehat{R}_1}+B^{(1)}_{v_1\ell(\widehat{R}_1)-u_1}\Big)\times\Big(x_{\widehat{R}_2}+B^{(2)}_{v_2\ell(\widehat{R}_2)-u_2}\Big)$. By Lemma 2.3(iv) and (2.1), $R^\sharp\subset(\widehat{R}_1\times\widehat{R}_2)\subsetneq\bar{R}^*$ and $R^\sharp\subset\widetilde{\Omega}''$ which is deduced from the fact that $\widetilde{R}_1\times\widehat{R}_2\subset\widetilde{\Omega}''$ and $\widehat{R}_1\subset\widetilde{R}_1$. For any $R\in m(\widetilde{\Omega})$ and $x\in\bar{R}^*$,

$$\mathcal{M}_s(\chi_{\widetilde{\Omega}''})(x)\ge\frac{1}{|\bar{R}^*|}\int_{\bar{R}^*}\chi_{\widetilde{\Omega}''}(y)\,dy>\frac{|R^\sharp|}{|\bar{R}^*|}=b_1^{-2u_1}b_2^{-2u_2}\eta_0,$$

which implies that $\bigcup_{R\in m(\widetilde{\Omega})}\bar{R}^*\subset\widetilde{\Omega}'''$. From this, $w\in\mathcal{A}_q(\vec{A})$ and the boundedness of $\mathcal{M}_s$ on $L^q_w(\mathbb{R}^n\times\mathbb{R}^m)$ (see Proposition 2.10(ii)), it follows that

$$w\Bigg(\bigcup_{R\in m(\widetilde{\Omega})}\bar{R}^*\Bigg)\le w(\widetilde{\Omega}''')\lesssim w(\Omega).$$

Thus, (4.40) holds.

Then for $w\in\mathcal{A}_\infty(\vec{A})$, $p\in(0,\,1]$ and $q\in[2,\,\infty)\cap(q_w,\,\infty)$, by Hölder's inequality, Theorem 3.4, (4.40) and Definition 4.2(II), we obtain

$$\begin{aligned}\iint_{\bigcup_{R\in m(\widetilde{\Omega})}\bar{R}^*}[\vec{S}_\psi(a)(x)]^p w(x)\,dx &\le\Bigg[w\Bigg(\bigcup_{R\in m(\widetilde{\Omega})}\bar{R}^*\Bigg)\Bigg]^{1-p/q}\|\vec{S}_\psi(a)\|^p_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}\\ &\lesssim[w(\Omega)]^{1-p/q}\|a\|^p_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}\\ &\lesssim 1.\end{aligned}\tag{4.41}$$

Step 3. Estimate $\vec{S}_\psi(a)$ on the complement of a "finite" expansion of the support of a.

Set $\bar{R}_1 \equiv x_{R_1} + B^{(1)}_{v_1(\ell(R_1)-1)+u_1+5\sigma_1}$ and $\bar{R}_2 \equiv x_{R_2} + B^{(2)}_{v_2(\ell(R_2)-1)+u_2+5\sigma_2}$. Then by $a = \sum_{R\in m(\widetilde{\Omega})} a_R$ in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ as in Definition 4.2 and the monotonicity of the ℓ^p-norm with $p \in (0,1]$, we obtain

$$
\begin{aligned}
&\iint_{\left(\bigcup_{R\in m(\widetilde{\Omega})} \bar{R}^*\right)^{\complement}} [\vec{S}_\psi(a)(x)]^p w(x)\,dx \\
&\quad \le \sum_{R\in m(\widetilde{\Omega})} \iint_{(\bar{R}^*)^{\complement}} [\vec{S}_\psi(a_R)(x)]^p w(x)\,dx \\
&\quad \le \sum_{R\in m(\widetilde{\Omega})} \left[\iint_{(\bar{R}_1^*)^{\complement}\times \bar{R}_2} + \iint_{(\bar{R}_1^*)^{\complement}\times (\bar{R}_2)^{\complement}} + \iint_{\bar{R}_1\times(\bar{R}_2^*)^{\complement}} + \iint_{(\bar{R}_1)^{\complement}\times(\bar{R}_2^*)^{\complement}}\right] [\vec{S}_\psi(a_R)(x)]^p w(x)\,dx \\
&\quad \equiv \sum_{R\in m(\widetilde{\Omega})} (\mathrm{K}_1 + \mathrm{K}_2 + \mathrm{K}_3 + \mathrm{K}_4).
\end{aligned}
\tag{4.42}
$$

Step 4. Pointwise estimate of $\vec{S}_\psi(a_R)$ on $(\bar{R}_1)^{\complement} \times \bar{R}_2^*$.

Let $\gamma_1(R) \equiv \ell(\widehat{R}_1) - \ell(R_1)$, $\bar{R}^*_{1,\,k_1} \equiv x_{R_1} + B^{(1)}_{v_1(\ell(\widehat{R}_1)-1-k_1)+u_1+5\sigma_1}$ for $k_1 \in \mathbb{N}$, and $\bar{R}^*_{1,0} \equiv \bar{R}^*_1$. We will prove in this step that for all $k_1 \in \mathbb{Z}_+$ and $x = (x_1,\, x_2)$ with $x_1 \in \bar{R}^*_{1,\,k_1+1} \setminus \bar{R}^*_{1,\,k_1}$ and $x_2 \in \bar{R}_2$,

$$
\vec{S}_\psi(a_R)(x) \lesssim b_1^{[k_1-\gamma_1(R)]v_1(s_1+1)\zeta_{1,\,-}} b_1^{-v_1(\ell(\widehat{R}_1)-k_1)} \int_{R_1''} S_{\psi^{(2)}}(a_R(z_1,\,\cdot))(x_2)\,dz_1, \tag{4.43}
$$

where $S_{\psi^{(2)}}$ is the Lusin-area function with respect to the second variable and s_1 as in Definition 4.2.

Let $L(R_1) \equiv v_1[\ell(R_1)-1] + u_1 + 3\sigma_1$. We now estimate $a_R * \psi_{j_1,\,j_2}(x-y)$ by considering two cases, where x is as in (4.43), $j_1,\, j_2 \in \mathbb{Z}$ and $y \in B^{(1)}_{j_1} \times B^{(2)}_{j_2}$.

Case I. $j_1 > L(R_1)$. For any $z_1 \in R_1'' \equiv x_{R_1} + B^{(1)}_{L(R_1)}$, we have

$$
z_1' \equiv A_1^{-j_1} z_1 \in A_1^{-j_1} R_1'' = A_1^{-j_1} x_{R_1} + B^{(1)}_{L(R_1)-j_1} \equiv \widetilde{R}_1''.
$$

Then, by $j_1 > L(R_1)$ and (2.4), we have

$$
\left|z_1' - A_1^{-j_1} x_{R_1}\right| \lesssim b_1^{[L(R_1)-j_1]\zeta_{1,\,-}}. \tag{4.44}
$$

Let

$$
\mathcal{P}_{w_1}(z_1) \equiv \sum_{|\alpha_1|\le s_1} \frac{1}{\alpha_1!} \partial^{\alpha_1}\psi^{(1)}(w_1)(z_1 - w_1)^{\alpha_1}
$$

be the Taylor polynomial of $\psi^{(1)}$ about $w_1 \in \mathbb{R}^{n_1}$ of degree s_1. Since $\operatorname{supp} a_R \subset R''$ and a_R has vanishing moments up to order s_1 for the first variable, by Taylor's remainder theorem, we have

$$
\begin{aligned}
&|a_R * \psi_{j_1,\,j_2}(x-y)| \\
&\quad = \left| \int_{R_1''} \left(a_R *_2 \psi^{(2)}_{j_2}(x_2-y_2)\right)(z_1) b_1^{-j_1} \psi^{(1)}\left(A_1^{-j_1}(x_1-y_1-z_1)\right)\,dz_1 \right| \\
&\quad = \left| \int_{\widetilde{R}_1''} \left(a_R *_2 \psi^{(2)}(x_2-y_2)\right)\left(A_1^{j_1} z_1\right)\left[\psi^{(1)} - \mathcal{P}_{A_1^{-j_1}(x_1-y_1-x_{R_1})}\right]\left(A_1^{-j_1}(x_1-y_1) - z_1\right)\,dz_1 \right| \\
&\quad \lesssim \mathcal{D} \int_{\widetilde{R}_1''} \left|\left(a_R *_2 \psi^{(2)}_{j_2}(x_2-y_2)\right)\left(A_1^{j_1} z_1\right)\right| \left|\left(A_1^{-j_1} x_{R_1} - z_1\right)\right|^{s_1+1}\,dz_1,
\end{aligned}
$$

where

$$
\mathcal{D} \equiv \sup_{|\alpha_1|=s_1+1} \sup_{\xi_1 \in A_1^{-j_1} x_{R_1} + B^{(1)}_{L(R_1)-j_1}} \left|\partial^{\alpha_1}\psi^{(1)}\left(A_1^{-j_1}(x_1-y_1) - \xi_1\right)\right|.
$$

Since $A_1^{-j_1}x_1 - A_1^{-j_1}x_{R_1} \notin B^{(1)}_{L(R_1)+v_1[\gamma_1(R)-k_1]+2\sigma_1-j_1}$, by (2.2), we know

$$A_1^{j_1}x_1 - \xi_1 \notin B^{(1)}_{L(R_1)+v_1[\gamma_1(R)-k_1]+\sigma_1-j_1}.$$

Thus, if $j_1 \le L(R_1) + v_1[\gamma_1(R) - k_1]$, by $A_1^{-j_1}y_1 \in B_0^{(1)}$ and (2.2), we have $A_1^{-j_1}(x_1 - y_1) - \xi_1 \notin B^{(1)}_{L(R_1)+v_1[\gamma_1(R)-k_1]-j_1}$. This together with $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^n)$ and (2.3) yields that

$$\begin{aligned}\mathcal{D} &\lesssim \sup_{\xi_1 \in A_1^{-j_1}x_{R_1}+B^{(1)}_{L(R_1)-j_1}} \left[1 + \rho_1\big(A_1^{-j_1}(x_1 - y_1) - \xi_1\big)\right]^{-N_1} \\ &\lesssim \left[1 + b_1^{L(R_1)+v_1(\gamma_1(R)-k_1)-j_1}\right]^{-N_1}\end{aligned} \tag{4.45}$$

for any given $N_1 > 0$. The same estimate also holds trivially for $j_1 > L(R_1) + v_1[\gamma_1(R) - k_1]$ since $\mathcal{D} \lesssim 1$. Combining (4.44) through (4.45) yields

$$|a_R * \psi_{j_1,j_2}(x-y)| \lesssim I(j_1) \int_{R_1''} \left|\left(a_R *_2 \psi^{(2)}_{j_2}(x_2-y_2)\right)(z_1)\right| \, dz_1, \tag{4.46}$$

where $I(j_1) \equiv b_1^{-j_1}\left[1 + b_1^{L(R_1)+v_1(\gamma_1(R)-k_1)-j_1}\right]^{-N_1} b_1^{(s_1+1)[L(R_1)-j_1]\zeta_{1,-}}$. Observe also that by choosing $N_1 > s_1 + 2$, which implies that $N_1 > (s_1 + 1)\zeta_{1,-} + 1$, we have

$$\sum_{j_1 > L(R_1)} I(j_1)^2 \lesssim b_1^{-2v_1(\ell(\widehat{R}_1)-k_1)} b_1^{2[k_1-\gamma_1(R)]v_1(s_1+1)\zeta_{1,-}}. \tag{4.47}$$

Case II. $j_1 \le L(R_1)$. In this case, for $x_1 - x_{R_1} \notin B^{(1)}_{L(R_1)+v_1(\gamma_1(R)-k_1)+2\sigma_1}$ and $z_1 - x_{R_1} \in B^{(1)}_{L(R_1)}$, since $-v_1, -\gamma_1(R), k_1 \ge 0$ and $j_1 \le L(R_1)$, by (2.2), we obtain $A_1^{-j_1}(x_1 - z_1) \notin B^{(1)}_{L(R_1)+v_1(\gamma_1(R)-k_1)+\sigma_1-j_1}$. From this, $A_1^{-j_1}y_1 \in B_0^{(1)}$ and (2.2), we deduce $A_1^{-j_1}(x_1 - y_1 - z_1) \notin B^{(1)}_{L(R_1)+v_1(\gamma_1(R)-k_1)-j_1}$, and hence,

$$\rho_1\big(A_1^{-j_1}(x_1 - y_1 - z_1)\big) \ge b_1^{L(R_1)+v_1(\gamma_1(R)-k_1)-j_1}. \tag{4.48}$$

Choosing $N_1 \ge s_1 + 2$, we have $b_1 b_1^{(s_1+1)\zeta_{1,-}} \le b_1^{N_1}$. Since $\operatorname{supp} a_R \subset R''$ and $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^n)$, by (4.48), we have

$$\begin{aligned}&|a_R * \psi_{j_1,j_2}(x-y)| \\ &\quad= \left|\int_{R_1''} \left(a_R *_2 \psi^{(2)}_{j_2}(x_2-y_2)\right)(z_1) b_1^{-j_1}\psi_1^{(1)}\left(A_1^{-j_1}(x_1-y_1-z_1)\right) dz_1\right| \\ &\quad\lesssim b_1^{-j_1} b_1^{-N_1[L(R_1)+v_1(\gamma_1(R)-k_1)-j_1]} \int_{R_1''} \left|\left(a_R *_2 \psi^{(2)}_{j_2}(x_2-y_2)\right)(z_1)\right| dz_1 \\ &\quad\lesssim I(j_1) \int_{R_1''} \left|\left(a_R *_2 \psi^{(2)}_{j_2}(x_2-y_2)\right)(z_1)\right| dz_1,\end{aligned} \tag{4.49}$$

where $I(j_1) \equiv b_1^{-v_1(\ell(\widehat{R}_1)-k_1)} b_1^{-[L(R_1)+v_1(\gamma_1(R)-k_1)-j_1](s_1+1)\zeta_{1,-}}$. Observe also that we have

$$\sum_{j_1 \le L(R_1)} [I(j_1)]^2 \lesssim b_1^{-2v_1(\ell(\widehat{R}_1)-k_1)} b_1^{2[k_1-\gamma_1(R)]v_1(s_1+1)\zeta_{1,-}}. \tag{4.50}$$

Therefore, (4.43) follows by (4.46), (4.47), (4.49), (4.50), and Minkowski's inequality

$$\begin{aligned}
&\vec{S}_\psi(a_R)(x)\\
&\lesssim \left\{\left(\sum_{j_1\in\mathbb{Z}}[I(j_1)]^2\right)\sum_{j_2\in\mathbb{Z}} b_2^{-j_2}\int_{B^{(2)}_{j_2}}\left[\int_{R_1''}\left|\left(a_R *_2 \psi^{(2)}_{j_2}(x_2-y_2)\right)(z_1)\right|\,dz_1\right]^2 dy_2\right\}^{1/2}\\
&\lesssim b_1^{-v_1(\ell(\widehat{R}_1)-k_1)} b_1^{[k_1-\gamma_1(R)]v_1(s_1+1)\zeta_{1,\,-}}\int_{R_1''} S_{\psi^{(2)}}(a_R(z_1,\,\cdot))(x_2)\,dz_1.
\end{aligned}$$

Step 5. Estimate for K_1.

