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Extrapolation of discrete Triebel-Lizorkin spaces

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This paper is dedicated to Hans Triebel on the occasion of his 75th birthday

We show an extrapolation formula for discrete Triebel-Lizorkin spaces which extends a formula of Cwikel and Nilsson [14] to quasi-Banach lattices. This is done in the general setting of anisotropic Triebel-Lizorkin spaces associated with expansive dilations and doubling measures on $\mathbb{R}^n$ introduced by the author [4], [5]. Our main result is new even in the standard dyadic setting.

1 Introduction

In this paper we prove an extrapolation formula for discrete Triebel-Lizorkin spaces $\dot{\mathbf{f}}_p^{\alpha,q}$. These spaces were introduced and studied in the influential papers by Frazier and Jawerth [16]–[18]. Their importance stems from the fact that they describe wavelet coefficients of functions/distributions in Triebel-Lizorkin spaces $\dot{\mathbf{F}}_p^{\alpha,q}$. In turn, Triebel-Lizorkin spaces form a unifying class of function spaces encompassing many well-studied classical function spaces such as Lebesgue spaces, Hardy spaces, the Lipschitz spaces, and the space BMO.

Suppose that $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$. Let $\mathcal{Q}$ denote the collection of all dyadic cubes $Q = 2^j([0,1]^n + k)$, where $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$. The discrete Triebel-Lizorkin sequence space $\dot{\mathbf{f}}_p^{\alpha,q}$ is defined as the collection of all complex-valued sequences $w = \{w_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|w\|_{\dot{\mathbf{f}}_p^{\alpha,q}} = \left\| \left(\sum_{Q\in\mathcal{Q}} \left(|Q|^{-\alpha}|w_Q|\tilde{\chi}_Q\right)^q \right)^{1/q} \right\|_{L^p} < \infty, \tag{1.1}$$

where $\tilde{\chi}_Q = |Q|^{-1/2}\chi_Q$ is the L^2-normalized characteristic function of the dilated cube Q. However, in the special case $p = \infty$, the above formula needs to be replaced by the following localized definition introduced by Frazier and Jawerth [18],

$$\|w\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}} = \left(\sup_{P\in\mathcal{Q}} \frac{1}{|P|} \int_P \sum_{Q\in\mathcal{Q},\ Q\subset P} \left(|Q|^{-\alpha}|w_Q|\tilde{\chi}_Q\right)^q dx \right)^{1/q} < \infty. \tag{1.2}$$

Moreover, the case $q = \infty$ requires an obvious modification due to the presence of ℓ^∞ norm. The main result of this paper is the following extrapolation theorem for $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces.

Theorem 1.1 *Suppose that $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1 \leq \infty$, $0 < q_0, q_1 \leq \infty$, and $0 < \theta < 1$. Then, for any $w \in \dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}$ we have*

$$\|w\|^\theta_{\dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}} \asymp \sup\left\{ \left\| |w|^\theta |v|^{1-\theta} \right\|_{\dot{\mathbf{f}}_p^{\alpha,q}} : \|v\|_{\dot{\mathbf{f}}_{p_0}^{\alpha_0,q_0}} \leq 1 \right\}, \tag{1.3}$$

where $(1/p, 1/q, \alpha) = (1-\theta)(1/p_0, 1/q_0, \alpha_0) + \theta(1/p_1, 1/q_1, \alpha_1)$.

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The formula (1.3) is a lesser known cousin of the well-known interpolation formula for the Calderón product of $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces, see [18, Theorem 8.2],

$$\|w\|_{\dot{\mathbf{f}}_p^{\alpha,q}} \asymp \|w\|_{(\dot{\mathbf{f}}_{p_0}^{\alpha_0,q_0})^{1-\theta}(\dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1})^{\theta}} = \inf\Big\{\|v_0\|_{\dot{\mathbf{f}}_{p_0}^{\alpha_0,q_0}}^{1-\theta}\|v_1\|_{\dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}}^{\theta} : |w| = |v_0|^{1-\theta}|v_1|^{\theta}\Big\}. \tag{1.4}$$

The interpolation formula (1.4) enables the recovery of the norms of intermediate $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces once the norms at endpoints are known. The remarkable feature of the extrapolation formula (1.3) is that it makes possible the recovery of the norm of $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces for the range of parameters beyond the two known endpoint norms. The extrapolation formula is not only a curiosity, but it has found applications in the study of rearrangement operators of the Haar system on dyadic Hardy spaces in the work of Geiss, Müller, and Pillwein [20].

An abstract interpolation space version of extrapolation formula (1.3) was established by Cwikel and Nilsson [14]. Their result [14, Theorem 3.5] can be restated as follows; the precise definitions are given in Definitions 3.1 and 3.2.

Theorem 1.2 (Cwikel, Nilsson) *Let (X_0, X_1) be a couple of saturated Banach lattices, and let $X_0^{1-\theta}X_1^{\theta}$ be their Calderón product for some $0 < \theta < 1$. If X_1 has the Fatou property, then for any $x \in X_1$,*

$$\|x\|_{X_1}^{\theta} = \sup\Big\{\big\||u|^{1-\theta}|x|^{\theta}\big\|_{X_0^{1-\theta}X_1^{\theta}} : \|u\|_{X_0} \le 1\Big\}. \tag{1.5}$$

Note that (1.5) has the equality of norms rather than equivalence of norms as in (1.3) and (1.4). In the Banach setting, when $1 \le p_0, p_1, q_0, q_1 \le \infty$, Theorem 1.2 can be shown to imply Theorem 1.1. However, Theorem 1.2 does not apply in the quasi-Banach setting where a more concrete approach is needed. In fact, the proof of Theorem 1.2 in [14] relies on [14, Theorem 1.5] saying that a saturated Banach lattice X has the Fatou property if and only if $X = X''$, where X' is the Köthe dual of X. Since $X = X''$ is known to fail for $X = \dot{\mathbf{f}}_p^{\alpha,q}$ when $p < 1$ or $q < 1$, see Theorem 2.4, this illustrates the difficulty of quasi-Banach setting where duality techniques generally do not work.

