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Littlewood–Paley characterization and duality of weighted anisotropic product Hardy spaces ☆

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A B S T R A C T

The authors study anisotropic product Hardy spaces $H^p_w(\vec{A})$ associated with a pair $\vec{A} := (A_1, A_2)$ of expansive dilations and a class of product Muckenhoupt weights $\mathcal{A}_\infty(\vec{A})$ on $\mathbb{R}^n \times \mathbb{R}^m$. This article is a continuation of their earlier work in Bownik et al. (2010) [8]. The authors establish the Littlewood–Paley g-function characterization and the φ-transform characterization of $H^p_w(\vec{A})$, $0 < p \leqslant 1$. The authors also introduce the weighted anisotropic product Campanato space $\mathcal{L}_{p,w}(\vec{A})$ and establish its φ-transform characterization. As an application, the authors identify the dual space of $H^p_w(\vec{A})$ with $\mathcal{L}_{p,w}(\vec{A})$. The results of this article improve the existing results for weighted product Hardy spaces on $\mathbb{R} \times \mathbb{R}$ and are new even in the weighted isotropic setting.

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1. Introduction

Due to the request in applications of analysis such as PDEs, harmonic analysis and approximation theory, there were several efforts of extending classical function spaces arising in harmonic analysis from Euclidean spaces to other domains and anisotropic settings; see, for example, [1,3,2,4,10,15,11,16,20,31,35–37,24,38,39]. Calderón and Torchinsky initiated the study of Hardy spaces associated with anisotropic dilations in [10,11,9]. Recently, a theory of anisotropic Hardy spaces and their weighted theory were developed by Bownik et al. in [1,7].

Another direction is the development of the theory of Hardy spaces on product domains initiated by Gundy and Stein [23]. In particular, Chang and Fefferman [12,13] characterized the classical product Hardy spaces via atoms. Fefferman [19], Krug [26] and Zhu [43] established the weighted theory of the classical product Hardy spaces, and Sato [29,30] established parabolic Hardy spaces on product domains. It was also proved that the classical product Hardy spaces are good substitutes of product Lebesgue spaces when $p \in (0,1]$; see, for example, [17–19,30,32].

Let $\vec{A} := (A_1, A_2)$ be a pair of expansive dilations and $\mathcal{A}_\infty(\vec{A})$ the corresponding class of product Muckenhoupt weights on $\mathbb{R}^n \times \mathbb{R}^m$ (see Definition 2.5 below). Recently, a theory of the weighted anisotropic product Hardy spaces $H^p_w(\vec{A})$ associated with expansive dilations and product Muckenhoupt weights was established in [8]. In particular, the Hardy spaces $H^p_w(\vec{A})$ were characterized in terms of the Lusin-area function and the atomic decompositions. Moreover, the boundedness on $H^p_w(\vec{A})$ was obtained in [28] for a class of anisotropic singular integrals on $\mathbb{R}^n \times \mathbb{R}^m$, whose kernels are adapted to $\vec{A}$ in the sense of Bownik (see [1]) and have vanishing moments defined via bump functions in the sense of Stein (see [33]).

In this article we continue our study by establishing the Littlewood–Paley characterization and the duality of weighted anisotropic product Hardy spaces. Our first result (see Proposition 2.8 below) shows the equivalence of the Lusin-area function definition of the space $H^p_w(\vec{A})$ for tempered distributions in $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ with tempered distributions in $\mathcal{S}'_0(\mathbb{R}^n \times \mathbb{R}^m)$ vanishing weakly at infinity. Here, $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ is the dual space of the set of all Schwartz functions with all vanishing moments (see Section 2 below). This seemingly inconsequential result enables us to establish the φ-transform characterization (see Theorem 2.12 below) and the Littlewood–Paley g-function characterization (see Theorem 2.14 below) of the Hardy space $H^p_w(\vec{A})$. We also introduce the weighted anisotropic product Campanato space $\mathcal{L}_{p,w}(\vec{A})$ (see Definition 2.10 below) and establish its φ-transform characterization (see also Theorem 2.12 below). In the final part of this article, we identify the dual space of $H^p_w(\vec{A})$ with $\mathcal{L}_{p,w}(\vec{A})$ in Theorem 2.16 below. This improves the result of Krug and Torchinsky [27] which describes the duals of the classical weighted product Hardy spaces $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ when the weights w satisfy Muckenhoupt's $A_r(\mathbb{R} \times \mathbb{R})$ condition on rectangles and $2/r < p \leqslant 1$. Moreover, the dual spaces in [27] have quite different description from $\mathcal{L}_{p,w}(\vec{A})$, and the method employed by Krug and Torchinsky [27] is based on the atomic decomposition characterization of $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$.

To achieve our targets, one key tool is the discrete Calderón reproducing formulae (see Lemma 2.3 below), which is a discrete variant, via dilated cubes introduced by Bownik and Ho [5], of the Calderón reproducing formulae in [8, Proposition 2.16]. Motivated by Frazier and Jawerth [21], Bownik [3] and Bownik and Ho [5], to obtain the Littlewood–Paley g-function characterization of $H^p_w(\vec{A})$, we invoke a weighted anisotropic product variant of the Plancherel–Pôlya inequality (see Lemmas 3.13(i) and 3.15(i) below) and the boundedness of the almost diagonal operators on the discrete weighted anisotropic product Hardy space (see Lemma 3.17 below).

Notice that the φ-transform characterization of $\mathcal{L}_{p,w}(\vec{A})$ closely connects with dilated cubes of Bownik–Ho associated to A_1 and A_2. Although dilated cubes nicely reflect the properties of expansive dilations, they have a critical defect, that is, dilated cubes with different levels have no nested property, which makes it impossible to establish Journé's covering lemma for these dilated cubes. To overcome this difficulty, we invoke the dyadic cubes of Christ [14] for general spaces of homogeneous type in the sense of Coifman and Weiss [15]. To be precise, we establish some subtle relations in Lemma 3.10 below between dilated cubes and dyadic cubes, which further induce some important relations between the sequence spaces $\ell_{p,w}(\vec{A})$ defined via dilated cubes and $\dot{\ell}_{p,w}(\vec{A})$ defined via dyadic cubes (see Lemma 3.12 below). Applying the nested property of dyadic cubes of Christ (see also Lemma 3.7 below) and Journé's covering lemma in [8, Lemma 4.9], we establish a weighted anisotropic product variant of the Plancherel–Pôlya inequality on $\ell_{p,w}(\vec{A})$ (see Lemmas 3.13(ii) and 3.15(ii) below), which, together with some standard argument (see, for example, the proof of Bownik [3, Theorem 3.12]), yields the φ-transform characterization of $\ell_{p,w}(\vec{A})$. Applying the φ-transform characterizations of $H^p_w(\vec{A})$ and $\mathcal{L}_{p,w}(\vec{A})$ together with some ideas from Wang [40] and Frazier and Jawerth [21], we then prove that the dual space of $H^p_w(\vec{A})$ is just $\mathcal{L}_{p,w}(\vec{A})$. We particularly point out that Lemma 3.13 below plays a key role, whose proof is quite geometrical in the sense that we prove this lemma via subtly classifying the dyadic cubes of Christ in [14] (see also Lemma 3.7 below) and its associated Journé's covering lemma in [8, Lemma 4.9].

The main results of this article are stated in Section 2 and their proofs are given in Section 3 below.

Finally, we make some conventions on symbols. Throughout this article, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. Constants with subscripts, such as C_0, do not change in different occurrences. The *symbol* $A \lesssim B$ means that $A \leqslant CB$ and the *symbol* $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. Denote by $\sharp E$ the *cardinality of the set* E. For any $p \in [1,\infty]$, we denote by p' its *conjugate index*, namely, $1/p + 1/p' = 1$. We also let $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and $\mathbb{Z}^n_+ := (\mathbb{Z}_+)^n$. For any $a, b \in \mathbb{R}$, we denote $\min\{a, b\}$ and $\max\{a, b\}$, respectively, by $a \wedge b$ and $a \vee b$. If E is a subset of $\mathbb{R}^n$, we denote by χ_E its *characteristic function*. For any multi-index $\gamma := (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n_+$, let $|\gamma| := \gamma_1 + \cdots + \gamma_n$, and $\partial^\gamma := (\frac{\partial}{\partial \xi_1})^{\gamma_1} \cdots (\frac{\partial}{\partial \xi_n})^{\gamma_n}$. We also denote $(\overbrace{0, \dots, 0}^{n\ \text{times}})$ by the *symbol* $\vec{0}_n$.

2. Main results

We begin with some notions. Let $m, n \in \mathbb{Z}$. In what follows, for convenience, we often let $n_1 := n$ and $n_2 := m$.

Definition 2.1. Let $i \in \{1,2\}$. A real $n_i \times n_i$ matrix A_i is called an *expansive dilation*, shortly a *dilation*, if $\min_{\lambda \in \sigma(A_i)} |\lambda| > 1$, where $\sigma(A_i)$ denotes the *set of all eigenvalues of* A_i. A *quasi-norm* associated with expansive matrix A_i is a Borel measurable mapping $\rho_{A_i} : \mathbb{R}^{n_i} \to [0,\infty)$, for simplicity, denoted as ρ_i, such that

(i) $\rho_i(x_i) > 0$ for $x_i \neq 0$;
(ii) $\rho_i(A_i x_i) = b_i \rho_i(x_i)$ for $x_i \in \mathbb{R}^{n_i}$, where $b_i := |\det A_i|$;
(iii) $\rho_i(x_i + y_i) \leqslant H_i[\rho_i(x_i) + \rho_i(y_i)]$ for all $x_i, y_i \in \mathbb{R}^{n_i}$, where $H_i \geqslant 1$ is a constant.

Throughout the whole article, we always let A_1 and A_2 be expansive dilations, respectively, on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, and ρ_1 and ρ_2 the corresponding quasi-norms. Such ρ_1 and ρ_2 indeed exist; see [1, p. 8]. Let $i \in \{1,2\}$. The set $\mathcal{Q}_i$ of *dilated cubes* of $\mathbb{R}^{n_i}$ is defined by

$$\mathcal{Q}_i := \{Q_i := A_i^{j_i}([0,1)^{n_i} + k_i):\ j_i \in \mathbb{Z},\ k_i \in \mathbb{Z}^{n_i}\}.$$

For any $Q_i := A_i^{j_i}([0,1)^{n_i} + k_i)$, let $x_{Q_i} := A_i^{j_i} k_i$ be the "*lower-left corner*" of Q_i. It is easy to see that, for any fixed $j_i \in \mathbb{Z}$, $\{Q_i := A_i^{j_i}([0,1)^{n_i} + k_i):\ k_i \in \mathbb{Z}^{n_i}\}$ is a partition of $\mathbb{R}^{n_i}$. Denote by $\mathcal{R} := \mathcal{Q}_1 \times \mathcal{Q}_2$ the *set of all dilated rectangles*.

For any function $\varphi^{(i)}$ on $\mathbb{R}^{n_i}$, φ on $\mathbb{R}^n \times \mathbb{R}^m$, $j_i \in \mathbb{Z}$, $k_i \in \mathbb{Z}^{n_i}$, $Q_i := A_i^{-j_i}([0,1)^{n_i} + k_i)$ and $Q := Q_1 \times Q_2$, let $\varphi_{j_i}^{(i)}(x_i) := b_i^{j_i} \varphi^{(i)}(A_i^{j_i} x_i)$ for all $x_i \in \mathbb{R}^{n_i}$,

$$\varphi_{j_1,j_2}(x) := b_1^{j_1} b_2^{j_2} \varphi(A_1^{j_1} x_1, A_2^{j_2} x_2) \quad \text{for all } x := (x_1,x_2) \in \mathbb{R}^n \times \mathbb{R}^m,$$

and, correspondingly,

$$\varphi_{Q_i}^{(i)}(x_i) := |Q_i|^{\frac{1}{2}} \varphi_{j_i}^{(i)}(x_i - x_{Q_i}), \qquad \varphi_Q(x) := |Q|^{\frac{1}{2}} \varphi_{j_1,j_2}(x_1 - x_{Q_1}, x_2 - x_{Q_2}), \tag{2.1}$$

where $|\cdot|$ means the *Lebesgue measure* on $\mathbb{R}^{n_i}$ or $\mathbb{R}^n \times \mathbb{R}^m$, respectively.

Denote by $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ the *set of all Schwartz functions on* $\mathbb{R}^n \times \mathbb{R}^m$ and by $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ its *topological dual space*. As in [22], we let

$$\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m) := \left\{\phi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m):\ \int\limits_{\mathbb{R}^{n_i}} \phi(x_1,x_2) x_i^{\alpha_i}\, dx_i = 0,\ \alpha_i \in \mathbb{Z}_+^{n_i},\ i \in \{1,2\}\right\}.$$

We consider $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ as a subspace of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, including the topology. Thus, $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ is a *complete metric space* (see, for example, [34, p. 21, (3.7)]).

Equivalently, $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ can be defined as a set of $\phi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ such that the *semi-norms*

$$\|\phi\|_M^* := \sup_{|\gamma|\leqslant M} \sup_{\xi\in\mathbb{R}^n\times\mathbb{R}^m} \left|\partial^\gamma \widehat{\phi}(\xi)\right| \prod_{i=1}^{2} \left(|\xi_i|^M + |\xi_i|^{-M}\right) < \infty$$

for all $M \in \mathbb{Z}_+$ (see [5, p. 1479]). The semi-norms $\{\|\cdot\|_M^*\}_{M\in\mathbb{Z}_+}$ generate a topology of a locally convex space on $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ which coincides with the topology of $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ as a subspace of a locally convex space $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$. Let $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ be the *topological dual space* of $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ with the weak-∗ topology.

For any $N \in \mathbb{Z}_+$, let $\mathcal{S}_N(\mathbb{R}^n)$ be the *set of all* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *satisfying* $\int_{\mathbb{R}^n} \varphi(x)x^\alpha\,dx = 0$ *for any* $\alpha \in \mathbb{Z}_+^n$ *with* $|\alpha| \leqslant N$. Given two functions $\phi^{(i)}$ on $\mathbb{R}^{n_i}$, $i \in \{1,2\}$, define $\phi := \phi^{(1)} \otimes \phi^{(2)}$ by $\phi(x_1,x_2) := \phi^{(1)}(x_1)\phi^{(2)}(x_2)$ for all $(x_1,x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Recall that

$$\vec{0}_{n_i} := (\overbrace{0,\ldots,0}^{n_i \text{ times}}).$$

Definition 2.2. Let $\mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ be the *set of all functions of the form* $\varphi := \varphi^{(1)} \otimes \varphi^{(2)}$ *with* $\varphi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$, $i \in \{1,2\}$, *such that*

(i) $\operatorname{supp}\widehat{\varphi^{(i)}} \subset [-\pi,\pi]^{n_i} \setminus \{\vec{0}_{n_i}\}$, *and*
(ii) $\sup_{j\in\mathbb{Z}} |\widehat{\varphi^{(i)}}((A_i^*)^j\xi_i)| > 0$ for all $\xi_i \in \mathbb{R}^{n_i} \setminus \{\vec{0}_{n_i}\}$, where A_i^* denotes the *transpose* of A_i.

Suppose that $\varphi, \psi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. The pair (φ,ψ) is called an *admissible pair of dual frame wavelets* if, in addition to (i) and (ii),

(iii) $\sum_{j\in\mathbb{Z}} \widehat{\varphi^{(i)}}((A_i^*)^j\xi_i)\overline{\widehat{\psi^{(i)}}((A_i^*)^j\xi_i)} = 1$ for all $\xi_i \in \mathbb{R}^{n_i} \setminus \{\vec{0}_{n_i}\}$.

We should point out that such φ and ψ indeed exist. Indeed, by [5, Lemma 3.6], for any $\varphi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, there exists some $\psi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ such that (φ,ψ) is an admissible pair of dual frame wavelets.

The following Calderón reproducing formulae are product variants of [5, Lemmas 2.6 and 2.8], which play an important role in the whole article.

Lemma 2.3.

(i) *Let* $\phi := \phi^{(1)} \otimes \phi^{(2)}$, *where, for* $i \in \{1,2\}$, $\phi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ *satisfies that* $\operatorname{supp}\widehat{\phi^{(i)}}$ *is compact and bounded away from the origin and, for all* $\xi_i \in \mathbb{R}^{n_i} \setminus \{\vec{0}_{n_i}\}$,

$$\sum_{j_i\in\mathbb{Z}} \widehat{\phi^{(i)}}\left(\left(A_i^*\right)^{j_i}\xi_i\right) = 1. \tag{2.2}$$

Then, for any $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ (*resp.* $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$),

$$f = \sum_{j_1, j_2 \in \mathbb{Z}} f * \phi_{j_1, j_2} \tag{2.3}$$

holds true in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ (*resp.* $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$).

(ii) *Let* (φ, ψ) *be an admissible pair of dual frame wavelets as in Definition 2.2. For any* $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ (*resp.* $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$),

$$f = \sum_{R \in \mathcal{R}} \langle f, \varphi_R \rangle \psi_R \tag{2.4}$$

holds true in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ (*resp.* $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$), *where* φ_R *and* ψ_R *are as in* (2.1).

The proof of Lemma 2.3 is given in Section 3. Based on the Calderón reproducing formulae, we can establish some new equivalent characterizations of weighted anisotropic product Hardy spaces in [8].

We first recall the weight class of Muckenhoupt associated with A introduced in [5].

Definition 2.4. Let $p \in [1, \infty)$, A be a dilation and w a non-negative measurable function on $\mathbb{R}^n$. Let $b := |\det A|$. The function w is said to belong to the *weight class* $\mathcal{A}_p(\mathbb{R}^n; A)$ *of Muckenhoupt*, if there exists a positive constant C such that, when $p \in (1, \infty)$,

$$\sup_{x \in \mathbb{R}^n} \sup_{k \in \mathbb{Z}} \left\{ b^{-k} \int_{B_\rho(x, b^k)} w(y)\, dy \right\} \left\{ b^{-k} \int_{B_\rho(x, b^k)} \left[w(y) \right]^{-1/(p-1)} dy \right\}^{p-1} \leqslant C$$

and, when $p = 1$,

$$\sup_{x \in \mathbb{R}^n} \sup_{k \in \mathbb{Z}} \left\{ b^{-k} \int_{B_\rho(x, b^k)} w(y)\, dy \right\} \left\{ \operatorname*{ess\,sup}_{y \in B_\rho(x, b^k)} \left[w(y) \right]^{-1} \right\} \leqslant C;$$

and the minimal constant C as above is denoted by $C_{p,A,n}(w)$. Here, for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, $B_\rho(x, b^k) := \{ y \in \mathbb{R}^n \colon \rho(x - y) < b^k \}$.

Define

$$\mathcal{A}_\infty(\mathbb{R}^n; A) := \bigcup_{1 \leqslant p < \infty} \mathcal{A}_p(\mathbb{R}^n; A).$$

Definition 2.5. For $i \in \{1, 2\}$, let A_i be a dilation on $\mathbb{R}^{n_i}$ and $\vec{A} := (A_1, A_2)$. Let $p \in (1, \infty)$ and w be a non-negative measurable function on $\mathbb{R}^n \times \mathbb{R}^m$. The function w is said to be in the *weight class* $\mathcal{A}_p(\vec{A})$ *of Muckenhoupt*, if $w(x_1, \cdot) \in \mathcal{A}_p(A_2)$ for almost every $x_1 \in \mathbb{R}^n$ and $\operatorname{ess\,sup}_{x_1\in\mathbb{R}^n} C_{p,A_2,m}(w(x_1,\cdot)) < \infty$, and $w(\cdot,x_2) \in \mathcal{A}_p(A_1)$ for almost every $x_2 \in \mathbb{R}^m$ and $\operatorname{ess\,sup}_{x_2\in\mathbb{R}^m} C_{p,A_1,n}(w(\cdot,x_2)) < \infty$. In what follows, let

$$C_{q,\vec{A},n,m}(w) := \max\Big\{\operatorname*{ess\,sup}_{x_1\in\mathbb{R}^n} C_{p,A_2,m}\big(w(x_1,\cdot)\big),\ \operatorname*{ess\,sup}_{x_2\in\mathbb{R}^m} C_{p,A_1,n}\big(w(\cdot,x_2)\big)\Big\}.$$

Define

$$\mathcal{A}_\infty(\vec{A}) := \bigcup_{1<p<\infty} \mathcal{A}_p(\vec{A}).$$

The above product anisotropic weights also satisfy similar basic properties of the classical weights; see [8, Proposition 2.10] for more details.

Recall that a distribution $f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ is said to *vanish weakly at infinity* if, for any $\varphi^{(1)} \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$, $f * \varphi_{k_1,k_2} \to 0$ in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ as $k_1, k_2 \to -\infty$, where $\varphi := \varphi^{(1)} \otimes \varphi^{(2)}$; see [8]. Denote by $\mathcal{S}'_0(\mathbb{R}^n \times \mathbb{R}^m)$ the *set of all* $f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ *vanishing weakly at infinity*.