Since $s_1 \ge \lfloor (q_w/p-1)\zeta_{1,\,-}^{-1}\rfloor$, there exists $r\in(q_w,\,q]$ such that $p(s_1+1)\zeta_{1,\,-}+p-r>0$ and $w\in\mathcal{A}_r(\vec{A})$. Recall that $\mathcal{M}^{(1)}$ denotes the Hardy-Littlewood maximal operator on $\mathbb{R}^n$. Then, by (4.43), $\operatorname{supp} a_R\subset R''$, $w\in\mathcal{A}_r(\vec{A})$, the $L^r_w(\mathbb{R}^n)$-boundedness of $\mathcal{M}^{(1)}$, Theorem 3.2 and Hölder's inequality, we obtain

$$\begin{aligned}
&b_1^{-[k_1-\gamma_1(R)]v_1(s_1+1)\zeta_{1,\,-}}\left[\int_{\bar{R}^*_{1,\,k_1+1}\setminus\bar{R}^*_{1,\,k_1}}\int_{\bar{R}_2}[\vec{S}_\psi(a_R)(x)]^r w(x)\,dx\right]^{1/r}\\
&\quad\lesssim b_1^{-v_1(\ell(\widehat{R}_1)-k_1)}\left\{\int_{\bar{R}^*_{1,\,k_1+1}\setminus\bar{R}^*_{1,\,k_1}}\int_{\bar{R}_2}\left[\int_{R_1''} S_{\psi^{(2)}}(a_R(z_1,\,\cdot))(x_2)\,dz_1\right]^r w(x)\,dx\right\}^{1/r}\\
&\quad\lesssim \left\{\int_{\bar{R}^*_{1,\,k_1+1}\setminus\bar{R}^*_{1,\,k_1}}\int_{\bar{R}_2}\left[\mathcal{M}^{(1)}\left(S_{\psi^{(2)}}(a_R)(x_2)\right)(x_1)\right]^r w(x)\,dx\right\}^{1/r}\\
&\quad\lesssim \left\{\int_{\mathbb{R}^n}\int_{\bar{R}_2}\left(S_{\psi^{(2)}}(a_R(x_1,\,\cdot))(x_2)\right)^r w(x)\,dx\right\}^{1/r}\\
&\quad\lesssim \left\{\int_{R''}|a_R(x)|^r w(x)\,dx\right\}^{1/r}\\
&\quad\lesssim \|a_R\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}[w(R)]^{1/r-1/q}.
\end{aligned}$$

From this, $v_1<0$, $p(s_1+1)\zeta_{1,\,-}+p-r>0$ and $w\in\mathcal{A}_r(\vec{A})$, Hölder's inequality and Lemma 2.3(iv), it follows that

$$\begin{aligned}
\mathrm{K}_1 &= \sum_{k_1=0}^\infty\int_{\bar{R}^*_{1,\,k_1+1}\setminus\bar{R}^*_{1,\,k_1}}\int_{\bar{R}_2}[\vec{S}_\psi(a_R)(x)]^p w(x)\,dx\\
&\le \sum_{k_1=0}^\infty\left[w\left(\bar{R}^*_{1,\,k_1+1}\times\bar{R}_2\right)\right]^{1-p/r}\left[\int_{\bar{R}^*_{1,\,k_1+1}\setminus\bar{R}^*_{1,\,k_1}}\int_{\bar{R}_2}[\vec{S}_\psi(a_R)(x)]^r w(x)\,dx\right]^{p/r}\\
&\lesssim \sum_{k_1=0}^\infty b_1^{v_1(\gamma_1(R)-k_1)(r-p)}[w(R)]^{1-p/r} b_1^{p[k_1-\gamma_1(R)]v_1(s_1+1)\zeta_{1,\,-}}\|a_R\|^p_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}[w(R)]^{p/r-p/q}\\
&\lesssim [w(R)]^{1-p/q}\|a_R\|^p_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)} b_1^{v_1\gamma_1(R)[r-p-p(s_1+1)\zeta_{1,\,-}]}.
\end{aligned}\tag{4.51}$$

Step 6. Estimate for $\sum_{R\in m(\widetilde{\Omega})}(\mathrm{K}_1+\mathrm{K}_2)$.

Observe that the integral in K_2 is on the domain $(\bar{R}_1)^{\complement}\times(\bar{R}_2^*)^{\complement}$ and the integral in K_1 is on the domain $(\bar{R}_1)^{\complement}\times\bar{R}_2^*$. Thus, applying the ideas used in the estimate of K_1 on the first variable to both variables of K_2, we also have

$$\mathrm{K}_2\lesssim [w(R)]^{1-p/q}\|a_R\|^p_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)} b_1^{v_1\gamma_1(R)[r-p-p(s_1+1)\zeta_{1,\,-}]}.$$

Take $h_1(t)\equiv t^{a_1}$ for $t\in(0,1)$ and $a_1\equiv p+p(s_1+1)\zeta_{1,\,-}-r$. Then, by $a_1>0$, we obtain that $\sum_{j\ge0} jh_1(t^j)^{q/(q-p)}<\infty$. By Lemma 2.3(iv), we have

$$b_1^{v_1\gamma_1(R)[r-p-p(s_1+1)\zeta_{1,\,-}]}\sim h_1\left(\frac{|R_1|}{|\widehat{R}_1|}\right).$$

From this, Definition 4.2(II), Hölder's inequality, Lemma 4.9 and Proposition 2.10(ii) with $w \in \mathcal{A}_q(\vec{A})$, it follows that

$$\begin{aligned}\sum_{R\in m(\widetilde{\Omega})}(\mathrm{K}_1+\mathrm{K}_2) &\lesssim \sum_{R\in m(\widetilde{\Omega})}\|a_R\|^p_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}[w(R)]^{1-p/q}h_1\left(\frac{|R_1|}{|\widehat{R}_1|}\right)\\ &\lesssim \left\{\sum_{R\in m(\widetilde{\Omega})}\|a_R\|^q_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}\right\}^{p/q}\left\{\sum_{R\in m(\widetilde{\Omega})}w(R)h_1\left(\frac{|R_1|}{|\widehat{R}_1|}\right)^{\frac{q}{q-p}}\right\}^{1-p/q}\\ &\lesssim [w(\Omega)]^{p/q(1-q/p)}\big[w\big(\widetilde{\Omega}\big)\big]^{1-p/q}\\ &\lesssim 1.\end{aligned}$$

Step 7. Estimate for $\sum_{R\in m(\widetilde{\Omega})}(\mathrm{K}_3+\mathrm{K}_4)$.

To estimate K_3 and K_4, notice that if $I_i^{(1)}\times R_2 \in m\big(\widetilde{\Omega}\big)$ for $i=1,\,2$, then either ${\mathrm{I}_1}^{(1)}={\mathrm{I}_2}^{(2)}$ or ${\mathrm{I}_1}^{(1)}\cap{\mathrm{I}_2}^{(1)}=\emptyset$. Recall that for any $R_1\times R_2\in m\big(\widetilde{\Omega}\big)$, then $\widehat{R}_2=\widehat{R}_2(\widetilde{R}_1)$, where $\widetilde{R}_1\times R_2\in m_1\big(\widetilde{\Omega}'\big)$ and $\widetilde{R}_1\supset R_1$. Thus, we have

$$\begin{aligned}\sum_{R\in m(\widetilde{\Omega})}w(R)h_2\left(\frac{|R_2|}{|\widehat{R}_2|}\right)^{\frac{q}{q-p}} &\le \sum_{\widetilde{R}_1\times R_2\in m_1(\widetilde{\Omega}')}\ \sum_{\substack{R=R_1\times R_2\in m(\widetilde{\Omega})\\ R_1\subset\widetilde{R}_1}}w(R)h_2\left(\frac{|R_2|}{|\widehat{R}_2|}\right)^{\frac{q}{q-p}}\\ &\le \sum_{\widetilde{R}_1\times R_2\in m_1(\widetilde{\Omega}')}w(\widetilde{R}_1\times R_2)h_2\left(\frac{|R_2|}{|\widehat{R}_2|}\right)^{\frac{q}{q-p}},\end{aligned}$$

where $h_2(t)\equiv t^{a_2}$ for $t\in(0,\,1)$ and $a_2\equiv p+p(s_2+1)\zeta_{2,\,-}-r$. From this, Lemma 4.9 and an argument similar to the estimate for $\sum_{R\in m(\widetilde{\Omega})}(\mathrm{K}_1+\mathrm{K}_2)$, we deduce $\sum_{R\in m(\widetilde{\Omega})}(\mathrm{K}_3+\mathrm{K}_4)\lesssim 1$. This together with (4.41) implies (4.39) and thus completes the proof of Lemma 4.8. □

Lemma 4.10 *Let the assumptions be as in Theorem* 4.5. *Then* $H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})\subset\mathcal{S}'_{\infty,\,w}(\mathbb{R}^n\times\mathbb{R}^m)$.

To prove Lemma 4.10, for $i=1,\,2$ and $N_i\in\mathbb{Z}_+$, we let $\vec{N}\equiv(N_1,\,N_2)$. Set

$$\mathscr{S}_{N_i}(\mathbb{R}^{n_i})\ \equiv\ \left\{\varphi^{(i)}\in\mathcal{S}(\mathbb{R}^{n_i})\ :\ \big\|\varphi^{(i)}\big\|_{\mathscr{S}_{N_i}(\mathbb{R}^{n_i})}\equiv\sup_{x_i\in\mathbb{R}^{n_i}}\ \sup_{|\alpha_i|\le N_i}\big|\partial^{\alpha_i}\varphi^{(i)}(x_i)\big|[1+\rho_i(x_i)]^{N_i}\le 1\right\},$$

and denote by $\mathscr{S}_{\vec{N}}(\mathbb{R}^n\times\mathbb{R}^m)$ the collection of all φ such that $\varphi(x)=\varphi^{(1)}(x_1)\varphi^{(2)}(x_2)$ for all $x=(x_1,\,x_2)\in\mathbb{R}^n\times\mathbb{R}^m$ and all $\varphi^{(i)}\in\mathscr{S}_{N_i}(\mathbb{R}^{n_i})$.

For any $f\in\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^m)$ and $x\in\mathbb{R}^n\times\mathbb{R}^m$, we define the *grand maximal function* $\mathcal{M}_{\vec{N}}(f)(x)$ of f by

$$\mathcal{M}_{\vec{N}}(f)(x)\equiv\sup_{\varphi\in\mathscr{S}_{\vec{N}}(\mathbb{R}^n\times\mathbb{R}^m)}\ \sup_{k_1,\,k_2\in\mathbb{Z}}|f*\varphi_{k_1,\,k_2}(x)|.$$

Notice that if $N_1,\,N_2\ge 2$, then for all locally integrable functions f and $x\in\mathbb{R}^n\times\mathbb{R}^m$, $\mathcal{M}_{\vec{N}}(f)(x)\lesssim\mathcal{M}_s(f)(x)$. Thus if $w\in\mathcal{A}_p(\vec{A})$ with $p\in(1,\,\infty)$, then $\mathcal{M}_{\vec{N}}$ is bounded on $L^p_w(\mathbb{R}^n\times\mathbb{R}^m)$. Moreover, we have the following proposition.

Proposition 4.11 *Let the assumptions be as in Theorem* 4.5. *If* $N_i\ge s_i+2$ *for* $i=1,\,2$, *then* $\mathcal{M}_{\vec{N}}$ *is bounded from* $H^{p,\,q,\,\vec{s}}_w(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})$ *to* $L^p_w(\mathbb{R}^n\times\mathbb{R}^m)$.

Then Lemma 4.10 follows from Proposition 4.11.

P r o o f of L e m m a 4.10. Fix $\varphi\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^m)$. Let $\varphi_y(x)=\varphi(x+y)$ for all $x\in\mathbb{R}^n\times\mathbb{R}^m$ and $y\in B_0^{(1)}\times B_0^{(2)}$. Notice that there exists a positive constant C, depending on φ, such that $C\varphi_y\in\mathscr{S}_{\vec{N}}(\mathbb{R}^n\times\mathbb{R}^m)$

for all $y \in B_0^{(1)} \times B_0^{(2)}$. If a is $(p,\,q,\,\vec{s})_w$-atom, then for $j_1,\,j_2 \in \mathbb{N}$ and $w \in \mathcal{A}_q(\vec{A})$, by Proposition 4.11 and Proposition 2.10(i), we have

$$\begin{aligned}|a * \varphi_{j_1,\,j_2}(x)|^p &\lesssim \inf_{y \in B_{j_1}^{(1)} \times B_{j_2}^{(2)}} [\mathcal{M}_{\vec{N}}(a)(x-y)]^p \\ &\lesssim \frac{1}{w(x + B_{j_1}^{(1)} \times B_{j_2}^{(2)})} \int_{x + B_{j_1}^{(1)} \times B_{j_2}^{(2)}} [\mathcal{M}_{\vec{N}}(a)(y)]^p w(y)\,dy \\ &\le C_{x,\,w} b_1^{\frac{-j_1}{q}} b_2^{-\frac{j_2}{q}},\end{aligned}$$

where $C_{x,\,w}$ is a positive constant independent of j_1 and j_2, and the atom a. If $f = \sum_{k \in \mathbb{Z}} \lambda_k a_k$ in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$, where a_k is $(p,\,q,\,\vec{s})_w$-atom and $\sum_{k \in \mathbb{Z}} |\lambda_k|^p < \infty$, then

$$|f * \varphi_{j_1,\,j_2}(x)|^p \le C_{x,\,w} b_1^{\frac{-j_1}{q}} b_2^{-\frac{j_2}{q}} \sum_{k \in \mathbb{Z}} |\lambda_k|^p \longrightarrow 0$$

as $j_1,\,j_2 \to \infty$, which completes the proof of Lemma 4.10. $\square$

Finally we prove Proposition 4.11.

P r o o f o f P r o p o s i t i o n 4.11. The proof of Proposition 4.11 is similar to that of Lemma 4.8. By a reason similar to that used in Step 1 of the proof of Lemma 4.8, it suffices to show that $\|\mathcal{M}_{\vec{N}}(a)\|_{L_w^p(\mathbb{R}^n \times \mathbb{R}^m)} \lesssim 1$ for all $(p,\,q,\,\vec{s})_w$-atoms a.

Assuming that $a = \sum_{R \in m(\widetilde{\Omega})} a_R$ is a $(p,\ q,\ \vec{s})_w$-atom associated with open set Ω with $w(\Omega) < \infty$ as in Definition 4.2. Let all the notation be as in the proof of Lemma 4.8. Similarly to the proof of (4.41), using the $L_w^q(\mathbb{R}^n \times \mathbb{R}^m)$-boundedness of $\mathcal{M}_{\vec{N}}$ (see Proposition 2.10(ii)), we have

$$\int_{\bigcup_{R \in m(\widetilde{\Omega})} \bar{R}^*} [\mathcal{M}_{\vec{N}}(a)(x)]^p w(x)\,dx \lesssim 1.$$

And similarly to the proof of (4.42), we write

$$\begin{aligned}&\iint_{\left(\bigcup_{R \in m(\widetilde{\Omega})} \bar{R}^*\right)^{\complement}} [\mathcal{M}_{\vec{N}}(a)(x)]^p w(x)\,dx \\ &\le \sum_{R \in m(\widetilde{\Omega})} \left[\iint_{(\bar{R}_1^*)^{\complement} \times \bar{R}_2} + \iint_{(\bar{R}_1^*)^{\complement} \times (\bar{R}_2)^{\complement}} + \iint_{\bar{R}_1 \times (\bar{R}_2^*)^{\complement}} + \iint_{(\bar{R}_1)^{\complement} \times (\bar{R}_2^*)^{\complement}}\right] [\mathcal{M}_{\vec{N}}(a_R)(x)]^p w(x)\,dx \\ &\equiv \sum_{R \in m(\widetilde{\Omega})} (\mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + \mathrm{J}_4).\end{aligned}$$

For any $\psi \in \mathscr{S}_{\vec{N}}(\mathbb{R}^n \times \mathbb{R}^m)$, $x_1 \in \bar{R}^*_{1,\,k_1+1} \setminus \bar{R}^*_{1,\,k_1}$ with $k_1 \in \mathbb{Z}_+$, $x_2 \in \bar{R}_2$ and $y \in B_{j_1}^{(1)} \times B_{j_2}^{(2)}$ with $j_1,\,j_2 \in \mathbb{Z}$, similarly to the proofs of Case I and Case II in the proof of Lemma 4.8, we have that

(I) if $j_1 > L(R_1)$, then $|a_R * \psi_{j_1,\,j_2}(x-y)|$ has the same upper estimate as in (4.46) and

$$\begin{aligned}&\sup_{j_1 > L(R_1)} b_1^{-j_1} \left(1 + b_1^{L(R_1) + v_1(\gamma_1(R) - k_1) - j_1}\right)^{-N_1} b_1^{(s_1+1)[L(R_1) - j_1]\zeta_{1,\,-}} \\ &\lesssim b_1^{-v_1(\ell(\widehat{R}_1) - k_1)} b_1^{[k_1 - \gamma_1(R)]v_1(s_1+1)\zeta_{1,\,-}}.\end{aligned}$$

Here, unlike the calculation of (4.47), we only need $N_1 \ge s_1 + 2$;

(II) if $j_1 \le L(R_1)$, then $|a_R * \psi_{j_1,\,j_2}|$ has the same upper estimate as in (4.49) and

$$\begin{aligned}&\sup_{j_1 \le L(R_1)} b_1^{-v_1(\ell(\widehat{R}_1) - k_1)} b_1^{-[L(R_1) + v_1(\gamma_1(R) - k_1) - j_1](s_1+1)\zeta_{1,\,-}} \\ &\lesssim b_1^{-v_1(\ell(\widehat{R}_1) - k_1)} b_1^{[k_1 - \gamma_1(R)]v_1(s_1+1)\zeta_{1,\,-}}.\end{aligned}$$

Then similarly to the estimate of (4.43), by $N_1 \geq s_1 + 2$, we have

$$\mathcal{M}_{\vec{N}}(a_R)(x) \lesssim b_1^{-v_1(\ell(\widehat{R}_1)-k_1)} b_1^{[k_1-\gamma_1(R)]v_1(s_1+1)\zeta_{1,-}} \int_{R_1''} \mathcal{M}_{N_2}^{(2)}(a_R(z_1, \cdot))(x_2)\, dz_1,$$

where

$$\mathcal{M}_{N_2}^{(2)}(g)(x_2) = \sup_{\psi^{(2)} \in \mathscr{S}_{N_2}(\mathbb{R}^{n_2})} \sup_{k_2 \in \mathbb{Z}} \left| \left(\psi_{k_2}^{(2)} * g \right)(x_2) \right|.$$

Observing that for $s \in (1, \infty)$ and $\nu \in \mathcal{A}_s(A_2)$, $\mathcal{M}_{N_2}^{(2)}$ is bounded on $L_\nu^s(\mathbb{R}^n)$. Then similarly to the estimate of (4.51), we obtain

$$\mathrm{J}_1 \lesssim [w(R)]^{1-p/q} \|a_R\|_{L_w^q(\mathbb{R}^n \times \mathbb{R}^m)}^p b_1^{v_1\gamma_1(R)[r-p-p(s_1+1)\zeta_{1,-}]}.$$

Also, similarly to the proof in Step 6 of the proof of Lemma 4.8, we have

$$\mathrm{J}_2 \lesssim [w(R)]^{1-p/q} \|a_R\|_{L_w^q(\mathbb{R}^n \times \mathbb{R}^m)}^p b_1^{v_1\gamma_1(R)[r-p-p(s_1+1)\zeta_{1,-}]},$$

and $\sum_{R \in m(\widetilde{\Omega})}(\mathrm{J}_1 + \mathrm{J}_2) \lesssim 1$. Finally, similarly to the proof in Step 7 of the proof of Lemma 4.8, we obtain $\sum_{R \in m(\widetilde{\Omega})}(\mathrm{J}_3 + \mathrm{J}_4) \lesssim 1$, which completes the proof of Proposition 4.11. □

Remark 4.12 Let $w \in \mathcal{A}_\infty(\vec{A})$ and $(p, q, s)_w$ be an admissible triplet. By Proposition 4.11 and Theorem 4.11, for $N_i \geq s_i + 2$ with $i = 1, 2$, we obtain the boundedness of $\mathcal{M}_{\vec{N}}$ from $H_w^p(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ to $L_w^p(\mathbb{R}^n \times \mathbb{R}^m)$.