We shall prove that Theorem 1.1 holds in a much more general setting than the classical dyadic case described above. Our favorite setting involves general expansive dilations and doubling measures on $\mathbb{R}^n$. A systematic study of Besov and Triebel-Lizorkin spaces in this setting has been undertaken in [3]–[6]. In Section 2 we survey the highlights of this setting which are especially relevant to this paper. In the final section we give the proof of our main result. We use the convention that the symbol $x \lesssim y$ means that there exists a constant $C > 0$ such that $x \le Cy$, and the symbol $x \asymp y$ means that $x \lesssim y$ and $x \gtrsim y$.

2 Anisotropic Triebel-Lizorkin spaces

Triebel-Lizorkin spaces in the isotropic setting are well-studied unifying class of function spaces. The usual isotropic structure can be replaced by more general non-isotropic dilations which produces as a result anisotropic variants of the classical function spaces. Among many others we mention here parabolic Hardy spaces [11], [12], Besov and Triebel-Lizorkin spaces for diagonal dilations [1], [15], [27]–[31]. The other possible direction is the study of weighted Besov and Triebel-Lizorkin spaces associated with general Muckenhoupt A_∞ weights, see [7]–[9], [26]. One should also add that a significant portion of the theory of function spaces can also be done on very general domains such as spaces of homogeneous type introduced by Coifman and Weiss [13]; for example, see [21]–[25]. However, this high level of generality imposes restrictions on possible values of the integrability exponent p, i.e., $p > 1 - \delta$ for some possibly small $\delta > 0$.

To strike a balance between a high level generality and often diminishing range of possible results, we consider the class of non-isotropic dilation structures associated with expansive dilations, which includes previously considered diagonal setting. In the context of Hardy H^p spaces this goal was achieved by the author in [2], where it was demonstrated that significant portion of a real-variable isotropic H^p theory extends to such anisotropic setting. Analogous extensions to anisotropic Besov and Triebel-Lizorkin spaces with doubling measures were studied in [3]–[6]. These studies show that the isotropic methods of dyadic φ-transforms of Frazier and Jawerth [16], [18], [19] extend to non-isotropic setting associated with general expansive dilations. In particular, weighted anisotropic Triebel-Lizorkin and Besov spaces were characterized by their wavelet transform coefficients and smooth atomic and molecular decompositions of these spaces were established. In addition, the localized version of Triebel-Lizorkin spaces in the endpoint case of $p = \infty$ was developed in [4]. Finally, the duality and interpolation results about anisotropic Triebel-Lizorkin spaces were obtained in [5].

We start by recalling basic definitions and properties of the Euclidean spaces associated with general expansive dilations.

2.1 Anisotropic setting

We say that a real $n \times n$ matrix is *expansive* if all of its eigenvalues λ satisfy $|\lambda| > 1$. A quasi-norm associated with an expansive matrix A is a Borel measurable mapping $\rho_A : \mathbb{R}^n \to [0, \infty)$ satisfying

$$\begin{aligned} \rho_A(x) &> 0, && \text{for} \quad x \neq 0, \\ \rho_A(Ax) &= |\det A| \rho_A(x) && \text{for} \quad x \in \mathbb{R}^n, \\ \rho_A(x+y) &\leq H(\rho_A(x) + \rho_A(y)) && \text{for} \quad x, y \in \mathbb{R}^n, \end{aligned} \tag{2.1}$$

where $H \geq 1$ is a constant. The corresponding ρ_A-*balls* are defined as

$$B_{\rho_A}(x, r) = \{y \in \mathbb{R}^n : \rho_A(x - y) < r\}, \quad x \in \mathbb{R}^n, \quad r > 0.$$

We say that a non-negative Borel measure μ on $\mathbb{R}^n$ is ρ_A-*doubling* if there exists $C > 0$ such that

$$\mu(B_{\rho_A}(x, |\det A| r)) \leq C \mu(B_{\rho_A}(x, r)) \quad \text{for all} \quad x \in \mathbb{R}^n, \quad r > 0. \tag{2.2}$$

Let $\mathcal{Q}$ be the collection of all *dilated cubes*

$$\mathcal{Q} = \left\{ Q = A^j([0,1]^n + k) : j \in \mathbb{Z},\ k \in \mathbb{Z}^n \right\}$$

adapted to the action of a dilation A. Obviously, if $A = 2Id$ we obtain the usual collection of *dyadic cubes*. The scale of a dilated cube $Q = A^j([0,1]^n + k) \in \mathcal{Q}$ is defined as $\operatorname{scale}(Q) = j$. Alternatively, $\operatorname{scale}(Q) = \log_{|\det A|} |Q|$. The *tent* $\mathcal{T}(P)$ over $P \in \mathcal{Q}$ is defined as

$$\mathcal{T}(P) = \{Q \in \mathcal{Q} : |Q \cap P| > 0 \text{ and } \operatorname{scale}(Q) \leq \operatorname{scale}(P)\}.$$

2.2 Anisotropic function spaces

We start by recalling the definition of anisotropic $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces in the range $0 < p < \infty$. Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ be the Schwartz class, $\mathcal{S}'$ the space of tempered distributions, and $\mathcal{P}$ the subspace of polynomials.