Let $\Phi := \Phi^{(1)} \otimes \Phi^{(2)}$ with $\Phi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ satisfying $\widehat{\Phi^{(i)}}(\vec{0}_{n_i}) = 0$, $i \in \{1,2\}$. For any $f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ and all $x \in \mathbb{R}^n \times \mathbb{R}^m$, the *anisotropic product Lusin-area function* of f is defined by

$$\vec{S}_\Phi(f)(x) := \Big\{\sum_{k_1,k_2\in\mathbb{Z}} b_1^{k_1} b_2^{k_2} \int_{B_{\rho_1}(x_1,b_1^{-k_1})\times B_{\rho_2}(x_2,b_2^{-k_2})} \big|f * \Phi_{k_1,k_2}(y)\big|^2\,dy\Big\}^{\frac{1}{2}}.$$

The weighted anisotropic product Hardy space $H_w^p(\vec{A})$ was defined via the anisotropic product Lusin-area function in [8] as follows. The class of allowable test functions in [8] was somewhat restricted; see the following Definition 2.6. Later, we shall deduce, from Theorem 2.14, that this restriction can be relaxed to $\Phi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$.

Definition 2.6. Let $\Psi := \Psi^{(1)} \otimes \Psi^{(2)}$ and $\Phi := \Phi^{(1)} \otimes \Phi^{(2)}$ be such that $\Psi^{(i)}, \Phi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$, $i \in \{1,2\}$, satisfying

(i) $\operatorname{supp}\Psi^{(i)} \subset B_{\rho_i}(\vec{0}_{n_i},1) := \{x_i \in \mathbb{R}^{n_i}\colon\ \rho_i(x_i) < 1\}$, $\Psi^{(i)} \in \mathcal{S}_{N_i}(\mathbb{R}^{n_i})$, where N_i is some fixed non-negative integer, and $\widehat{\Psi}^{(i)}(\xi) \geqslant C > 0$ for $\xi \in \{x_i \in \mathbb{R}^{n_i}\colon\ a_i \leqslant \rho_i(x_i) \leqslant b_i\}$, where $0 < a_i < b_i < 1$ are constants;
(ii) $\operatorname{supp}\widehat{\Phi^{(i)}}$ is compact and bounded away from the origin;
(iii) for all $\xi_i \in \mathbb{R}^{n_i} \setminus \{\vec{0}_{n_i}\}$,

$$\sum_{j\in\mathbb{Z}} \widehat{\Psi^{(i)}}\big(\big(A_i^*\big)^j\xi_i\big)\widehat{\Phi^{(i)}}\big(\big(A_i^*\big)^j\xi_i\big) = 1.$$

We should point out that such pairs (Ψ,Φ) indeed exist by virtue of [8, Proposition 2.14].

Definition 2.7. Let $p \in (0,1]$, $w \in \mathcal{A}_\infty(\vec{A})$ and Φ be as in Definition 2.6.

(i) (See [8].) The *weighted anisotropic product Hardy space* $H^p_w(\vec{A})$ is defined by

$$H^p_w(\vec{A}) := \bigl\{ f \in \mathcal{S}'_0(\mathbb{R}^n \times \mathbb{R}^m)\colon \ \|f\|_{H^p_w(\vec{A})} := \bigl\|\vec{S}_\Phi(f)\bigr\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} < \infty \bigr\}.$$

(ii) The *weighted anisotropic product Hardy space* $\widetilde{H}^p_w(\vec{A})$ is defined via replacing $\mathcal{S}'_0(\mathbb{R}^n \times \mathbb{R}^m)$ in (i) by $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$.

The following theorem shows that $H^p_w(\vec{A})$ and $\widetilde{H}^p_w(\vec{A})$ are equivalent in some sense.

Proposition 2.8. *Let $w \in \mathcal{A}_\infty(\vec{A})$ and $p \in (0,1]$. Then $H^p_w(\vec{A}) = \widetilde{H}^p_w(\vec{A})$ in the following sense: if $f \in H^p_w(\vec{A})$, then $f \in \widetilde{H}^p_w(\vec{A})$ and there exists a positive constant C, independent of f, such that $\|f\|_{\widetilde{H}^p_w(\vec{A})} \leqslant C\|f\|_{H^p_w(\vec{A})}$. Conversely, if $f \in \widetilde{H}^p_w(\vec{A})$, then there exists a unique extension $\breve{f} \in \mathcal{S}'_0(\mathbb{R}^n \times \mathbb{R}^m)$ such that, for all $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, $\langle \breve{f}, \varphi\rangle = \langle f, \varphi\rangle$ and there exists a positive constant C, independent of f, such that $\|\breve{f}\|_{H^p_w(\vec{A})} \leqslant C\|f\|_{\widetilde{H}^p_w(\vec{A})}$.*

The proof of Proposition 2.8 is given in Section 3.

Now let us introduce two kinds of weighted anisotropic product Hardy spaces defined, respectively, via the Littlewood–Paley g-function.

Definition 2.9. Let $p \in (0,\infty)$, $w \in \mathcal{A}_\infty(\vec{A})$ and $\varphi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$.

The *weighted anisotropic product Hardy space* $\ddot{H}^p_w(\vec{A})$ is defined, via the Littlewood–Paley g-function, to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ such that

$$\|f\|_{\ddot{H}^p_w(\vec{A})} := \left\| \left\{ \sum_{j_1,j_2\in\mathbb{Z}} |\varphi_{j_1,j_2} * f|^2 \right\}^{\frac{1}{2}} \right\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} < \infty,$$

and the corresponding *discrete weighted anisotropic product Hardy space* $\ddot{h}^p_w(\vec{A})$ is defined to be the set of all complex-valued sequences $s := \{s_R\}_{R\in\mathcal{R}}$ such that

$$\|s\|_{\ddot{h}^p_w(\vec{A})} := \left\| \left\{ \sum_{R\in\mathcal{R}} |s_R|^2|R|^{-1}\chi_R \right\}^{\frac{1}{2}} \right\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} < \infty.$$

For any $Q \in \mathcal{R}$ with $Q := Q_1 \times Q_2 := A_1^{j_1}([0,1)^n + k_1) \times A_2^{j_2}([0,1)^m + k_2)$, where $j_1, j_2 \in \mathbb{Z}$ and $k_1 \in \mathbb{Z}^n, k_2 \in \mathbb{Z}^m$, let the *symbol*

$$\operatorname{scale}(Q) := \bigl(\operatorname{scale}(Q_1), \operatorname{scale}(Q_2)\bigr) := (j_1, j_2).$$

The weighted anisotropic product Campanato spaces are defined as follows, which are weighted variants of anisotropic Campanato spaces on $\mathbb{R}^n$ in [1], and are proved to be the dual spaces of weighted anisotropic Hardy spaces in Section 3.

Definition 2.10. Let $p \in (0,1]$, $w \in \mathcal{A}_\infty(\vec{A})$ and $\varphi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$.

(i) The *space* $\mathcal{L}_{p,w}(\vec{A})$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ such that

$$\|f\|_{\mathcal{L}_{p,w}(\vec{A})} := \left\{ \sup_{w(\Omega)<\infty} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \int_\Omega \sum_{j_1,j_2\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R},\, R\subset\Omega \\ \mathrm{scale}(R)=(-j_1,-j_2)}} \left|\varphi_{j_1,j_2} * f(x)\right|^2 \times \frac{|R|^2}{[w(R)]^2} \chi_R(x) w(x)\, dx \right\}^{\frac{1}{2}} < \infty,$$

where Ω runs over all open sets in $\mathbb{R}^n \times \mathbb{R}^m$ with $w(\Omega) < \infty$.

(ii) The corresponding *sequence space* $\ell_{p,w}(\vec{A})$ is defined to be the set of all complex-valued sequences $s := \{s_R\}_{R\in\mathcal{R}}$ such that

$$\|s\|_{\ell_{p,w}(\vec{A})} := \left\{ \sup_{w(\Omega)<\infty} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{R\in\mathcal{R},\, R\subset\Omega} |s_R|^2 \frac{|R|}{w(R)} \right\}^{\frac{1}{2}} < \infty,$$

where Ω runs over all open sets in $\mathbb{R}^n \times \mathbb{R}^m$ with $w(\Omega) < \infty$.

Definition 2.11. Let $\varphi := \varphi^{(1)} \otimes \varphi^{(2)}$ and $\psi := \psi^{(1)} \otimes \psi^{(2)}$, with $\varphi^{(i)}, \psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ for $i \in \{1,2\}$, such that $\operatorname{supp} \widehat{\varphi^{(i)}}$ and $\operatorname{supp} \widehat{\psi^{(i)}}$ are compact and bounded away from the origin. The *φ-transform* S_φ is the map taking each $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ to the sequence $S_\varphi f := \{(S_\varphi f)_R\}_{R\in\mathcal{R}}$ defined by $(S_\varphi(f))_R := \langle f, \varphi_R\rangle$. The *inverse φ-transform* T_ψ is the map taking the sequence $s := \{s_R\}_{R\in\mathcal{R}}$ to $T_\psi s := \sum_{R\in\mathcal{R}} s_R \psi_R$.

Theorem 2.12. *Let $p \in (0,\infty)$ and $w \in \mathcal{A}_\infty(\vec{A})$. The φ-transform $T_\psi : \ddot{h}^p_w(\vec{A}) \to \ddot{H}^p_w(\vec{A})$ and the inverse transform $S_\varphi : \ddot{H}^p_w(\vec{A}) \to \ddot{h}^p_w(\vec{A})$ are bounded. Moreover, if (ψ,φ) is an admissible pair of dual frame wavelets as in Definition 2.2, then the map $T_\psi \circ S_\varphi$ is an identity on $\ddot{H}^p_w(\vec{A};\varphi) = \ddot{H}^p_w(\vec{A};\widetilde{\varphi})$.*

The above results also hold if $\ddot{H}^p_w(\vec{A})$ and $\ddot{h}^p_w(\vec{A})$ are replaced, respectively, by $\mathcal{L}_{p,w}(\vec{A})$ and $\ell_{p,w}(\vec{A})$ for $p \in (0,1]$.

The proof of Theorem 2.12 is given in Section 3.

Then, by Theorem 2.12, with proofs similar to those of [3, Corollaries 3.13 and 3.14], we can obtain that the space $\mathcal{L}_{p,w}(\vec{A})$, equipped with $\|\cdot\|_{\mathcal{L}_{p,w}(\vec{A})}$, is well defined and complete as follows, the details being omitted.

Corollary 2.13. *Let $p \in (0,\infty)$ and $w \in \mathcal{A}_\infty(\vec{A})$. The space $\ddot{H}^p_w(\vec{A})$ is well defined in the following sense that, for any $\varphi, \widetilde{\varphi} \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, their associated quasi-norms, respectively, in $\ddot{H}^p_w(\vec{A};\varphi)$ and $\ddot{H}^p_w(\vec{A};\widetilde{\varphi})$ are equivalent, namely, there exist positive constants C_1 and C_2 such that, for all $f \in \ddot{H}^p_w(\vec{A})$,*

$$C_1\|f\|_{\ddot{H}^p_w(\vec{A};\widetilde{\varphi})} \leqslant \|f\|_{\ddot{H}^p_w(\vec{A};\varphi)} \leqslant C_2\|f\|_{\ddot{H}^p_w(\vec{A};\widetilde{\varphi})}.$$

When $p \in (0,1]$, the space $\mathcal{L}_{p,w}(\vec{A})$ is also well defined in the above sense. Moreover, the spaces $\ddot{H}^p_w(\vec{A})$ and $\mathcal{L}_{p,w}(\vec{A})$, equipped respectively with $\|\cdot\|_{\ddot{H}^p_w(\vec{A})}$ and $\|\cdot\|_{\mathcal{L}_{p,w}(\vec{A})}$, are also complete.

We have the following equivalences on the product Hardy spaces $\widetilde{H}^p_w(\vec{A})$ and $\ddot{H}^p_w(\vec{A})$.

Theorem 2.14. *Let $w \in \mathcal{A}_\infty(\vec{A})$ and $p \in (0,1]$. Then $f \in \widetilde{H}^p_w(\vec{A})$ if and only if $f \in \ddot{H}^p_w(\vec{A})$. Moreover, their corresponding quasi-norms are equivalent.*

The proof of Theorem 2.14 is given in Section 3.

To show that $\mathcal{L}_{p,w}(\vec{A})$ is the dual space of $\ddot{H}^p_w(\vec{A})$, we first establish the duality between their corresponding sequence spaces.

Proposition 2.15. *Let $w \in \mathcal{A}_\infty(\vec{A})$ and $p \in (0,1]$. Then $(\ddot{h}^p_w(\vec{A}))^* = \ell_{p,w}(\vec{A})$ in the following sense: for any $t \in \ell_{p,w}(\vec{A})$, the map*

$$L_t(h) := \langle t, h\rangle := \sum_{R\in\mathcal{R}} t_R \bar{h}_R$$

for any $h \in \ddot{h}^p_w(\vec{A})$ defines a continuous linear functional on $\ddot{h}^p_w(\vec{A})$ with norm

$$\|L_t\|_{(\ddot{h}^p_w(\vec{A}))^*} \leqslant C\|t\|_{\ell_{p,w}(\vec{A})},$$

where C is some positive constant, independent of t. Conversely, every $L \in (\ddot{h}^p_w(\vec{A}))^*$ is of this form for some $t \in \ell_{p,w}(\vec{A})$ with norm $\|t\|_{\ell_{p,w}(\vec{A})} \leqslant C\|L\|_{(\ddot{h}^p_w(\vec{A}))^*}$, where C is some positive constant, independent of L.

The proof of Proposition 2.15 is also given in Section 3. Applying Proposition 2.15 and Theorem 2.12, we can prove that $\mathcal{L}_{p,w}(\vec{A})$ is the dual space of $\ddot{H}^p_w(\vec{A})$ as follows.

Theorem 2.16. *Let $w \in \mathcal{A}_\infty(\vec{A})$ and $p \in (0,1]$. Then $(\ddot{H}^p_w(\vec{A}))^* = \mathcal{L}_{p,w}(\vec{A})$ in the following sense: there exists a positive constant C such that, for any $g \in \mathcal{L}_{p,w}(\vec{A})$, there exists a linear functional $L_g(f) := \langle f, g\rangle$ initially defined on $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, which has a uniquely continuous extension to $\ddot{H}^p_w(\vec{A})$ and $\|L_g\|_{(\ddot{H}^p_w(\vec{A}))^*} \leqslant C\|g\|_{\mathcal{L}_{p,w}(\vec{A})}$. Conversely, there exists a positive constant C such that every continuous linear functional L on $\ddot{H}^p_w(\vec{A})$ can be written as $L = L_g$ with some $g \in \mathcal{L}_{p,w}(\vec{A})$ and $\|g\|_{\mathcal{L}_{p,w}(\vec{A})} \leqslant C\|L\|_{(\ddot{H}^p_w(\vec{A}))^*}$.*

The proof of Theorem 2.16 is given in Section 3.

We shall finish the article by giving another equivalent description of the duals of anisotropic weighted Hardy spaces, which itself is quite interesting.

Definition 2.17. Let $p \in (0,1]$, $w \in \mathcal{A}_\infty(\vec{A})$ and $\varphi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. The *space* $\widetilde{\mathcal{L}}_{p,w}(\vec{A})$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ such that

$$\|f\|_{\widetilde{\mathcal{L}}_{p,w}(\vec{A})} := \left\{ \sup_{w(\Omega)<\infty} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \int_{\Omega} \sum_{j_1,j_2\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R},\, R\subset\Omega \\ \mathrm{scale}(R)=(-j_1,-j_2)}} \left|\varphi_{j_1,j_2} * f(x)\right|^2 \right.$$
$$\left. \times \frac{|R|}{w(R)} \chi_R(x)\, dx \right\}^{\frac{1}{2}} < \infty,$$

where Ω runs over all open sets in $\mathbb{R}^n \times \mathbb{R}^m$ with $w(\Omega) < \infty$.

Comparing with the definition of $\mathcal{L}_{p,w}(\vec{A})$, an interesting phenomenon appearing in the definition of $\widetilde{\mathcal{L}}_{p,w}(\vec{A})$ is that the integral in Definition 2.17 is not weighted. However, both spaces are equivalent as follows.

Corollary 2.18. *Let $w \in \mathcal{A}_\infty(\vec{A})$ and $p \in (0,1]$. Then $\mathcal{L}_{p,w}(\vec{A}) = \widetilde{\mathcal{L}}_{p,w}(\vec{A})$ with equivalent norms.*

Finally, we shall comment about the proof of Corollary 2.18. By adapting the proof of Theorem 2.12, we show that Theorem 2.12 also holds with $\mathcal{L}_{p,w}(\vec{A})$ replaced by $\widetilde{\mathcal{L}}_{p,w}(\vec{A})$, $p \in (0,1]$, albeit with the same sequence space $\ell_{p,w}(\vec{A})$. Once this is shown, Corollary 2.18 follows immediately, the details being omitted.

3. Proofs of main results

We first introduce some notation associated to expansive dilations.

Definition 3.1. Let A be an expansive dilation on $\mathbb{R}^n$ and $\sigma(A)$ the *set of all eigenvalues of* A. If A is diagonalizable over $\mathbb{C}$, then take $\lambda_- := \min_{\lambda\in\sigma(A)} |\lambda|$ and $\lambda_+ := \max_{\lambda\in\sigma(A)} |\lambda|$. Otherwise, let λ_- and λ_+ be some *positive real numbers such that* $1 < \lambda_- < \min_{\lambda\in\sigma(A)} |\lambda|$ *and* $\lambda_+ > \max_{\lambda\in\sigma(A)} |\lambda|$. Set $\zeta_+ := \frac{\ln \lambda_+}{\ln b}$ and $\zeta_- := \frac{\ln \lambda_-}{\ln b}$.

The following inequalities concerning A, ρ and the Euclidean norm $|\cdot|$ established in [1, Section 2] are used in the whole article:

$$\left[\rho(x)\right]^{\zeta_-} \lesssim |x| \lesssim \left[\rho(x)\right]^{\zeta_+} \quad \text{for all } \rho(x) \geqslant 1, \tag{3.1}$$
$$\left[\rho(x)\right]^{\zeta_+} \lesssim |x| \lesssim \left[\rho(x)\right]^{\zeta_-} \quad \text{for all } \rho(x) \leqslant 1, \tag{3.2}$$
$$b^{j\zeta_-}|x| \lesssim \left|A^j x\right| \lesssim b^{j\zeta_+}|x| \quad \text{for all } j \geqslant 0, \quad \text{and} \tag{3.3}$$
$$b^{j\zeta_+}|x| \lesssim \left|A^j x\right| \lesssim b^{j\zeta_-}|x| \quad \text{for all } j \leqslant 0. \tag{3.4}$$

Let $i \in \{1,2\}$. For dilation A_i on $\mathbb{R}^{n_i}$, let $\lambda_{i,-}$, $\lambda_{i,+}$, $\zeta_{i,-}$ and $\zeta_{i,+}$ be associated to A_i as above.

Proof of Lemma 2.3. To prove this lemma, we borrow some ideas from the proofs of [5, Lemma 2.8], [41, Lemma 2.1] and [42, Lemma 2.1].

(i) For any $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, $M \in \mathbb{Z}_+$ and $j_1, j_2 \in \mathbb{Z}$, let us first show that

$$\mathrm{I} := \sum_{j_1, j_2 \in \mathbb{Z}} \|f * \phi_{j_1,j_2}\|_M^* < \infty$$

by considering the following four cases.