5 Weighted finite atomic Hardy spaces

In this section we establish finite atomic decomposition of the anisotropic product Hardy spaces.

Definition 5.1 Let $w \in \mathcal{A}_\infty(\vec{A})$, q_w be as in (2.7) and $(p, q, \vec{s})_w$ be an admissible triplet as in Definition 4.2. Let a be a $(p, q, \vec{s})_w$-atom associated with an open set Ω. We say a is a $(p, q, \vec{s})_w^*$*-atom* if $a \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, Ω is bounded, and there exist only finitely many $R \in m(\widetilde{\Omega})$ such that $a_R \neq 0$.

The *weighted finite Hardy space* $H_{w,\,\mathrm{fin}}^{p,\,q,\,\vec{s}}(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ is defined to be the space of all functions $f = \sum_{j=1}^k \lambda_j a_j$, where $k \in \mathbb{N}$, $\{a_j\}_{j=1}^k$ are $(p, q, \vec{s})_w^*$-atoms and $\{\lambda_j\}_{j=1}^k \subset \mathbb{C}$. The norm of f is defined by $\|f\|_{H_{w,\,\mathrm{fin}}^{p,\,q,\,\vec{s}}(\mathbb{R}^n \times \mathbb{R}^m;\,\vec{A})} \equiv \inf\left\{\left(\sum_{j=1}^k |\lambda_j|^p\right)^{1/p}\right\}$, where the infimum is taken over all the above finite decompositions of f.

The main result of this section is as follows.

Theorem 5.2 *Let* $w \in \mathcal{A}_\infty(\vec{A})$*,* q_w *be as in* (2.7)*,* $(p, q, \vec{s})_w$ *be an admissible triplet as in Definition* 4.2*. Then,*

(i) $H_{w,\,\mathrm{fin}}^{p,\,q,\,\vec{s}}(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$ *is dense in* $H_w^p(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$.

(ii) *Moreover, if* $\vec{s} \equiv (s_1, s_2)$ *satisfies*

$$s_1 > [(q_w/p) - 1 + (q_w/p)(v_2/v_1)(\log_{b_1} b_2)]\zeta_{1,-}^{-1} - 1 \textit{ and} \tag{5.1}$$

$$s_2 > [(q_w/p) - 1 + (q_w/p)(v_1/v_2)(\log_{b_2} b_1)]\zeta_{2,-}^{-1} - 1, \tag{5.2}$$

then $\|f\|_{H_{w,\,\mathrm{fin}}^{p,\,q,\,\vec{s}}(\mathbb{R}^n \times \mathbb{R}^m;\,\vec{A})} \sim \|f\|_{H_w^p(\mathbb{R}^n \times \mathbb{R}^m;\,\vec{A})}$ *for all* $f \in H_{w,\,\mathrm{fin}}^{p,\,q,\,\vec{s}}(\mathbb{R}^n \times \mathbb{R}^m; \vec{A})$.

Remark 5.3 Notice that comparing with the non-product case (see [6,33,40]), we need additional assumptions (5.1) and (5.2) on vanishing moments of atoms in Theorem 5.2(ii). This is due to the fact that the product Hardy space is not just a product of one-parameter Hardy spaces.

To prove Theorem 5.2, we need the following auxiliary lemma, which generalizes Lemma 2 and Lemma 4 in Appendix (III) of [29]. Lemma 5.4 below can be also deduced with some effort from [5, Lemma 6.3].

Lemma 5.4 *Let A be a dilation on $\mathbb{R}^n$, $s \in \mathbb{Z}_+$ and $M \in [0, \infty)$.*

(i) If $g \in \mathcal{S}(\mathbb{R}^n)$ and $\psi \in \mathcal{S}_s(\mathbb{R}^n)$, then there exists a positive constant C such that for all $k \in \mathbb{Z} \setminus \mathbb{N}$ and all $x \in \mathbb{R}^n$, $|(g * \psi_k)(x)| \le C b^{k(s+1)\zeta_-}[1 + \rho(x)]^{-M}$.

(ii) If $g \in \mathcal{S}_s(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$, then there exists a positive constant C such that for all $k \in \mathbb{Z}_+$ and all $x \in \mathbb{R}^n$, $|(g * \psi_k)(x)| \le C b^{-k[(s+1)\zeta_- + 1]}[1 + b^{-k}\rho(x)]^{-M}$.

Proof. To prove (i), let $k \in \mathbb{Z} \setminus \mathbb{N}$. Since $\psi \in \mathcal{S}_s(\mathbb{R}^n)$, for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned}(g * \psi_k)(x) &= \left[\int_{\rho(y) \le \rho(x)/(2b^\sigma)} + \int_{\rho(y) > \rho(x)/(2b^\sigma)}\right] \psi_k(y) \left[g(x-y) - \sum_{|\alpha| \le s} \frac{\partial^\alpha g(x)}{\alpha!}(-y)^\alpha\right] dy \\ &\equiv \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

For I_1, since $g \in \mathcal{S}(\mathbb{R}^n)$, by Taylor's remainder theorem, we have

$$\begin{aligned}\left|g(x-y) - \sum_{|\alpha| \le s} \frac{1}{\alpha!}\partial^\alpha g(x)(-y)^\alpha\right| &\lesssim |y|^{s+1} \sup_{|\alpha| = s+1,\ \rho(z) \le \rho(x)/(2b^\sigma)} |\partial^\alpha g(x-z)| \\ &\lesssim |y|^{s+1} \sup_{\rho(z) \le \rho(x)/(2b^\sigma)} [1 + \rho(x-z)]^{-M}.\end{aligned}$$

This together with $\rho(x-z) \ge \rho(x)/b^\sigma - \rho(z) \ge \rho(x)/(2b^\sigma)$, (2.3), (2.4), $k \le 0$ and $\psi \in \mathcal{S}_s(\mathbb{R}^n)$, yields

$$\begin{aligned}|\mathrm{I}_1| &\lesssim [1 + \rho(x)]^{-M} \left\{\int_{\rho(y) \le 1} \rho(y)^{(s+1)\zeta_-} |\psi_k(y)|\, dy + \int_{\rho(y) > 1} \rho(y)^{(s+1)\zeta_+} |\psi_k(y)|\, dy\right\} \\ &\lesssim b^{k(s+1)\zeta_-}[1 + \rho(x)]^{-M} \left\{\int_{\mathbb{R}^n} \left[\rho(y)^{(s+1)\zeta_-} + \rho(y)^{(s+1)\zeta_+}\right] |\psi(y)|\, dy\right\} \\ &\lesssim b^{k(s+1)\zeta_-}[1 + \rho(x)]^{-M}.\end{aligned}$$

For I_2, if $\rho(x) > 1$, since $g \in \mathcal{S}(\mathbb{R}^n)$ and $k \le 0$, by Taylor's remainder theorem, (2.3) and (2.4), we have

$$\begin{aligned}|\mathrm{I}_2| &\lesssim \int_{\rho(y) > \rho(x)/(2b^\sigma)} |y|^{s+1} \sup_{|\alpha| = s+1} \|\partial^\alpha g\|_{L^\infty(\mathbb{R}^n)} |\psi_k(y)|\, dy \\ &\lesssim \int_{\rho(x)/(2b^\sigma) < \rho(y) < 1} \rho(y)^{(s+1)\zeta_-} |\psi_k(y)|\, dy + \int_{\rho(x)/(2b^\sigma) < \rho(y),\ \rho(y) > 1} \rho(y)^{(s+1)\zeta_+} |\psi_k(y)|\, dy \\ &\lesssim b^{k(s+1)\zeta_-} \int_{\rho(y) > b^{-k}\rho(x)/(2b^\sigma)} \left[\rho(y)^{(s+1)\zeta_-} + \rho(y)^{(s+1)\zeta_+}\right] |\psi(y)|\, dy \\ &\lesssim b^{k(s+1)\zeta_-} \rho(x)^{-M}.\end{aligned}$$

By this and $\rho(x) > 1$, we have $|\mathrm{I}_2| \lesssim b^{k(s+1)\zeta_-}[1 + \rho(x)]^{-M}$. For $\rho(x) \le 1$, similarly to the above estimate, we obtain $|\mathrm{I}_2| \lesssim b^{k(s+1)\zeta_-} \lesssim b^{k(s+1)\zeta_-}[1 + \rho(x)]^{-M}$. Combining above estimates for I_1 and I_2 completes the proof Lemma 5.4(i).

To prove (ii), we observe the identity $g * \psi_k = (g_{-k} * \psi)_k$. Thus, if $k \in \mathbb{Z}_+$, then (i) with the roles of g and ψ exchanged yields

$$|g * \psi_k(x)| = |(g_{-k} * \psi)_k(x)| \lesssim b^{-k(s+1)\zeta_-} \left[1 + \rho\left(A^{-k}x\right)\right]^{-M} b^{-k},$$

which completes the proof of Lemma 5.4. □

By Lemma 5.4 and an argument similar to the proof of [11, Lemma 2.2], we have the following estimates. We leave the details to the reader.

Lemma 5.5 *For $i = 1,\ 2$, let A_i be a dilation on $\mathbb{R}^{n_i}$, let $s_i \in \mathbb{Z}_+$ and let $M_i \in [0,\ \infty)$. Suppose that $f \in \mathcal{S}_{s_1,\, s_2}(\mathbb{R}^n \times \mathbb{R}^m)$, $\varphi^{(1)} \in \mathcal{S}_{s_1}(\mathbb{R}^n)$, $\varphi^{(2)} \in \mathcal{S}_{s_2}(\mathbb{R}^n)$ and $\varphi_{t_1,\, t_2}(x) \equiv \varphi^{(1)}_{t_1}(x_1)\varphi^{(2)}_{t_2}(x_2)$ for all $t_1,\ t_2 \in \mathbb{Z}$ and $x = (x_1,\ x_2) \in \mathbb{R}^n \times \mathbb{R}^m$. Then there exists $C > 0$ such that $|(\varphi_{t_1,\, t_2} * f)(x)|/C$ is bounded for all $x \in \mathbb{R}^n \times \mathbb{R}^m$ by:*

$$
\begin{aligned}
&\prod_{i=1}^{2} b_i^{t_i(s_i+1)\zeta_{i,-}}[1+\rho_i(x_i)]^{-M_i} \quad \text{if} \quad t_1,\ t_2 \le 0,\\
&b_1^{t_1(s_1+1)\zeta_{1,-}} b_2^{-t_2[(s_2+1)\zeta_{2,-}+1]}[1+\rho_1(x_1)]^{-M_1}\left[1+b_2^{-t_2}\rho_2(x_2)\right]^{-M_2} \quad \text{if} \quad t_1 \le 0,\ \ t_2 \ge 0,\\
&b_1^{-t_1[(s_1+1)\zeta_{1,-}+1]} b_2^{t_2(s_2+1)\zeta_{2,-}}\left[1+b_1^{-t_1}\rho_1(x_1)\right]^{-M_1}[1+\rho_2(x_2)]^{-M_2} \quad \text{if} \quad t_1 \ge 0,\ \ t_2 \le 0,\\
&\prod_{i=1}^{2}\left[1+b_i^{-t_i}\rho_i(x_i)\right]^{-M_i} b_i^{-t_i[(s_i+1)\zeta_{i,-}+1]} \quad \text{if} \quad t_1,\ \ t_2 \ge 0.
\end{aligned}
$$

We now turn to the proof of Theorem 5.2.

Proof of Theorem 5.2. We first show (i). Let the notation be as in the proof of the Lemma 4.6. For $f \in H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A})$, by (4.8), we have

$$
f = \sum_{k\in\mathbb{Z}} \lambda_k a_k = \sum_{k\in\mathbb{Z}} \lambda_k \sum_{P\in m(\widetilde{\Omega}_k)} a_P = \sum_{k\in\mathbb{Z}} \lambda_k \sum_{P\in m(\widetilde{\Omega}_k)} \sum_{R\in\mathcal{R}_k,\ R^*=P} \lambda_k^{-1} e_R \tag{5.3}
$$

in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$. For $N,\ L \in \mathbb{N}$ and $k \in \mathbb{Z}$, let $\mathcal{R}_{k,L} \equiv \{R \in \mathcal{R}_k :\ |\ell(R_i)| \le L,\ i = 1,\ 2\}$ and $f_{N,\, L} \equiv \sum_{|k|\le N} \lambda_k a_{k,\, L}$, where $a_{k,\, L} \equiv \sum_{P\in m(\widetilde{\Omega}_k)} a_{P,\, L}$, $a_{P,\, L} \equiv \sum_{R\in\mathcal{R}_{k,\, L},\ R^*=P} \lambda_k^{-1} e_R$ if $\{R \in \mathcal{R}_{k,\, L} :\ R^* = P\} \neq \emptyset$ and otherwise $a_{P,\, L} = 0$.

On the other hand, notice that Ω_k is a bounded set. In fact, let $M_i > 0$ satisfying that $(s_i+1)\zeta_{i,-} - M_i > 0$. Observing that $1 + \rho_i(x_i) \le b_i^{t_i} + \rho_i(x_i) \sim b_i^{t_i} + \rho_i(y_i)$ for $y_i \in x_i + B^{(i)}_{t_i}$ and $t_i \in \mathbb{Z}_+$, by Lemma 5.5, we have

$$
\begin{aligned}
&[\vec{S}_\psi(f)(x_1,\ x_2)]^2\\
&\quad\lesssim \Bigg\{\int_{-\infty}^{0}\int_{-\infty}^{0}\int_{y_1\in x_1+B^{(1)}_{t_1}}\int_{y_2\in x_2+B^{(2)}_{t_2}} \frac{b_1^{2t_1(s_1+1)\zeta_{1,-}} b_2^{2t_2(s_2+1)\zeta_{2,-}}}{[1+\rho_1(y_1)]^{2M_1}[1+|\rho_2(y_2)]^{2M_2}}\\
&\qquad + \int_{0}^{\infty}\int_{-\infty}^{0}\int_{y_1\in x_1+B^{(1)}_{t_1}}\int_{y_2\in x_2+B^{(2)}_{t_2}} \frac{b_1^{-2t_1(s_1+1)\zeta_{1,-}-2t_1} b_2^{2t_2(s_2+1)\zeta_{2,-}}}{[1+b_1^{-t_1}\rho_1(y_1)]^{2M_1}[1+|\rho_2(y_2)]^{2M_2}}\\
&\qquad + \int_{-\infty}^{0}\int_{0}^{\infty}\int_{y_1\in x_1+B^{(1)}_{t_1}}\int_{y_2\in x_2+B^{(2)}_{t_2}} \frac{b_1^{2t_1(s_1+1)\zeta_{1,-}} b_2^{-2t_2(s_2+1)\zeta_{2,-}-2t_2}}{[1+\rho_1(y_1)]^{2M_1}[1+b_2^{-t_2}\rho_2(y_2)]^{2M_2}}\\
&\qquad + \int_{0}^{\infty}\int_{0}^{\infty}\int_{y_1\in x_1+B^{(1)}_{t_1}}\int_{y_2\in x_2+B^{(2)}_{t_2}} \frac{b_1^{-2t_1(s_1+1)\zeta_{1,-}-2t_1} b_2^{-2t_2(s_2+1)\zeta_{2,-}-2t_2}}{[1+b_1^{-t_1}\rho_1(y_1)]^{2M_1}[1+b_2^{-t_2}\rho_2(y_2)]^{2M_2}}\Bigg\}\, dy_1\, dy_2\, \frac{d\sigma(t_1)}{b_1^{t_1}}\, \frac{d\sigma(t_2)}{b_2^{t_2}}\\
&\quad\lesssim [1+\rho_1(x_1)]^{-2M_1}[1+\rho_2(x_2)]^{-2M_2}.
\end{aligned}
$$

Thus for any $k \in \mathbb{Z}$, Ω_k is a bounded set in $\mathbb{R}^n \times \mathbb{R}^m$ and so is $\widetilde{\Omega}_k$.