Definition 2.1 For $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, and a ρ_A-doubling measure μ, we define the *anisotropic Triebel-Lizorkin space* $\dot{\mathbf{F}}_p^{\alpha,q} = \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ as the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha,q}} = \left\| \left(\sum_{j \in \mathbb{Z}} \left(|\det A|^{j\alpha} |f * \varphi_j| \right)^q \right)^{1/q} \right\|_{L^p(\mu)} < \infty, \tag{2.3}$$

where $\varphi_j(x) = |\det A|^j \varphi(A^j x)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (2.4), (2.5)

$$\operatorname{supp} \hat{\varphi} := \overline{\{\xi \in \mathbb{R}^n : \hat{\varphi}(\xi) \neq 0\}} \subset [-\pi, \pi]^n \setminus \{0\}, \tag{2.4}$$

$$\sup_{j \in \mathbb{Z}} |\hat{\varphi}((A^*)^j \xi)| > 0 \quad \text{for all} \quad \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.5}$$

The discrete Triebel-Lizorkin sequence space $\dot{\mathbf{f}}_p^{\alpha,q} = \dot{\mathbf{f}}_p^{\alpha,q}(A, \mu)$ is defined as the collection of all complex-valued sequences $s = \{s_Q\}_{Q \in \mathcal{Q}}$ such that

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}} = \left\| \left(\sum_{Q \in \mathcal{Q}} (|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q)^q \right)^{1/q} \right\|_{L^p(\mu)} < \infty, \tag{2.6}$$

where $\tilde{\chi}_Q = |Q|^{-1/2} \chi_Q$ is the L^2-normalized characteristic function of the dilated cube Q.

In the special case $p = \infty$ the above definition is unsatisfactory. Frazier and Jawerth [18] had proposed a localized definition of the norm when $p = \infty$ by considering averages only over small scales.

Definition 2.2 For $\alpha \in \mathbb{R}$, $0 < q \le \infty$, and a ρ_A-doubling measure μ, we define the *anisotropic Triebel-Lizorkin space* $\dot{\mathbf{F}}^{\alpha,q}_\infty = \dot{\mathbf{F}}^{\alpha,q}_\infty(\mathbb{R}^n, A, \mu)$ as the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that,

$$\|f\|_{\dot{\mathbf{F}}^{\alpha,q}_\infty(\mathbb{R}^n,A,\mu)} = \sup_{P\in\mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \sum_{j=-\operatorname{scale}(P)}^{\infty} \big(|\det A|^{j\alpha} |f * \varphi_j(x)|\big)^q d\mu(x) \right)^{1/q} < \infty, \tag{2.7}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (2.4) and (2.5). The sequence space, $\dot{\mathbf{f}}^{\alpha,q}_\infty = \dot{\mathbf{f}}^{\alpha,q}_\infty(A,\mu)$ is the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|s\|_{\dot{\mathbf{f}}^{\alpha,q}_\infty(A,\mu)} = \sup_{P\in\mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \sum_{Q\in\mathcal{T}(P)} \big(|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q(x)\big)^q d\mu(x) \right)^{1/q} < \infty. \tag{2.8}$$

One of the main results about these spaces is due to Frazier and Jawerth [18], who proved in the dyadic setting that sequence spaces $\dot{\mathbf{f}}^{\alpha,q}_p$ describe the coefficients of wavelet expansions of elements in $\dot{\mathbf{F}}^{\alpha,q}_p$. This result also yields the equivalence of quasi-norms (2.3) and (2.7) regardless of the choice of φ. The author [4] proved that the same holds in the non-isotropic setting for general expansive dilations.

Theorem 2.3 *Suppose that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a Parseval wavelet such that the support of $\hat{\varphi}$ is bounded and bounded away from* 0. *That is,*

$$\|f\|^2 = \sum_{Q\in\mathcal{Q}} |\langle f, \varphi_Q\rangle|^2 \quad \textit{for all} \quad f \in L^2(\mathbb{R}^n),$$

where $\varphi_Q(x) = |\det A|^{j/2}\varphi(A^j x - k)$ for $Q = A^{-j}([0,1]^n + k)$. Let S_φ be the wavelet analysis transform mapping $f \in \mathcal{S}'/\mathcal{P}$ to the sequence $S_\varphi f = \{\langle f, \varphi_Q\rangle\}_{Q\in\mathcal{Q}}$. Let T_φ be the wavelet synthesis transform mapping $s = \{s_Q\}_{Q\in\mathcal{Q}}$ to $T_\varphi s = \sum_{Q\in\mathcal{Q}} s_Q \varphi_Q$. Then, the operators $S_\varphi : \dot{\mathbf{F}}^{\alpha,q}_p \to \dot{\mathbf{f}}^{\alpha,q}_p$ and $T_\varphi : \dot{\mathbf{f}}^{\alpha,q}_p \to \dot{\mathbf{F}}^{\alpha,q}_p$ are bounded and $T_\varphi S_\varphi = Id$.

The following duality result is a manifestation of the fact that the endpoint space $\dot{\mathbf{F}}^{\alpha,q}_\infty$ is a natural extension of $\dot{\mathbf{F}}^{\alpha,q}_p$ spaces for $p < \infty$. Theorem 2.4 is an extension of the well-known isotropic results of Triebel [28], Frazier and Jawerth [18], and Verbitsky [32] who has filled a gap in the case of $(p,q) \in (1,\infty) \times (0,1)$. For convenience, we state it in the unweighted case where the duality pairing takes more natural form than in the weighted case, see [5, Corollary 4.7 and Theorem 4.8].

Theorem 2.4 *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q < \infty$. Then, the dual space of discrete Triebel-Lizorkin space is*

$$\big(\dot{\mathbf{f}}^{\alpha,q}_p(\mathbb{R}^n, A)\big)^* = \begin{cases} \dot{\mathbf{f}}^{-\alpha,q'}_{p'}(\mathbb{R}^n, A), & 1 \le p < \infty, \\ \dot{\mathbf{f}}^{-\alpha+(1/p-1),\infty}_{\infty}(\mathbb{R}^n, A), & 0 < p < 1. \end{cases}$$

A similar duality also holds for $\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A)$ spaces.

Many other results about $\dot{\mathbf{F}}^{\alpha,q}_p$ spaces in the anisotropic setting can be found [4], [5]. Here, we shall only state two additional facts which are employed in Section 3. These are [5, Lemma 3.1] and [5, Corollary 3.4 and Theorem 3.6], resp.