We first assume that $j_1 \geqslant 0$ and $j_2 < 0$. Let $\beta := (\beta_1, \beta_2)$ with $\beta_1 \in \mathbb{Z}_+^n$ and $\beta_2 \in \mathbb{Z}_+^m$. Since $\operatorname{supp} \widehat{\phi^{(i)}}$ is compact and bounded away from the origin, then there exists a positive constant C such that $\operatorname{supp} \widehat{\phi^{(i)}} \subset \{\xi_i \in \mathbb{R}^{n_i}\colon 1/C \leqslant |\xi_i| \leqslant C\}$, $i \in \{1,2\}$. Moreover, noticing that $\widehat{\phi_{j_i}^{(i)}}(\xi_i) = \widehat{\phi^{(i)}}((A_i^*)^{-j_i}\xi_i)$, by [1, (3.13)] for $j_1 \geqslant 0$ and a similar proof for $j_2 < 0$, we conclude that, for any $M \in \mathbb{Z}_+$,

$$\begin{aligned} &\sup_{|\beta_1|=M} \big\|\partial^{\beta_1} \widehat{\phi_{j_1}^{(1)}}\big\|_{L^\infty(\mathbb{R}^n)} \lesssim 1 \quad \text{and} \\ &\sup_{|\beta_2|=M} \big\|\partial^{\beta_2} \widehat{\phi_{j_2}^{(2)}}\big\|_{L^\infty(\mathbb{R}^m)} \lesssim (\lambda_{2,+})^{-j_2 M} \sup_{|\beta_2|=M} \big\|\partial^{\beta_2} \widehat{\phi^{(2)}}\big\|_{L^\infty(\mathbb{R}^m)}. \end{aligned} \tag{3.5}$$

Therefore, by (3.5), we know that

$$\begin{aligned} &\|f * \phi_{j_1,j_2}\|_M^* \\ &\quad = \sup_{\xi \in \mathbb{R}^n \times \mathbb{R}^m} \sup_{|\beta| \leqslant M} \big|\partial^\beta (\widehat{f * \phi_{j_1,j_2}})(\xi)\big| \big(|\xi_1|^M + |\xi_1|^{-M}\big)\big(|\xi_2|^M + |\xi_2|^{-M}\big) \\ &\quad \leqslant \sup_{\xi \in \mathbb{R}^n \times \mathbb{R}^m} \sup_{|\beta| \leqslant M} \big|\partial^\beta \widehat{f}(\xi)\big| \sup_{|\beta_1| \leqslant M} \prod_{i=1}^2 \big|\partial^{\beta_i} \widehat{\phi_{j_i}^{(i)}}(\xi_i)\big| \big(|\xi_i|^M + |\xi_i|^{-M}\big) \\ &\quad \lesssim (\lambda_{2,+})^{-j_2 M} \sup_{\substack{1/C \leqslant |(A_1^*)^{-j_1}\xi_1| \leqslant C \\ 1/C \leqslant |(A_2^*)^{-j_2}\xi_2| \leqslant C}} \sup_{|\beta| \leqslant M} \big|\partial^\beta \widehat{f}(\xi)\big| \big(|\xi_1|^M + |\xi_1|^{-M}\big)\big(|\xi_2|^M + |\xi_2|^{-M}\big) \\ &\quad =: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4, \end{aligned}$$

where

$$\mathrm{I}_1 \sim (\lambda_{2,+})^{-j_2 M} \sup_{\substack{1/C \leqslant |(A_1^*)^{-j_1}\xi_1| \leqslant C \\ 1/C \leqslant |(A_2^*)^{-j_2}\xi_2| \leqslant C}} \sup_{|\beta| \leqslant M} \big|\partial^\beta \widehat{f}(\xi)\big| |\xi_1|^M |\xi_2|^{-M},$$

and, similarly, I_2, I_3 and I_4 are defined via replacing $|\xi_1|^M|\xi_2|^{-M}$ in the definition of I_1, respectively, by $|\xi_1|^M|\xi_2|^M$, $|\xi_1|^{-M}|\xi_2|^M$ and $|\xi_1|^{-M}|\xi_2|^{-M}$.

We shall only estimate I_1 since the other estimates are similar. Let d be the minimal positive integer such that $(\lambda_{2,-})^d/\lambda_{2,+} > 1$. Thus, by (3.3) and (3.4), we have

$$\begin{aligned}
\mathrm{I}_1 &\sim (\lambda_{2,+})^{-j_2M} \sup_{\substack{1/C\leqslant|(A_1^*)^{-j_1}\xi_1|\leqslant C\\ 1/C\leqslant|(A_2^*)^{-j_2}\xi_2|\leqslant C}} \sup_{|\beta|\leqslant M} |\xi_1|^{-1-dM}|\xi_2|^{dM+1}\big|\partial^\beta \widehat{f}(\xi)\big|\big(|\xi_1|/|\xi_2|\big)^{(d+1)M+1} \\
&\lesssim (\lambda_{2,+})^{-j_2M} \sup_{\substack{1/C\leqslant|\xi_1|\leqslant C\\ 1/C\leqslant|\xi_2|\leqslant C}} \big|\big(A_1^*\big)^{j_1}\xi_1\big|^{-dM-1}\big|\big(A_2^*\big)^{j_2}\xi_2\big|^{dM+1}\|f\|^*_{(d+1)M+1} \\
&\lesssim (\lambda_{1,-})^{-j_1(dM+1)}\left(\frac{(\lambda_{2,-})^d}{\lambda_{2,+}}\right)^{j_2M}(\lambda_{2,-})^{j_2}\|f\|^*_{(d+1)M+1} \\
&\lesssim (\lambda_{1,-})^{-j_1(dM+1)}(\lambda_{2,-})^{j_2}\|f\|^*_{(d+1)M+1}.
\end{aligned}$$

Hence, $\sum_{j_1\geqslant 0,\, j_2<0}\mathrm{I}_1 \lesssim \|f\|^*_{(d+1)M+1}$. Similarly, we have

$$\sum_{j_1\geqslant 0,\, j_2<0}(\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4)\lesssim \|f\|^*_{(d+1)M+1}.$$

Thus,

$$\sum_{j_1\geqslant 0,\, j_2<0}\|f*\phi_{j_1,j_2}\|^*_M \lesssim \|f\|^*_{(d+1)M+1},$$

which is the desired estimate for this case.

In the remaining cases when $j_1\geqslant 0$ and $j_2\geqslant 0$, or $j_1<0$ and $j_2\geqslant 0$, or $j_1<0$ and $j_2<0$, we obtain similar estimates with d replaced by

$$\widetilde{d} := \min\bigl\{\ell\in\mathbb{N}:\ (\lambda_{i,-})^\ell/\lambda_{i,+}>1,\ i\in\{1,2\}\bigr\}.$$

Thus, combining these estimates, we obtain

$$\sum_{j_1,j_2\in\mathbb{Z}}\|f*\phi_{j_1,j_2}\|^*_M \lesssim \|f\|^*_{(\widetilde{d}+1)M+1}. \tag{3.6}$$

This implies that the series $\sum_{j_1,j_2\in\mathbb{Z}} f*\phi_{j_1,j_2}$ converges unconditionally in the semi-norms of $\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$.

Let

$$f_0 := \sum_{j_1,j_2\in\mathbb{Z}}\phi_{j_1,j_2}*f\in\mathcal{S}_\infty\bigl(\mathbb{R}^n\times\mathbb{R}^m\bigr).$$

For all $\xi\in(\mathbb{R}^n\times\mathbb{R}^m)\setminus\{(\xi_1,\xi_2)\in\mathbb{R}^n\times\mathbb{R}^m:\ \xi_1=\vec{0}_n \text{ or } \xi_2=\vec{0}_m\}$, by (2.2), we have $\sum_{j_1,j_2\in\mathbb{Z}}\widehat{\phi}((A_1^*)^{j_1}\xi_1,(A_2^*)^{j_2}\xi_2)=1$. Thus, we obtain

$$\widehat{f_0} = \sum_{j_1,j_2\in\mathbb{Z}}\widehat{\phi_{j_1,j_2}*f} = \sum_{j_1,j_2\in\mathbb{Z}}\widehat{\phi}_{j_1,j_2}\widehat{f} = \widehat{f}$$

in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$. Noticing that the Fourier transform is a homeomorphism of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ onto itself (see, for example, [34]), we obtain $f_0 = f$. Thus, (2.3) holds for any $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$.

A standard duality argument shows that (2.3) also holds for any $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Indeed, let $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Since $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ is a dual, endowed with the weak-$*$ topology, of the locally convex space $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ if and only if there exist a positive constant C_f and $M \in \mathbb{Z}_+$ such that, for all $\phi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\left|\langle f, \phi\rangle\right| \leqslant C_f \|\phi\|_M^*.$$

This observation and (3.6) further imply that, for all $\theta \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\left|\left\langle f, \sum_{j_1,j_2\in\mathbb{Z}} \widetilde{\phi}_{j_1,j_2} * \theta\right\rangle\right| \leqslant C_f \sum_{j_1,j_2\in\mathbb{Z}} \|\widetilde{\phi}_{j_1,j_2} * \theta\|_M^* \lesssim \|\phi\|^*_{(\widetilde{d}+1)M+1},$$

where $\widetilde{\phi}(\cdot) := \overline{\phi(-\cdot)}$. From this and the completeness of $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, it follows that $\sum_{j_1,j_2\in\mathbb{Z}} \phi_{j_1,j_2} * f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Thus, for all $\theta \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, it holds true that

$$\left\langle \sum_{j_1,j_2\in\mathbb{Z}} \phi_{j_1,j_2} * f, \theta\right\rangle = \left\langle f, \sum_{j_1,j_2\in\mathbb{Z}} \widetilde{\phi}_{j_1,j_2} * \theta\right\rangle = \langle f, \theta\rangle.$$

This finishes the proof of part (i) of Lemma 2.3.

(ii) Let (φ, ψ) be an admissible triplet of dual frame wavelets in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ as in Definition 2.2. For $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ (resp. $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$), by (2.3) with $\phi := \psi * \widetilde{\varphi}$, we conclude that

$$f = \sum_{j_1,j_2\in\mathbb{Z}} \psi_{j_1,j_2} * \widetilde{\varphi}_{j_1,j_2} * f \tag{3.7}$$

in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ (resp. $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$).

For $\vec{A} := (A_1, A_2)$, $j_1, j_2 \in \mathbb{Z}$ and $\vec{k} := (k_1, k_2) \in \mathbb{Z}^n \times \mathbb{Z}^m$, let

$$\vec{A}_{j_1,j_2} := \begin{pmatrix} A_1^{j_1} & 0_{n\times m} \\ 0_{m\times n} & A_2^{j_2} \end{pmatrix}, \qquad \vec{A}^*_{j_1,j_2} := \begin{pmatrix} (A_1^*)^{j_1} & 0_{n\times m} \\ 0_{m\times n} & (A_2^*)^{j_2} \end{pmatrix} \quad \text{and} \quad \vec{k} := \begin{pmatrix} k_1 \\ k_2 \end{pmatrix},$$

where $0_{n\times m}$ denotes the $n \times m$ matrix with all entries 0, and $0_{m\times n}$ is similarly defined. Let $g := \widetilde{\varphi}_{j_1,j_2} * f$. We first claim that, for all $j_1, j_2 \in \mathbb{Z}$ and $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ (resp. $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$),

$$g * \psi_{j_1,j_2}(\cdot) = \sum_{k_1\in\mathbb{Z}^n,\, k_2\in\mathbb{Z}^m} b_1^{-j_1} b_2^{-j_2} g(\vec{A}_{-j_1,-j_2}\vec{k})\psi_{j_1,j_2}(\cdot - \vec{A}_{-j_1,-j_2}\vec{k}) \tag{3.8}$$

in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ (resp. $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$). Assuming this claim for the moment, combining this with (3.7), we see that (2.4) holds in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ (resp. $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$).

Now let us first prove the claim (3.8) for $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. By $\widehat{\varphi_{j_i}^{(i)}}(\cdot) = \widehat{\varphi^{(i)}}((A_i^*)^{-j_i}\cdot)$ and $\operatorname{supp}\widehat{\varphi^{(i)}} \subset ([-\pi,\pi]^{n_i} \setminus \{\vec{0}_{n_i}\})$ with $i \in \{1,2\}$, we see that $g \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ with $\operatorname{supp}\widehat{g} \subset \vec{A}^*_{j_1,j_2}([-\pi,\pi]^{n+m})$. Then, by using the Fourier orthonormal basis

$$\left\{\frac{b_1^{-j_1/2}b_2^{-j_2/2}}{(2\pi)^{(n+m)/2}}e^{-i\langle \vec{A}_{-j_1,-j_2}\vec{k},\xi\rangle}\right\}_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m}$$

of $L^2(\vec{A}^*_{j_1,j_2}([-\pi,\pi]^{n+m}))$, we know that, for all $\xi \in \vec{A}^*_{j_1,j_2}([-\pi,\pi]^{n+m})$,

$$\widehat{g}(\xi) = \sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} \frac{b_1^{-j_1}b_2^{-j_2}}{(2\pi)^{n+m}}\left[\int_{\vec{A}_{j_1,j_2}([-\pi,\pi]^{n+m})} \widehat{g}(y)e^{i\langle \vec{A}_{-j_1,-j_2}\vec{k},y\rangle}\,dy\right]e^{-i\langle \vec{A}_{-j_1,-j_2}\vec{k},\xi\rangle}$$

in $L^2(\vec{A}^*_{j_1,j_2}([-\pi,\pi]^{n+m}))$. Since $\operatorname{supp}\widehat{g} \subset \vec{A}^*_{j_1,j_2}([-\pi,\pi]^{n+m})$, $\vec{A}^*_{j_1,j_2}([-\pi,\pi]^{n+m})$ can be replaced by $\mathbb{R}^n \times \mathbb{R}^m$ in the above integral. Thus, by the Fourier inversion formula, we find that, for any $\xi \in \vec{A}^*_{j_1,j_2}([-\pi,\pi]^{n+m})$,

$$\widehat{g}(\xi) = \sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} b_1^{-j_1}b_2^{-j_2}g(\vec{A}_{-j_1,-j_2}\vec{k})e^{-i\langle \vec{A}_{-j_1,-j_2}\vec{k},\xi\rangle}$$

in $L^2(\vec{A}^*_{j_1,j_2}([-\pi,\pi]^{n+m}))$. Noticing that $\operatorname{supp}\widehat{\psi}_{j_1,j_2} \subset \vec{A}^*_{j_1,j_2}([-\pi,\pi]^{n+m})$, we can replace $\widehat{g}$ by its periodic extension without altering the product $\widehat{g}\widehat{\psi_{j_1,j_2}}$. Using $g * \psi_{j_1,j_2} = (\widehat{g}\widehat{\psi_{j_1,j_2}})^\vee$ with $f^\vee(\cdot) := \widehat{f}(-\cdot)$, we obtain

$$\begin{aligned}(g*\psi_{j_1,j_2})(x) &= \sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} b_1^{-j_1}b_2^{-j_2}g(\vec{A}_{-j_1,-j_2}\vec{k})\big(e^{-i\langle \vec{A}_{-j_1,-j_2}\vec{k},\xi\rangle}\widehat{\psi_{j_1,j_2}}(\xi)\big)^\vee(x)\\ &= \sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} b_1^{-j_1}b_2^{-j_2}g(\vec{A}_{-j_1,-j_2}\vec{k})\psi_{j_1,j_2}(x-\vec{A}_{-j_1,-j_2}\vec{k}) \qquad (3.9)\end{aligned}$$

holds in $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ and hence pointwise.

To prove that (3.9) holds in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, we claim that, for any $M \in \mathbb{Z}_+$ and $\vec{k} \in \mathbb{Z}^n \times \mathbb{Z}^m$,

$$\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} b_1^{-j_1}b_2^{-j_2}\big|g(\vec{A}_{-j_1,-j_2}\vec{k})\big|\big\|\psi_{j_1,j_2}(\cdot-\vec{A}_{-j_1,-j_2}\vec{k})\big\|_M^* < \infty. \qquad (3.10)$$

Assuming the claim (3.10) for the moment, combining this with the completeness of $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, we see that

$$\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} b_1^{-j_1}b_2^{-j_2}g(\vec{A}_{-j_1,-j_2}\vec{k})\psi_{j_1,j_2}(\cdot-\vec{A}_{-j_1,-j_2}\vec{k}) \in \mathcal{S}_\infty\big(\mathbb{R}^n \times \mathbb{R}^m\big).$$

By this and (3.9), we find that (3.8) holds in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. This shows that (2.4) holds true in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$.

The proof of (3.10) is similar to the estimate of I in (i). For simplicity, we only prove the claim (3.10) when $j_1 \geqslant 0$ and $j_2 < 0$. In this case, for any $M \in \mathbb{Z}_+$ and $\vec{k} \in \mathbb{Z}^n \times \mathbb{Z}^m$, by the chain rule and [1, (3.13)] for $j_1 \geqslant 0$ and a similar proof for $j_2 < 0$, we have

$$\begin{aligned}
&\big\|\psi_{j_1,j_2}(\cdot - \vec{A}_{-j_1,-j_2}\vec{k})\big\|_M^* \\
&\quad = \sup_{\xi\in\mathbb{R}^n\times\mathbb{R}^m}\sup_{|\gamma|\leqslant M}\big|\partial^\gamma\big[e^{-i\langle\vec{A}_{-j_1,-j_2}\vec{k},\xi\rangle}\widehat{\psi}_{j_1,j_2}(\xi)\big]\big|\prod_{i=1}^2\big(|\xi_i|^M + |\xi_i|^{-M}\big) \\
&\quad \lesssim \sup_{\xi\in\mathbb{R}^n\times\mathbb{R}^m}\sup_{|\gamma|\leqslant M}\big|\partial^\gamma\big(e^{-i\langle\vec{A}_{-j_1,-j_2}\vec{k},\xi\rangle}\big)\big|\|\psi_{j_1,j_2}\|_M^* \\
&\quad \lesssim \|\psi_{j_1,j_2}\|_M^*\sup_{|\gamma_1|\leqslant M}\big[(\lambda_{1,-})^{-j_1}|k_1|\big]^{|\gamma_1|}\sup_{|\gamma_2|\leqslant M}\big[(\lambda_{2,+})^{-j_2}|k_2|\big]^{|\gamma_2|} \\
&\quad \lesssim \|\psi_{j_1,j_2}\|_M^*|k_1|^M\big[(\lambda_{2,+})^{-j_2}|k_2|\big]^M.
\end{aligned}$$

From this and $|g(\vec{A}_{-j_1,-j_2}y)| \lesssim \prod_{i=1}^2[1+\rho_i(A_i^{-j_i}y_i)]^{-(2+M\zeta_{i,+})}$ with $\zeta_{i,+}$ as in Definition 3.1, it follows that

$$\begin{aligned}
&\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m}\big\|b_1^{-j_1}b_2^{-j_2}g(\vec{A}_{-j_1,-j_2}\vec{k})\psi_{j_1,j_2}(\cdot - \vec{A}_{-j_1,-j_2}\vec{k})\big\|_M^* \\
&\quad \lesssim \|\psi_{j_1,j_2}\|_M^*\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m}\big|g(\vec{A}_{-j_1,-j_2}\vec{k})\big||k_1|^M\big[(\lambda_{2,+})^{-j_2}|k_2|\big]^M \\
&\quad \lesssim \|\psi_{j_1,j_2}\|_M^*(\lambda_{2,+})^{-j_2M}\prod_{i=1}^2\int_{\mathbb{R}^{n_i}}\frac{|y_i|^M}{[1+\rho_i(A_i^{-j_i}y_i)]^{2+M\zeta_{i,+}}}\,dy_i \lesssim \|\psi_{j_1,j_2}\|_M^*,
\end{aligned}$$

which is a desired estimate and hence shows the above claim (3.10).

It remains to prove that (3.8) also holds true for any $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Let $g := \widetilde{\varphi}_{j_1,j_2} * f$. It is well known that g is a slowly increasing C^∞ function on $\mathbb{R}^n \times \mathbb{R}^m$ (see, for example, [34]).

For $\delta > 0$, let $g_\delta(\cdot) := \gamma(\delta\cdot)g(\cdot)$, where $\gamma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ satisfies $\gamma(\vec{0}_n, \vec{0}_m) = 1$ and $\operatorname{supp}\widehat{\gamma}$ is compact. Then $g_\delta \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$. If $\delta > 0$ is sufficiently small, we further have $\operatorname{supp}\widehat{g_\delta} \subset (\vec{A}^*_{j_1,j_2}([-\pi,\pi]^{n+m}))$. By the already shown part of (3.8), we know that

$$\begin{aligned}
\psi_{j_1,j_2} * g_\delta(\cdot) &= b_1^{-j_1}b_2^{-j_2}\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m}g_\delta(\vec{A}_{-j_1,-j_2}\vec{k})\psi_{j_1,j_2}(\cdot - \vec{A}_{-j_1,-j_2}\vec{k}) \\
&= b_1^{-j_1}b_2^{-j_2}\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m}g_\delta(\cdot - \vec{A}_{-j_1,-j_2}\vec{k})\psi_{j_1,j_2}(\vec{A}_{-j_1,-j_2}\vec{k}) \qquad (3.11)
\end{aligned}$$

holds in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$.

Assume that g is at most polynomially increasing with order $M \in \mathbb{Z}_+$. Since $\psi_{j_1,j_2} \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, then, for any fixed $x \in \mathbb{R}^n \times \mathbb{R}^m$, we have

$$\begin{aligned}
&\big|g_\delta(x - \vec{A}_{-j_1,-j_2}\vec{k})\psi_{j_1,j_2}(\vec{A}_{-j_1,-j_2}\vec{k})\big| \\
&\quad \leqslant C_{\gamma,j_1,j_2}|x - \vec{A}_{-j_1,-j_2}\vec{k}|^M\big(1 + |\vec{A}_{-j_1,-j_2}\vec{k}|\big)^{-(M+n+m+1)}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} b_1^{-j_1}b_2^{-j_2}|x - \vec{A}_{-j_1,-j_2}\vec{k}|^M\big(1 + |\vec{A}_{-j_1,-j_2}\vec{k}|\big)^{-(M+n+m+1)} \\
&\leqslant C_{\gamma,j_1,j_2}\int_{\mathbb{R}^n\times\mathbb{R}^m}|x - y|^M\big(1 + |y|\big)^{-(M+n+m+1)}\,dy < \infty.
\end{aligned}$$

By applying the Lebesgue dominated convergence theorem and taking the limit as $\delta \to 0$ in (3.11), we conclude that (3.8) converges pointwise.