Therefore, for any $N \in \mathbb{N}$ and $k = -N,\ \ldots,\ N$, $a_{k,\, L}$ is a $(p,\ q,\ \vec{s})^*_w$-atom associated with the bounded open set Ω_k and thus $f_{N,\, L} \in H^{p,\, q,\, \vec{s}}_{w,\, \mathrm{fin}}(\mathbb{R}^n \times \mathbb{R}^m;\ \vec{A})$.

Observe that for any $\epsilon > 0$, there exists an integer $N_\epsilon > 0$ such that $\left(\sum_{|k|>N_\epsilon} |\lambda_k|^p\right)^{1/p} < \epsilon$. Moreover, for $k = -N_\epsilon,\ \ldots,\ N_\epsilon$, similarly to the estimate for (4.11), we have

$$
\|a_k - a_{k,\, L}\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)} \lesssim \lambda_k^{-1} \left\| \left(\sum_{P\in m(\widetilde{\Omega}_k)} \sum_{\substack{R\notin\mathcal{R}_{k,\, L}\\ R^*=P}} c_R^2 \chi_{R\cap(\Omega_k\setminus\Omega_{k+1})} \right)^{1/2} \right\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)},
$$

which together with (4.12) implies that $\|a_k - a_{k,\,L}\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)} \to 0$ as $L\to\infty$. Similarly to the estimate of (4.14), we also have $\sum_{P\in m(\widetilde{\Omega}_k)} \|a_P - a_{P,\,L}\|^q_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)} \to 0$ as $L\to\infty$. Thus there exists an integer $L_\epsilon > 0$ such that $\frac{(2(N_\epsilon+1))^{1/p}}{\epsilon}(a_k - a_{k,\,L_\epsilon})$ is a $(p,\,q,\,s)_w$-atom. Therefore,

$$
\begin{aligned}
&\|f - f_{N_\epsilon,\,L_\epsilon}\|_{H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}\\
&\quad\lesssim \left\{\sum_{|k|>N_\epsilon} |\lambda_k|^p\right\}^{1/p} + \left\|\sum_{|k|\le N_\epsilon} \lambda_k (a_k - a_{k,\,L_\epsilon})\right\|_{H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}\\
&\quad\lesssim \epsilon + \|f\|_{H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}\left[\sum_{|k|\le N_\epsilon} \|a_k - a_{k,\,L_\epsilon}\|^p_{H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}\right]^{1/p}\\
&\quad\lesssim \epsilon\left(1 + \|f\|_{H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}\right),
\end{aligned}
$$

which gives (i).

Now we prove (ii). From Definition 5.1 and Theorem 4.5, we automatically deduce $\|f\|_{H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})} \lesssim \|f\|_{H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}$. Thus, to show (ii), it suffices to prove that for all $f\in H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})$,

$$
\|f\|_{H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})} \lesssim \|f\|_{H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}.
$$

Let $f\in H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})$. Since $f\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^m)$, by Lemma 2.15, we know that (5.3) also holds in $L^q(\mathbb{R}^n\times\mathbb{R}^m)$ and hence, pointwise. Assume that $\operatorname{supp} f\subset B^{(1)}_{h_1}\times B^{(2)}_{h_2}$ for certain $h_1,\ h_2\in\mathbb{Z}$. By homogeneity, we further assume that $\|f\|_{H^p_w(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})} \equiv \|\vec{S}_\psi(f)\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} = 1$, where ψ is as in Proposition 2.16.

Let $i=1,\,2$. For certain given $N\in\mathbb{N}$ which will be determined later, set $D_i\equiv -v_iN + u_i + \sigma_i$; then we choose certain $M_0\in\mathbb{N}$, depending on N, such that $d_i\equiv v_i(M_0N-1)+u_i+\sigma_i$ satisfies $d_i(s_i+1)\zeta_{i,\,-}\le -D_i[1+(s_i+1)\zeta_{i,\,-}]$. We first assume that N is large enough such that $D_i > h_i$. Then, by the definition of R_+ in (4.1), we know that there exist finite dyadic rectangles R, whose collection is denoted by $\mathcal{R}^N$, such that

$$
R_+\cap\left\{B^{(1)}_{D_1+\sigma_1}\times B^{(2)}_{D_2+\sigma_2}\times[d_1,\ D_1)\times[d_2,\ D_2)\right\}\ne\emptyset. \tag{5.4}
$$

From now on, we adopt the notation in the proof of Lemma 4.6 again. Observe that for each $R\in\mathcal{R}^N$, there exists a unique $k\in\mathbb{Z}$ such that $R\in\mathcal{R}_k$, and we denote by J_N the set of all such k's.

Let $\widetilde{a}_{P,\,N}\equiv\lambda_k^{-1}\sum_{R\in\mathcal{R}_k\cap\mathcal{R}^N,\,R^*=P} e_R$ if $\{R\in\mathcal{R}_k\cap\mathcal{R}^N:\ R^*=P\}\ne\emptyset$ and otherwise $\widetilde{a}_{P,\,N}=0$. Let $\widetilde{a}_{k,N}\equiv\sum_{P\in m(\widetilde{\Omega}_k)}\widetilde{a}_{P,\,N}$. Then similarly to the proof of Lemma 4.6, we know that $\widetilde{a}_{k,\,N}$ is a $(p,\,q,\,\vec{s})_w$-atom which is a finite linear combination of particles $\widetilde{a}_{P,\,N}$. Obviously, $\widetilde{a}_{P,\,N}$ is also a finite linear combination of e_R and hence is smooth. This further implies that $\widetilde{a}_{k,\,N}$ is a $(p,\,q,\,\vec{s})^*_w$-atom. Let $f_N\equiv\sum_{k\in J_N}\lambda_k\widetilde{a}_{k,\,N}$ and $g_N\equiv f-f_N$. Then $f_N\in H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n\times\mathbb{R}^m;\ \vec{A})$ and $\|f_N\|^p_{H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}\le\sum_{k\in\mathbb{Z}}|\lambda_k|^p\lesssim 1$.

So it remains to prove $g_N\in H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n\times\mathbb{R}^m;\ \vec{A})$ and $\|g_N\|^p_{H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}\lesssim 1$. In fact, we will prove that there exists a positive constant $\widetilde{C}$, independent of f and N, such that $\widetilde{C}g_N$ is a $(p,\,q,\,s)^*_w$-atom, which implies $\|g_N\|^p_{H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n\times\mathbb{R}^m;\,\vec{A})}\lesssim 1$.

Obviously, $g_N\in\mathcal{S}_{s_1,\,s_2}(\mathbb{R}^n\times\mathbb{R}^m)$. Noticing that if $R\in\mathcal{R}^N$, then by (5.4), $\ell(R_i)\in(-N,\ M_0N)$. By this, (2.1) and Lemma 2.3(iv), we further obtain

$$
R_i'\equiv x_{R_i}+B^{(i)}_{v_i(\ell(R_i)-1)+u_i+2\sigma_i}\subset x_{R_i}+B^{(i)}_{D_i+\sigma_i},
$$

which together with $R\subset R'$ and (5.4) yields that $(x_{R_i}+B^{(i)}_{D_i+\sigma_i})\cap B^{(i)}_{D_i+\sigma_i}\ne\emptyset$. Then by (4.9) and (2.1), we obtain

$$
\operatorname{supp} f_N\subset\bigcup_{R\in\mathcal{R}^N} R'\subset\left(B^{(1)}_{D_1+3\sigma_1}\times B^{(2)}_{D_2+3\sigma_2}\right).
$$

From this, $\operatorname{supp} f \subset B^{(1)}_{h_1} \times B^{(2)}_{h_2}$ and $D_i > h_i$, it follows that $\operatorname{supp} g_N \subset \left(B^{(1)}_{D_1+3\sigma_1} \times B^{(2)}_{D_2+3\sigma_2}\right)$.

We now claim that there exists an $N_0 \in \mathbb{N}$, depending on f, w, m, n, A_1 and A_2, such that for all $N \geq N_0$,

$$\|g_N\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)} \leq \left[w\left(B^{(1)}_{D_1} \times B^{(2)}_{D_2}\right)\right]^{1/q-1/p}. \tag{5.5}$$

Now we prove that there exists a positive constant $\widetilde{C}$, independent of f and N, such that $\widetilde{C}g_N$ is a $(p,\, q,\, \vec{s})^*_w$-atom.

In fact, by Lemma 2.3(i), there exist certain $P_i \in \mathcal{Q}^{(i)}$ and $x_{i,\,0} \in \mathbb{R}^{n_i}$ satisfying that $x_{i,\,0} \in P_i \cap B^{(i)}_{D_i+\sigma_i}$ and $v_i\ell(P_i) + u_i < D_i + \sigma_i \leq v_i[\ell(P_i) - 1] + u_i$. For this P, let P'' be as in Definition 4.2(I). Then $P \equiv P_1 \times P_2 \subset B^{(1)}_{D_1+3\sigma_1} \times B^{(2)}_{D_2+3\sigma_2} \subset P''$. To see this, for any $x_i \in P_i$, since $x_{i,\,0} \in P_i \cap B^{(i)}_{D_i+\sigma_i}$ and $v_i\ell(P_i) + u_i < D_i + \sigma_i$, using Lemma 2.3(iv) and (2.1), we obtain

$$\begin{aligned} x_i \in x_{P_i} + B^{(i)}_{v_i\ell(P_i)+u_i} &\subset x_{i,\,0} + B^{(i)}_{v_i\ell(P_i)+u_i} + B^{(i)}_{v_i\ell(P_i)+u_i} \\ &\subset B^{(i)}_{D_i+\sigma_i} + B^{(i)}_{D_i+\sigma_i} + B^{(i)}_{D_i+\sigma_i} \subset B^{(i)}_{D_i+3\sigma_i}, \end{aligned}$$

which implies that $P \subset B^{(1)}_{D_1+3\sigma_1} \times B^{(2)}_{D_2+3\sigma_2}$. For any $x_i \in B^{(i)}_{D_i+3\sigma_i}$, since $D_i + \sigma_i \leq v_i[\ell(P_i) - 1] + u_i$ and $x_{i,\,0} \in P_i \cap B^{(i)}_{D_i+\sigma_i}$, by Lemma 2.3(iv) and (2.1), we have

$$\begin{aligned} x_i - x_{P_i} \in B^{(i)}_{D_i+3\sigma_i} + x_{i,\,0} + B^{(i)}_{v_i\ell(P_i)+u_i} &\subset B^{(i)}_{v_i[\ell(P_i)-1]+u_i+2\sigma_i} + B^{(i)}_{v_i[\ell(P_i)-1]+u_i+\sigma_i} \\ &\subset B^{(i)}_{v_i[\ell(P_i)-1]+u_i+3\sigma_i}, \end{aligned}$$

which implies that $B^{(1)}_{D_1+3\sigma_1} \times B^{(2)}_{D_2+3\sigma_2} \subset P''$.

Let $\Omega \equiv B^{(1)}_{D_1+3\sigma_1} \times B^{(2)}_{D_2+3\sigma_2}$ and $\widetilde{\Omega}$ be as in (4.3). Obviously, Ω is an open bounded set. Noticing that $P \subset \Omega$, then we have $P \subset \widetilde{\Omega}$. Thus, there exists a dyadic rectangle $P^\star \in m(\widetilde{\Omega})$ such that $P \subset P^\star$. Moreover, since $P \subset P^\star$, similarly to the proof of $\Omega \subset P''$, we have that $\Omega \subset (P^\star)''$. For $R \in m(\widetilde{\Omega})$, let $a_R = g_N$ if $R = P^\star$ and $a_R = 0$ if $R \in m(\widetilde{\Omega})$ and $R \neq P^\star$. By the vanishing moment satisfied by g_N and (5.5) together with Proposition 2.10(i), we know that $\widetilde{C}g_N$ is a $(p,\, q,\, s)^*_w$-atom associated with Ω for certain positive constant $\widetilde{C}$ independent of f and N.

Finally, we establish the estimate (5.5). Since $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, by (4.4), (5.3) and

$$\left|\left\{B^{(1)}_{D_1+\sigma_1} \times B^{(2)}_{D_2+\sigma_2} \times [d_1,\, D_1) \times [d_2,\, D_2)\right\} \setminus \left(\bigcup_{R\in\mathcal{R}^N} R_+\right)\right| = 0,$$

together with the observation that for two different rectangles R and S, then $R_+ \cap S_+ = \emptyset$, we have that for all $x \in B^{(1)}_{D_1+3\sigma_1} \times B^{(2)}_{D_2+3\sigma_2}$,

$$\begin{aligned} |g_N(x)| &= \left|\sum_{R\in\mathcal{R}} e_R(x) - \sum_{R\in\mathcal{R}^N} e_R(x)\right| \\ &\lesssim \Bigg[\int_{[d_1,\, D_1)^\complement\times\mathbb{R}} \int_{\mathbb{R}^n\times\mathbb{R}^m} + \int_{\mathbb{R}\times[d_2,\, D_2)^\complement} \int_{\mathbb{R}^n\times\mathbb{R}^m} \\ &\qquad + \int_{[d_1,\, D_1)\times[d_2,\, D_2)} \int_{\mathbb{R}^n\times\left(B^{(2)}_{D_2+\sigma_2}\right)^\complement} + \int_{[d_1,\, D_1)\times[d_2,\, D_2)} \int_{\left(B^{(1)}_{D_1+\sigma_1}\right)^\complement\times\mathbb{R}^m}\Bigg] \\ &\qquad \times |\theta_{t_1,\, t_2}(x-y)(\psi_{t_1,\, t_2} * f)(y)|\, dy\, d\sigma(t_1)\, d\sigma(t_2) \\ &\equiv \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + \mathrm{J}_4. \end{aligned}$$

Recall that $\theta = \theta^{(1)}\theta^{(2)} \in \mathcal{S}_{s_1,\, s_2}(\mathbb{R}^n \times \mathbb{R}^m)$ with $\operatorname{supp}\theta^{(1)} \subset B^{(1)}_0$ and $\operatorname{supp}\theta^{(2)} \subset B^{(2)}_0$. Notice that if $x_i - y_i \in B^{(i)}_{t_i}$, then for all $t_i \in \mathbb{Z}$, we have

$$1 + b_i^{-t_i}\rho(x_i) \sim 1 + b_i^{-t_i}\rho(y_i). \tag{5.6}$$

Let $M_i > 1$ for $i = 1, 2$. Since $f, \psi = \psi^{(1)}\psi^{(2)}, \theta = \theta^{(1)}\theta^{(2)} \in \mathcal{S}_{s_1, s_2}(\mathbb{R}^n \times \mathbb{R}^m)$, by Lemma 5.5 and (5.6) , we have

$$\begin{aligned}
\mathrm{J}_2 &\lesssim \left(\int_0^\infty \frac{b_1^{-t_1[(s_1+1)\zeta_{1,-}+1]}}{[1+b_1^{-t_1}\rho_1(x_1)]^{M_1}}\, d\sigma(t_1) + \int_{-\infty}^0 \frac{b_1^{t_1(s_1+1)\zeta_{1,-}}}{[1+\rho_1(x_1)]^{M_1}}\, d\sigma(t_1) \right) \\
&\quad \times \left(\int_{-\infty}^{d_2} \frac{b_2^{t_2(s_2+1)\zeta_{2,-}}}{[1+\rho_2(x_2)]^{M_2}}\, d\sigma(t_2) + \int_{D_2}^\infty \frac{b_2^{-t_2[(s_2+1)\zeta_{2,-}+1]}}{[1+b_2^{-t_2}\rho_2(x_2)]^{M_2}}\, d\sigma(t_2) \right) \\
&\lesssim b_2^{d_2(s_2+1)\zeta_{2,-}} + b_2^{-D_2[(s_2+1)\zeta_{2,-}+1]} \\
&\lesssim b_2^{-D_2[1+(s_2+1)\zeta_{2,-}]}.
\end{aligned}$$

The last inequality is a consequence of our stipulation that $d_2(s_2+1)\zeta_{2,-} \le -D_2[1+(s_2+1)\zeta_{2,-}]$. Moreover, by the assumptions (5.1), (5.2) and that $(p, q, \vec{s})_w$ is an admissible triplet, there exists $\kappa > 0$ such that $(s_i+1)\zeta_{i,-} + 1 - (q_w+\kappa)/p > 0$ for $i = 1, 2$, and

$$b_1^* \equiv b_1^{v_1[(s_1+1)\zeta_{1,-}+1-(q_w+\kappa)/p]} b_2^{-v_2(q_w+\kappa)/p} < 1, \tag{5.7}$$

$$b_2^* \equiv b_1^{-v_1(q_w+\kappa)/p} b_2^{v_2[(s_2+1)\zeta_{2,-}+1-(q_w+\kappa)/p]} < 1. \tag{5.8}$$

Thus, by (5.8), $\operatorname{supp} g_N \subset B^{(1)}_{D_1+3\sigma_1} \times B^{(2)}_{D_2+3\sigma_2}$ and Proposition 2.10(i) with $w \in \mathcal{A}_{q_w+\kappa}(\vec{A})$, if we choose N large enough, we further obtain

$$\begin{aligned}
\|\mathrm{J}_2\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)} &\le C\left[w\left(B^{(1)}_{D_1}\times B^{(2)}_{D_2}\right)\right]^{1/q} b_2^{-D_2[1+(s_2+1)\zeta_{2,-}]} \\
&\le C\left[w\left(B^{(1)}_{D_1}\times B^{(2)}_{D_2}\right)\right]^{1/q-1/p} (b_2^*)^N \\
&\le \left[w\left(B^{(1)}_{D_1}\times B^{(2)}_{D_2}\right)\right]^{1/q-1/p},
\end{aligned}$$

where C is a positive constant, which is the desired estimate.