Lemma 2.5 *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is a ρ_A-doubling measure. Fix $\varepsilon > 0$. Suppose that for each $Q \in \mathcal{Q}$, $E_Q \subset Q$ is a Borel set with $\mu(E_Q)/\mu(Q) > \varepsilon$. Then, for any $s = \{s_Q\}$*

$$\|s\|_{\dot{\mathbf{f}}^{\alpha,q}_p(A,\mu)} \asymp \left\| \left(\sum_{Q\in\mathcal{Q}} \big(|Q|^{-\alpha}|s_Q|\tilde{\chi}_{E_Q}\big)^q \right)^{1/q} \right\|_{L^p(\mu)}, \quad 0 < p < \infty, \tag{2.9}$$

$$\|s\|_{\dot{\mathbf{f}}^{\alpha,q}_\infty(A,\mu)} \asymp \sup_{P\in\mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \sum_{Q\in\mathcal{T}(P)} \big(|Q|^{-\alpha}|s_Q|\tilde{\chi}_{E_Q}(x)\big)^q d\mu(x) \right)^{1/q}, \quad p = \infty, \tag{2.10}$$

where $\tilde{\chi}_{E_Q} = |Q|^{-1/2}\chi_{E_Q}$ is the normalized characteristic function of E_Q.

Theorem 2.6 *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is a ρ_A-doubling measure. Fix $0 < \varepsilon < 1$. Then, for any $s = \{s_Q\}$*

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)} \asymp \inf \left\{ \left\| \left(\sum_{Q\in\mathcal{Q}} \left(|Q|^{-\alpha}|s_Q|\tilde{\chi}_{E_Q}\right)^q \right)^{1/q} \right\|_{L^p(\mu)} : E_Q \subset Q,\ \mu(E_Q)/\mu(Q) > \varepsilon \right\}. \tag{2.11}$$

Moreover, for any $0 < r < \infty$ we have the equivalence of norms when $p = \infty$,

$$\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}} \asymp \sup_{P\in\mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \left(\sum_{Q\in\mathcal{T}(P)} \left(|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q(x)\right)^q \right)^{r/q} d\mu(x) \right)^{1/r}. \tag{2.12}$$

3 Interpolation and extrapolation of $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces

In this section we shall prove Theorem 1.1 in the general setting of expansive dilations A and ρ_A-doubling measures μ. A reader more familiar with the isotropic setting might choose for the simplicity that $A = 2Id$ and μ is the Lebesgue measure on $\mathbb{R}^n$. We start by defining relevant properties of quasi-Banach lattices and the Calderón product [10].

Definition 3.1 Suppose that ν is a positive measure on Ω, and X is a quasi-Banach space of ν-measurable functions on Ω, which are identified if equal ν-a.e.

(i) We say that X is a *quasi-Banach lattice* on Ω if for any ν-measurable functions f, g

$$f \in X \quad \text{and} \quad |g(x)| \le |f(x)|\ \nu\text{-a.e.} \implies g \in X \quad \text{and} \quad \|g\|_X \le \|f\|_X.$$

(ii) X is *saturated* if for every measurable set $E \subset \Omega$ with $\mu(E) > 0$, there exists a measurable subset $F \subset E$ such that $\chi_F \in X$.

(iii) X has the Fatou property if for a.e. pointwise increasing sequence $\{f_k\}_{k=1}^\infty$ of non-negative functions in X with $\sup_k \|f_k\|_X < \infty$, the function f defined by $f(\omega) = \lim_{k\to\infty} f_k(\omega)$ is in X and $\|f\|_X = \lim_{k\to\infty} \|f_k\|_X$.

Definition 3.2 Suppose that X_0 and X_1 are quasi-Banach lattices on Ω. Given $0 < \theta < 1$, define the *Calderón product* $X_0^{1-\theta}X_1^\theta$ as the collection of all ν-measurable functions u satisfying

$$\begin{aligned}\|u\|_{X_0^{1-\theta}X_1^\theta} = \inf \big\{ M > 0 : &|u(x)| \le M|v_0(x)|^{1-\theta}|v_1(x)|^\theta\ \nu\text{-a.e.} \\ &\text{for some} \quad \|v_0\|_{X_0} \le 1 \quad \text{and} \quad \|v_1\|_{X_1} \le 1 \big\} < \infty.\end{aligned}$$

Equivalently, the Calderón product quasi-norm can be defined as

$$\|u\|_{X_0^{1-\theta}X_1^\theta} = \inf \left\{ \|v_0\|_{X_0}^{1-\theta}\|v_1\|_{X_1}^\theta : |u(x)| = |v_0(x)|^{1-\theta}|v_1(x)|^\theta\ \nu\text{-a.e.} \right\}.$$

Theorem 3.3, which was shown in [5, Theorem 6.1], generalizes the result of Frazier and Jawerth [18, Theorem 8.2] on Calderón products of $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces.

Theorem 3.3 *Suppose that $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1 \le \infty$, $0 < q_0, q_1 \le \infty$, and μ is a ρ_A-doubling measure. Then, for any $0 < \theta < 1$, we can identify the Calderón product as*

$$\left(\dot{\mathbf{f}}_{p_0}^{\alpha_0,q_0}(A,\mu)\right)^{1-\theta}\left(\dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}(A,\mu)\right)^\theta = \dot{\mathbf{f}}_p^{\alpha,q}(A,\mu), \tag{3.1}$$

with equivalent quasi-norms, where

$$(1/p, 1/q, \alpha) = (1-\theta)(1/p_0, 1/q_0, \alpha_0) + \theta(1/p_1, 1/q_1, \alpha_1). \tag{3.2}$$

It is worth emphasizing that Theorem 3.3 also covers the endpoint cases $p_0 = \infty$ or $p_1 = \infty$. The same remarkable feature is true for our main result, Theorem 3.4, which establishes the extrapolation formula for discrete Triebel-Lizorkin spaces. In particular, Theorem 3.4 implies that the Triebel-Lizorkin norm in the case $p_1 = \infty$ can be recovered from the corresponding norms with finite integrability parameters $p, p_0 < \infty$. This is yet another manifestation of the fact that the definition of the Triebel-Lizorkin spaces when $p = \infty$, which was historically troublesome until a satisfactory definition Frazier and Jawerth [18], is a natural continuation of the better understood scale $0 < p < \infty$.