Notice that, for any $\theta \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, since $\psi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, by [8, Lemma 5.5], we have

$$\big|\big\langle\psi_{j_1,j_2}(\cdot - \vec{A}_{-j_1,-j_2}\vec{k}), \theta\big\rangle\big| \leqslant C_{j_1,j_2}\prod_{i=1}^{2}\big[1 + \rho_i(k_i)\big]^{-2-M\zeta_{i,+}}.$$

From this and $|g_\delta(\vec{A}_{-j_1,-j_2}\vec{k})| \leqslant C_\gamma(1 + |\vec{A}_{-j_1,-j_2}\vec{k})^M \leqslant C_{\gamma,j_1,j_2}[1 + \rho_i(k_i)]^{M\zeta_{i,+}}$, it follows that

$$\begin{aligned}
&\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} b_1^{-j_1}b_2^{-j_2}\big|g_\delta(\vec{A}_{-j_1,-j_2}\vec{k})\big|\big|\big\langle\psi_{j_1,j_2}(\cdot - \vec{A}_{-j_1,-j_2}), \theta\big\rangle\big| \\
&\leqslant C_{\gamma,j_1,j_2}\prod_{i=1}^{2}\int_{\mathbb{R}^{n_i}}\big[1 + \rho_i(y_i)\big]^{-2}\,dy_i < \infty.
\end{aligned}$$

This observation, together with (3.11) and the Lebesgue dominated convergence theorem, implies that, for any $\theta \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\begin{aligned}
\langle\psi_{j_1,j_2} * g, \theta\rangle &= \lim_{\delta\to 0}\langle\psi_{j_1,j_2} * g_\delta, \theta\rangle \\
&= \lim_{\delta\to 0}\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} b^{-j}g_\delta(\vec{A}_{-j_1,-j_2}\vec{k})\big\langle\psi_{j_1,j_2}(\cdot - \vec{A}_{-j_1,-j_2}\vec{k}), \theta\big\rangle \\
&= \sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} b_1^{-j_1}b_2^{-j_2}g\big(\vec{A}^{-j_1,-j_2}\vec{k}\big)\big\langle\psi_{j_1,j_2}(\cdot - \vec{A}_{-j_1,-j_2}\vec{k}), \theta\big\rangle.
\end{aligned}$$

Thus, (3.8) holds in $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, which completes the proof of Lemma 2.3. □

Proof of Proposition 2.8. Noticing that $\mathcal{S}_0'(\mathbb{R}^n \times \mathbb{R}^m) \subset \mathcal{S}_\infty'(\mathbb{R}^n \times \mathbb{R}^m)$, we obviously have $H^p_w(\vec{A}) \subset \widetilde{H}^p_w(\vec{A})$.

Conversely, let (Ψ, Φ) be as in Definition 2.6. For any $f \in \widetilde{H}^p_w(\vec{A})$, by Lemma 2.3(i) with $\phi := \Psi * \Phi$, we obtain $f = \sum_{j_1,j_2\in\mathbb{Z}} \Psi_{j_1,j_2} * \Phi_{j_1,j_2} * f$ in $\mathcal{S}_\infty'(\mathbb{R}^n \times \mathbb{R}^m)$. This result serves as a replacement of [8, Proposition 2.16] which is applicable only to elements in $\mathcal{S}_0'(\mathbb{R}^n \times \mathbb{R}^m)$. By repeating the proof of [8, Lemma 4.6] with this modification, we obtain the atomic decomposition $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $\mathcal{S}_\infty'(\mathbb{R}^n \times \mathbb{R}^m)$, where $\{a_j\}_{j\in\mathbb{N}}$ are $(p, q, \vec{s})_w$-atoms as in [8, Definition 4.2], $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and $(\sum_{j\in\mathbb{N}} |\lambda_j|^p)^{1/p} \lesssim \|f\|_{\widetilde{H}^p_w(\vec{A})}$. Now, for any $\varphi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, by the proof of [8, Lemma 4.10] (see [8, p. 424]), we know that

$$\left| \int\limits_{\mathbb{R}^n\times\mathbb{R}^m} a_j(x)\varphi(x)\,dx \right| = \left| a_j * \widetilde{\varphi}(0) \right| \lesssim 1,$$

where $\widetilde{\varphi}(\cdot) := \varphi(-\cdot)$.

Thus, if we define

$$\langle \breve{f}, \varphi \rangle := \sum_{j\in\mathbb{N}} \lambda_j \int\limits_{\mathbb{R}^n\times\mathbb{R}^m} a_j(x)\varphi(x)\,dx,$$

then

$$\left|\langle \breve{f}, \varphi \rangle\right| \lesssim \sum_{j\in\mathbb{N}} |\lambda_j| \lesssim \|f\|_{\widetilde{H}^p_w(\vec{A})}.$$

Therefore, $\breve{f} \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$, $\breve{f} = f$ in $\mathcal{S}_\infty'(\mathbb{R}^n \times \mathbb{R}^m)$ and $\breve{f} = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$, which, together with [8, Theorem 4.5], implies that $\breve{f} \in H^p_w(\vec{A})$ and $\|\breve{f}\|_{H^p_w(\vec{A})} \lesssim \|f\|_{\widetilde{H}^p_w(\vec{A})}$.

Now let us prove that the extension is unique. Assume that there exist two extensions $\breve{f}_1, \breve{f}_2 \in H^p_w(\vec{A})$ with $\breve{f}_1 = \breve{f}_2 = f$ in $\mathcal{S}_\infty'(\mathbb{R}^n \times \mathbb{R}^m)$. We need to show that

$$g := \breve{f}_1 - \breve{f}_2 = 0 \in \mathcal{S}'\big(\mathbb{R}^n \times \mathbb{R}^m\big).$$

Set $\mathscr{E} := \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m\colon\ x_1 = \vec{0}_n \text{ or } x_2 = \vec{0}_m\}$. Let us first prove $\operatorname{supp}\widehat{g} \subset \mathscr{E}$. Take any $x \in (\mathbb{R}^n \times \mathbb{R}^m) \setminus \mathscr{E}$ and sufficiently small positive numbers δ_1 and δ_2 such that

$$\big(B_{\rho_1}(x_1, \delta_1) \times B_{\rho_2}(x_2, \delta_2)\big) \cap \mathscr{E} = \emptyset.$$

Then, for any $\varphi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ with $\operatorname{supp}\varphi \subset (B_{\rho_1}(x_1, \delta_1) \times B_{\rho_2}(x_2, \delta_2))$, we have $\langle \widehat{g}, \varphi \rangle = 0$. Indeed, for any $(x_1', x_2') \in \mathbb{R}^n \times \mathbb{R}^m$ and $\alpha_1 \in \mathbb{Z}_+^n, \alpha_2 \in \mathbb{Z}_+^m$, we see that

$$\partial^{\alpha_1}\varphi\big(\vec{0}_n, x_2'\big) = \partial^{\alpha_2}\varphi\big(x_1', \vec{0}_m\big) = 0,$$

which implies that $\widehat{\varphi} \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. From this and $g = 0$ in $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, it follows that $\langle \widehat{g}, \varphi\rangle = \langle g, \widehat{\varphi}\rangle = 0$. Thus, $\operatorname{supp}\widehat{g} \subset \mathscr{E}$.

Finally, let us prove $g = 0$ in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$. Since $\breve{f}_1, \breve{f}_2 \in H^p_w(\vec{A})$, we have $g \in H^p_w(\vec{A})$. By this and Lemma 4.10 in [8], we conclude that $g \in \mathcal{S}'_0(\mathbb{R}^n \times \mathbb{R}^m)$. Let $\phi := \phi^{(1)} \otimes \phi^{(2)}$ be as in Lemma 2.3(i). Then, there exist two integers k_i and ℓ_i such that $k_i \geqslant \ell_i$ and $\operatorname{supp}\widehat{\phi^i} \subset B_{\rho_i}(\vec{0}_{n_i}, b_i^{k_i}) \setminus B_{\rho_i}(\vec{0}_{n_i}, b_i^{\ell_i})$, $i \in \{1,2\}$. By the Calderón reproducing formula [8, Lemma 2.15], we have $g = \sum_{j_1,j_2\in\mathbb{Z}} g * \phi_{j_1,j_2}$ with convergence in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$. Therefore, for any $\varphi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, we obtain

$$\langle g, \varphi\rangle = \sum_{j_1,j_2\in\mathbb{Z}} \langle g * \phi_{j_1,j_2}, \varphi\rangle = \sum_{j_1,j_2\in\mathbb{Z}} \big\langle \widehat{g}(\cdot)\widehat{\phi}\big(\big(A_1^*\big)^{-j_1}\cdot, \big(A_2^*\big)^{-j_2}\cdot\big), \widehat{\varphi}(-\cdot)\big\rangle. \tag{3.12}$$

Observe that, for any $j_1, j_2 \in \mathbb{Z}$,

$$\begin{aligned}&\operatorname{supp}\widehat{\phi}\big(\big(A_1^*\big)^{-j_1}\cdot, \big(A_2^*\big)^{-j_2}\cdot\big)\\ &\quad\subset \big[B_{\rho_1}\big(\vec{0}_n, b_1^{k_1+j_1}\big) \setminus B_{\rho_1}\big(\vec{0}_n, b_1^{\ell_1+j_1}\big)\big] \times \big[B_{\rho_2}\big(\vec{0}_m, b_2^{k_2+j_2}\big) \setminus B_{\rho_2}\big(\vec{0}_m, b_2^{\ell_2+j_2}\big)\big].\end{aligned}$$

From this and $\operatorname{supp}\widehat{g} \subset \mathscr{E}$, it follows that, for any $j_1, j_2 \in \mathbb{Z}$,

$$(\operatorname{supp}\widehat{g}) \cap \big(\operatorname{supp}\widehat{\phi}\big(\big(A_1^*\big)^{-j_1}\cdot, \big(A_2^*\big)^{-j_2}\cdot\big)\big) = \emptyset.$$

Combining this with (3.12), we obtain $\langle g, \varphi\rangle = 0$. This finishes the proof of Proposition 2.8. $\square$

The proof of Theorem 2.12 needs a series of technical lemmas. First, we need to show that the φ-transform T_ψ as in Definition 2.11 is well defined, respectively, on $\ddot{h}^p_w(\vec{A})$ and $\ell_{p,w}(\vec{A})$. Let us begin with the following technical lemma.

Lemma 3.2. *Let $\Phi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and $\Psi := \psi^{(1)} \otimes \psi^{(2)}$, where $\psi^{(i)} \in \mathcal{S}_\infty(\mathbb{R}^{n_i})$, $i \in \{1,2\}$. For any positive constants L_1 and L_2, there exist positive integers N_1 and N_2 and positive constant C, depending only on L_1 and L_2, such that, for all $P, Q \in \mathcal{R}$*

$$\big|\langle \Psi_Q, \Phi_P\rangle\big| \leqslant C\|\Psi\|_{N_1,N_2}\|\Phi\|_{N_1,N_2} \prod_{i=1}^2 \bigg\{\bigg[1 + \frac{\rho_i(x_{Q_i} - x_{P_i})}{|Q_i| \vee |P_i|}\bigg]^{-L_i} \bigg(\frac{|Q_i|}{|P_i|} \wedge \frac{|P_i|}{|Q_i|}\bigg)^{L_i}\bigg\},$$

where

$$\|\Psi\|_{N_1,N_2} := \sup_{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^m} \sup_{|\gamma_1|\leqslant N_1,\, |\gamma_2|\leqslant N_2} \big(1 + |x_1|\big)^{N_1}\big(1 + |x_2|\big)^{N_2}\big|\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\Psi(x_1, x_2)\big|.$$

Proof. To prove this lemma, we need the estimate [3, (3.18)], namely, for any $\varphi, \psi \in \mathcal{S}_\infty(\mathbb{R}^n)$ and positive constant L, there exist a positive constant C and a positive integer N, depending only on L, such that, for all $P, Q \in \mathcal{Q}_1$,

$$\left|\langle\varphi_Q,\phi_P\rangle\right| \leqslant C\|\varphi\|_N\|\phi\|_N\left(1+\frac{\rho(x_Q-x_P)}{|Q|\vee|P|}\right)^{-L}\left(\frac{|Q|}{|P|}\wedge\frac{|P|}{|Q|}\right)^{L}, \tag{3.13}$$

where $\|\varphi\|_N := \sup_{x\in\mathbb{R}^n,\,|\gamma|\leqslant N}[1+|x|]^N|\partial^\gamma\varphi(x)|$.

For any $P,Q\in\mathcal{R},\Phi\in\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$ and $\Psi := \psi^{(1)}\otimes\psi^{(2)}$ with $\psi^{(i)}\in\mathcal{S}_\infty(\mathbb{R}^{n_i})$, $i\in\{1,2\}$, let $\Psi_Q := \psi^{(1)}_{Q_1}\otimes\psi^{(2)}_{Q_2}$ and Φ_P be as in (2.1). Moreover, for any $x := (x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^m$ and $P := P_1\times P_2\in\mathcal{R}$ with $P_i := A_i^{-j_i}([0,1)^{n_i}+k_i)$, $k_i\in\mathbb{Z}_i^{n_i}$ $i\in\{1,2\}$, we let

$$\Phi_{P_2}(x_1,x_2) := |P_2|^{-1/2}\Phi\big(x_1,A_2^{j_2}x_2-k_2\big).$$

Then it is easy to show that

$$\big\langle\psi^{(2)}_{Q_2},\Phi_P\big\rangle := \int_{\mathbb{R}^m}\psi^{(2)}_{Q_2}(x_2)\overline{\Phi_P(x_1,x_2)}\,dx_2 =: \big\langle\psi^{(2)}_{Q_2},\Phi_{P_2}\big\rangle_{P_1}\in\mathcal{S}_\infty\big(\mathbb{R}^n\big).$$

Consequently, using the fact that $\Phi_{P_2}(x_1,\cdot)\in\mathcal{S}_\infty(\mathbb{R}^m)$ for all $x_1\in\mathbb{R}^n$ and (3.13) twice (resp. with dilated cubes of $\mathbb{R}^n$ and $\mathbb{R}^m$), for any positive constants L_1 and L_2, there exist positive integers N_1 and N_2, depending only on L_1 and L_2, such that

$$\begin{aligned}
\left|\langle\Psi_Q,\Phi_P\rangle\right| &= \left|\big\langle\psi^{(1)}_{Q_1},\big\langle\psi^{(2)}_{Q_2},\Phi_{P_2}\big\rangle_{P_1}\big\rangle\right| \\
&\lesssim \big\|\psi^{(1)}\big\|_{N_1}\big\|\big\langle\psi^{(2)}_{Q_2},\Phi_{P_2}\big\rangle\big\|_{N_1}\left(1+\frac{\rho_1(x_{Q_1}-x_{P_1})}{|Q_1|\vee|P_1|}\right)^{-L_1}\left(\frac{|Q_1|}{|P_1|}\wedge\frac{|P_1|}{|Q_1|}\right)^{L_1} \\
&\lesssim \big\|\psi^{(1)}\big\|_{N_1}\sup_{x_1\in\mathbb{R}^n}\sup_{|\gamma_1|\leqslant N_1}\big(1+|x_1|\big)^{N_1}\big\|\psi^{(2)}\big\|_{N_2}\big\|\partial^{\gamma_1}_{x_1}\Phi(x_1,\cdot)\big\|_{N_2} \\
&\quad\times\prod_{i=1}^2\left\{\left[1+\frac{\rho_i(x_{Q_i}-x_{P_i})}{|Q_i|\vee|P_i|}\right]^{-L_i}\left(\frac{|Q_i|}{|P_i|}\wedge\frac{|P_i|}{|Q_i|}\right)^{L_i}\right\} \\
&\sim \|\Psi\|_{N_1,N_2}\|\Phi\|_{N_1,N_2}\prod_{i=1}^2\left\{\left[1+\frac{\rho_i(x_{Q_i}-x_{P_i})}{|Q_i|\vee|P_i|}\right]^{-L_i}\left(\frac{|Q_i|}{|P_i|}\wedge\frac{|P_i|}{|Q_i|}\right)^{L_i}\right\},
\end{aligned}$$

which completes the proof of Lemma 3.2. □

The following technical lemma is just [8, Proposition 2.10(i)].

Lemma 3.3. *(See [8].) Let $q\in(1,\infty)$ and $w\in\mathcal{A}_q(\vec{A})$. Then there exists a positive constant C such that, for all $x\in\mathbb{R}^n\times\mathbb{R}^m$ and $k_i\in\mathbb{Z}_+$, $\ell_i\in\mathbb{Z}$ with $i\in\{1,2\}$,*

$$\begin{aligned}
\frac{w(B_{\rho_1}(x_1,b_1^{k_1+\ell_1})\times B_{\rho_2}(x_2,b_2^{k_2+\ell_2}))}{w(B_{\rho_1}(x_1,b_1^{\ell_1})\times B_{\rho_2}(x_2,b_2^{\ell_2}))} &\leqslant C\frac{|B_{\rho_1}(x_1,b_1^{k_1+\ell_1})\times B_{\rho_2}(x_2,b_2^{k_2+\ell_2})|^q}{|B_{\rho_1}(x_1,b_1^{\ell_1})\times B_{\rho_2}(x_2,b_2^{\ell_2})|^q} \\
&\sim \big[b_1^{k_1}b_2^{k_2}\big]^q.
\end{aligned}$$

For any $w \in \mathcal{A}_\infty(\vec{A})$, the *critical index* of w is defined by

$$q_w := \inf\{q \in (1,\infty):\ w \in \mathcal{A}_q(\vec{A})\}. \tag{3.14}$$

Obviously, $q_w \in [1,\infty)$ and if $q_w \in (1,\infty)$, then $w \notin \mathcal{A}_{q_w}(\vec{A})$. Moreover, Johnson and Neugebauer [25, p. 254] gave an example of $w \notin \mathcal{A}_1(2I_{n\times n})$ such that $q_w = 1$ (see also [8]), where $I_{n\times n}$ denotes the $n\times n$ identity matrix.

Lemma 3.4. *Let $w \in \mathcal{A}_\infty(\vec{A})$ with q_w as in (3.14), $q \in (q_w,\infty)$ and $\delta \in \mathbb{R}$. Then there exist positive constants L_1, L_2 and C, depending on δ, such that, for all $j_1, j_2 \in \mathbb{Z}$,*

$$\sum_{R\in\mathcal{R},\,\mathrm{scale}(R)=(j_1,j_2)} \frac{[w(R)]^\delta}{\prod_{i=1}^2[1+\rho_i(x_{R_i})/(1\vee|R_i|)]^{L_i}} \leqslant C\prod_{i=1}^2 b_i^{(2q|\delta|+1)|j_i|}.$$

Proof. The proof of this lemma follows along the lines of its one parameter variant [3, Lemma 2.11]. The key observation is that the measure $w(x)\,dx$ is doubling with respect to the action of the 2-parameter group of dilations $\vec{A}_{j_1,j_2}$. Indeed, by Lemma 3.3, there exists a positive constant C such that, for all $x \in \mathbb{R}^n\times\mathbb{R}^m$, $r_1, r_2 > 0$, and $k_1, k_2 \in \mathbb{Z}_+$,

$$w\left(\prod_{i=1}^2 B_{\rho_i}\big(x_i,(b_i)^{k_i}r_i\big)\right) \leqslant Cb_1^{k_1q}b_2^{k_2q}w\left(\prod_{i=1}^2 B_{\rho_i}(x_i,r_i)\right).$$

This practically means that w is a product q-doubling measure albeit with a positive constant C. In particular, for any dilated rectangles $P := P_1\times P_2, R := R_1\times R_2 \in \mathcal{R}$ of the form $P_i := A_i^{j_i}([0,1)^{n_i}+k_i)$ and $R_i = A_i^{j_i}([0,1)^{n_i}+l_i)$, we have the following analogue of [3, (2.6)], namely,

$$w(R) \lesssim \prod_{i=1}^2\big[1+\rho_i(k_i-l_i)\big]^q w(P). \tag{3.15}$$

Mimicking the proof of [3, Lemma 2.11] by considering four cases that $j_1, j_2 \geqslant 0$, $j_1, j_2 < 0$, $j_1 \geqslant 0$, $j_2 < 0$ and $j_1 < 0$, $j_2 \geqslant 0$, we then obtain the desired estimate in Lemma 3.4. $\square$

Lemma 3.5. *Suppose that $w \in \mathcal{A}_\infty(\vec{A}), p \in (0,\infty)$ and $\psi := \psi^{(1)}\otimes\psi^{(2)}$, where $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$, $i \in \{1,2\}$, satisfies that $\operatorname{supp}\widehat{\psi^{(i)}}$ is compact and bounded away from the origin. Then, the inverse φ-transform $T_\psi : \ddot{h}^p_w(\vec{A}) \to \mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$ is well defined and continuous. The same result holds if $\ddot{h}^p_w(\vec{A})$ is replaced by $\ell_{p,w}(\vec{A})$, $p \in (0,1]$.*

Proof. For any $s \in \ddot{h}^p_w(\vec{A})$, by the definition of $\ddot{h}^p_w(\vec{A})$, we know that, for all $Q \in \mathcal{R}$,

$$|s_Q| \leqslant \|s\|_{\ddot{h}^p_w(\vec{A})}|Q|^{\frac12}\big[w(Q)\big]^{-1/p}.$$

Applying Lemma 3.2 for any $\phi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and $P := [0,1)^n \times [0,1)^m$, we see that, for all $Q \in \mathcal{R}$,

$$\left|\langle \psi_Q, \phi\rangle\right| \lesssim \|\phi\|_{N_1,N_2} \prod_{i=1}^{2} \left[1 + \frac{\rho_i(x_{Q_i})}{1 \vee |Q_i|}\right]^{-L_i} \left(|Q_i| \wedge |Q_i|^{-1}\right)^{L_i}.$$

Combining the above estimates with Lemma 3.4, we then find that, for sufficiently large L_1 and L_2,

$$\begin{aligned}
&\sum_{Q\in\mathcal{R}} |s_Q|\left|\langle \psi_Q, \phi\rangle\right| \\
&\quad\lesssim \|\phi\|_{N_1,N_2}\|s\|_{\ddot{h}^p_w(\vec{A})} \sum_{j_1,j_2\in\mathbb{Z}} \sum_{\mathrm{scale}(Q)=(j_1,j_2)} \prod_{i=1}^{2} b_i^{j_i/2-|j_i|L_i} \frac{[w(Q)]^{-1/p}}{[1+\frac{\rho_i(x_{Q_i})}{1\vee|Q_i|}]^{L_i}} \\
&\quad\lesssim \|\phi\|_{N_1,N_2}\|s\|_{\ddot{h}^p_w(\vec{A})} \sum_{j_1,j_2\in\mathbb{Z}} \prod_{i=1}^{2} b_i^{j_i/2+|j_i|(2q/p+1)-|j_i|L_i} \lesssim \|\phi\|_{N_1,N_2}\|s\|_{\ddot{h}^p_w(\vec{A})},
\end{aligned}$$

where $q \in (q_w, \infty)$. Thus, by the definition of $T_\psi s$, we conclude that, for all $\phi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\langle T_\psi s, \phi\rangle = \sum_{Q\in\mathcal{R}} s_Q \langle \psi_Q, \phi\rangle.$$

Moreover, for all $\phi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, it holds true that

$$\left|\langle T_\psi s, \phi\rangle\right| \lesssim \|\phi\|_{N_1,N_2}\|s\|_{\ddot{h}^p_w(\vec{A})},$$

which implies that $T_\psi : \ddot{h}^p_w(\vec{A}) \to \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ is continuous.