For J_4, observe that if $y_1 \in \left(B^{(1)}_{D_1+\sigma}\right)^{\complement}$, $t_1 \le D_1$ and $\theta^{(1)}_{t_1}(x_1-y_1) \ne 0$, then by (2.2), we have

$$x_1 \in y_1 + B^{(1)}_{t_1} \subset \left(B^{(1)}_{D_1}\right)^{\complement} \tag{5.9}$$

and thus $\rho_1(x_1) \ge b_1^{D_1}$. Let $M_1 \in (1, (s_1+1)\zeta_{1,-}+1)$ and $M_2 > 1$. Then by (5.9) and an argument similar to the estimate of J_2, we have

$$\begin{aligned}
\mathrm{J}_4 &\lesssim \left[\int_0^{D_1} \frac{b_1^{-t_1[(s_1+1)\zeta_{1,-}+1]}}{[1+b_1^{-t_1}\rho_1(x_1)]^{M_1}}\, d\sigma(t_1) + \int_{d_1}^0 \frac{b_1^{t_1(s_1+1)\zeta_{1,-}}}{[1+\rho_1(x_1)]^{M_1}}\, d\sigma(t_1) \right] \\
&\quad \times \left[\int_{d_2}^0 \frac{b_2^{t_2(s_2+1)\zeta_{2,-}}}{[1+\rho_2(x_2)]^{M_2}}\, d\sigma(t_2) + \int_0^{D_2} \frac{b_2^{-t_2[(s_2+1)\zeta_{2,-}+1]}}{[1+b_2^{-t_2}\rho_2(x_2)]^{M_2}}\, d\sigma(t_2) \right] \\
&\lesssim b_1^{-D_1[(s_1+1)\zeta_{1,-}+1]}.
\end{aligned}$$

Moreover, by (5.7), Proposition (2.10)(i) with $w \in \mathcal{A}_{q_w+\kappa}(\vec{A})$ and an argument similar to the estimate of $\|\mathrm{J}_2\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}$, if we choose N large enough, we then have

$$\|\mathrm{J}_4\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)} \le \left[w\left(B^{(1)}_{D_1}\times B^{(2)}_{D_2}\right)\right]^{1/q-1/p}.$$

By symmetry, we have similar estimates for $\|\mathrm{J}_1\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}$ and $\|\mathrm{J}_3\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)}$, which gives (5.5) and hence completes the proof of Theorem 5.2. □

6 Applications

We first recall that a *quasi-Banach space* $\mathcal{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is non-negative, non-degenerate (i. e., $\|f\|_{\mathcal{B}} = 0$ if and only if $f = 0$), homogeneous, and obeys the quasi-triangle inequality, i. e., there exists a positive constant K no less than 1 such that for all $f,\, g \in \mathcal{B}$, $\|f+g\|_{\mathcal{B}} \le K(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}})$.

Recall that the following notion of γ-quasi-Banach spaces was first introduced in [64].

Definition 6.1 Let $\gamma \in (0,\, 1]$. A quasi-Banach space $\mathcal{B}_\gamma$ with the quasi-norm $\|\cdot\|_{\mathcal{B}_\gamma}$ is called a γ-*quasi-Banach space* if $\|f+g\|_{\mathcal{B}_\gamma}^\gamma \le \|f\|_{\mathcal{B}_\gamma}^\gamma + \|g\|_{\mathcal{B}_\gamma}^\gamma$ for all $f,\, g \in \mathcal{B}_\gamma$.

Notice that any Banach space is a 1-quasi-Banach space, and the quasi-Banach spaces ℓ^γ, $L^\gamma_w(\mathbb{R}^n \times \mathbb{R}^m)$ and $H^\gamma_w(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A})$ with $\gamma \in (0,\, 1)$ are typical γ-quasi-Banach spaces. Moreover, according to the Aoki-Rolewicz theorem (see [2], [31, p. 66] or [48]), any quasi-Banach space is essentially a γ-quasi-Banach space, where $\gamma \equiv [\log_2(2K)]^{-1}$.

For any given γ-quasi-Banach space $\mathcal{B}_\gamma$ with $\gamma \in (0,\, 1]$ and a linear space $\mathcal{Y}$, an operator T from $\mathcal{Y}$ to $\mathcal{B}_\gamma$ is called $\mathcal{B}_\gamma$-*sublinear* if for all $f,\, g \in \mathcal{Y}$ and $\lambda,\, \nu \in \mathbb{C}$, we have

$$\|T(\lambda f + \nu g)\|_{\mathcal{B}_\gamma} \le \Big(|\lambda|^\gamma \|T(f)\|_{\mathcal{B}_\gamma}^\gamma + |\nu|^\gamma \|T(g)\|_{\mathcal{B}_\gamma}^\gamma\Big)^{1/\gamma}$$

and $\|T(f) - T(g)\|_{\mathcal{B}_\gamma} \le \|T(f-g)\|_{\mathcal{B}_\gamma}$. The notion of B_γ-sublinear operators was first introduced in [63].

We remark that if T is linear, then T is $\mathcal{B}_\gamma$-sublinear. Moreover, if $\mathcal{B}_\gamma$ is a space of functions, T is sublinear in the classical sense and $T(f) \ge 0$ for all $f \in \mathcal{Y}$, then T is also $\mathcal{B}_\gamma$-sublinear.

Theorem 6.2 *Let $w \in \mathcal{A}_\infty(\vec{A})$, q_w as in* (2.7) *and $(p,\, q,\, \vec{s})_w$ an admissible triplet. Let $\gamma \in [p,\, 1]$ and $\mathcal{B}_\gamma$ be a γ-quasi-Banach space. Suppose that $T : H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A}) \to \mathcal{B}_\gamma$ is a $\mathcal{B}_\gamma$-sublinear operator such that*

$$\sup\{\|T(a)\|_{\mathcal{B}_\gamma} :\ a \text{ is any } (p,\, q,\, \vec{s})^*_w\text{-atom}\} < \infty. \tag{6.1}$$

Then there exists a unique bounded $\mathcal{B}_\gamma$-sublinear operator $\widetilde{T}$ from $H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A})$ to $\mathcal{B}_\gamma$ which extends T.

Proof. Without loss of generality, we may also assume that $\vec{s}$ satisfies (5.1) and (5.2). For every $f \in H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A})$, by Theorem 5.2(ii), there exist $\{\lambda_j\}_{j=1}^\ell \subset \mathbb{C}$ and $(p,\, q,\, \vec{s})^*_w$-atoms $\{a_j\}_{j=1}^\ell$ such that $f = \sum_{j=1}^\ell \lambda_j a_j$ pointwise and $\sum_{j=1}^\ell |\lambda_j|^p \lesssim \|f\|^p_{H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A})}$. Then by (6.1), we have

$$\|T(f)\|_{\mathcal{B}_\gamma} \lesssim \left[\sum_{j=1}^\ell |\lambda_j|^p\right]^{1/p} \lesssim \|f\|_{H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A})}.$$

Since $H^{p,\,q,\,\vec{s}}_{w,\,\mathrm{fin}}(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A})$ is dense in $H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A})$ by Theorem 5.2(i), a density argument gives the desired result. This finishes the proof of Theorem 6.2. $\square$

Remark 6.3 If T is a bounded $\mathcal{B}_\gamma$-sublinear operator from $H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A})$ to $\mathcal{B}_\gamma$, then it is clear that for all admissible triplet $(p,\, q,\, \vec{s})_w$, T maps all $(p,\, q,\, \vec{s})^*_w$-atoms into uniformly bounded elements of $\mathcal{B}_\gamma$. Thus the condition (6.1) of Theorem 6.2 is also necessary.

Motivated by Theorem 1 in [25], we introduce the rectangular atoms in the current setting and then derive the boundedness of sublinear operators from their behavior on rectangular atoms.

Definition 6.4 Let $w \in \mathcal{A}_\infty(\vec{A})$ and q_w be as in (2.7) and $(p,\, q,\, \vec{s})_w$ be an admissible triplet as in Definition 4.2. For $R \in \mathcal{R}$, a function a_R is said to be a *rectangular* $(p,\, q,\, \vec{s})_w$*-atom* if

(i) a_R is supported on $R'' = R''_1 \times R''_2$, where $R''_i \equiv x_{R_i} + B^{(i)}_{v_i(\ell(R_i)-1)+u_i+3\sigma_i}$, $i = 1,\, 2$;

(ii) $\int_{\mathbb{R}^n} a_R(x_1,\, x_2) x_1^\alpha\, dx_1 = 0$ for all $|\alpha| \le s_1$ and almost all $x_2 \in \mathbb{R}^m$, and $\int_{\mathbb{R}^m} a_R(x_1,\, x_2) x_2^\beta\, dx_2 = 0$ for all $|\beta| \le s_2$ and almost all $x_1 \in \mathbb{R}^n$;

(iii) $\|a\|_{L^q_w(\mathbb{R}^n \times \mathbb{R}^m)} \lesssim [w(R)]^{1/q-1/p}$.

Let $i = 1, 2$. For any $R_i \in \mathcal{Q}^{(i)}$ and $k \in \mathbb{Z}_+$, set $R_{i,k} \equiv x_{R_i} + B^{(i)}_{v_i(\ell(R_i)-1)+u_i+5\sigma_i+k}$.

The following corollary is very useful in the study of boundedness of operators in $H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A})$.

Corollary 6.5 *Let $w \in \mathcal{A}_\infty(\vec{A})$, q_w as in (2.7) and $(p,\, q_1,\, \vec{s})_w$ an admissible triplet. Let T be a bounded sublinear operator from $L^{q_1}_w(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^{q_0}_w(\mathbb{R}^n \times \mathbb{R}^m)$, where $q_0 \in [q_1,\, \infty)$. Let $q \in [p,\, 2)$ be such that $1/q - 1/p = 1/q_0 - 1/q_1$. If there exist positive constants C, ϵ such that for all $k \in \mathbb{Z}_+$ and all rectangular $(p,\, q_1,\, \vec{s})_w$-atoms a_R,*

$$\int_{(R_{1,k}\times R_{2,k})^{\complement}} |T(a_R)(x)|^q w(x)\, dx \le C \min\left\{b_1^{-k\epsilon},\, b_2^{-k\epsilon}\right\}, \tag{6.2}$$

then T uniquely extends to a bounded operator from $H^p_w(\mathbb{R}^n \times \mathbb{R}^m;\, \vec{A})$ to $L^q_w(\mathbb{R}^n \times \mathbb{R}^m)$.

Proof. Let all the notation be as in the proof of Lemma 4.8. To show Corollary 6.5, by Theorem 6.2, we only need to show that for all $(p,\, q_1,\, \vec{s})^*_w$-atoms $a = \sum_{R\in m(\widetilde{\Omega})} a_R$, $\|Ta\|_{L^q_w(\mathbb{R}^n\times\mathbb{R}^m)} \lesssim 1$.

Recall that $\eta_0 \equiv b_1^{v_1-5\sigma_1} b_2^{v_2-5\sigma_2}$. For each $R = R_1 \times R_2 \in m(\widetilde{\Omega})$, let $\widehat{R}_1 \equiv \widehat{R}_1(R_2)$ being the "longest" dyadic cube containing R_1 such that $\left|(\widehat{R}_1 \times R_2) \cap \widetilde{\Omega}\right| > \eta_0 \left|\widehat{R}_1 \times R_2\right|$. Let

$$\widetilde{\Omega}' \equiv \left\{x \in \mathbb{R}^n \times \mathbb{R}^m :\, \mathcal{M}_s\big(\chi_{\widetilde{\Omega}}\big)(x) > b_1^{-2u_1} b_2^{-2u_2}\eta_0\right\}.$$

For any given $\widetilde{R}_1 \times R_2 \in m(\widetilde{\Omega}')$ and $\widetilde{R}_1 \supset R_1$, let $\widehat{R}_2 \equiv \widehat{R}_2(\widetilde{R}_1)$ being the "longest" dyadic cube containing R_2 such that $\left|(\widetilde{R}_1 \times \widehat{R}_2) \cap \widetilde{\Omega}'\right| > \eta_0 \left|\widetilde{R}_1 \times \widehat{R}_2\right|$.

Let $\gamma_1(R) \equiv \gamma_1(R,\, \widetilde{\Omega}) \equiv \ell(\widehat{R}_1) - \ell(R_1)$ and $\gamma_2(\widetilde{R}_1 \times R_2) \equiv \gamma_2(\widetilde{R}_1 \times R_2,\, \widetilde{\Omega}') \equiv \ell(\widehat{R}_2) - \ell(R_2)$. Then by Lemma 4.9, for any $\delta > 0$, we obtain

$$\sum_{R\in m(\widetilde{\Omega})} b_1^{\gamma_1(R)\delta} w(R) \lesssim w(\Omega) \tag{6.3}$$

and

$$\sum_{R\in m_1(\widetilde{\Omega}')} b_2^{\gamma_2(R_1\times R_2)\delta} w(R) \lesssim w(\Omega). \tag{6.4}$$

Set

$$\bar{R}^* \equiv \bar{R}_1^* \times \bar{R}_2^* \equiv \left(x_{R_1} + B^{(1)}_{v_1(\ell(\widehat{R}_1)-1)+u_1+5\sigma_1}\right) \times \left(x_{R_2} + B^{(2)}_{v_2(\ell(\widehat{R}_2)-1)+u_2+5\sigma_2}\right).$$

By the argument for (4.40) and the $L^{q_0}_w(\mathbb{R}^n \times \mathbb{R}^m)$-boundedness of $\mathcal{M}_s$ (see Proposition 2.10(ii)), we have

$$\bigcup_{R\in m(\widetilde{\Omega})} \bar{R}^* \subset \widetilde{\Omega}''' \quad \text{and} \quad w\big(\widetilde{\Omega}'''\big) \lesssim w(\Omega). \tag{6.5}$$

From this, $1/q - 1/p = 1/q_0 - 1/q_1$ and the size condition of a, together with Hölder's inequality and the boundedness of T from $L^{q_1}_w(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^{q_0}_w(\mathbb{R}^n \times \mathbb{R}^m)$, we deduce that

$$\begin{aligned}\left\{\int_{\widetilde{\Omega}'''} |T(a)(x)|^q w(x)\, dx\right\}^{1/q} &\lesssim \left\{\int_{\widetilde{\Omega}'''} |T(a)(x)|^{q_0} w(x)\, dx\right\}^{1/q_0} w\big(\widetilde{\Omega}'''\big)^{1/q-1/q_0}\\ &\lesssim \|a\|_{L^{q_1}_w(\mathbb{R}^n\times\mathbb{R}^m)} w(\Omega)^{1/p-1/q_1}\\ &\lesssim 1.\end{aligned}$$

It remains to prove that $\int_{(\widetilde{\Omega}''')^{\complement}} |T(a)(x)|^q\, dx \lesssim 1$. Without loss of generality, we may assume that $q \le 1$. The proof of the case $q \in (1,\, 2)$ is similar and we omit the details. Since $q \le 1$ and $a = \sum_{R\in m(\widetilde{\Omega})} a_R$, by (6.5), we obtain

$$
\begin{aligned}
&\int_{(\widetilde{\Omega}''')^{\complement}} |T(a)(x)|^q w(x)\,dx \\
&\le \sum_{R\in m(\widetilde{\Omega})} \int_{(\widetilde{\Omega}''')^{\complement}} |T(a_R)(x)|^q w(x)\,dx \\
&\le \sum_{R\in m(\widetilde{\Omega})} \left[\int_{(\mathbb{R}^n\setminus \bar{R}_1^*)\times\mathbb{R}^m} + \int_{\mathbb{R}^n\times(\mathbb{R}^m\setminus \bar{R}_2^*)}\right] |T(a_R)(x)|^q w(x)\,dx \\
&\equiv \mathrm{E}_1 + \mathrm{E}_2.
\end{aligned}
$$

Since $a_R[w(R)]^{1/q_1-1/p}\|a_R\|^{-1}_{L^{q_1}_w(\mathbb{R}^n\times\mathbb{R}^m)}$ is a rectangular $(p,\, q_1,\, \vec{s})_w$-atom, by (6.2), we have

$$\int_{(\mathbb{R}^m\setminus \bar{R}_1^*)\times\mathbb{R}^n} |T(a_R)(x)|^q w(x)\,dx \lesssim \|a_R\|^q_{L^{q_1}_w(\mathbb{R}^n\times\mathbb{R}^m)} [w(R)]^{1-q/q_0} b_1^{\epsilon\gamma_1(R)}.$$

From this, $1/q_1 - 1/p = 1/q_0 - 1/q$, Hölder's inequality, the size condition of a and (6.3), it follows that

$$
\begin{aligned}
\mathrm{E}_1 &\lesssim \left\{\sum_{R\in m(\widetilde{\Omega})} \|a_R\|^{q_1}_{L^{q_1}_w(\mathbb{R}^n\times\mathbb{R}^m)}\right\}^{q/q_1} \left\{\sum_{R\in m(\widetilde{\Omega})} [w(R)]^{\frac{q_1(q_0-q)}{q_0(q_1-q)}} b_1^{\frac{q_1\epsilon\gamma_1(R)}{q_1-q}}\right\}^{1-q/q_1} \\
&\lesssim [w(\Omega)]^{q(1/q_1-1/p)}[w(\Omega)]^{q(1/q_1-1/q_0)} \left\{\sum_{R\in m(\widetilde{\Omega})} w(R) b_1^{q_1\epsilon\gamma_1(R)/(q_1-q)}\right\}^{1-q/q_1} \\
&\lesssim 1.
\end{aligned}
$$

Similarly, by (6.4), we obtain $\mathrm{E}_2 \lesssim 1$. This finishes the proof of Corollary 6.5. □

Appendix

In this appendix, we give the proof of Proposition 3.6 by establishing a more general version, namely, Theorem A.3 below. Let $\mathcal{B}$ be a Banach space and $L^\infty_c(\mathbb{R}^n,\, \mathcal{B})$ the set of $f \in L^\infty(\mathbb{R}^n,\, \mathcal{B})$ with compact support. Through the whole appendix, we use $\mathcal{B}_1$ and $\mathcal{B}_2$ to denote two *Banach spaces*.