Theorem 3.4 *Suppose that* $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1 \le \infty$, $0 < q_0, q_1 \le \infty$, $0 < \theta < 1$, *and* μ *is a* ρ_A*-doubling measure. Then, for any* $w \in \dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}$ *we have*

$$\|w\|^{\theta}_{\dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}} \asymp \sup\Big\{ \big\| |w|^{\theta} |v|^{1-\theta} \big\|_{\dot{\mathbf{f}}_p^{\alpha,q}} : \|v\|_{\dot{\mathbf{f}}_{p_0}^{\alpha_0,q_0}} \le 1 \Big\}, \tag{3.3}$$

where the relevant parameters are constrained by (3.2).

Proof. For simplicity, let $X_0 = \dot{\mathbf{f}}_{p_0}^{\alpha_0,q_0}(A,\mu)$, $X_\theta = \dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$, and $X_1 = \dot{\mathbf{f}}_{p_1}^{\alpha_1,q_1}(A,\mu)$. The inequality $\gtrsim$ in (3.3) follows immediately from Theorem 3.3.

Indeed, suppose that $w = \{w_Q\} \in X_1$, and $v = \{v_Q\} \in X_0$ with $\|v\|_{X_0} \le 1$. Then, by Theorem 3.3 and Definition 3.2

$$\big\| |w|^{\theta} |v|^{1-\theta} \big\|_{X_\theta} \asymp \big\| |w|^{\theta} |v|^{1-\theta} \big\|_{X_0^{1-\theta} X_1^{\theta}} \le \|v\|^{1-\theta}_{X_0} \|w\|^{\theta}_{X_1} \le \|w\|^{\theta}_{X_1}.$$

The proof of the opposite inequality of (3.3) requires more work and uses a stopping time argument. Moreover, it must be split into several cases since $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces have a different definition when $p = \infty$.

Case 1: $\boldsymbol{p_0, p_1 < \infty}$. Suppose that $w \in X_1$. We claim that there exists $v \in X_0$ and a universal constant $C > 0$ such that

$$\|v\|_{X_0} \le C \|w\|^{p_1/p_0}_{X_1}, \tag{3.4}$$

$$\big\| |w|^{\theta} |v|^{1-\theta} \big\|_{X_\theta} \ge \frac{1}{C} \|w\|^{p_1/p}_{X_1}. \tag{3.5}$$

Assuming (3.4) and (3.5) for the moment, yields

$$\|v\|^{1-\theta}_{X_0} \|w\|^{\theta}_{X_1} \lesssim \|w\|^{\theta+(1-\theta)p_1/p_0}_{X_1} = \|w\|^{p_1/p}_{X_1} \lesssim \big\| |w|^{\theta} |v|^{1-\theta} \big\|_{X_\theta}.$$

Hence, after renormalizing $\|v\|_{X_0} = 1$, we obtain $\lesssim$ in (3.3).

First, consider the subcase of $q_0, q_1 < \infty$. We shall temporarily assume that $p_0/q_0 \le p_1/q_1$; the opposite case follows by an easy adaptation. For $k \in \mathbb{Z}$, define

$$\Omega_k = \left\{ x \in \mathbb{R}^n : \left(\sum_Q \big(|Q|^{-\alpha_1} |w_Q| \tilde{\chi}_Q(x)\big)^{q_1} \right)^{1/q_1} > 2^k \right\}.$$

$$\mathcal{Q}_k = \{Q \in \mathcal{Q} : \mu(Q \cap \Omega_k) \ge \mu(Q)/2 \text{ and } \mu(Q \cap \Omega_{k+1}) < \mu(Q)/2\}.$$

Note that the families $\mathcal{Q}_k$, $k \in \mathbb{Z}$, are pairwise disjoint, and if $Q \notin \bigcup_{k\in\mathbb{Z}} \mathcal{Q}_k$, then $w_Q = 0$. In this case set $v_Q = 0$. Otherwise, if $Q \in \mathcal{Q}_k$ for some $k \in \mathbb{Z}$, then we set

$$v_Q = 2^{k\delta} |Q|^u |w_Q|^{q_1/q_0}, \quad \text{where} \quad \delta = p_1/p_0 - q_1/q_0 \ge 0, \quad u = \alpha_0 + 1/2 - (\alpha_1 + 1/2) q_1/q_0. \tag{3.6}$$

To prove (3.4), we use Lemma 2.5 with sets $E_Q = Q \cap \Omega_k$, $Q \in \mathcal{Q}_k$,

$$\begin{aligned}
\|v\|^{p_0}_{X_0} &\lesssim \int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_k} |Q|^{-(\alpha_0+1/2)q_0} |Q|^{uq_0} 2^{k\delta q_0} |w_Q|^{q_1} \chi_{E_Q} \right)^{p_0/q_0} d\mu \\
&= \int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} 2^{k\delta q_0} \chi_{\Omega_k} \sum_{Q\in\mathcal{Q}_k} \big(|Q|^{-\alpha_1} |w_Q| \tilde{\chi}_Q\big)^{q_1} \right)^{p_0/q_0} d\mu \\
&\lesssim \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{Q}} \big(|Q|^{-\alpha_1} |w_Q| \tilde{\chi}_Q\big)^{q_1} \right)^{(1+\delta q_0/q_1)p_0/q_0} d\mu = \|w\|^{p_1}_{X_1}.
\end{aligned}$$

Here, in the penultimate step we used the fact that $\delta \geq 0$ and

$$2^{k\delta q_0}\chi_{\Omega_k} \leq \left(\sum_{Q\in\mathcal{Q}} \left(|Q|^{-\alpha_1}|w_Q|\tilde{\chi}_Q\right)^{q_1}\right)^{\delta q_0/q_1}.$$