Now, for any $s \in \ell_{p,w}(\vec{A})$, by the definition of $\ell_{p,w}(\vec{A})$, we know that

$$|s_Q| \leqslant \|s\|_{\ell_{p,w}(\vec{A})} \left[w(Q)\right]^{1/p} |Q|^{-1/2} \quad \text{for all } Q \in \mathcal{R}.$$

Then, repeating the above proof for $\ddot{h}^p_w(\vec{A})$, we obtain the desired results for T_ψ on $\ell_{p,w}(\vec{A})$. This finishes the proof of Lemma 3.5. □

Motivated by [3, Definition 3.9], we introduce the notion of majorant sequences.

Definition 3.6. Given a complex-valued sequence $s := \{s_R\}_{R\in\mathcal{R}}$ and $r, \lambda > 0$, define its majorant sequence $s^*_{r,\lambda} := \{(s^*_{r,\lambda})_R\}_{R\in\mathcal{R}}$ by

$$\left(s^*_{r,\lambda}\right)_R := \left\{ \sum_{P\in\mathcal{R},\, \mathrm{scale}(P)=\mathrm{scale}(R)} \frac{|s_P|^r}{\prod_{i=1}^{2}[1+|R_i|^{-1}\rho_i(x_{R_i}-x_{P_i})]^\lambda} \right\}^{1/r}.$$

The spaces $\mathcal{L}_{p,w}(\vec{A})$, $\ell_{p,w}(\vec{A})$, and the sequences $\{(s^*_{r,\lambda})_Q\}_{Q\in\mathcal{R}}$ can also be defined "equivalently" via generalized dyadic rectangles associated to $\vec{A}$ in some sense. Precisely, let us begin with recalling the dyadic cubes associated to A introduced in [8, Lemma 2.3], which is a slight variant of [14, Theorem 11].

Lemma 3.7. *Let A be a dilation. Then there exists a set*

$$\dot{\mathcal{Q}}_1 := \{\dot{Q}^k_\alpha \subset \mathbb{R}^n:\ k\in\mathbb{Z},\ \alpha\in \mathrm{I}_k\}$$

of open subsets, where I_k is some index set, such that

(i) $|\mathbb{R}^n \setminus (\bigcup_\alpha \dot{Q}^k_\alpha)| = 0$ *for each fixed k and* $\dot{Q}^k_\alpha \cap \dot{Q}^k_\beta = \emptyset$ *if* $\alpha\neq\beta$;
(ii) *for any α, β, k, ℓ with $\ell \geqslant k$, either* $\dot{Q}^k_\alpha\cap\dot{Q}^\ell_\beta=\emptyset$ *or* $\dot{Q}^\ell_\alpha\subset\dot{Q}^k_\beta$;
(iii) *for each (ℓ,β) and each $k<\ell$, there exists a unique α such that* $\dot{Q}^\ell_\beta\subset\dot{Q}^k_\alpha$;
(iv) *there exist some negative integer v and positive integer u such that, for all $\dot{Q}^k_\alpha$ with $k\in\mathbb{Z}$ and $\alpha\in\mathrm{I}_k$, there exists $c_{\dot{Q}^k_\alpha}\in\dot{Q}^k_\alpha$ satisfying that, for all $x\in\dot{Q}^k_\alpha$,*

$$B_\rho\big(c_{\dot{Q}^k_\alpha}, b^{vk-u}\big)\subset \dot{Q}^k_\alpha\subset B_\rho\big(x, b^{vk+u}\big).$$

In what follows, for convenience, we call $\{\dot{Q}^k_\alpha\}_{k\in\mathbb{Z},\alpha\in\mathrm{I}_k}$ in Lemma 3.7 *dyadic cubes*. Also, for any dyadic cube $\dot{Q}^k_\alpha$ with $k\in\mathbb{Z}$ and $\alpha\in\mathrm{I}_k$, we always define $\ell(\dot{Q}^k_\alpha):=k$ to be its *level*.

Let A_i be a dilation on $\mathbb{R}^{n_i}$, and $\dot{\mathcal{Q}}_i$, $\ell(\dot{Q}_i)$, v_i, u_i *the same as in Lemma 3.7 corresponding to A_i for $i\in\{1,2\}$*. Let $\dot{\mathcal{R}}:=\dot{\mathcal{Q}}_1\times\dot{\mathcal{Q}}_2$. For $\dot{R}\in\dot{\mathcal{R}}$, we always write $\dot{R}:=\dot{R}_1\times\dot{R}_2$ with $\dot{R}_i\in\dot{\mathcal{Q}}_i$ and call $\dot{R}$ a *dyadic rectangle*. Moreover, we let $\ell(\dot{R}):=(\ell(\dot{R}_1),\ell(\dot{R}_2))$, and $\ell(\dot{R})\leqslant\ell(\dot{P})$ always means that $\ell(\dot{R}_i)\leqslant\ell(\dot{P}_i)$, $i\in\{1,2\}$.

Definition 3.8. For two sets $E_1\subset\mathbb{R}^n$ and $E_2\subset\mathbb{R}^m$, let $\prod_{i=1}^2 E_i := E_1\times E_2$. For any locally integrable function f on $\mathbb{R}^n\times\mathbb{R}^m$, the *strong maximal function* $\mathcal{M}_s(f)$ of f is defined by setting, for all $x\in\mathbb{R}^n\times\mathbb{R}^m$,

$$\mathcal{M}_s(f)(x):=\sup_{y\in\mathbb{R}^n\times\mathbb{R}^m,\, r_1,r_2>0}\ \sup_{x\in\prod_{i=1}^2 B_{\rho_i}(y_i,r_i)}\frac{1}{|\prod_{i=1}^2 B_{\rho_i}(y_i,r_i)|}\int_{\prod_{i=1}^2 B_{\rho_i}(y_i,r_i)}\big|f(z)\big|\,dz.$$

The following lemma comes from [3, Lemma 2.9(a)].

Lemma 3.9. *There exists a positive integer $\tau_i:=\tau_i(A_i,n_i)$ for all $Q_i:=A^{j_i}((0,1]^{n_i}+k_i)$ with $j_i\in\mathbb{Z}$ and $k_i\in\mathbb{Z}^{n_i}$, $i\in\{1,2\}$, such that*

$$B_{\rho_i}\big(c_{Q_i}, b_i^{j_i-\tau_i}\big)\subset Q_i\subset B_{\rho_i}\big(c_{Q_i}, b_i^{j_i+\tau_i}\big).$$

In what follows, for any $\alpha\in\mathbb{R}$, we denote by $\lfloor\alpha\rfloor$ the *maximal integer not more than* α. Recall that $\sharp E$ denotes the *cardinality* of the set E.

Lemma 3.10.

(i) *For any* $\dot{R}\in\dot{\mathcal{R}}$, *let* $U_{\dot{R}}:=U_{\dot{R}_1}\times U_{\dot{R}_2}$ *with*

$$U_{\dot{R}_i}:=\{R_i\in\mathcal{Q}_i\colon\ R_i\cap\dot{R}_i\neq\emptyset,\ \ell(\dot{R}_i)=\lfloor(\mathrm{scale}(R_i)-u_i)/v_i\rfloor\},\quad i\in\{1,2\}.$$

For any $R\in\mathcal{R}$, *let* $U_R:=U_{R_1}\times U_{R_2}$ *with*

$$U_{R_i}:=\{\dot{R}_i\in\dot{\mathcal{Q}}_i\colon\ R_i\cap\dot{R}_i\neq\emptyset,\ \ell(\dot{R}_i)=\lfloor(\mathrm{scale}(R_i)-u_i)/v_i\rfloor\},\quad i\in\{1,2\}.$$

Then there exists a positive integer $\widetilde{N}$ *such that, for all* $R\in\mathcal{R}$ *and* $\dot{R}\in\dot{\mathcal{R}}$, $\sharp U_{\dot{R}}+\sharp U_R\leqslant\widetilde{N}$. *Moreover, for all* $w\in\mathcal{A}_\infty(\vec{A})$, $R\in U_{\dot{R}}$ *and* $\dot{R}\in U_R$, $w(\dot{R})\sim w(R)$.

(ii) *There exists a positive constant* $\eta_0:=\eta_0(\vec{A},n,m)\in(0,1)$ *such that, for any open set* $\Omega\subset\mathbb{R}^n\times\mathbb{R}^m$,

$$\bigcup_{R\in\mathcal{R},\,R\subset\Omega}\ \bigcup_{\dot{R}\in U_R}\dot{R}\subset\Omega^{(0)}:=\{x\in\mathbb{R}^n\times\mathbb{R}^m\colon\ \mathcal{M}_s(\chi_\Omega)(x)>\eta_0\}.$$

Proof. (i) For any $R\in U_{\dot{R}}$, let $\widetilde{x}_i\in R_i\cap\dot{R}_i$, $i\in\{1,2\}$. Then, for any $x_i\in R_i$, by Lemma 3.9, Lemma 3.7(iv) and $\mathrm{scale}(R_i)\leqslant v_i\ell(\dot{R}_i)+u_i$, we obtain

$$\rho_i(x_i-c_{\dot{R}_i})\leqslant H_i^2[\rho_i(x_i-c_{R_i})+\rho_i(c_{R_i}-\widetilde{x}_i)+\rho_i(\widetilde{x}_i-c_{\dot{R}_i})]\leqslant 3H_i^2b_i^{v_i\ell(\dot{R}_i)+u_i+\tau_i},$$

which implies that

$$\bigcup_{R_i\in U_{\dot{R}_i}}R_i\subset B_{\rho_i}(c_{\dot{R}_i},3H_i^2b_i^{v_i\ell(\dot{R}_i)+u_i+\tau_i}),\quad i\in\{1,2\}.\tag{3.16}$$

From this, $v_i[\ell(\dot{R}_i)+1]+u_i<\mathrm{scale}(R_i)\leqslant v_i\ell(\dot{R}_i)+u_i$ and Lemma 3.9, it follows that

$$\sharp U_{\dot{R}}\leqslant v_1v_2\frac{|\prod_{i=1}^2B_{\rho_i}(c_{\dot{R}_i},3H_i^2b_i^{v_i\ell(\dot{R}_i)+u_i+\tau_i})|}{\prod_{i=1}^2b_i^{v_i[\ell(\dot{R}_i)+1]+u_i-\tau_i}}\lesssim 1.$$

Moreover, by (3.16), Lemma 3.3 and Lemma 3.7(iv), we have

$$w(R)\leqslant w\left(\prod_{i=1}^2B_{\rho_i}(c_{\dot{R}_i},3H_i^2b_i^{v_i\ell(\dot{R}_i)+u_i+\tau_i})\right)\lesssim w\left(\prod_{i=1}^2B_{\rho_i}(c_{\dot{R}_i},b_i^{v_i\ell(\dot{R}_i)-u_i})\right)\lesssim w(\dot{R}).$$

The converse inequality also holds true via an argument similar to the above and hence $w(\dot{R})\sim w(R)$.

Similarly, for any $R\in\mathcal{R}$, we also have $\sharp U_R\lesssim 1$ and, for all $\dot{R}\in U_R$, $w(R)\sim w(\dot{R})$.

(ii) For any $R \in \mathcal{R}$ and $R \subset \Omega$ with $\mathrm{scale}(R) = (j_1, j_2)$, $\dot{R} \in U_R$ and $x \in \dot{R}$, by an estimate similar to that of (3.16), we have $\dot{R}_i \subset B_{\rho_i}(c_{R_i}, 3H_i^2 b_i^{j_i - v_i + \tau_i})$. From this and Lemma 3.9, it follows that

$$\begin{aligned} \mathcal{M}_s(\chi_\Omega)(x) &\geqslant \frac{|\prod_{i=1}^2 B_{\rho_i}(c_{R_i}, 3H_i^2 b_i^{j_i - v_i + \tau_i}) \cap \Omega|}{|\prod_{i=1}^2 B_{\rho_i}(c_{R_i}, 3H_i^2 b_i^{j_i - v_i + \tau_i})|} \\ &\geqslant \frac{|\prod_{i=1}^2 B_{\rho_i}(c_{R_i}, b_i^{j_i - \tau_i})|}{|\prod_{i=1}^2 B_{\rho_i}(c_{R_i}, 3H_i^2 b_i^{j_i - v_i + \tau_i})|} > \frac{1}{10} \prod_{i=1}^2 (H_i^2)^{-1} b_i^{v_i - 2\tau_i} =: \eta_0, \end{aligned}$$

which completes the proof of Lemma 3.10. □

Definition 3.11. For any $R \in \mathcal{R}$ and U_R as in Lemma 3.10(i), let $\chi_{U_R}(\dot{R})$ be equal to one if $\dot{R} \in U_R$ or else zero. For any $\dot{R} \in \dot{\mathcal{R}}$ and $U_{\dot{R}}$ as in Lemma 3.10(i), let $\chi_{U_{\dot{R}}}(R)$ be similarly defined.

Lemma 3.12.

(i) *For any complex-valued sequence $s := \{s_R\}_{R \in \mathcal{R}}$, its induced sequence $\dot{s}$ is defined by setting $\dot{s} := \{\dot{s}_{\dot{R}}\}_{\dot{R} \in \dot{\mathcal{R}}}$, where $\dot{s}_{\dot{R}} := \sum_{R \in U_{\dot{R}}} |s_R|$ with $U_{\dot{R}}$ as in Lemma 3.10(i). Then, for any $w \in \mathcal{A}_\infty(\vec{A})$ and $p \in (0,1]$, there exists a positive constant C such that $\|\dot{s}\|_{\dot{\ell}_{p,w}(\vec{A})} \leqslant C\|s\|_{\ell_{p,w}(\vec{A})}$, where the definition of the norm $\|\cdot\|_{\dot{\ell}_{p,w}(\vec{A})}$ is the same as $\|\cdot\|_{\ell_{p,w}(\vec{A})}$ but R and $\mathcal{R}$ are, respectively, replaced by $\dot{R}$ and $\dot{\mathcal{R}}$.*

(ii) *For any $\lambda \in (0,\infty)$, the majorant sequence $\{(\dot{s}^*_{2,\lambda})_{\dot{R}}\}_{\dot{R} \in \dot{\mathcal{R}}}$ of $\dot{s}$ is defined to be the same as in Definition 3.6 but R, $\mathcal{R}$, x_{R_i}, x_{P_i}, $\mathrm{scale}(P)$ and $\mathrm{scale}(R)$ are, respectively, replaced by $\dot{R}$, $\dot{\mathcal{R}}$, $c_{\dot{R}_i}$, $c_{\dot{P}_i}$, $\ell(\dot{P})$ and $\ell(\dot{R})$, $i \in \{1,2\}$. Then there exists a positive constant C such that, for all $R \in \mathcal{R}$ and $\dot{R} \in U_R$, $(s^*_{2,\lambda})_R \leqslant C(\dot{s}^*_{2,\lambda})_{\dot{R}}$.*

Proof. (i) Let $w \in \mathcal{A}_\infty(\vec{A})$. For any $\dot{R} \in \dot{\mathcal{R}}$, let $U_{\dot{R}}$ be as in Lemma 3.10(i). Then, for any $R \in U_{\dot{R}}$, by Lemma 3.10(i), we have

$$w(R) \sim w(\dot{R}) \quad \text{and} \quad |\dot{R}| \sim |R|. \tag{3.17}$$

Moreover, for any open set Ω of $\mathbb{R}^n \times \mathbb{R}^m$, similar to the proof of Lemma 3.10(ii), there exists a positive constant $\widetilde{\eta}_0 := \widetilde{\eta}_0(n, m, \vec{A}) \in (0,1)$ such that

$$\bigcup_{\dot{R} \in \dot{\mathcal{R}},\, \dot{R} \subset \Omega} \ \bigcup_{R \in U_{\dot{R}}} \subset \widetilde{\Omega}^{(0)}, \tag{3.18}$$

where

$$\widetilde{\Omega}^{(0)} := \{x \in \mathbb{R}^n \times \mathbb{R}^m \colon \mathcal{M}_s \chi_\Omega(x) > \widetilde{\eta}_0\}.$$

Furthermore, for any $q \in (q_w, \infty)$, by the $L^q_w(\mathbb{R}^n \times \mathbb{R}^m)$-boundedness of $\mathcal{M}_s$ (see [8, Proposition 2.10(ii)]), we also have $w(\widetilde{\Omega}^{(0)}) \lesssim w(\Omega)$. Therefore, for any open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ with $w(\Omega) < \infty$, by (3.17), (3.18), $w(\widetilde{\Omega}^{(0)}) \lesssim w(\Omega)$, $p \in (0,1]$ and Lemma 3.10, we know that

$$\begin{aligned}
&\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\dot{R}\in\dot{\mathcal{R}},\, \dot{R}\subset\Omega} |\dot{s}_{\dot{R}}|^2 \frac{|\dot{R}|}{w(\dot{R})} \\
&\quad\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\dot{R}\in\dot{\mathcal{R}},\, \dot{R}\subset\Omega} \sum_{R\in\mathcal{R}} |s_R|^2 \frac{|R|}{w(R)} \chi_{U_{\dot{R}}}(R) \\
&\quad\lesssim \frac{1}{[w(\widetilde{\Omega}^{(0)})]^{\frac{2}{p}-1}} \sum_{R\in\mathcal{R},\, R\subset\widetilde{\Omega}^{(0)}} |s_R|^2 \frac{|R|}{w(R)} \sum_{\dot{R}\in\dot{\mathcal{R}},\, \dot{R}\subset\Omega} \chi_{U_R}(\dot{R}) \lesssim \|s\|^2_{\ell_{p,w}(\vec{A})}.
\end{aligned}$$

From this and the arbitrariness of the open set Ω, it further follows that

$$\|\dot{s}\|_{\dot{\ell}_{p,w}(\vec{A})} \lesssim \|s\|_{\ell_{p,w}(\vec{A})}.$$

(ii) For any $\lambda \in (0,\infty)$, $R \in \mathcal{R}$ with $\mathrm{scale}(R) = (j_1, j_2)$ and any $\dot{R} \in U_R$, let us prove that $(s^*_{2,\lambda})_R \lesssim (\dot{s}^*_{2,\lambda})_{\dot{R}}$. To this end, we choose any other $P \in \mathcal{R}$ with $\mathrm{scale}(P) = \mathrm{scale}(R)$ and $\dot{P} \in U_P$ with $\ell(\dot{P}) = \ell(\dot{R})$. Let $x_{R_i} := A_i^{j_i} k_i$ and $x_{P_i} := A_i^{j_i} \widetilde{k}_i$ with $k_i, \widetilde{k}_i \in \mathbb{Z}^{n_i}$ and $k_i \neq \widetilde{k}_i$, $i \in \{1,2\}$. Then, by (3.1) and (3.2), there exists a constant c_i such that $\rho_i(k_i - \widetilde{k}_i) \geqslant b_i^{c_i}$ and hence $\rho_i(x_{R_i} - x_{P_i}) = b_i^{j_i} \rho(k_i - \widetilde{k}_i) \geqslant b_i^{j_i + c_i}$. Let $\breve{x}_i \in R_i \cap \dot{R}_i$ and $\widetilde{x}_i \in P_i \cap \dot{P}_i$. Therefore, for $i \in \{1,2\}$, by this estimate, Lemma 3.9, $v_i \ell(\dot{P}_i) + u_i < j_i - v_i$, and Lemma 3.7(iv), we have

$$\begin{aligned}
\rho_i(c_{\dot{R}_i} - c_{\dot{P}_i}) &\leqslant H_i^6 \big[\rho_i(c_{\dot{R}_i} - \breve{x}_i) + \rho_i(\breve{x}_i - c_{R_i}) + \rho_i(c_{R_i} - x_{R_i}) + \rho_i(x_{R_i} - x_{P_i}) \\
&\qquad + \rho_i(x_{P_i} - c_{\dot{P}_i}) + \rho_i(c_{\dot{P}_i} - \widetilde{x}_i) + \rho_i(\widetilde{x}_i - c_{\dot{P}_i})\big] \\
&\leqslant H_i^6 \big(3 b_i^{-v_i - c_i} + 3 b_i^{\tau_i - c_i} + 1\big) \rho_i(x_{R_i} - x_{P_i}).
\end{aligned}$$

Thus, for all $R \in \mathcal{R}$ and $\dot{R} \in U_R$, using the above estimate, Lemma 3.10(i) and (3.17), we obtain

$$\begin{aligned}
\big(s^*_{2,\lambda}\big)^2_R &= \sum_{\mathrm{scale}(P)=\mathrm{scale}(R)} \frac{|s_P|^2}{\prod_{i=1}^2 [1 + |R_i|^{-1} \rho_i(x_{R_i} - x_{P_i})]^\lambda} \\
&\lesssim \sum_{\mathrm{scale}(P)=\mathrm{scale}(R)} \sum_{\ell(\dot{P})=\ell(\dot{R})} \frac{|\dot{s}_{\dot{P}}|^2 \chi_{U_P}(\dot{P})}{\prod_{i=1}^2 [1 + |\dot{R}_i|^{-1} \rho_i(c_{\dot{R}_i} - c_{\dot{P}_i})]^\lambda} \lesssim \big(\dot{s}^*_{2,\lambda}\big)^2_{\dot{R}},
\end{aligned}$$

which completes the proof of Lemma 3.12. $\square$

The following lemma is a generalization of [3, Lemma 3.10] and [40, Theorem 1.2 and Lemma 3.1].