Definition A.1 An operator $\mathcal{T}$ is called a Calderón-Zygmund operator if $\mathcal{T}$ is bounded from $L^r(\mathbb{R}^n,\, \mathcal{B}_1)$ to $L^r(\mathbb{R}^n,\, \mathcal{B}_2)$ for certain fixed $r \in (1,\, \infty)$, and $\mathcal{T}$ has a distributional $\mathcal{L}(\mathcal{B}_1,\, \mathcal{B}_2)$-valued kernel $\mathcal{K}$ such that for all $f \in L^\infty_c(\mathbb{R}^n,\, \mathcal{B}_1)$ and $x \notin \operatorname{supp} f$,

$$\mathcal{T}(f)(x) = \int_{\mathbb{R}^n} \mathcal{K}(x,\, y) f(y)\,dy,$$

where $\mathcal{K}$ is a standard kernel in the following sense: there exist positive constants C and ϵ such that for all $x,\, y,\, z \in \mathbb{R}^n$ satisfying $\rho(z-y) \le b^{-2\sigma}\rho(x-y)$,

$$\|\mathcal{K}(x,\, y)\|_{L(\mathcal{B}_1,\, \mathcal{B}_2)} \le C/\rho(x-y) \tag{A.1}$$

and

$$\|\mathcal{K}(y,\, x) - \mathcal{K}(z,\, x)\|_{L(\mathcal{B}_1,\, \mathcal{B}_2)} + \|\mathcal{K}(x,\, y) - \mathcal{K}(x,\, z)\|_{L(\mathcal{B}_1,\, \mathcal{B}_2)} \le C\frac{\rho(z-y)^\epsilon}{\rho(x-y)^{1+\epsilon}}. \tag{A.2}$$

Let $L^{1,\,\infty}(\mathbb{R}^n,\, \mathcal{B})$ be the set of all $\mathcal{B}$-measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{1,\,\infty}(\mathbb{R}^n,\, \mathcal{B})} \equiv \sup_{\alpha>0} \alpha \left|\{x \in \mathbb{R}^n : \|f\|_{\mathcal{B}} > \alpha\}\right| < \infty.$$

Then by [34, Theorem 1.1], we have the following result.

Lemma A.2 *Let $p \in (1, \infty)$. Suppose that $\mathcal{T}$ is a Calderón-Zygmund operator. Then $\mathcal{T}$ is bounded from $L^p(\mathbb{R}^n, \mathcal{B}_1)$ to $L^p(\mathbb{R}^n, \mathcal{B}_2)$ and bounded from $L^1(\mathbb{R}^n, \mathcal{B}_1)$ to $L^{1,\infty}(\mathbb{R}^n, \mathcal{B}_2)$*

The following theorem is the main result of this appendix, which is a weighted version of Lemma A.2. This theorem extends [19, Theorems 7.11 and 7.12] to the weighted anisotropic settings and also has an independent interest.

Theorem A.3 *Suppose that $\mathcal{T}$ is Calderón-Zygmund operator. If $p \in (1, \infty)$ and $w \in \mathcal{A}_p(A)$, then $\mathcal{T}$ is bounded from $L^p_w(\mathbb{R}^n, \mathcal{B}_1)$ to $L^p_w(\mathbb{R}^n, \mathcal{B}_2)$, and if $w \in \mathcal{A}_1(A)$, then $\mathcal{T}$ is bounded from $L^1_w(\mathbb{R}^n, \mathcal{B}_1)$ to $L^{1,\infty}_w(\mathbb{R}^n, \mathcal{B}_2)$.*

The proof of Theorem A.3 follows from the procedure in [19]. Here we present some details for the convenience of readers.

To this end, we first introduce the dyadic maximal function in this setting. For any given $\mathcal{B}$-measurable function $f \in L^1_{\rm loc}(\mathbb{R}^n, \mathcal{B})$ and $x \in \mathbb{R}^n$, we define the dyadic maximal function by $M_d(f)(x) \equiv \sup_{k\in\mathbb{Z}} E_k(f)(x)$, where

$$E_k(f)(x) \equiv \sum_{Q\in\mathcal{Q}_k} \left(\frac{1}{|Q|}\int_Q \|f(y)\|_{\mathcal{B}}\, dy\right)\chi_Q(x)$$

and $\mathcal{Q}_k \equiv \left\{Q^k_\alpha : \alpha \in I_k\right\}$ denotes the set of dyadic cubes as in Lemma 2.3.

In fact, $E_k(f)$ is a discrete analog of an approximation of the identity. The following Proposition A.4 makes this precise, whose proof is similar to that of [19, Theorem 2.10] and we omit the details.

Proposition A.4 *(i) Let $p \in (1, \infty]$. The dyadic maximal function M_d is bounded from $L^1(\mathbb{R}^n, \mathcal{B})$ to $L^{1,\infty}(\mathbb{R}^n)$ and bounded from $L^p(\mathbb{R}^n, \mathcal{B})$ to $L^p(\mathbb{R}^n)$.*

(ii) If $f \in L^1_{\rm loc}(\mathbb{R}^n, \mathcal{B})$, then $\lim_{k\to\infty} E_k(f)(x) = \|f(x)\|_{\mathcal{B}}$ and $\|f(x)\|_{\mathcal{B}} \le M_d(f)(x)$ almost everywhere.

The following proposition provides the Calderón-Zygmund decomposition in our setting with a non-typical assumption on f instead of the usual $f \in L^1$. This adds an extra layer of difficulty to the standard arguments as in [19, Theorem 2.11].

Proposition A.5 *Given a $\mathcal{B}$-measurable function $f \in L^p_w(\mathbb{R}^n, \mathcal{B})$ for certain $p \in [1, \infty)$ and $w \in \mathcal{A}_p(A)$, and a positive number λ, then exists a sequence $\{Q_j\}_j \subset \mathcal{Q}$ of disjoint dyadic cubes such that*

(i) $\bigcup_j Q_j = \{x \in \mathbb{R}^n : M_d(f)(x) > \lambda\}$;

(ii) $\|f(x)\|_{\mathcal{B}} < \lambda$ for almost every $x \notin \cup_j Q_j$;

(iii) $\lambda < \frac{1}{|Q_j|}\int_{Q_j} \|f(x)\|_{\mathcal{B}}\, dx \le C\lambda$, where $C \ge 1$ is a constant independent of f and λ;

(iv) for any $Q \in \{Q_j\}_j$, there exists a unique $\widetilde{Q} \in \mathcal{Q}$ such that

$$Q \subset \widetilde{Q}, \quad \ell(\widetilde{Q}) = \ell(Q) - 1 \quad \textit{and} \quad \frac{1}{|\widetilde{Q}|}\int_{\widetilde{Q}} \|f(x)\|_{\mathcal{B}}\, dx < \lambda.$$

Proof. Let $p \in [1, \infty)$, $w \in \mathcal{A}_p(A)$ and $f \in L^p_w(\mathbb{R}^n)$. It is easy to see that $f \in L^1_{\rm loc}(\mathbb{R}^n, \mathcal{B})$. In fact, if $p > 1$, by $w \in \mathcal{A}_p(A)$, we have $w^{-p'/p} = w^{1-p'} \in \mathcal{A}_{p'}(A)$, which implies that $w^{-p'/p} \in L^1_{\rm loc}(\mathbb{R}^n)$, where $p' \in (1, \infty)$ satisfying $1/p' + 1/p = 1$. Then for any $k \in \mathbb{Z}$ and B_k, by Hölder's inequality, we have

$$\int_{B_k} \|f(x)\|_{\mathcal{B}}\, dx \le \|f\|_{L^p_w(\mathbb{R}^n,\mathcal{B})} \left\{\int_{B_k} [w(x)]^{-p/p'}\, dx\right\}^{1/p'} < \infty.$$

If $p = 1$, observing that $\sup_B \frac{1}{|B|}\int_B w(x)\, dx \sup_B [w(x)]^{-1} \lesssim 1$, we have

$$\int_{B_k} \|f(x)\|_{\mathcal{B}}\, dx \le \sup_{B_k} [w(x)]^{-1} \int_{B_k} \|f(x)\|_{\mathcal{B}} w(x)\, dx < \infty.$$

Moreover, we claim that for almost all $y \in \mathbb{R}^n$, we have $E_k(f)(y) \to 0$, as $k \to -\infty$. To see this, notice that for almost all $y \in \mathbb{R}^n$, by Lemma 2.1(i), there exists a unique dyadic cube $Q_{k,y} \in \mathcal{Q}_k$ for each $k \in \mathbb{Z}$ such that $y \in Q_{k,y}$.

Then for sufficient small $k \in \mathbb{Z}$, by $Q_{0,\,y} \subset Q_{k\,y}$, Proposition 2.1(i), Lemma 2.1(iii) and (iv), we have

$$\frac{w(Q_{k,\,y})}{w(Q_{0,\,y})} \gtrsim \frac{w(x_{Q_{k,\,y}} + B_{vk-u})}{w(x_{Q_{0,\,y}} + B_u)} \gtrsim \frac{|B_{vk-u}|^{1/p}}{|B_u|^{1/p}} \gtrsim b^{vk/p}.$$

From this, Hölder's inequality, $w \in \mathcal{A}_p(A)$ and $v < 0$, it follows that

$$\begin{aligned} E_k(f)(y) &\le \|f\|_{L^p_w(\mathbb{R}^n)} \frac{1}{|Q_{k,\,y}|} \left\{ \int_{Q_{k,\,y}} [w(x)]^{-p'/p}\,dx \right\}^{1/p'} \\ &\lesssim \|f\|_{L^p_w(\mathbb{R}^n)} [w(Q_{k,\,y})]^{-1/p} \longrightarrow 0 \quad \text{as} \quad k \longrightarrow -\infty. \end{aligned}$$

Thus, the claim holds.

For each $k \in \mathbb{Z}$, set

$$\Omega_k \equiv \{x \in \mathbb{R}^n : \; E_k(f)(x) > \lambda, \text{ and for all } j < k, E_j(f)(x) \le \lambda\}.$$

Then we have

$$\{x \in \mathbb{R}^n : M_d(f)(x) > \lambda\} = \bigcup_k \Omega_k.$$

Indeed, obviously, we have

$$\bigcup_k \Omega_k \subset \{x \in \mathbb{R}^n : M_d(f)(x) > \lambda\}.$$

On the other hand, for almost all $y \in \mathbb{R}^n$ such that $M_d(f)(y) > \lambda$, since $E_k(f)(y) \to 0$ as $k \to -\infty$, there exists a minimal $k_0 \in \mathbb{Z}$ such that $E_{k_0}(f)(y) > \lambda$ and for any $j < k_0$, $E_j(f)(y) \le \lambda$. Thus, we obtain $y \in \Omega_{k_0}$.

Moreover, observe that Ω_k can be covered by disjoint dyadic cubes for each $k \in \mathbb{Z}$. In fact, if $Q \cap \Omega_k \neq \emptyset$, then $Q \subset \Omega_k$ by the definition of $E_k f$. Also notice that $\{\Omega_k\}_k$ are disjoint with each other. By this and $\{x \in \mathbb{R}^n : M_d(f)(x) > \lambda\} = \bigcup_k \Omega_k$, we get (i).

By the definition of Ω_k and $\bigcup_k \Omega_k = \bigcup_j Q_j$, we obtain that (ii), (iv) and the first inequality of (iii) hold. Furthermore, for any $Q \in \{Q_j\}_j$, by (iv) and Lemma 2.1(iv), there exists a unique dyadic cube $\widetilde{Q} \supset Q$ such that $\ell(\widetilde{Q}) = \ell(Q) - 1$ and

$$\frac{1}{|Q|} \int_Q \|f(x)\|_{\mathcal{B}}\,dx \le \frac{|\widetilde{Q}|}{|Q|} \frac{1}{|\widetilde{Q}|} \int_{\widetilde{Q}} \|f(x)\|_{\mathcal{B}}\,dx \le C\lambda,$$

where $C \ge 1$ is a constant independent of f and λ. Thus, the second inequality of (iii) holds. This finishes the proof of Proposition A.5. $\square$

For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathcal{B})$ and $E \subset \mathbb{R}^n$, set $f_E \equiv \frac{1}{|E|} \int_E f(x)\,dx$, and define the sharp maximal function associated with dilation A by setting, for all $x \in \mathbb{R}^n$,

$$\mathcal{M}^\sharp f(x) \equiv \sup_{k \in \mathbb{Z},\, y \in \mathbb{R}^n} \sup_{x \in y + B_k} b^{-k} \int_{y+B_k} \|f(z) - f_{y+B_k}\|_{\mathcal{B}}\,dz.$$

Then by a similar argument to that used in [19, Proposition 6.4], we have the following result. We omit the details.

Proposition A.6 *For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathcal{B})$ and all $x \in \mathbb{R}^n$, $\mathcal{M}^\sharp f(x) \le M(\|f\|_{\mathcal{B}})(x)$, and*

$$\frac{1}{2}\mathcal{M}^\sharp f(x) \le \sup_{k \in \mathbb{Z},\, y \in \mathbb{R}^n} \sup_{x \in y + B_k} \inf_{a \in \mathcal{B}} b^{-k} \int_{y+B_k} \|f(z) - a\|_{\mathcal{B}}\,dz \le \mathcal{M}^\sharp f(x).$$

Based on this, we have the following conclusion.

Lemma A.7 *If* $p_0,\, p \in [1,\,\infty)$, $p_0 \le p$, $w \in \mathcal{A}_p(A)$ *and* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\,\mathcal{B})$ *such that* $M_d(f) \in L^{p_0}_w(\mathbb{R}^n)$, *then there exists a positive constant* C, *independent of* f, *such that*

$$\int_{\mathbb{R}^n} [M_d(f)(x)]^p w(x)\,dx \le C \int_{\mathbb{R}^n} [\mathcal{M}^\sharp(f)(x)]^p\, w(x)\,dx.$$

The proof of Lemma A.7 needs the following generalized "good-λ" inequality, which is an extension of [19, Lemma 7.10].

Lemma A.8 *Let* $p_0 \in [1,\,\infty)$ *and* $w \in \mathcal{A}_{p_0}(A)$. *Then there exists a positive constant* C_0 *such that for all* $f \in L^{p_0}_w(\mathbb{R}^n,\,\mathcal{B})$, $\gamma > 0$ *and* $\lambda > 0$,

$$w(\{x \in \mathbb{R}^n :\ M_d(f)(x) > 2\lambda,\, \mathcal{M}^\sharp(f)(x) \le \gamma\lambda\}) \le C_0 \gamma^{1/p} w(\{x \in \mathbb{R}^n :\ M_d(f)(x) > \lambda\}).$$

Proof. Fix $\lambda,\,\gamma > 0$. Since $f \in L^{p_0}_w(\mathbb{R}^n,\,\mathcal{B})$, by Proposition A.5 the set $\{x \in \mathbb{R}^n : M_d(f)(x) > \lambda\}$ can be written as the union of disjoint dilated cubes. To show Lemma A.8, it suffices to prove that if Q is one of such cubes, then $w(E) \lesssim \gamma^{1/p} w(Q)$, where $E \equiv \{x \in Q : M_d(f)(x) > 2\lambda,\, \mathcal{M}^\sharp(f)(x) < \gamma\lambda\}$. By Lemma 2.3 and Proposition 2.5(i), we have

$$\frac{w(E)}{w(Q)} \le \frac{w(E)}{w(x_Q + B_{v\ell(Q)-u})} \lesssim \frac{|E|^{1/p}}{|x_Q + B_{v\ell(Q)-u}|^{1/p}} \lesssim \frac{|E|^{1/p}}{|Q|^{1/p}},$$

where u and v are the same as in Lemma 2.3(iv). Therefore, to finish the proof of Lemma A.8, we only need to prove $|E| \lesssim \gamma|Q|$. By Proposition A.5(iv), there exists $\widetilde{Q} \in \mathcal{Q}$ such that $\ell(\widetilde{Q}) = \ell(Q) - 1$, $Q \subset \widetilde{Q}$ and

$$\frac{1}{|\widetilde{Q}|} \int_{\widetilde{Q}} \|f(x)\|_{\mathcal{B}}\,dx < \lambda. \tag{A.3}$$

Furthermore, if $x \in Q$ and $M_d(f)(x) > 2\lambda$, then there exist certain $k_0 \in \mathbb{Z}$ and $Q_{k_0} \in \mathcal{Q}_{k_0}$ such that $E_{k_0}(f)(x) > 2\lambda$, namely, $\frac{1}{|Q_{k_0}|}\int_{Q_{k_0}} \|f(y)\|_{\mathcal{B}}\,dx > 2\lambda$, Proposition A.5(iv) further implies that $Q_{k_0} \subset Q$. Therefore, for such x, we have

$$M_d(f\chi_Q)(x) \ge E_{k_0}(f\chi_Q)(x) = \frac{1}{|Q_{k_0}|}\int_{Q_{k_0}} \|f\chi_Q(y)\|_{\mathcal{B}}\,dy > 2\lambda,$$

from which and (A.3), it follows that

$$\begin{aligned} M_d((f - f_{\widetilde{Q}})\chi_Q)(x) &\ge M_d(f\chi_Q)(x) - M_d(f_{\widetilde{Q}}\chi_Q)(x)\\ &\ge M_d(f\chi_Q)(x) - \frac{1}{|\widetilde{Q}|}\int_{\widetilde{Q}} \|f(y)\|_{\mathcal{B}}\,dy\\ &> \lambda, \end{aligned}$$

where we used the fact that

$$\left\|\int_\Omega f(x)\,dx\right\|_{\mathcal{B}} \le \int_\Omega \|f(x)\|_{\mathcal{B}}\,dx \tag{A.4}$$

for all measurable sets Ω and integrable functions f on Ω; see [31, 65]. Therefore, $E \subset \big\{x \in \mathbb{R}^n : M_d\big((f - f_{\widetilde{Q}})\chi_Q\big)(x) > \lambda\big\}$.