To prove (3.5), we use a similar argument as above after redefining $E_Q = Q\cap(\Omega_{k+1})^c$, $Q \in \mathcal{Q}_k$. By the identities $(\alpha+1/2)q - (1-\theta)uq = (\alpha_1+1/2)q_1$ and $(\theta + (1-\theta)q_1/q_0)q = q_1$, we have

$$\begin{aligned}
\big\||w|^\theta|v|^{1-\theta}\big\|_{X_\theta}^p &= \int_{\mathbb{R}^n}\left(\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_k}|Q|^{-(\alpha+1/2)q}|w_Q|^{\theta q}2^{(1-\theta)k\delta q}|Q|^{(1-\theta)uq}|w_Q|^{(1-\theta)qq_1/q_0}\chi_Q\right)^{p/q}d\mu\\
&\geq \int_{\mathbb{R}^n}\left(\sum_{k\in\mathbb{Z}}2^{(1-\theta)k\delta q}\chi_{(\Omega_{k+1})^c}\sum_{Q\in\mathcal{Q}_k}\left(|Q|^{-\alpha_1}|w_Q|\tilde{\chi}_Q\right)^{q_1}\right)^{p/q}d\mu\\
&\gtrsim \int_{\mathbb{R}^n}\left(\sum_{Q\in\mathcal{Q}}\left(|Q|^{-\alpha_1}|w_Q|\tilde{\chi}_{E_Q}\right)^{q_1}\right)^{(1+(1-\theta)\delta q/q_1)p/q}d\mu\\
&= \int_{\mathbb{R}^n}\left(\sum_{Q\in\mathcal{Q}}\left(|Q|^{-\alpha_1}|w_Q|\tilde{\chi}_{E_Q}\right)^{q_1}\right)^{p_1/q_1}d\mu \gtrsim \|w\|_{X_1}^{p_1}.
\end{aligned}$$

The last step follows by Lemma 2.5 and the third step by the fact that

$$2^{k+1}\chi_{(\Omega_{k+1})^c} \geq \left(\sum_{Q\in\mathcal{Q}_k}\left(|Q|^{-\alpha_1}|w_Q|\tilde{\chi}_{E_Q}\right)^{q_1}\right)^{1/q_1}, \tag{3.7}$$

In the complementary case $p_0/q_0 \geq p_1/q_1$ we have $\delta \leq 0$ and it suffices to switch the definitions of sets E_Q in the proofs of (3.4) and (3.5). The details are left to the reader.

To prove (3.4) and (3.5) in the subcase of $q_0 = \infty$ requires only a minor adaptation of the above arguments. For $Q \in \mathcal{Q}_k$, $k \in \mathbb{Z}$ we set

$$v_Q = 2^{k\delta}|Q|^u, \quad \text{where} \quad \delta = p_1/p_0,\ u = \alpha_0 + 1/2. \tag{3.8}$$

For $Q \notin \bigcup_{k\in\mathbb{Z}}\mathcal{Q}_k$, we set $v_Q = 0$. Then, by Lemma 2.5 with $E_Q = Q\cap\Omega_k$ we have

$$\begin{aligned}
\|v\|_{X_0}^{p_0} &\lesssim \int_{\mathbb{R}^n}\sup_{k\in\mathbb{Z}}\sup_{Q\in\mathcal{Q}_k}\left(2^{kp_1/p_0}\chi_{\Omega_k}(x)\chi_Q(x)\right)^{p_0}d\mu(x)\\
&\lesssim \int_{\mathbb{R}^n}\left(\sum_{Q\in\mathcal{Q}}\left(|Q|^{-\alpha_1}|w_Q|\tilde{\chi}_Q\right)^{q_1}\right)^{\delta p_0/q_1}d\mu = \|w\|_{X_1}^{p_1}.
\end{aligned}$$

Likewise, we also obtain (3.5) with obvious modifications.

On the other hand, the subcase $q_1 = \infty$ requires a truly separate treatment. For simplicity we shall assume that $q_0 < \infty$ since the case $q_0 = q_1 = \infty$ is dealt as above. For $k \in \mathbb{Z}$ define

$$\mathcal{Q}_k = \{Q\in\mathcal{Q} : 2^k < |Q|^{-(\alpha_1+1/2)}|w_Q| \leq 2^{k+1}\}, \quad \Omega_k = \bigcup_{Q\in\mathcal{Q}_k} Q.$$

We equip $\mathcal{Q}$ with *stacked below* partial order as in [4, Definition 6.3]. That is, for $Q, P \in \mathcal{Q}$, we define $Q \preccurlyeq P$, if there is a chain of cubes $Q = Q_0, Q_1, \ldots, Q_s = P \in \mathcal{Q}$ such that

$$\operatorname{scale}(Q_i) < \operatorname{scale}(Q_{i+1}) \quad \text{and} \quad |Q_i \cap Q_{i+1}| > 0 \quad \text{for all} \quad i = 0, \ldots, s-1.$$

Note that in the dyadic case $A = 2Id$, $\preccurlyeq$ is simply an inclusion $\subset$ relation. Let $\mathcal{Q}_k'$ be the collection of maximal dilated cubes in $\mathcal{Q}_k$. We need to use two facts about $\mathcal{Q}_k'$. By [4, Proposition 2.10] and the fact that $\|w\|_{X_1} < \infty$, for every $Q \in \mathcal{Q}_k$, there exists (not necessarily unique) $P \in \mathcal{Q}'_k$ such that $Q \preccurlyeq P$. Moreover, by [4, Lemma 6.5] and the the doubling property of μ we have

$$\mu(\Omega_k) \lesssim \mu(\Omega'_k), \quad \text{where} \quad \Omega'_k = \bigcup_{Q \in \mathcal{Q}'_k} Q.$$