Lemma 3.13. *Let $w \in \mathcal{A}_\infty(\vec{A})$ with q_w as in (3.14).*

(i) *If $p \in (0,\infty)$, $r \in (0,\infty)$ and $\lambda > q_w(\max\{1, r/2, r/p\})$, then there exists a positive constant C such that, for all $s := \{s_R\}_{R\in\mathcal{R}}$,*

$$\|s\|_{\ddot{h}^p_w(\vec{A})} \leqslant \|s^*_{r,\lambda}\|_{\ddot{h}^p_w(\vec{A})} \leqslant C\|s\|_{\ddot{h}^p_w(\vec{A})}.$$

(ii) *If $p \in (0,1]$ and $\lambda > 2q_w/p$, then there exists a positive constant C such that, for all $s := \{s_R\}_{R\in\mathcal{R}} \in \ell_{p,w}(\vec{A})$,*

$$\|s\|_{\ell_{p,w}(\vec{A})} \leqslant \|s^*_{2,\lambda}\|_{\ell_{p,w}(\vec{A})} \leqslant C\|s\|_{\ell_{p,w}(\vec{A})}.$$

Proof. (i) can be proved by an argument similar to that used in the proof of [3, Lemma 3.10]; see also the proof of [21, Lemma 2.3], the details being omitted.

To show (ii), let $w \in \mathcal{A}_\infty(\vec{A})$, $p \in (0,1]$ and $\lambda \in \mathbb{R}$ satisfy $\lambda > 2q_w/p$. Observe that the definition of $\ell_{p,w}(\vec{A})$ is defined via the mean value on open sets, while the corresponding one parameter space $\dot{f}^{0,2}_\infty(A;w)$ is defined via the mean value on dilated cubes. This makes the proof of (ii) quite different from that of [3, Lemma 3.10]. We need to use Journé's covering lemma under the setting of expansive dilations (see [8, Lemma 4.9]).

Obviously, for all $R \in \mathcal{R}$, $|s_R| \leqslant (s^*_{2,\lambda})_R$, which implies that $\|s\|_{\ell_{p,w}(\vec{A})} \leqslant \|s^*_{2,\lambda}\|_{\ell_{p,w}(\vec{A})}$. To prove (ii), we still need to show $\|s^*_{2,\lambda}\|_{\ell_{p,w}(\vec{A})} \lesssim \|s\|_{\ell_{p,w}(\vec{A})}$. Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ be any fixed open set satisfying $w(\Omega) < \infty$ and $\Omega^{(0)}$ as in Lemma 3.10. For $i \in \{0,1\}$, we define inductively the sets

$$\Omega^{(i+1)} := \{x \in \mathbb{R}^n \times \mathbb{R}^m:\ \mathcal{M}_s(\chi_{\Omega^{(i)}})(x) > \eta_1\},$$

where $\eta_1 := \eta_1(\vec{A}, n, m) \in (0,1)$ is a constant to be fixed later.

For any $\{s_R\}_{R\in\mathcal{R}}$, define $\{r_R\}_{R\in\mathcal{R}}$ and $\{t_R\}_{R\in\mathcal{R}}$ by letting $r_R := s_R$ if $R \subset \Omega^{(2)}$ or else $r_R := 0$ and $t_R := s_R - r_R$ for any $R \in \mathcal{R}$. Moreover, picking any $q \in (q_w, \infty)$, by the $L^q_w(\mathbb{R}^n \times \mathbb{R}^m)$-boundedness of $\mathcal{M}_s$, we have $w(\Omega^{(2)}) \sim w(\Omega)$. Thus, we obtain

$$\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}\\ R\subset\Omega}} \left|\left(s^*_{2,\lambda}\right)_R\right|^2 \frac{|R|}{w(R)} \lesssim \mathrm{I} + \mathrm{J},$$

where

$$\mathrm{I} := \frac{1}{[w(\Omega^{(2)})]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}\\ R\subset\Omega}} \left|\left(r^*_{2,\lambda}\right)_R\right|^2 \frac{|R|}{w(R)}$$

and

$$\mathrm{J} := \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}\\ R\subset\Omega}} \left|\left(t^*_{2,\lambda}\right)_R\right|^2 \frac{|R|}{w(R)}.$$

Estimate I. For any $l_1, l_2 \in \mathbb{Z}_+$ and $R \in \mathcal{R}$, let

$$M_{R,l_1,l_2} := \bigl\{P \in \mathcal{R}\colon \operatorname{scale}(P) = \operatorname{scale}(R),\ b_i^{l_i} \leqslant |R_i|^{-1}\rho_i(x_{P_i} - x_{R_i}) < b_i^{l_i+1},\ i \in \{1,2\}\bigr\}.$$

In the case that l_i for $i \in \{1,2\}$ is 0, the above condition is replaced by

$$|R_i|^{-1}\rho_i(x_{P_i} - x_{R_i}) < b_i.$$

Then we have

$$\mathrm{I} \leqslant \frac{1}{[w(\Omega)^{(2)}]^{\frac{2}{p}-1}} \sum_{j_1,j_2\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}\\ \operatorname{scale}(R)=(j_1,j_2)}} \sum_{l_1,l_2\in\mathbb{Z}_+} \sum_{P\in M_{R,l_1,l_2}} \frac{|r_P|^2|P|[w(R)]^{-1}}{\prod_{i=1}^2[1+|R_i|^{-1}\rho_i(x_{R_i}-x_{P_i})]^\lambda}.$$

Since $\lambda > 2q_w/p$ with $p \in (0,1]$, we choose $q \in (q_w,\infty)$ sufficiently close to q_w such that $\lambda > 1+q$. For any $P \in M_{R,l_1,l_2}$, by (3.15), we see that

$$w(R) \lesssim b_1^{ql_1} b_2^{ql_2} w(P).$$

Moreover, by an elementary lattice counting lemma (see [6, Lemma 2.8]), we conclude that $\sharp M_{R,l_1,l_2} \lesssim b_1^{l_1} b_2^{l_2}$. Thus,

$$\begin{aligned}
\mathrm{I} &\lesssim \frac{1}{[w(\Omega^{(2)})]^{\frac{2}{p}-1}} \sum_{j_1,j_2\in\mathbb{Z}} \sum_{\substack{P\in\mathcal{R}\\ \operatorname{scale}(P)=(j_1,j_2)}} |r_P|^2 \frac{|P|}{w(P)} \sum_{l_1,l_2\in\mathbb{Z}_+} b_1^{-l_1(\lambda-q-1)} b_2^{-l_2(\lambda-q-1)}\\
&\lesssim \frac{1}{[w(\Omega^{(2)})]^{\frac{2}{p}-1}} \sum_{P\subset\Omega^{(2)}} |s_P|^2 \frac{|P|}{w(P)} \lesssim \|s\|^2_{\ell_{p,w}(\vec{A})},
\end{aligned}$$

which is a desired estimate for I.

Estimate J. We need to show $\mathrm{J} \lesssim \|s\|_{\ell_{p,w}(\vec{A})}$. Notice that, for any $\dot{R} \in U_R$, by Lemma 3.10(i), $|\dot{R}| \sim| R|$ and $w(\dot{R}) \sim w(R)$. Moreover, by the $L^q_w(\mathbb{R}^n\times\mathbb{R}^m)$-boundedness of $\mathcal{M}_s$ with $q \in (q_w,\infty)$ (see [8, Proposition 2.10(ii)]), we have $w(\Omega^{(0)}) \lesssim w(\Omega)$. Also, observe that, for any $R \in \mathcal{R}$ and $\dot{R} \in \dot{\mathcal{R}}$, $R \in U_{\dot{R}}$ if and only if $\dot{R} \in U_R$, which, together with Lemma 3.10(i), further implies that $\sum_{R\in U_{\dot{R}}} \chi_{U_R}(\dot{R}) = \sum_{R\in U_{\dot{R}}} \chi_{U_{\dot{R}}}(R) \sim 1$. Using these facts, together with the trivial fact that, for any $R \in \mathcal{R}$, $\sum_{\dot{R}\in U_R} \chi_{U_R}(\dot{R}) \geqslant 1$, Lemmas 3.12(ii) and 3.10(ii), we obtain

$$\begin{aligned}
\mathrm{J} &\leqslant \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}\\ R\subset\Omega}} \sum_{\dot{R}\in U_R} \chi_{U_R}(\dot{R}) \bigl|(t^*_{2,\lambda})_R\bigr|^2 \frac{|R|}{w(R)}\\
&\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}\\ R\subset\Omega}} \sum_{\dot{R}\in U_R} \chi_{U_{\dot{R}}}(R) \bigl|(\dot{t}^*_{2,\lambda})_{\dot{R}}\bigr|^2 \chi_{U_R}(\dot{R}) \frac{|\dot{R}|}{w(\dot{R})}
\end{aligned}$$

$$\begin{aligned}&\lesssim \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \sum_{\dot{R}\subset\Omega^{(0)},\,\dot{R}\in\dot{\mathcal{R}}} \Bigg[\sum_{R\in U_{\dot{R}}} \chi_{U_R}(\dot{R})\Bigg] \big|(\dot{t}^*_{2,\lambda})_{\dot{R}}\big|^2 \frac{|\dot{R}|}{w(\dot{R})}\\ &\sim \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \sum_{\dot{R}\subset\Omega^{(0)},\,\dot{R}\in\dot{\mathcal{R}}} \big|(\dot{t}^*_{2,\lambda})_{\dot{R}}\big|^2 \frac{|\dot{R}|}{w(\dot{R})}. \qquad (3.19)\end{aligned}$$

Denote by $m_i(\Omega^{(0)})$ the *family of all dyadic rectangles* $\dot{R}\subset\Omega^{(0)}$ *which is maximal in the* $\mathbb{R}^{n_i}$ *"direction"*, where $i\in\{1,2\}$. Let $m(\Omega^{(0)}) := m_1(\Omega^{(0)})\cap m_2(\Omega^{(0)})$. Notice that, for any $\dot{R}\subset\Omega^{(0)}$, there exists at least one dyadic rectangle $\dot{P}\in m(\Omega^{(0)})$ such that $\dot{R}\subset\dot{P}$. Then, by (3.19), we know that

$$\mathrm{J} \lesssim \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \sum_{\dot{P}\in m(\Omega^{(0)})} \sum_{\dot{R}\in\dot{\mathcal{R}},\,\dot{R}\subset\dot{P}} \sum_{\substack{\dot{Q}\in\dot{\mathcal{R}}\\ \ell(\dot{Q})=\ell(\dot{R})}} \frac{|\dot{t}_{\dot{Q}}|^2|\dot{Q}|[w(\dot{R})]^{-1}}{\prod_{i=1}^2[1+|\dot{Q}_i|^{-1}\rho_i(c_{\dot{Q}_i}-c_{\dot{R}_i})]^\lambda}. \qquad (3.20)$$

We now need to obtain some subtle decompositions on $\dot{t}_{\dot{Q}}$. For any $\dot{P} := \dot{P}_1\times\dot{P}_2\in m(\Omega^{(0)})$, let $\dot{P}_{1,*}\supset\dot{P}_1$ be the maximal dyadic cube such that

$$\big|(\dot{P}_{1,*}\times\dot{P}_2)\cap\Omega^{(0)}\big| > 5H_1^4\eta_1 b_1^{2u_1}b_2^{2u_2}|\dot{P}_{1,*}\times\dot{P}_2|, \qquad (3.21)$$

where we choose $\eta_1\in(0,1)$ small enough such that $5H_1^4\eta_1 b_1^{2u_1}b_2^{2u_2}<1$. For $B_{\dot{P}_{1,*}} := B_{\rho_1}(c_{\dot{P}_{1,*}}, 3H_1^2 b_1^{v_1\ell(\dot{P}_{1,*})+u_1})$ and $U_{\dot{P}_{1,*}} := \{\dot{S}_1\in\dot{\mathcal{Q}}_1:\ \ell(\dot{S}_1)=\ell(\dot{P}_{1,*}),\ \dot{S}_1\cap B_{\dot{P}_{1,*}}\neq\emptyset\}$, using Lemma 3.7(iv), we see that

$$B_{U_1} := B_{\rho_1}\big(c_{\dot{P}_{1,*}}, 5H_1^4 b_1^{v_1\ell(\dot{P}_{1,*})+u_1}\big) \supset \bigcup_{\dot{S}_1\in U_{\dot{P}_{1,*}}} \dot{S}_1.$$

Then, for any $\dot{S}_1\in U_{\dot{P}_{1,*}}$ and $x\in\dot{S}_1\times\dot{P}_2$, by Lemma 3.7(iv) and (3.21), we have

$$\mathcal{M}_s(\chi_{\Omega^{(0)}})(x) \geqslant \frac{|(B_{U_1}\times B_{\rho_2}(c_{\dot{P}_2}, b_2^{v_2\ell(\dot{P}_2)+u_2}))\cap\Omega^{(0)}|}{|B_{U_1}\times B_{\rho_2}(c_{\dot{P}_2}, b_2^{v_2\ell(\dot{P}_2)+u_2})|} \geqslant \frac{|(\dot{P}_{1,*}\times\dot{P}_2)\cap\Omega^{(0)}|}{5H_1^4 b_1^{2u_1}b_2^{2u_2}|\dot{P}_{1,*}\times\dot{P}_2|} > \eta_1,$$

which implies that $(\bigcup_{\dot{S}_1\in U_{\dot{P}_{1,*}}}\dot{S}_1)\times\dot{P}_2\subset\Omega^{(1)}$.

On the other hand, for any $\dot{P}_{1,*}\times\dot{P}_2$, there exists a dyadic rectangle $\dot{P}_{1,\centerdot}\supset\dot{P}_{1,*}$ such that $\dot{P}_{1,\centerdot}\times\dot{P}_2\in m_1(\Omega^{(1)})$. Then, we further choose the maximal dyadic cube $\dot{P}_{2,*}\supset\dot{P}_2$ such that

$$\big|(\dot{P}_{1,\centerdot}\times\dot{P}_{2,*})\cap\Omega^{(1)}\big| > 35H_1^6H_2^4 b_1^{2u_1}b_2^{2u_2}\eta_1|\dot{P}_{1,\centerdot}\times\dot{P}_2|, \qquad (3.22)$$

where we choose $\eta_1\in(0,1)$ small enough such that $35H_1^6H_2^4 b_1^{2u_1}b_2^{2u_2}\eta_1<1$.

For $B_{\dot{P}_{2,*}} := B_{\rho_2}(c_{\dot{P}_{2,*}}, 3H_2^2 b_2^{v_2\ell(\dot{P}_{2,*})+u_2})$ and

$$U_{\dot{P}_{2,*}} := \{\dot{S}_2 \in \dot{\mathcal{Q}}_2 \colon \ \ell(\dot{S}_2) = \ell(\dot{P}_{2,*}), \ \dot{S}_2 \cap B_{\dot{P}_{2,*}} \neq \emptyset\},$$

using Lemma 3.7(iv), we find that

$$B_{U_2} := B_{\rho_2}\big(c_{\dot{P}_{2,*}}, 5H_2^4 b_2^{v_2\ell(\dot{P}_{2,*})+u_2}\big) \supset \bigcup_{\dot{S}_2 \in U_{\dot{P}_{2,*}}} \dot{S}_2.$$

Then, for any $x := (x_1, x_2)$ with $x_1 \in B_{\rho_1}(c_{\dot{P}_{1,\cdot}}, 7H_1^6 b_1^{v_1\ell(\dot{P}_{1,\cdot})+u_1})$ and $x_2 \in \dot{S}_2 \in U_{\dot{P}_{2,*}}$, by Lemma 3.7(iv) and (3.22), we have

$$\begin{aligned}
\mathcal{M}_s(\chi_{\Omega^{(1)}})(x) &\geqslant \frac{|(B_{\rho_1}(c_{\dot{P}_{1,\cdot}}, 7H_1^6 b_1^{v_1\ell(\dot{P}_{1,\cdot})+u_1}) \times B_{U_2}) \cap \Omega^{(1)}|}{|B_{\rho_1}(c_{\dot{P}_{1,\cdot}}, 7H_1^6 b_1^{v_1\ell(\dot{P}_{1,\cdot})+u_1}) \times B_{U_2}|} \\
&\geqslant \frac{|(\dot{P}_{1,\cdot} \times \dot{P}_{2,*}) \cap \Omega^{(1)}|}{35H_1^6 H_2^4 b_1^{2u_1} b_2^{2u_2} |\dot{P}_{1,\cdot} \times \dot{P}_{2,*}|} > \eta_1,
\end{aligned}$$

which, together with $\bigcup_{\dot{S}_1 \in U_{\dot{P}_{1,*}}} \dot{S}_1 \subset B_{U_1} \subset B_{\rho_1}(c_{\dot{P}_{1,\cdot}}, 7H_1^6 b_1^{v_1\ell(\dot{P}_{1,\cdot})+u_1})$, implies that

$$\bigcup_{\dot{S} \in U_{\dot{P}_{1,*}} \times U_{\dot{P}_{2,*}}} \dot{S} \subset \Omega^{(2)}.$$

Therefore, for any $\dot{Q} \in \dot{\mathcal{R}}$ and $\dot{P} := \dot{P}_1 \times \dot{P}_2 \in m(\Omega^{(0)})$ with $\dot{Q} := \dot{Q}_1 \times \dot{Q}_2 \not\subset \Omega^{(2)}$ and $\ell(\dot{Q}_i) \geqslant \ell(\dot{P}_i)$, $i \in \{1, 2\}$, by Lemma 3.7(ii), we obtain either

$$\dot{Q}_1 \cap \dot{S}_1 = \emptyset \quad \text{for all } \dot{S}_1 \in U_{\dot{P}_{1,*}}, \tag{3.23}$$

or $\dot{Q}_1 \subset \dot{S}_1$ for some $\dot{S}_1 \in U_{\dot{P}_{1,*}}$. Likewise, we either have

$$\dot{Q}_2 \cap \dot{S}_2 = \emptyset \quad \text{for all } \dot{S}_2 \in U_{\dot{P}_{2,*}}, \tag{3.24}$$

or $\dot{Q}_2 \subset \dot{S}_2$ for some $\dot{S}_2 \in U_{\dot{P}_{2,*}}$. Observe that either (3.23) or (3.24) must hold. Otherwise, there would exist $\dot{S} := \dot{S}_1 \times \dot{S}_2 \in U_{\dot{P}_{1,*}} \times U_{\dot{P}_{2,*}}$ such that $\dot{Q} \subset \dot{S} \subset \Omega^{(2)}$. This is a contradiction with $\dot{Q} \not\subset \Omega^{(2)}$. Define two *sequences* $\{\widetilde{t}_{\dot{Q}}\}_{\dot{Q} \in \dot{\mathcal{R}}}$ and $\{\check{t}_{\dot{Q}}\}_{\dot{Q} \in \dot{\mathcal{R}}}$, respectively, by setting $\widetilde{t}_{\dot{Q}} := \dot{s}_{\dot{Q}}$ if $\dot{Q}_1 \cap (\bigcup_{\dot{S}_1 \in U_{\dot{P}_{1,*}}} \dot{S}_1) = \emptyset$ with $\ell(\dot{Q}) \geqslant \ell(\dot{P})$ or else $\widetilde{t}_{\dot{Q}} := 0$, and $\check{t}_{\dot{Q}} := \dot{s}_{\dot{Q}}$ if $\dot{Q}_2 \cap (\bigcup_{\dot{S}_2 \in U_{\dot{P}_{2,*}}} \dot{S}_2) = \emptyset$ with $\ell(\dot{Q}) \geqslant \ell(\dot{P})$ or else $\check{t}_{\dot{Q}} := 0$. Notice that, by (3.20), if $\dot{Q} \in \dot{\mathcal{R}}$ appears in the sum of J, then $\dot{Q} \not\subset \Omega^{(2)}$ and $\ell(\dot{Q}) \geqslant \ell(\dot{P})$ for some $\dot{P} \in m(\Omega^{(0)})$. This observation, together with (3.20) again, yields that $\mathrm{J} \lesssim \mathrm{J}_1 + \mathrm{J}_2$, where

$$\mathrm{J}_1 := \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \sum_{\dot{P} \in m(\Omega^{(0)})} \sum_{\dot{R} \in \dot{\mathcal{R}},\, \dot{R} \subset \dot{P}} \sum_{\substack{\dot{Q} \in \dot{\mathcal{R}} \\ \ell(\dot{Q}) = \ell(\dot{R})}} \frac{|\widetilde{t}_{\dot{Q}}|^2 |\dot{Q}| [w(\dot{R})]^{-1}}{\prod_{i=1}^{2} [1 + |\dot{Q}_i|^{-1} \rho_i(c_{\dot{Q}_i} - c_{\dot{R}_i})]^{\lambda}}$$

and

$$\mathrm{J}_2 := \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \sum_{\dot{P} \in m(\Omega^{(0)})} \sum_{\dot{R} \in \dot{\mathcal{R}},\, \dot{R} \subset \dot{P}} \sum_{\substack{\dot{Q} \in \dot{\mathcal{R}} \\ \ell(\dot{Q}) = \ell(\dot{R})}} \frac{|\check{t}_{\dot{Q}}|^2 |\dot{Q}| [w(\dot{R})]^{-1}}{\prod_{i=1}^{2} [1 + |\dot{Q}_i|^{-1} \rho_i(c_{\dot{Q}_i} - c_{\dot{R}_i})]^{\lambda}}.$$