Moreover, by $\ell(\widetilde{Q}) = \ell(Q) - 1$, Proposition A.4(i) and Lemma 2.1, we have

$$\begin{aligned}
&\left|\left\{x \in \mathbb{R}^n : M_d\big((f - f_{\widetilde{Q}})\chi_Q\big)(x) > \lambda\right\}\right| \\
&\quad\lesssim \frac{1}{\lambda}\int_Q \left\|f(x) - f_{\widetilde{Q}}\right\|_{\mathcal{B}} dx \\
&\quad\lesssim \frac{1}{\lambda}\int_Q \left\|f(x) - f_{x_{\widetilde{Q}}+B_{v\ell(\widetilde{Q})+u}}\right\|_{\mathcal{B}} dx + \frac{1}{\lambda}\left\|f_{\widetilde{Q}} - f_{x_{\widetilde{Q}}+B_{v\ell(\widetilde{Q})+u}}\right\|_{\mathcal{B}} \\
&\quad\lesssim \frac{|\widetilde{Q}|}{\lambda}\frac{1}{b^{v\ell(\widetilde{Q})+u}}\int_{x_{\widetilde{Q}}+B_{v\ell(\widetilde{Q})+u}} \left\|f(x) - f_{x_{\widetilde{Q}}+B_{v\ell(\widetilde{Q})+u}}\right\|_{\mathcal{B}} dx \\
&\quad\lesssim \frac{|Q|}{\lambda}\inf_{x\in Q}\mathcal{M}^\sharp(f)(x).
\end{aligned} \tag{A.5}$$

If the set E is empty, there is nothing to prove. Otherwise, there exists certain $x \in Q$ such that $\mathcal{M}^\sharp(f)(x) < \gamma\lambda$, which together with (A.5) further implies that $|E| \lesssim \gamma|Q|$. This finishes the proof of Lemma A.8. $\square$

Proof of Lemma A.7. For $N > 0$, let

$$\mathrm{I_N} \equiv \int_0^N p\lambda^{p-1} w(\{x \in \mathbb{R}^n : M_d(f)(x) > \lambda\})\, d\lambda.$$

The assumptions that $p_0 \le p$ and $M_d(f) \in L_w^{p_0}(\mathbb{R}^n)$ imply that $\mathrm{I_N} < \infty$. Then, by Lemma A.8,

$$\begin{aligned}
\mathrm{I_N} &= 2^p \int_0^{N/2} p\lambda^{p-1} w(\{x \in \mathbb{R}^n : M_d(f)(x) > 2\lambda\})\, d\lambda \\
&\le 2^p \int_0^{N/2} p\lambda^{p-1}\big[w(\{x \in \mathbb{R}^n : M_d(f)(x) > 2\lambda,\ \mathcal{M}^\sharp(f)(x) \le \gamma\lambda\}) \\
&\qquad\qquad + w(\{x \in \mathbb{R}^n : \mathcal{M}^\sharp(f)(x) > \gamma\lambda\})\big]\, d\lambda \\
&\le C_0 2^p \gamma^{1/p}\mathrm{I_N} + \frac{2^p}{\gamma^p}\int_0^{N\gamma/2} p\lambda^{p-1} w(\{x \in \mathbb{R}^n : \mathcal{M}^\sharp(f)(x) > \lambda\})\, d\lambda.
\end{aligned}$$

Choose γ such that $C_0 2^p \gamma^{1/p} = 1/2$. Thus, we obtain

$$\mathrm{I_N} \le \frac{2^{p+1}}{\gamma^p}\int_0^{N\gamma/2} p\lambda^{p-1} w(\{x \in \mathbb{R}^n : \mathcal{M}^\sharp(f)(x) > \lambda\})\, d\lambda,$$

which implies the desired conclusion of the lemma. This finishes the proof of Lemma A.7. $\square$

Lemma A.9 *If $\mathcal{T}$ is a Calderón-Zygmund operator as in Definition A.1, then for each $s \in (1, \infty)$, there exists a positive constant C_s such that for all $f \in L_c^\infty(\mathbb{R}^n, \mathcal{B}_1)$ and $x \in \mathbb{R}^n$,*

$$\mathcal{M}^\sharp(\mathcal{T}(f))(x) \le C_s[M(\|f\|_{\mathcal{B}}^s)(x)]^{1/s},$$

where M is the Hardy-Littlewood maximal operator.

Proof. Fix $s \in (1, \infty)$. For any given $x \in \mathbb{R}^n$, pick $y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ such that $x \in y + B_k$. By Proposition A.6, to complete the proof of Lemma A.9, it suffices to find an element $a \in \mathcal{B}_2$ such that

$$b^{-k}\int_{y+B_k} \|\mathcal{T}(f)(z) - a\|_{\mathcal{B}_2}\, dz \lesssim [M(\|f\|_{\mathcal{B}_1}^s)(x)]^{1/s}.$$

Decompose f as $f = f_1 + f_2$, where $f_1 = f\chi_{y+B_{k+2\sigma}}$. Now let $a \equiv \mathcal{T}(f_2)(x)$. By Definition A.1 and $f \in L_c^\infty(\mathbb{R}^n, \mathcal{B}_1)$, we have that $a \in \mathcal{B}_2$ and

$$
\begin{aligned}
&b^{-k}\int_{y+B_k}\|\mathcal{T}(f)(z)-a\|_{\mathcal{B}_2}\,dz\\
&\quad\le b^{-k}\int_{y+B_k}\|\mathcal{T}(f_1)(z)\|_{\mathcal{B}_2}\,dz+b^{-k}\int_{y+B_k}\|\mathcal{T}(f_2)(z)-a\|_{\mathcal{B}_2}\,dz\\
&\quad\equiv \mathrm{I}+\mathrm{II}.
\end{aligned}
$$

By Hölder's inequality and the boundedness of $\mathcal{T}$ from $L^s(\mathbb{R}^n,\,\mathcal{B}_1)$ to $L^s(\mathbb{R}^n,\,\mathcal{B}_2)$ (see Lemma A.2), we then have

$$
\begin{aligned}
\mathrm{I}&\lesssim\left\{b^{-k}\int_{y+B_k}\|\mathcal{T}(f_1)(z)\|^s_{\mathcal{B}_2}\,dz\right\}^{1/s}\\
&\lesssim\left\{b^{-k-2\sigma}\int_{y+B_{k+2\sigma}}\|f(z)\|^s_{\mathcal{B}_1}\,dz\right\}^{1/s}\\
&\lesssim[M(\|f\|^s_{\mathcal{B}_1})(x)]^{1/s}.
\end{aligned}
$$

Moreover, if $x-y\in B_k$, $y-z\in B_k$ and $\alpha-y\in B^{\complement}_{k+2\sigma}$, by (2.1) and (2.2), we obtain $\rho(z-x)\le b^{-\sigma}\rho(x-\alpha)$ and $\rho(x-\alpha)\ge b^{k+\sigma}$. From this, (A.2) and Hölder's inequality, it follows that

$$
\begin{aligned}
\mathrm{II}&\lesssim b^{-k}\int_{y+B_k}\int_{\mathbb{R}^n\setminus(y+B_{k+2\sigma})}\|\mathcal{K}(z,\,\alpha)-\mathcal{K}(x,\,\alpha)\|_{L(\mathcal{B}_1,\,\mathcal{B}_2)}\|f(\alpha)\|_{\mathcal{B}_1}\,d\alpha\,dz\\
&\lesssim b^{-k}\int_{y+B_k}\int_{\rho(x-\alpha)\ge b^{k+\sigma}}\frac{[\rho(z-x)]^{\epsilon}}{[\rho(x-\alpha)]^{1+\epsilon}}\|f(\alpha)\|_{\mathcal{B}_1}\,d\alpha\,dz\\
&\lesssim b^{-k}\int_{y+B_k}b^{k\epsilon}\sum_{j=0}^{\infty}\int_{b^{k+\sigma+j+1}\le\rho(x-\alpha)<b^{k+2\sigma+j}}b^{-(k+j)(1+\epsilon)}\|f(\alpha)\|_{\mathcal{B}_1}\,d\alpha\,dz\\
&\lesssim\sum_{j=0}^{\infty}b^{-j\epsilon}b^{k+2\sigma+j+1}\int_{y+B_{k+2\sigma+j+1}}\|f(\alpha)\|_{\mathcal{B}_1}\,d\alpha\\
&\lesssim[M(\|f\|^s_{\mathcal{B}_1})(x)]^{1/s}.
\end{aligned}
$$

Combining the estimates of I and II yields the desired result and thus finishes the proof Lemma A.9. □

Proof of Theorem A.3. We first prove that $\mathcal{T}$ is bounded from $L^p_w(\mathbb{R}^n,\,\mathcal{B}_1)$ to $L^p_w(\mathbb{R}^n,\,\mathcal{B}_2)$ when $p\in(1,\,\infty)$ and $w\in\mathcal{A}_p(A)$. By [57, Lemma 8, p. 5], there exists $r\in(1,\,p)$ such that $w\in\mathcal{A}_{p/r}(A)$. Since $L^\infty_c(\mathbb{R}^n,\,\mathcal{B}_1)$ is dense in $L^p_w(\mathbb{R}^n,\,\mathcal{B}_1)$ (see [34, Remark 2.2]), then we only need to prove the conclusions of Theorem A.3 by assuming that $f\in L^\infty_c(\mathbb{R}^n,\,\mathcal{B}_1)$. Observe that if $\mathcal{T}(f)\in L^p_w(\mathbb{R}^n,\,\mathcal{B}_2)$, then by Proposition A.4(ii), Lemma A.7, Lemma A.9 and Proposition 2.5(ii), we have

$$
\begin{aligned}
\int_{\mathbb{R}^n}\|\mathcal{T}(f)(x)\|^p_{\mathcal{B}_2}w(x)\,dx&\le\int_{\mathbb{R}^n}[M_d(\mathcal{T}(f))(x)]^p w(x)\,dx\\
&\lesssim\int_{\mathbb{R}^n}[\mathcal{M}^\sharp(\mathcal{T}(f))(x)]^p w(x)\,dx\\
&\lesssim\int_{\mathbb{R}^n}[M(\|f\|^r_{\mathcal{B}_1})(x)]^{p/r}w(x)\,dx\\
&\lesssim\int_{\mathbb{R}^n}\|f(x)\|^p_{\mathcal{B}_1}w(x)\,dx.
\end{aligned}
$$

Now we turn to prove $\mathcal{T}(f)\in L^p_w(\mathbb{R}^n,\,\mathcal{B}_2)$. Since $f\in L^\infty_c(\mathbb{R}^n,\,\mathcal{B}_1)$, we assume that $\operatorname{supp} f\subset B_{k_0}$ for certain $k_0\in\mathbb{Z}$. Write

$$
\|\mathcal{T}(f)\|^p_{L^p_w(\mathbb{R}^n,\,\mathcal{B}_2)}=\left\{\int_{B_{k_0+\sigma}}+\int_{B^{\complement}_{k_0+\sigma}}\right\}\|\mathcal{T}(f)(x)\|^p_{\mathcal{B}_2}w(x)\,dx=\mathrm{I}+\mathrm{II}.
$$

By [57, p. 7], there exists $\eta \in (1, \infty)$ such that w satisfies the reverse Hölder's inequality, which implies that $w \in L^{\eta}_{\mathrm{loc}}(\mathbb{R}^n)$. This combined with Hölder's inequality and Lemma A.2 yields that $\mathrm{I} < \infty$.

For $x \in (B_{k_0+\sigma})^{\complement}$ and $y \in B_{k_0}$, we have $x - y \in B_{k_0}^{\complement}$ and $\rho(x) \lesssim \rho(x-y) + \rho(y) \lesssim \rho(x-y)$. By this, $f \in L^{\infty}_c(\mathbb{R}^n, \mathcal{B}_1)$, (A.4) and (A.1), we have

$$\|\mathcal{T}(f)(x)\|_{\mathcal{B}_2} \le \int_{\mathbb{R}^n} \|f(y)\|_{\mathcal{B}_1} \|\mathcal{K}(x, y)\|_{L(\mathcal{B}_1, \mathcal{B}_2)}\, dy \lesssim \int_{B_{k_0}} \frac{1}{\rho(x-y)}\, dy \lesssim \rho(x)^{-1}.$$

Therefore,

$$\mathrm{II} \lesssim \sum_{j=k_0}^{\infty} \int_{B_{\sigma+j+1} \setminus B_{\sigma+j}} \rho(x)^{-p} w(x)\, dx \lesssim \sum_{j=k_0}^{\infty} b^{-jp} w(B_{\sigma+j+1}).$$

By $w \in \mathcal{A}_{p/s}(A)$ and Proposition 2.5(i), we have $w(B_{\sigma+j+1}) \lesssim b^{jp/s} w(B_{k_0})$, which together with $s \in (1, \infty)$ implies that II is finite. Thus, $T(f) \in L^p_w(\mathbb{R}^n, \mathcal{B}_2)$, which completes the proof of the boundedness of T from $L^p_w(\mathbb{R}^n, \mathcal{B}_1)$ to $L^p_w(\mathcal{B}_2)$.

Finally, we prove that $\mathcal{T}$ is bounded from $L^1_w(\mathbb{R}^n, \mathcal{B}_1)$ to $L^{1,\infty}_w(\mathbb{R}^n, \mathcal{B}_2)$. Fix $\lambda > 0$ and $f \in L^{\infty}_c(\mathbb{R}^n, \mathcal{B}_1)$. By Proposition A.5, there exists a sequence $\{Q_j\}_j$ of disjoint dilated cubes such that the conclusions (i)-(iv) of Proposition A.5 hold. Then we write $f = g + b$, where

$$g(x) \equiv \begin{cases} f(x), & x \in \mathbb{R}^n \setminus \bigcup_j Q_j, \\ \dfrac{1}{|Q_j|} \displaystyle\int_{Q_j} f(y)\, dy, & x \in Q_j, \end{cases}$$

and $b(x) \equiv \sum_j b_j(x)$ with

$$b_j(x) = \left\{ f(x) - \frac{1}{|Q_j|} \int_{Q_j} f(y)\, dy \right\} \chi_{Q_j}(x).$$

Thus by Proposition A.5 and (A.4), we obtain

$$\|g(x)\|_{\mathcal{B}_1} \le \lambda \ \text{ for almost all } \ x \in \mathbb{R}^n, \ \operatorname{supp} b \subset \bigcup_j Q_j \ \text{ and } \ \int_{Q_j} b(x)\, dx = 0. \tag{A.6}$$

So the estimate of $w(\{x \in \mathbb{R}^n : \|\mathcal{T}(f)(x)\|_{\mathcal{B}_2} > 2\lambda\})$ is reduced to those of $w(\{x \in \mathbb{R}^n : \|\mathcal{T}(g)(x)\|_{\mathcal{B}_2} > \lambda\})$ and $w(\{x \in \mathbb{R}^n : \|\mathcal{T}(b)(x)\|_{\mathcal{B}_2} > \lambda\})$. Notice that $w \in \mathcal{A}_1(A)$ implies $w \in \mathcal{A}_2(A)$ and thus, $\mathcal{T}$ is bounded from $L^2_w(\mathbb{R}^n, \mathcal{B}_1)$ to $L^2_w(\mathbb{R}^n, \mathcal{B}_2)$ as already proved above in this proof. Then by (A.6), we have

$$\begin{aligned} w(\{x \in \mathbb{R}^n : \|\mathcal{T}(g)(x)\|_{\mathcal{B}_2} > \lambda\}) &\le \frac{1}{\lambda^2} \int_{\mathbb{R}^n} \|\mathcal{T}(g)(x)\|^2_{\mathcal{B}_2} w(x)\, dx \\ &\lesssim \frac{1}{\lambda^2} \int_{\mathbb{R}^n} \|g(x)\|^2_{\mathcal{B}_1} w(x)\, dx \\ &\lesssim \frac{1}{\lambda} \int_{\mathbb{R}^n} \|g(x)\|_{\mathcal{B}_1} w(x)\, dx. \end{aligned}$$

To obtain a desired estimate for $\mathcal{T}(g)$, we still need to show that

$$\int_{\mathbb{R}^n} \|g(x)\|_{\mathcal{B}_1} w(x)\, dx \lesssim \int_{\mathbb{R}^n} \|f(x)\|_{\mathcal{B}_1}\, w(x)\, dx.$$

Notice that for all $x \in \mathbb{R}^n \setminus \bigcup_j Q_j$, we have $g(x) = f(x)$. On each Q_j, by (A.4) and $w \in \mathcal{A}_1(A)$, we have

$$\int_{Q_j} \|g(x)\|_{\mathcal{B}_1} w(x)\, dx \le \int_{Q_j} \frac{1}{|Q_j|} \int_{Q_j} \|f(y)\|_{\mathcal{B}_1}\, dy\, w(x)\, dx \lesssim \int_{Q_j} \|f(y)\|_{\mathcal{B}_1} w(y)\, dy.$$

Since $\{Q_j\}_j$ are disjoint, we further have

$$w(\{x \in \mathbb{R}^n : \|\mathcal{T}(g)(x)\|_{\mathcal{B}_2} > \lambda\}) \lesssim \int_{Q_j} \|f(y)\|_{\mathcal{B}_1} w(y)\, dy,$$

which completes the estimate for $\mathcal{T}(g)$.