Define the sequence $w' = \{w'_Q\}$ by $w'_Q = w_Q$ if $Q \in \mathcal{Q}'_k$ for some $k \in \mathbb{Z}$, and $w'_Q = 0$ otherwise. Therefore, we have

$$\begin{aligned}
\|w\|_{X_1}^{p_1} &= \int_{\mathbb{R}^n} \sup_{k \in \mathbb{Z}} \sup_{Q \in \mathcal{Q}_k} \left(|Q|^{-(\alpha_1+1/2)}|w_Q|\right)^{p_1} \chi_Q(x) d\mu(x) \\
&\asymp \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} 2^{kp_1} \chi_{\Omega_k} d\mu \\
&\lesssim \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} 2^{kp_1} \chi_{\Omega'_k}(x) d\mu \\
&\lesssim \int_{\mathbb{R}^n} \sup_{k \in \mathbb{Z}} 2^{kp_1} \chi_{\Omega'_k}(x) d\mu \asymp \|w'\|_{X_1}^{p_1}.
\end{aligned}$$

For $Q \in \mathcal{Q}'_k$, $k \in \mathbb{Z}$ we set

$$v_Q = 2^{k\delta}|Q|^u, \quad \text{where} \quad \delta = p_1/p_0,\ u = \alpha_0 + 1/2.$$

For $Q \notin \bigcup_{k\in\mathbb{Z}} \mathcal{Q}'_k$, we set $v_Q = 0$. Since the cubes in $\mathcal{Q}'_k$ are disjoint we have

$$\begin{aligned}
\|v\|_{X_0}^{p_0} &= \int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}'_k} \left(|Q|^{-(\alpha_0+1/2)+u} 2^{k\delta}\right)^{q_0} \chi_Q \right)^{p_0/q_0} d\mu \\
&= \int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} 2^{k\delta q_0} \chi_{\Omega'_k} \right)^{p_0/q_0} d\mu \\
&\lesssim \int_{\mathbb{R}^n} \left(\sup_{k \in \mathbb{Z}} 2^{k\delta q_0} \chi_{\Omega'_k} \right)^{p_0/q_0} d\mu \\
&= \int_{\mathbb{R}^n} \sup_{k \in \mathbb{Z}} 2^{kp_1} \chi_{\Omega'_k} d\mu \asymp \|w'\|_{X_1}^{p_1} \le \|w\|_{X_1}^{p_1}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
\big\||w|^\theta |v|^{1-\theta}\big\|_{X_\theta}^p &= \int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}'_k} \left(|Q|^{-(\alpha+1/2)}|w_Q|^\theta 2^{(1-\theta)k\delta}|Q|^{(1-\theta)u}\right)^q \chi_Q \right)^{p/q} d\mu \\
&\gtrsim \int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} 2^{k((1-\theta)\delta+\theta)q} \chi_{\Omega'_k} \right)^{p/q} d\mu \\
&\ge \int_{\mathbb{R}^n} \sup_{k \in \mathbb{Z}} 2^{kp_1} \chi_{\Omega'_k} d\mu \asymp \|w'\|_{X_1}^{p_1} \asymp \|w\|_{X_1}^{p_1}.
\end{aligned}$$

This completes the proof of case $p_0, p_1 < \infty$.

Case 2: $\boldsymbol{p_0 = \infty}$ and $\boldsymbol{p_1 < \infty}$. For brevity, we shall only consider the subcase of $q_0, q_1 < \infty$. Suppose that $w \in X_1$. By homogeneity of the formula (3.3), we can assume that $\|w\|_{X_1} = 1$. We claim that there exists $v \in X_0$ and a universal constant $C > 0$ such that

$$\|v\|_{X_0} \le C, \tag{3.9}$$

$$\big\||w|^\theta |v|^{1-\theta}\big\|_{X_\theta} \ge 1/C. \tag{3.10}$$

Assuming (3.9) and (3.10) for the moment, yields

$$\|v\|_{X_0}^{1-\theta}\|w\|_{X_1}^{\theta} \lesssim 1 \lesssim \big\||w|^{\theta}|v|^{1-\theta}\big\|_{X_\theta}.$$

Hence, after renormalizing $\|v\|_{X_0} = 1$, we obtain $\lesssim$ in (3.3).

Define the sequence $v = \{v_Q\}$ by (3.6) as in Case 1 with the understanding that $p_1/p_0 = 0$. To prove (3.9), we use Theorem 2.6 with the sets $E_Q = Q \cap (\Omega_{k+1})^c$, where $Q \in \mathcal{Q}_k$,

$$\begin{aligned}\|v\|_{X_0} &\lesssim \left\| \left(\sum_{Q\in\mathcal{Q}} \big(|Q|^{-\alpha_0}|v_Q|\tilde{\chi}_{E_Q}\big)^{q_0} \right)^{1/q_0} \right\|_{L^\infty} \\ &\lesssim \left\| \left(\sum_{k\in\mathbb{Z}} 2^{k\delta q_0}\chi_{(\Omega_{k+1})^c} \sum_{Q\in\mathcal{Q}_k} \big(|Q|^{-\alpha_1}|w_Q|\tilde{\chi}_Q\big)^{q_1} \right)^{1/q_0} \right\|_{L^\infty} \\ &\lesssim \left\| \left(\sum_{Q\in\mathcal{Q}} \big(|Q|^{-\alpha}|w_Q|\tilde{\chi}_Q\big)^{q_1} \right)^{(1+\delta q_0/q_1)/q_0} \right\|_{L^\infty} = 1.\end{aligned}$$

Here, in the penultimate step we used (3.7) and the fact that $\delta = -q_1/q_0 < 0$. The proof of (3.10) is done in the same way as in Case 1 because $\dot{\mathbf{f}}_{\infty}^{\alpha,q}$ norms are not involved in (3.10). The only needed change is a switch in a definition of sets $E_Q = Q \cap \Omega_k$, where $Q \in \mathcal{Q}_k$, due to the fact that $\delta < 0$. Finally, the special subcases of $q_0 = \infty$ or $q_1 = \infty$ are dealt in a similar fashion as in Case 1; the details are left to the reader.