Estimate J_1**.** Let us first classify those cubes "$\dot{Q}$" in J_1 by the definition of $\widetilde{t}_{\dot{Q}}$. In what follows, let $\widetilde{U}_{\dot{P}_{1,*}}$ denote the *union of all the dyadic cubes in* $U_{\dot{P}_{1,*}}$. For any $\dot{P} := \dot{P}_1 \times \dot{P}_2 \in m(\Omega^{(0)})$, we claim that

$$\begin{aligned} &\{\dot{Q} := \dot{Q}_1 \times \dot{Q}_2 \in \dot{\mathcal{R}}\colon\ \dot{Q}_1 \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset,\ \ell(\dot{Q}) \geqslant \ell(\dot{P})\} \\ &\quad \subset \bigcup_{\{\dot{P}' := \dot{P}'_1 \times \dot{P}'_2 \in \dot{\mathcal{R}}\colon\ \ell(\dot{P}') = \ell(\dot{P}),\ \dot{P}'_1 \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset\}} \{\dot{Q} \in \dot{\mathcal{R}}\colon\ \dot{Q} \subset \dot{P}'\}. \end{aligned} \tag{3.25}$$

Indeed, for a fixed $\dot{P} \in m(\Omega^{(0)})$ and any $\dot{Q}_1 \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset$ with $\ell(\dot{Q}_1) \geqslant \ell(\dot{P}_1)$, by (i) and (iii) of Lemma 3.7, there exists a unique $\dot{P}'_{1,*} \in \dot{\mathcal{Q}}_1$ such that $\ell(\dot{P}'_{1,*}) = \ell(\dot{P}_{1,*})$, $\dot{Q}_1 \subset \dot{P}'_{1,*}$ and $\dot{P}'_{1,*} \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset$. Furthermore, by Lemma 3.7(iii) again, there exists a unique $\dot{P}'_1 \in \dot{\mathcal{Q}}_1$ such that $\ell(\dot{P}'_1) = \ell(\dot{P}_1)$ and $\dot{Q}_1 \subset \dot{P}'_1$. Then, we have $\dot{Q}_1 \subset \dot{P}'_{1,*} \cap \dot{P}'_1$. From this, $\ell(\dot{P}'_{1,*}) \leqslant \ell(\dot{P}'_1)$ and Lemma 3.7(iii), it follows that $\dot{P}'_1 \subset \dot{P}'_{1,*}$ and hence $\dot{P}'_1 \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset$. By Lemma 3.7(iii), there also exists a unique $\dot{P}'_2 \in \dot{\mathcal{Q}}_2$ such that $\dot{Q}_2 \subset \dot{P}'_2$ and $\ell(\dot{P}'_2) = \ell(\dot{P}_2)$. Then we have $\dot{Q} \subset \dot{P}' := \dot{P}'_1 \times \dot{P}'_2$ satisfying $\ell(\dot{P}') = \ell(\dot{P})$ and $\dot{P}'_1 \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset$, which shows the above claim.

For any $\dot{P}_2 \in \dot{\mathcal{Q}}_2$, let $B_{\dot{P}_2} := B_{\rho_2}(c_{\dot{P}_2}, 3H_2^2 b_2^{v_2 \ell(\dot{P}_2) + u_2})$ and

$$U_{\dot{P}_2} := \{\dot{S}_2 \in \dot{\mathcal{Q}}_2\colon\ \ell(\dot{S}_2) = \ell(\dot{P}_2),\ \dot{S}_2 \cap B_{\dot{P}_2} \neq \emptyset\}.$$

Denote by $\widetilde{U}_{\dot{P}_2}$ the *union of all cubes in* $U_{\dot{P}_2}$. Define two sets of dyadic cubes:

$$W_{\dot{P},1} := \{\dot{P}' := \dot{P}'_1 \times \dot{P}'_2 \in \dot{\mathcal{R}}\colon\ \ell(\dot{P}') = \ell(\dot{P}),\ \dot{P}'_1 \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset,\ \dot{P}'_2 \cap \widetilde{U}_{\dot{P}'_2} = \emptyset\}$$

and

$$W_{\dot{P},2} := \{\dot{P}' := \dot{P}'_1 \times \dot{P}'_2 \in \dot{\mathcal{R}}\colon\ \ell(\dot{P}') = \ell(\dot{P}),\ \dot{P}'_1 \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset,\ \dot{P}'_2 \subset \widetilde{U}_{\dot{P}'_2}\}.$$

Then, for any $\dot{P} \in m(\Omega^{(0)})$, by (i) and (ii) of Lemma 3.7, we rewrite (3.25) as

$$\begin{aligned}&\bigl\{\dot{Q} := \dot{Q}_1 \times \dot{Q}_2 \in \dot{\mathcal{R}}:\ \dot{Q}_1 \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset,\ \ell(\dot{Q}) \geqslant \ell(\dot{P})\bigr\}\\ &\quad\subset \Bigl(\bigcup_{\dot{P}' \in W_{\dot{P},1}} \bigl\{\dot{Q} \in \dot{\mathcal{R}}:\ \dot{Q} \subset \dot{P}'\bigr\}\Bigr) \cup \Bigl(\bigcup_{\dot{P}' \in W_{\dot{P},2}} \bigl\{\dot{Q} \in \dot{\mathcal{R}}:\ \dot{Q} \subset \dot{P}'\bigr\}\Bigr). \qquad (3.26)\end{aligned}$$

Notice that, for any $\dot{P} \in m(\Omega^{(0)})$ and $\dot{P}_1' \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset$ with $\ell(\dot{P}_1') = \ell(\dot{P}_1)$, by $\dot{P}_1 \subset \dot{P}_{1,*} \subset B_{\dot{P}_{1,*}} := B_{\rho_1}(c_{\dot{P}_{1,*}}, 3H_1^2 b_1^{v_1\ell(\dot{P}_{1,*})+u_1}) \subset \widetilde{U}_{\dot{P}_{1,*}}$ and $\dot{P}_1' \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset$, we know that

$$\begin{aligned}\rho_1(c_{\dot{P}_1'} - c_{\dot{P}_1}) &\geqslant \frac{\rho_1(c_{\dot{P}_1'} - c_{\dot{P}_{1,*}})}{H_1} - \rho_1(c_{\dot{P}_{1,*}} - c_{\dot{P}_1})\\ &\geqslant \frac{3H_1^2 b_1^{v_1\ell(\dot{P}_{1,*})+u_1}}{H_1} - b_1^{v_1\ell(\dot{P}_{1,*})u_1} \geqslant 2H_1 b_1^{v_1\ell(\dot{P}_{1,*})+u_1}.\end{aligned}$$

Similarly, for any $\dot{P} \in m(\Omega^{(0)})$ and $\dot{P}_2' \cap \widetilde{U}_{\dot{P}_2} = \emptyset$ with $\ell(\dot{P}_2') = \ell(\dot{P}_2)$, we also have

$$\rho_2(c_{\dot{P}_2'} - c_{\dot{P}_2}) \geqslant 2H_2 b_2^{v_2\ell(\dot{P}_2)+u_2}.$$

Thus, we obtain

$$\begin{aligned}W_{\dot{P},1} \subset \bigl\{&\dot{P}' := \dot{P}_1' \times \dot{P}_2' \in \dot{\mathcal{R}}:\ \ell(\dot{P}') = \ell(\dot{P}),\\ &\rho_1(c_{\dot{P}_1'} - c_{\dot{P}_1}) \geqslant 2H_1 b_1^{v_1\ell(\dot{P}_{1,*})+u_1},\ \rho_2(c_{\dot{P}_2'} - c_{\dot{P}_2}) \geqslant 2H_2 b_2^{v_2\ell(\dot{P}_2)+u_2}\bigr\} \qquad (3.27)\end{aligned}$$

and

$$\begin{aligned}W_{\dot{P},2} \subset \bigl\{&\dot{P}' := \dot{P}_1' \times \dot{P}_2' \in \dot{\mathcal{R}}:\ \ell(\dot{P}') = \ell(\dot{P}),\\ &\rho_1(c_{\dot{P}_1'} - c_{\dot{P}_1}) \geqslant 2H_1 b_1^{v_1\ell(\dot{P}_{1,*})+u_1},\ \dot{P}_2' \subset \widetilde{U}_{\dot{P}_2}\bigr\}. \qquad (3.28)\end{aligned}$$

Let $\gamma_1(\dot{P}) := \ell(\dot{P}_{1,*}) - \ell(\dot{P}_1)$. For any $k_1, k_2 \in \mathbb{Z}_+$, let

$$\begin{aligned}U_{\dot{P},k_1,k_2} := \bigl\{&\dot{P}' \in \dot{\mathcal{R}}:\ \ell(\dot{P}') = \ell(\dot{P}),\ \rho_1(c_{\dot{P}_1} - c_{\dot{P}_1'}) \sim b_1^{v_1[\ell(\dot{P}_1)+\gamma_1(\dot{P})]+k_1},\\ &\rho_2(c_{\dot{P}_2} - c_{\dot{P}_2'}) \sim b_2^{v_2\ell(\dot{P}_2)+k_2}\bigr\}\end{aligned}$$

and

$$U_{\dot{P},k_1} := \bigl\{\dot{P}' \in \dot{\mathcal{R}}:\ \ell(\dot{P}') = \ell(\dot{P}),\ \rho_1(c_{\dot{P}_1} - c_{\dot{P}_1'}) \sim b_1^{v_1[\ell(\dot{P}_1)+\gamma_1(\dot{P})]+k_1},\ \dot{P}_2' \subset \widetilde{U}_{\dot{P}_2}\bigr\},$$

where $\rho_1(c_{\dot{P}_1} - c_{\dot{P}_1'}) \sim b_1^{v_1[\ell(\dot{P}_1)+\gamma_1(\dot{P})]+k_1}$ and $\rho_2(c_{\dot{P}_2} - c_{\dot{P}_2'}) \sim b_2^{v_2\ell(\dot{P}_2)+k_1}$ mean, respectively, that

$$2H_1 b_1^{v_1[\ell(\dot P_1)+\gamma_1(\dot P)]+u_1+k_1} \leqslant \rho_1(c_{\dot P_1}-c_{\dot P_1'}) < 2H_1 b_1^{v_1[\ell(\dot P_1)+\gamma_1(\dot P)]+u_1+k_1+1}$$

and

$$2H_2 b_2^{v_2\ell(\dot P_2)+u_2+k_2} \leqslant \rho_2(c_{\dot P_2}-c_{\dot P_2'}) < 2H_2 b_2^{v_2\ell(\dot P_2)+u_2+k_2+1}.$$

Thus, by this, (3.27) and (3.28), we conclude that

$$W_{\dot P,1} \subset \bigcup_{k_1,k_2\in\mathbb{Z}_+} U_{\dot P,k_1,k_2} \quad \text{and} \quad W_{\dot P,2} \subset \bigcup_{k_1\in\mathbb{Z}_+} U_{\dot P,k_1}.$$

Hence, for any $j_1, j_2 \in \mathbb{Z}_+$ and $\dot P \in m(\Omega^{(0)})$, using above two decompositions and (3.26), we obtain

$$\begin{aligned}
&\{\dot Q \in \dot{\mathcal R}:\ \ell(\dot Q) = \ell(\dot P) + (j_1,j_2),\ \dot Q_1 \cap \widetilde U_{\dot P_{1,*}} = \emptyset\}\\
&\subset \left(\bigcup_{k_1,k_2\in\mathbb{Z}_+}\ \bigcup_{\dot P'\in U_{\dot P,k_1,k_2}} \{\dot Q \in \dot{\mathcal R}:\ \ell(\dot Q) = \ell(\dot P)+(j_1,j_2),\ \dot Q \subset \dot P'\}\right)\\
&\quad\cup \left(\bigcup_{k_1\in\mathbb{Z}_+}\ \bigcup_{\dot P'\in U_{\dot P,k_1}} \{\dot Q \in \dot{\mathcal R}:\ \ell(\dot Q_1) = \ell(\dot P_1)+j_1,\ \dot Q \subset \dot P'\}\right)\\
&=: \mathrm{V}_{\dot P,j_1,j_2} \cup \mathrm{V}_{\dot P,j_1}.
\end{aligned} \tag{3.29}$$

From this and

$$\sum_{\dot R\in\dot{\mathcal R},\ \dot R\subset\dot P} = \sum_{j_1,j_2\in\mathbb{Z}_+} \sum_{\substack{\dot R\in\dot{\mathcal R},\ \dot R\subset\dot P\\ \ell(\dot R)=\ell(\dot P)+(j_1,j_2)}} = \sum_{j_1\in\mathbb{Z}_+} \sum_{\substack{\dot R\in\dot{\mathcal R},\ \dot R\subset\dot P\\ \ell(\dot R_1)=j_1+\ell(\dot P_1)}},$$

it follows that $\mathrm{J}_1 \lesssim \mathrm{J}_1^{(1)} + \mathrm{J}_1^{(2)}$, where

$$\begin{aligned}
\mathrm{J}_1^{(1)} :=& \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}}\\
&\times \sum_{\dot P\in m(\Omega^{(0)})} \sum_{j_1,j_2\in\mathbb{Z}_+} \sum_{\substack{\dot R\in\dot{\mathcal R},\ \dot R\subset\dot P\\ \ell(\dot R)=\ell(\dot P)+(j_1,j_2)}} \sum_{\dot Q\in\mathrm{V}_{\dot P,j_1,j_2}} \frac{|\widetilde t_{\dot Q}|^2|\dot Q|[w(\dot R)]^{-1}}{\prod_{i=1}^2[1+|\dot Q_i|^{-1}\rho_i(c_{\dot Q_i}-c_{\dot R_i})]^\lambda}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{J}_1^{(2)} :=& \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}}\\
&\times \sum_{\dot P\in m(\Omega^{(0)})} \sum_{j_1\in\mathbb{Z}_+} \sum_{\substack{\dot R\in\dot{\mathcal R},\ \dot R\subset\dot P\\ \ell(\dot R_1)=j_1+\ell(\dot P_1)}} \sum_{\dot Q\in\mathrm{V}_{\dot P,j_1}} \frac{|\widetilde t_{\dot Q}|^2|\dot Q|[w(\dot R)]^{-1}}{\prod_{i=1}^2[1+|\dot Q_i|^{-1}\rho_i(c_{\dot Q_i}-c_{\dot R_i})]^\lambda}.
\end{aligned}$$

Then let us now estimate $\mathrm{J}_1^{(1)}$ and $\mathrm{J}_1^{(2)}$, respectively.

Estimate $\mathrm{J}_1^{(1)}$. Since $\lambda > 2q_w/p+1$, we choose $q \in (q_w, \infty)$ to be close enough to q_w such that $\lambda > 2q/p+1$. For any $\dot{P} \in m(\Omega^{(0)})$, $\dot{R} \subset \dot{P}$ with $\ell(\dot{R}) = \ell(\dot{P}) + (j_1, j_2)$ and $\dot{Q} \in \mathrm{V}_{\dot{P},j_1,j_2}$, there exists a unique $\dot{P}' \in U_{\dot{P},k_1,k_2}$ for some $k_1, k_2 \in \mathbb{Z}_+$ such that $\dot{P}' \supset \dot{Q}$ and $\ell(\dot{Q}) = \ell(\dot{P}') + (j_1, j_2)$. Then, by Lemma 3.7(iv) and the definition of $U_{\dot{P},k_1,k_2}$, we see that $\dot{P}, \dot{P}' \subset B_{\rho_1}(c_{\dot{P}_1}, 3H_1^2 b_1^{v_1[\ell(\dot{P}_1)+\gamma_1(\dot{P})]+u_1+k_1+1}) \times B_{\rho_2}(c_{\dot{P}_2}, 3H_2^2 b_2^{v_2\ell(\dot{P}_2)+u_2+k_2+1})$. From this, $\dot{R} \subset \dot{P}$, $\dot{Q} \subset \dot{P}'$, $\ell(\dot{Q}) = \ell(\dot{R}) = \ell(\dot{P})+(j_1,j_2) = \ell(\dot{P}')+(j_1,j_2)$, Lemma 3.7(iv) and Lemma 3.3 with $w \in \mathcal{A}_q(\vec{A})$, it follows that

$$\begin{aligned}
w(\dot{R}) &\gtrsim w\left(\prod_{i=1}^{2} B_{\rho_i}\left(c_{\dot{P}_i}, b_i^{v_i[\ell(\dot{P}_i)+j_i]-u_i}\right)\right) \\
&\gtrsim b_1^{qv_1j_1} b_2^{qv_2j_2} w\left(\prod_{i=1}^{2} B_{\rho_i}\left(c_{\dot{P}_i}, b_i^{v_i\ell(\dot{P}_i)-u_i}\right)\right) \\
&\gtrsim b_1^{qv_1j_1} b_2^{qv_2j_2} w(\dot{P}) \\
&\gtrsim b_1^{q\{-v_1[\gamma_1(\dot{P})-j_1]-k_1\}} b_2^{-q(k_2-v_2j_2)} \\
&\quad \times w\big(B_{\rho_1}\left(c_{\dot{P}_1}, 3H_1^2 b_1^{v_1[\ell(\dot{P}_1)+\gamma_1(\dot{P})]+u_1+k_1+1}\right) \\
&\quad \times B_{\rho_2}\left(c_{\dot{P}_1}, 3H_2^2 b_2^{v_2\ell(\dot{P}_2)+u_2+k_2+1}\right)\big) \\
&\gtrsim b_1^{q\{-v_1[\gamma_1(\dot{P})-j_1]-k_1\}} b_2^{-q(k_2-v_2j_2)} w\big(\dot{P}'\big) \\
&\gtrsim b_1^{q\{-v_1[\gamma_1(\dot{P})-j_1]-k_1\}} b_2^{-q(k_2-v_2j_2)} w(\dot{Q})
\end{aligned} \tag{3.30}$$

and, similarly,

$$w\big(\dot{P}'\big) \lesssim b_1^{q[v_1\gamma_1(\dot{P})+k_1]} b_2^{qk_2} w(\dot{P}). \tag{3.31}$$

Moreover, for any $j_1, j_2, k_1, k_2 \in \mathbb{Z}_+$, by Lemma 3.7, we conclude that

$$\begin{aligned}
\sharp\big\{\dot{R} \in \dot{\mathcal{R}}\colon\ \dot{R} \subset \dot{P},\ \ell(\dot{R}) = \ell(\dot{P}) + (j_1, j_2)\big\} \lesssim b_1^{-v_1j_1} b_2^{-v_2j_2} \quad \text{and} \\
\sharp U_{\dot{P},k_1,k_2} \lesssim b_1^{v_1\gamma_1(\dot{P})+k_1} b_2^{k_2}.
\end{aligned} \tag{3.32}$$