On the other hand, set $Q_j^* \equiv x_{Q_j} + B_{v\ell(Q_j)+u+2\sigma}$, where u, x_Q, v and $\ell(Q_j)$ are as in Lemma 2.3. Then we obtain

$$w(\{x \in \mathbb{R}^n : \|\mathcal{T}(b)(x)\|_{\mathcal{B}_2} > \lambda\}) \le w\left(\bigcup_j Q_j^*\right) + w\left(\left\{x \in \mathbb{R}^n \setminus \bigcup_j Q_j^* : \|\mathcal{T}(b)(x)\|_{\mathcal{B}_2} > \lambda\right\}\right).$$

Since $w \in \mathcal{A}_1(A)$, by Proposition 2.5, Lemma 2.3, Proposition A.6(iv) and the definition of $\mathcal{A}_1(A)$, we have

$$w\left(\bigcup_j Q_j^*\right) \le \sum_j w(Q_j^*) \lesssim \sum_j |Q_j| \frac{w(Q_j^*)}{|Q_j^*|} \lesssim \frac{1}{\lambda} \sum_j \int_{Q_j} \|f(y)\|_{\mathcal{B}_1}\, w(y)\, dy \lesssim \frac{1}{\lambda} \|f\|_{L^1_w(\mathbb{R}^n,\, \mathcal{B}_1)}.$$

Moreover, from the fact that b_j has zero average on Q_j, and (A.4), it follows that

$$\begin{aligned}
w&\left(\left\{x \in \mathbb{R}^n \setminus \bigcup_j Q_j^* : \|\mathcal{T}(b)(x)\| > \lambda\right\}\right)\\
&\le \frac{1}{\lambda} \sum_j \int_{\mathbb{R}^n \setminus \bigcup_j Q_j^*} \|\mathcal{T}(b_j)(x)\|_{\mathcal{B}_2}\, w(x)\, dx\\
&= \frac{1}{\lambda} \sum_j \int_{\mathbb{R}^n \setminus \bigcup_j Q_j^*} \left| \int_{Q_j} [\mathcal{K}(x,\, y) - \mathcal{K}(x,\, x_{Q_j})] b_j(y)\, dy \right| w(x)\, dx\\
&\le \frac{1}{\lambda} \sum_j \int_{Q_j} \int_{\mathbb{R}^n \setminus \bigcup_j Q_j^*} \|\mathcal{K}(x,\, y) - \mathcal{K}(x,\, x_{Q_j})\|_{L(\mathcal{B}_1,\, \mathcal{B}_2)}\, w(x)\, dx\, \|b_j(y)\|_{\mathcal{B}_1}\, dy.
\end{aligned} \tag{A.7}$$

Observe that $x - x_{Q_j} \in B^{\complement}_{v\ell(Q_j)+u+2\sigma}$ and $y - x_{Q_j} \in B_{v\ell(Q_j)+u}$ imply that $\rho(y - x_{Q_j}) \le b^{-3\sigma} \rho(x - x_{Q_j})$. Then by (A.2), for all $y \in Q_j$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus \bigcup_j Q_j^*} \|\mathcal{K}(x,\, y) - \mathcal{K}(x,\, x_{Q_j})\|_{L(\mathcal{B}_1,\, \mathcal{B}_2)}\, w(x)\, dx\\
&\quad \lesssim \int_{\mathbb{R}^n \setminus \bigcup_j Q_j^*} \frac{\rho(y - x_{Q_j})^{\epsilon}}{\rho(x - x_{Q_j})^{1+\epsilon}}\, w(x)\, dx\\
&\quad \lesssim \sum_{k=0}^{\infty} b^{-k\epsilon} \frac{1}{b^{v\ell(Q_j)+u+2\sigma+k+1}} \int_{B_{v\ell(Q_j)+u+2\sigma+k+1}} w(x)\, dx\\
&\quad \lesssim M(w)(y).
\end{aligned}$$

From this, $w \in \mathcal{A}_1(A)$,

$$\int_{Q_j} \|b_j(y)\|_{\mathcal{B}_1}\, w(y)\, dy = \int_{Q_j} \|b(y)\|_{\mathcal{B}_1}\, w(y) dy \le \int_{Q_j} (\|f(y)\|_{\mathcal{B}_1} + \|g(y)\|_{\mathcal{B}_1})\, w(y)\, dy$$

and (A.7), it follows that

$$w\left(\left\{x \in \mathbb{R}^n \setminus \bigcup_j Q_j^* : \|\mathcal{T}(b)(x)\|_{\mathcal{B}_2} > \lambda\right\}\right) \lesssim \frac{1}{\lambda} \sum_j \int_{Q_j} \|b_j(y)\|_{\mathcal{B}_1}\, M(w)(y)\, dy \lesssim$$

$$\lesssim \frac{1}{\lambda}\sum_j \int_{Q_j} \left(\|f(y)\|_{\mathcal{B}_1} + \|g(y)\|_{\mathcal{B}_1}\right) w(y)\,dy$$
$$\lesssim \frac{1}{\lambda}\sum_j \int_{Q_j} \|f(y)\|_{\mathcal{B}_1}\, w(y)\,dy$$
$$\lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n} \|f(y)\|_{\mathcal{B}_1}\, w(y)\,dy.$$

This finishes the proof of Theorem A.3. □

Acknowledgements Marcin Bownik was partially supported by the NSF grant DMS-0653881 and Dachun Yang was supported by the National Natural Science Foundation (Grant No. 10871025) of China.

References

[1] K. F. Andersen and R. T. John, Weighted inequalities for vector-valued maximal functions and singular integrals, Studia Math. **69**, 19–31 (1980/81).
[2] T. Aoki, Locally bounded linear topological spaces, Proc. Imp. Acad. Tokyo **18**, 588–594 (1942).
[3] M. Bownik, Anisotropic Hardy Spaces and Wavelets, Memoirs of the American Mathematical Society Vol. 164 (Amer. Math. Soc., Providence, RI, 2003).
[4] M. Bownik, Boundedness of operators on Hardy spaces via atomic decompositions, Proc. Amer. Math. Soc. **133**, 3535–3542 (2005).
[5] M. Bownik and K.-P. Ho, Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces, Trans. Amer. Math. Soc. **358**, 1469–1510 (2006).
[6] M. Bownik, B. Li, D. Yang, and Y. Zhou, Weighted anisotropic Hardy spaces and their applications in boundedness of sublinear operators, Indiana Univ. Math. J. **57**, 3065–3100 (2008).
[7] C. Cabrelli, M. T. Lacey, U. Molter, and J. Pipher, Variations on the theme of Journé's lemma, Houston J. Math. **32**, 833–861 (2006).
[8] A.-P. Calderón, An atomic decomposition of distributions in parabolic H^p spaces, Adv. Math. **25**, 216–225 (1977).
[9] A.-P. Calderón and A. Torchinsky, Parabolic maximal functions associated with a distribution, Adv. Math. **16**, 1–64 (1975).
[10] A.-P. Calderón and A. Torchinsky, Parabolic maximal functions associated with a distribution. II, Adv. Math. **24**, 101–171 (1977).
[11] D.-C. Chang, D. Yang, and Y. Zhou, Boundedness of sublinear operators on product Hardy spaces and its application, J. Math. Soc. Japan **62**, 321–353 (2010).
[12] S.-Y. A. Chang and R. Fefferman, A continuous version of duality of H^1 with BMO on the bidisc, Ann. of Math. (2) **112**, 179–201 (1980).
[13] S.-Y. A. Chang and R. Fefferman, The Calderón-Zygmund decomposition on product domains, Amer. J. Math. **104**, 455–468 (1982).
[14] S.-Y. A. Chang and R. Fefferman, Some recent developments in Fourier analysis and H^p-theory on product domains, Bull. Amer. Math. Soc. (N.S.) **12**, 1–43 (1985).
[15] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, Colloq. Math. **60/61**, 601–628 (1990).
[16] R. R. Coifman, A real variable characterization of H^p, Studia Math. **51**, 269–274 (1974).
[17] R. R. Coifman, P.-L. Lions, Y. Meyer, and P. Semmes, Compensated compactness and Hardy spaces, J. Math. Pures Appl. (9) **72**, 247–286 (1993).
[18] R. R. Coifman and G. Weiss, Analyse Harmonique Non-commutative sur Certains Espaces Homogènes, Lecture Notes in Mathematics Vol. 242 (Springer, Berlin, 1971).
[19] J. Duoandikoetxea, Fourier Analysis (American Mathematical Society, Providence, R.I., 2001).
[20] E. B. Fabes and N. M. Rivière, Singular intervals with mixed homogeneity, Studia Math. **27**, 19–38 (1966).
[21] E. B. Fabes and N. M. Rivière, Symbolic calculus of kernels with mixed homogeneity, in: Singular Integrals, edited by A. P. Caldern, Proceedings of Symposia in Pure Mathematics Vol. 10 (Amer. Math. Soc., Providence, R.I., 1967) pp. 106–127.
[22] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math. **129**, 137–193 (1972).
[23] R. Fefferman, Singular integrals on product H^p spaces, Rev. Mat. Iberoamericana **1**, 25–31 (1985).
[24] R. Fefferman, Calderón-Zygmund theory for product domains: H^p spaces, Proc. Nat. Acad. Sci. U.S.A. **83**, 840–843 (1986).
[25] R. Fefferman, Harmonic analysis on product spaces, Ann. of Math. (2) **126**, 109–130 (1987).

[26] R. Fefferman, A_p weights and singular integrals, Amer. J. Math. **110**, 975–987 (1988).
[27] R. Fefferman and E. M. Stein, Singular integrals on product spaces, Adv. Math. **45**, 117–143 (1982).
[28] G. B. Folland and E. M. Stein, Hardy Spaces on Homogeneous Group, Mathematical Notes Vol. 28 (Princeton University Press, Princeton, N.J., 1982).
[29] M. Frazier, B. Jawerth, and G. Weiss, Littlewood-Paley Theory and the Study of Function Spaces, CBMS Regional Conference Series in Mathematics Vol. 79 (Amer. Math. Soc., Providence, R. I., 1991).
[30] J. García-Cuerva and J. L. Rubio de Francia, Weighted Norm Inequalities and Related Topics (North-Holland, Amsterdam, 1985).
[31] L. Grafakos, Classical Fourier Analysis, Second Edition, Graduate Texts in Mathematics Vol. 249 (Springer, New York, 2008).
[32] L. Grafakos, Modern Fourier Analysis, Second Edition, Graduate Texts in Mathematics Vol. 250 (Springer, New York, 2008).
[33] L. Grafakos, L. Liu, and D. Yang, Maximal function characterizations of Hardy spaces on RD-spaces and their applications, Sci. China Ser. A **51**, 2253–2284 (2008).
[34] L. Grafakos, L. Liu, and D. Yang, Vector-valued singular integular integrals and maximal functions on spaces of homogeneous type, Math. Scand. **104**, 296–310 (2009).
[35] R. F. Gundy and E. M. Stein, H^p theory for the poly-disc, Proc. Nat. Acad. Sci. **76**, 1026–1029 (1979).
[36] R. Johnson and C. J. Neugebauer, Homeomorphisms preserving A_p, Rev. Math. Iberoamericana **3**, 249–273 (1987).
[37] J.-L. Journé, A covering lemma for product spaces, Proc. Amer. Math. Soc. **96**, 593–598 (1986).
[38] D. Krug, A weighted version of the atomic decomposition for H^p (bi-halfspace), Indiana Univ. Math. J. **37**, 277–300 (1988).
[39] B. Li, M. Bownik, D. Yang, and Y. Zhou, Anisotropic singular integrals in product spaces, arXiv: 0903.4720.
[40] S. Meda, P. Sjögren, and M. Vallarino, On the H^1-L^1 boundedness of operators, Proc. Amer. Math. Soc. **136**, 2921–2931 (2008).
[41] Y. Meyer and R. R. Coifman, Wavelets. Calderón-Zygmund and Multilinear Operators (Cambridge University Press, Cambridge, 1997).
[42] Y. Meyer, M. H. Taibleson, and G. Weiss, Some functional analytic properties of the spaces B_q generated by blocks, Indiana Univ. Math. J. **34**, 493–515 (1985).
[43] S. Müller, Hardy space methods for nonlinear partial differential equations, Tatra Mt. Math. Publ. **4**, 159–168 (1994).
[44] A. Nagel and E. M. Stein, The $\bar{\partial}_b$-complex on decoupled boundaries in $\mathbb{C}^n$, Ann. of Math. (2) **164**, 649–713 (2006).
[45] J. Pipher, Journé's covering Lemma and its extension to higher dimensions, Duke Math. J. **53**, 683–690 (1986).
[46] F. Ricci and J. Verdera, Duality in spaces of finite linear combinations of atoms, Trans. Amer. Math. Soc., to appear.
[47] N. M. Rivière, Singular integrals and multiplier operators, Ark. Mat. **9**, 243–278 (1971).
[48] S. Rolewicz, Metric Linear Spaces, Second edition (PWN—Polish Scientific Publishers, Warsaw; D. Reidel Publishing Co., Dordrecht, 1984).
[49] S. Sato, Lusin functions on product spaces, Tohoku Math. J. (2) **39**, 41–59 (1987).
[50] S. Sato, An atomic decomposition for parabolic H^p spaces on product domains, Proc. Amer. Math. Soc. **104**, 185–192 (1988).
[51] S. Sato, Weighted inequalities on product domains, Studia Math. **92**, 59–72 (1989).
[52] H.-J. Schmeisser and H. Triebel, Topics in Fourier Analysis and Function Spaces (John Wiley & Sons Ltd., Chichester, 1987).
[53] A. Seeger and T. Tao, Sharp Lorentz space estimates for rough operators, Math. Ann. **320**, 381–415 (2001).
[54] S. Semmes, A primer on Hardy spaces, and some remarks on a theorem of Evans and Müller, Comm. Partial Differential Equations **19**, 277–319 (1994).
[55] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals (Princeton Univ. Press, Princeton, N.J., 1993).
[56] E. M. Stein and G. Weiss, On the theory of harmonic functions of several variables. I. The theory of H^p-spaces, Acta Math. **103**, 25–62 (1960).
[57] J.-O. Strömberg and A. Torchinsky, Weighted Hardy Spaces (Springer-Verlag, Berlin, 1989).
[58] M. H. Taibleson and G. Weiss, The molecular characterization of certain Hardy spaces. Representation theorems for Hardy spaces, Astérisque **77**, 67–149 (1980).
[59] H. Triebel, Theory of Function Spaces (Birkhäuser Verlag, Basel, 1983).
[60] H. Triebel, Theory of Function Spaces II (Birkhäuser Verlag, Basel, 1992).
[61] H. Triebel, Theory of Function Spaces III (Birkhäuser Verlag, Basel, 2006).
[62] K. Yabuta, A remark on the $(H^1,\ L^1)$ boundedness, Bull. Fac. Sci. Ibaraki Univ. Ser. A **25**, 19–21 (1993).
[63] D. Yang and Y. Zhou, Boundedness of sublinear operators in Hardy spaces on RD-spaces via atoms, J. Math. Anal. Appl. **339**, 622–635 (2008).
[64] D. Yang and Y. Zhou, A boundedness criterion via atoms for linear operators in Hardy spaces, Constr. Approx. **29**, 207–218 (2009).
[65] K. Yosida, Functional Analysis (Springer-Verlag, Berlin, 1995).
[66] X. Zhu, Atomic decomposition for weighted H^p spaces on product domains, Sci. China Ser. A **35**, 158–168 (1992).