Case 3: $p_0 < \infty$ and $p_1 = \infty$. For brevity, we shall only consider the subcase of $q_0, q_1 < \infty$. Suppose that $w \in X_1$. By homogeneity of the formula (3.3) we can assume that $\|w\|_{X_1} = 1$. As in Case 2 it suffices to show that there exists $v \in X_0$ such that (3.9) and (3.10) hold.

By (2.12) for any $0 < r < \infty$, there exists $P \in \mathcal{Q}$ such that

$$\begin{aligned}1 = \|w\|_{X_1}^r &\asymp \frac{1}{\mu(P)} \int_P \left(\sum_{Q\in\mathcal{T}(P)} \big(|Q|^{-\alpha_1}|w_Q|\tilde{\chi}_Q\big)^{q_1} \right)^{r/q_1} d\mu \\ &\asymp \frac{1}{\mu(P)} \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{T}(P)} \big(|Q|^{-\alpha_1}|w_Q|\tilde{\chi}_Q\big)^{q_1} \right)^{r/q_1} d\mu.\end{aligned} \tag{3.11}$$

The last step is trivial if the collection of dilated cubes $\mathcal{Q}$ is nested as in the dyadic case. Otherwise, it can be deduced as a consequence of the following two facts. By [4, Lemma 6.5] there exists $\eta > 0$ such that

$$\bigcup_{Q\in\mathcal{T}(P)} Q \subset \bigcup_{|k-k_0|<\eta} A^{j_0}([0,1]^n + k), \quad \text{where} \quad P = A^{j_0}([0,1]^n + k_0).$$

On the other hand, by [4, Proposition 2.10] we have

$$\mu(A^{j_0}([0,1]^n + k)) \asymp \mu(P) \quad \text{for} \quad |k - k_0| < \eta.$$

Thus, the integral over $\mathbb{R}^n$ in (3.11) can be split over a finite union of "neighboring" cubes of P with the same scale. As a consequence, the integral over $\mathbb{R}^n$ is controlled by the corresponding integral over P.

Define the sequence $v = \{v_Q\}$ by

$$v_Q = \mu(P)^{-1/p_0}|Q|^u|w_Q|^{q_1/q_0} \quad \text{for} \quad Q \in \mathcal{T}(P), \quad \text{where} \quad u = \alpha_0 + 1/2 - (\alpha_1 + 1/2)q_1/q_0, \tag{3.12}$$

and $v_Q = 0$ otherwise.

To prove (3.9), we use (3.11) with $r = p_0 q_1/q_0$

$$\begin{aligned}\|v\|_{X_0}^{p_0} &= \frac{1}{\mu(P)} \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{T}(P)} |Q|^{-(\alpha_0+1/2)q_0}|Q|^{uq_0}|w_Q|^{q_1}\chi_Q \right)^{p_0/q_0} d\mu \\ &= \frac{1}{\mu(P)} \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{T}(P)} \big(|Q|^{-\alpha_1}|w_Q|\tilde{\chi}_Q\big)^{q_1} \right)^{p_0/q_0} d\mu \asymp 1.\end{aligned}$$

To prove (3.10) we also use (3.11) with $r = pq_1/q$

$$\begin{aligned}
&\big\||w|^{\theta}|v|^{1-\theta}\big\|_{X_\theta}^{p}\\
&\quad = \frac{1}{\mu(P)^{(1-\theta)p/p_0}}\int_{\mathbb{R}^n}\left(\sum_{Q\in\mathcal{T}(P)}\left(|Q|^{-(\alpha+1/2)+(1-\theta)u}|w_Q|^{\theta+(1-\theta)q_1/q_0}\chi_Q\right)^{q}\right)^{p/q} d\mu\\
&\quad = \frac{1}{\mu(P)}\int_{\mathbb{R}^n}\left(\sum_{Q\in\mathcal{T}(P)}\left(|Q|^{-\alpha_1}|w_Q|\tilde{\chi}_Q\right)^{q_1}\right)^{p/q} d\mu \asymp 1.
\end{aligned}$$

This shows Case 3 in the subcase $q_0, q_1 < \infty$. The special case $q_0 = \infty$ or $q_1 = \infty$ is again left to the reader.

Case 4: $p_0 = p_1 = \infty$. This case is actually much simpler than the previous ones since it can be shown by direct calculations. Again, we shall only consider the subcase $q_0, q_1 < \infty$. Suppose that $w \in X_1$ with $\|w\|_{X_1} = 1$. It suffices to show that there exists $v \in X_0$ such that (3.9) and (3.10) hold. Define the sequence $v = \{v_Q\}$ by

$$v_Q = |Q|^{u}|w_Q|^{q_1/q_0}, \quad \text{for} \quad Q \in \mathcal{Q}, \quad \text{where} \quad u = \alpha_0 + 1/2 - (\alpha_1 + 1/2)q_1/q_0. \tag{3.13}$$

Then, by the definition of $\dot{\mathbf{f}}_{\infty}^{\alpha,q}$ norm

$$\begin{aligned}
\|v\|_{X_0} &= \sup_{P\in\mathcal{Q}}\left(\frac{1}{\mu(P)}\int_P\sum_{Q\in\mathcal{Q}}|Q|^{-(\alpha_0+1/2)q_0}|Q|^{uq_0}|w_Q|^{q_1}\chi_Q\, d\mu\right)^{1/q_0}\\
&= \sup_{P\in\mathcal{Q}}\left(\frac{1}{\mu(P)}\int_P\sum_{Q\in\mathcal{Q}}\left(|Q|^{-\alpha_1}|w_Q|\tilde{\chi}_Q\right)^{q_1} d\mu\right)^{1/q_0} = 1.
\end{aligned}$$

A similar calculation shows that $\big\||w|^{\theta}|v|^{1-\theta}\big\|_{X_\theta}^{p} = 1$. This completes Case 4 and the proof of Theorem 3.4. □

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