Furthermore, for any $\dot{P} \in m(\Omega^{(0)})$, $\dot{R} \subset \dot{P}$, $\dot{P}' \in U_{\dot{P},k_1,k_2}$ with any $k_1, k_2 \in \mathbb{Z}_+$ and $\dot{Q} \subset \dot{P}'$, by $\dot{P}_1 \subset B_{\dot{P}_{1,*}}$ and $\dot{P}'_1 \cap B_{\dot{P}_{1,*}} = \emptyset$, and (ii) and (iv) of Lemma 3.7, we find that $\dot{P}'_1 \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset$, $\dot{Q} \subset \dot{P}'$ and

$$\begin{aligned}
\rho_1(c_{\dot{P}'_1} - c_{\dot{P}_1}) &\leqslant H_1^2\big[\rho_1(c_{\dot{P}'_1} - c_{\dot{Q}_1}) + \rho_1(c_{\dot{Q}_1} - c_{\dot{R}_1}) + \rho_1(c_{\dot{R}_1} - c_{\dot{P}_1})\big] \\
&\leqslant H_1^2\big[2b_1^{v_1\ell(\dot{P}_{1,*})+u_1} + \rho_1(c_{\dot{Q}_1} - c_{\dot{R}_1})\big],
\end{aligned}$$

which, together with

$$\rho_1(c_{\dot Q_1}-c_{\dot R_1}) \geqslant \frac{\rho_1(c_{\dot Q_1}-c_{\dot P_{1,*}})}{H_1}-\rho_1(c_{\dot P_{1,*}}-c_{\dot R_1}) \geqslant 2H_1 b_1^{v_1\ell(\dot P_{1,*})+u_1},$$

implies that

$$\rho_1(c_{\dot P_1'}-c_{\dot P_1}) \lesssim \rho_1(c_{\dot R_1}-c_{\dot Q_1}). \tag{3.33}$$

Similarly, for any $\dot P \in m(\Omega^{(0)})$, $\dot R \subset \dot P$, $\dot P' \in U_{\dot P,k_1,k_2}$ and $\dot Q \subset \dot P'$, we also have

$$\rho_2(c_{\dot P_2'}-c_{\dot P_2}) \lesssim \rho_2(c_{\dot R_2}-c_{\dot Q_2}). \tag{3.34}$$

Then, by (3.29), (3.30), (3.31), (3.32), (3.33), (3.34), $p \in (0,1]$, $q \in (q_w,\infty)$ and $\lambda > 2q/p+1$, we know that

$$\begin{aligned}
\mathrm{J}_1^{(1)} &\lesssim \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \sum_{\dot P\in m(\Omega^{(0)})} \sum_{j_1,j_2\in\mathbb{Z}_+} b_1^{-v_1 j_1} b_2^{-v_2 j_2} \\
&\quad\times \sum_{k_1,k_2\in\mathbb{Z}_+} \sum_{\dot P'\in U_{\dot P,k_1,k_2}} b_1^{q[v_1\gamma_1(\dot P)+k_1](\frac{2}{p}-1)} b_2^{qk_2(\frac{2}{p}-1)} \left[\frac{w(\dot P)}{w(\dot P')}\right]^{\frac{2}{p}-1} \\
&\quad\times \sum_{\dot Q\subset \dot P'} \frac{|\widetilde{t}_{\dot Q}|^2|\dot Q| b_1^{q\{v_1[\gamma_1(\dot P)-j_1]+k_1\}} b_2^{q(k_2-v_2j_2)}[w(\dot Q)]^{-1}}{[b_1^{v_1[\gamma_1(\dot P)-j_1]+k_1} b_2^{k_2-v_2j_2}]^{\lambda}} \\
&\lesssim \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \left[\sum_{\dot P\in m(\Omega^{(0)})} w(\dot P) b_1^{v_1\gamma_1(\dot P)\frac{2q-p\lambda+p}{2-p}}\right]^{\frac{2}{p}-1} \|\dot s\|^2_{\dot\ell_{p,w}(\vec A)} \\
&\quad\times \prod_{i=1}^{2} \sum_{j_i\in\mathbb{Z}_+} b_i^{j_i v_i(\lambda-q-1)} \sum_{k_i\in\mathbb{Z}_+} b_i^{-k_i(\lambda-\frac{2q}{p}-1)} \lesssim \|\dot s\|^2_{\dot\ell_{p,w}(\vec A)},
\end{aligned} \tag{3.35}$$

where, in the last inequality, we used that $\sum_{\dot P\in m(\Omega^{(0)})} w(\dot P) b_1^{v_1\gamma_1(\dot P)\frac{2q-p\lambda+p}{2-p}} \lesssim w(\Omega^{(0)})$, which holds by Journé's covering lemma (see [8, Lemma 4.9]).

Estimate $\mathrm{J}_1^{(2)}$. For any $\dot Q_2$ and $k_2 \in \mathbb{Z}_+$, let

$$G_{\dot Q_2,k_2} := \{\dot R_2 \in \dot{\mathcal{Q}}_2\colon\ \ell(\dot R_2)=\ell(\dot Q_2),\ \rho_2(c_{\dot Q_2}-c_{\dot R_2}) \sim b_2^{v_2\ell(\dot Q_2)+k_2}\},$$

where $\rho_2(c_{\dot Q_2}-c_{\dot R_2}) \sim b_2^{v_2\ell(\dot Q_2)+k_2}$ always means that $\rho_2(c_{\dot Q_2}-c_{\dot R_2}) < b_2^{v_2\ell(\dot Q_2)+u_2+k_2}$ when $k_2=0$ and $b_2^{v_2\ell(\dot Q_2)+u_2+k_2-1} \leqslant \rho_2(c_{\dot Q_2}-c_{\dot R_2}) < b_2^{v_2\ell(\dot Q_2)+u_2+k_2}$ when $k_2 \geqslant 1$. Moreover, for any $\dot P \in m(\Omega^{(0)})$, by Lemmas 3.7 and 3.3, we have

$$\begin{aligned}
\sharp\{\dot R_1\colon\ \ell(\dot R_1)=\ell(\dot P_1)+j_1,\ \dot R_1\subset \dot P_1\} &\lesssim b_1^{-v_1j_1}, \\
\sharp G_{\dot Q_2,k_2} &\lesssim b_2^{k_2} \quad \text{and} \\
\sharp U_{\dot P,k_1} &\lesssim b_1^{v_1\gamma_1(\dot P)+k_1}.
\end{aligned} \tag{3.36}$$

For any $\dot P\in m(\Omega^{(0)})$, $\dot P'\in U_{\dot P,k_1}$, $\dot Q\subset \dot P'$ with $\ell(\dot Q_1)=\ell(\dot P_1)+j_1$, $\dot R\in\dot{\mathcal R}$ with $\dot R\subset\dot P$, $\ell(\dot R)=\ell(\dot P)+j_1$ and $\dot R_2\in G_{\dot Q_2,k_2}$, by $\dot P_1'\cap B_{\dot P_{1,*}}=\emptyset$, and Lemmas 3.7 and 3.3, we find that

$$\begin{aligned} w(\dot R)&\gtrsim b_1^{qv_1j_1}w(\dot P_1\times\dot R_2)\gtrsim b_1^{-q\{v_1[\gamma_1(\dot P)-j_1]+k_1\}}w(\dot P_1\times\dot R_2)\\ &\gtrsim b_1^{-q\{v_1[\gamma_1(\dot P)-j_1]+k_1\}}b_2^{-qk_2}w(\dot Q)\quad\text{and}\quad w(\dot P')\lesssim b_1^{q[v_1\gamma_1(\dot P)+k_1]}w(\dot P). \end{aligned}\tag{3.37}$$

Therefore, by (3.29), (3.33), (3.36), (3.37), $p\in(0,1]$, $q\in(q_w,\infty)$ and $\lambda>2q/p+1$, we conclude that

$$\begin{aligned} \mathrm{J}_1^{(2)}&\lesssim\frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}}\sum_{\dot P\in m(\Omega^{(0)})}\sum_{j_1\in\mathbb Z_+}b_1^{-v_1j_1}\\ &\quad\times\sum_{k_1\in\mathbb Z_+}\sum_{\dot P'\in U_{\dot P,k_1}}b_1^{(\frac{2}{p}-1)q[v_1\gamma_1(\dot P)+k_1]}\left[\frac{w(\dot P)}{w(\dot P')}\right]^{\frac{2}{p}-1}\\ &\quad\times\sum_{\dot Q\subset\dot P'}\sum_{k_2\in\mathbb Z_+}\sum_{\dot R_2\in G_{\dot Q_2,k_2}}b_1^{q\{v_1[\gamma_1(\dot P)-j_1]+k_1\}}b_2^{qk_2}\frac{|\widetilde t_{\dot Q}|^2|\dot Q|[w(\dot Q)]^{-1}}{[b_1^{v_1[\gamma_1(\dot P)-j_1]+k_1}b_2^{k_2}]^\lambda}\\ &\lesssim\frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}}\left[\sum_{\dot P\in m(\Omega^{(0)})}w(\dot P)b_1^{v_1\gamma_1(\dot P)\frac{2q-p\lambda+p}{2-p}}\right]^{\frac{2}{p}-1}\|\dot s\|^2_{\dot\ell_{p,w}(\vec A)}\\ &\quad\times\sum_{j_1\in\mathbb Z_+}b_1^{j_1v_1(\lambda-q-1)}\sum_{k_1\in\mathbb Z_+}b_1^{-k_1(\lambda-\frac{2q}{p}-1)}\sum_{k_2\in\mathbb Z_+}b_2^{-k_2(\lambda-q-1)}\\ &\lesssim\|\dot s\|^2_{\dot\ell_{p,w}(\vec A)}, \end{aligned}\tag{3.38}$$

where, in the last inequality, we used that $\sum_{\dot P\in m(\Omega^{(0)})}w(\dot P)b_1^{v_1\gamma_1(\dot P)\frac{2q-p\lambda+p}{2-p}}\lesssim w(\Omega^{(0)})$, which holds again by Journé's covering lemma (see [8, Lemma 4.9]).

Combining (3.35) and (3.38), we know that

$$\mathrm{J}_1\lesssim\mathrm{J}_1^{(1)}+\mathrm{J}_1^{(2)}\lesssim\|\dot s\|^2_{\dot\ell_{p,w}(\vec A)}.$$

Symmetrically, we also have $\mathrm{J}_2\lesssim\|\dot s\|^2_{\dot\ell_{p,w}(\vec A)}$. Combining the estimates of J_1 and J_2 and Lemma 3.12(i), we conclude that $\mathrm{J}\lesssim\|s\|^2_{\dot\ell_{p,w}(\vec A)}$, which completes the proof of Lemma 3.13. □

We also need to generalize Peetre's mean value inequality to our setting, which is an extension of [3, Lemma 8.3] and [21, Lemma A.4] from one parameter setting to two parameter setting. Since the proof of Lemma 3.14 is similar to that of [3, Lemma 8.3], we omit the details.

Lemma 3.14. *Let K be a compact subset of $\mathbb{R}^n \times \mathbb{R}^m$ and $\lambda \in (0,\infty)$. Suppose that $g \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ with $\operatorname{supp}\widehat{g} \subset K$. For any $\gamma \in \mathbb{N}$, define two sequences $\{a_Q\}_{Q\in\mathcal{R}}$ and $\{b_Q\}_{Q\in\mathcal{R}}$, respectively, by setting, for all $Q \in \mathcal{R}$,*

$$a_Q := \sup_{y\in Q}\bigl|g(y)\bigr| \quad \textit{and}$$

$$b_Q := \sup\Bigl\{\inf_{y\in P}\bigl|g(y)\bigr|\colon \operatorname{scale}(P) = \operatorname{scale}(Q) - (\gamma,\gamma),\ P\cap Q \neq \emptyset\Bigr\}. \tag{3.39}$$

Then, for any sufficiently large positive integer γ and $Q \in \mathcal{R}$ with $\operatorname{scale}(Q) = (0,0)$, $(a^*_{2,\lambda})_Q \sim (b^*_{2,\lambda})_Q$ with equivalent positive constants independent of g and Q.

Lemma 3.15. *Let $w \in \mathcal{A}_\infty(\vec{A})$ with q_w as in (3.14). Suppose $\varphi := \varphi^{(1)} \otimes \varphi^{(2)}$ with $\varphi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ satisfying that $\operatorname{supp}\varphi^{(i)}$ is compact and bounded away from the origin, where $i \in \{1,2\}$. For any $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and $\gamma \in \mathbb{Z}_+$, define the sequences $\sup(f) := \{\sup_Q(f)\}_{Q\in\mathcal{R}}$ and $\inf(f) := \{\inf_Q(f)\}_{Q\in\mathcal{R}}$ by setting, for all $Q \in \mathcal{R}$ with $\operatorname{scale}(Q) = (-j_1,-j_2)$,*

$$\sup{}_Q(f) := |Q|^{\frac12}\sup_{y\in Q}\bigl|\widetilde{\varphi}_{j_1,j_2} * f(y)\bigr| \quad \textit{and}$$

$$\inf{}_Q(f) := |Q|^{\frac12}\sup\Bigl\{\inf_{y\in P}\bigl|\widetilde{\varphi}_{j_1,j_2} * f(y)\bigr|\colon \operatorname{scale}(P) = (-j_1-\gamma,-j_2-\gamma),\ P\cap Q\neq\emptyset\Bigr\},$$

where $\widetilde{\varphi}(\cdot) = \overline{\varphi(-\cdot)}$.

(i) *If $p \in (0,\infty)$, then, for any sufficiently large $\gamma \in \mathbb{Z}_+$,*

$$\|f\|_{\ddot{H}^p_w(\vec{A})} \sim \bigl\|\sup(f)\bigr\|_{\ddot{h}^p_w(\vec{A})} \sim \bigl\|\inf(f)\bigr\|_{\ddot{h}^p_w(\vec{A})} \tag{3.40}$$

with equivalent positive constants independent of f.

(ii) *If $p \in (0,1]$, then (3.40) also holds with $\ddot{H}^p_w(\vec{A})$ and $\ddot{h}^p_w(\vec{A})$ replaced, respectively, by $\mathcal{L}_{p,w}(\vec{A})$ and $\ell_{p,w}(\vec{A})$.*

Proof. We shall only prove Lemma 3.15 for $\mathcal{L}_{p,w}(\vec{A})$ and $\ell_{p,w}(\vec{A})$; the proofs for the spaces $\ddot{H}^p_w(\vec{A})$ and $\ddot{h}^p_w(\vec{A})$ are similar to that of [3, Lemma 3.11], the details being omitted.

Let us first prove that, for all $f \in \mathcal{L}_{p,w}(\vec{A})$ with $p \in (0,1]$, $\|\inf(f)\|_{\ell_{p,w}(\vec{A})} \lesssim \|f\|_{\mathcal{L}_{p,w}(\vec{A})}$. For any fixed $\gamma \in \mathbb{Z}_+$, define the *sequence* $s := \{s_P\}_{P\in\mathcal{R}}$ by setting

$$s_P := |P|^{1/2}\inf_{y\in P}\bigl|\widetilde{\varphi}_{\ell_1,\ell_2} * f(y)\bigr|$$

for any $P \in \mathcal{R}$ with $\operatorname{scale}(P) := (-\ell_1,-\ell_2)$. Clearly, we have

$$|Q|^{-\frac12}\inf{}_Q(f) = \sup\bigl\{|P|^{-\frac12}|s_P|\colon P\cap Q\neq\emptyset,\ \operatorname{scale}(P) = \operatorname{scale}(Q) - (\gamma,\gamma)\bigr\}.$$

Fix $j_1, j_2 \in \mathbb{Z}$ and $Q \in \mathcal{R}$ with $\operatorname{scale}(Q) = (-j_1, -j_2)$. Suppose that $P, R \in \mathcal{R}$ satisfy

$$\operatorname{scale}(P) = \operatorname{scale}(R) = (-j_1 - \gamma, -j_2 - \gamma), \quad y \in P \cap Q \neq \emptyset, \ z \in R \cap Q \neq \emptyset. \tag{3.41}$$

Then, we have

$$\rho_i(x_{P_i} - x_{R_i}) \leqslant H_i^2 \big[\rho_i(x_{P_i} - y_i) + \rho_i(y_i - z_i) + \rho_i(z_i - x_{R_i})\big] \lesssim |Q_i|,$$

where $i \in \{1, 2\}$. Thus, for any $\lambda > 1$, we obtain

$$s_P \leqslant \big(s_{2,\lambda}^*\big)_R \prod_{i=1}^{2} \big[1 + |P_i|^{-1} \rho_i(x_{P_i} - x_{R_i})\big]^{\lambda/2} \lesssim b_1^{\gamma\lambda/2} b_2^{\gamma\lambda/2} \big(s_{2,\lambda}^*\big)_R. \tag{3.42}$$

Moreover, for any $Q \in \mathcal{R}$ with $\operatorname{scale}(Q) = (-j_1, -j_2)$, there exists a positive constant $C_1 > 1$ such that

$$\begin{aligned} \mathcal{U}_Q &:= \big\{R \in \mathcal{R}\colon \operatorname{scale}(R) \leqslant \operatorname{scale}(Q),\ R \cap Q \neq \emptyset\big\} \\ &\subset B_{\rho_1}\big(c_{Q_1}, C_1 b_1^{-j_1}\big) \times B_{\rho_2}\big(c_{Q_2}, C_1 b_2^{-j_2}\big) \end{aligned}$$

and $B_{\rho_1}(c_{Q_1}, C_1^{-1} b_1^{-j_1}) \times B_{\rho_2}(c_{Q_2}, C_1^{-1} b_2^{-j_2}) \subset Q$. Then, for any fixed open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, $Q \in \mathcal{R}$ with $Q \subset \Omega$, $\widetilde{\Omega} := \{x \in \mathbb{R}^n \times \mathbb{R}^m\colon \mathcal{M}_s(\chi_\Omega)(x) > C_1^{-4}\}$ and $x \in R \in \mathcal{U}_Q$, we have

$$\mathcal{M}_s(\chi_\Omega)(x) \geqslant \frac{|(B_{\rho_1}(x_{Q_1}, C_1 b_1^{j_1}) \times B_{\rho_2}(x_{Q_2}, C_1 b_2^{j_2})) \cap \Omega|}{|B_{\rho_1}(x_{Q_1}, C_1 b_1^{j_1}) \times B_{\rho_2}(x_{Q_2}, C_1 b_2^{j_2})|} > C_1^{-4},$$

which implies that

$$\bigcup_{R \in \mathcal{U}_Q} R \subset \widetilde{\Omega}. \tag{3.43}$$

Then, by (3.41), (3.42) and (3.43), we see that

$$\begin{aligned} &\sum_{\substack{Q \subset \Omega \\ \operatorname{scale}(Q) = (-j_1, -j_2)}} \big[\inf_Q(f) |Q|^{-\frac{1}{2}}\big]^2 \chi_Q \\ &\lesssim (b_1 b_2)^{\lambda\gamma} \sum_{\substack{Q \subset \Omega \\ \operatorname{scale}(Q) = (-j_1, -j_2)}} \sum_{\operatorname{scale}(P) = (-j_1 - \gamma, -j_2 - \gamma)} \big[\big(s_{2,\lambda}^*\big)_P |P|^{-\frac{1}{2}}\big]^2 \chi_P \chi_{\mathcal{U}_Q}(P) \chi_Q \\ &\lesssim (b_1 b_2)^{\lambda\gamma} \sum_{\substack{P \subset \widetilde{\Omega} \\ \operatorname{scale}(P) = (-j_1 - \gamma, -j_2 - \gamma)}} \big[\big(s_{2,\lambda}^*\big)_P |P|^{-\frac{1}{2}}\big]^2 \chi_P. \end{aligned}$$

Thus, for any open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ and $p \in (0,1]$, choosing $\lambda > 2q_w/p + 1$, by the above estimate, $w(\Omega) \sim w(\widetilde{\Omega})$ (by [8, Proposition 2.10(ii)]), $w(Q) \sim w(P)$ (by Lemma 3.3), $|P| \sim |Q|$ and Lemma 3.13, we find that

$$\begin{aligned}
&\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \int_{\Omega} \sum_{j_1,j_2\in\mathbb{Z}} \sum_{\substack{Q\subset\Omega\\ \operatorname{scale}(Q)=(-j_1,-j_2)}} \left[\inf_Q(f)|Q|^{-\frac{1}{2}}\chi_Q(x)\right]^2 \frac{|Q|^2}{[w(Q)]^2} w(x)\,dx \\
&\lesssim \frac{(b_1b_2)^{\lambda\gamma}}{[w(\widetilde{\Omega})]^{\frac{2}{p}-1}} \int_{\widetilde{\Omega}} \sum_{P\subset\widetilde{\Omega}} \left[\left(s_{2,\lambda}^*\right)_P |P|^{-\frac{1}{2}}\chi_P(x)\right]^2 \frac{|P|^2}{[w(P)]^2} w(x)\,dx \\
&\lesssim \left\|s_{2,\lambda}^*\right\|_{\ell_{p,w}(\vec{A})}^2 \lesssim \|s\|_{\ell_{p,w}(\vec{A})}^2 \\
&\lesssim \sup_{w(\Omega)<\infty} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \\
&\quad \times \int_{\Omega} \sum_{j_1,j_2\in\mathbb{Z}} \sum_{\substack{Q\subset\Omega\\ \operatorname{scale}(Q)=(-j_1,-j_2)}} \sum_{\substack{Q\cap P\neq\emptyset\\ \operatorname{scale}(P)=(-j_1-\gamma,-j_2-\gamma)}} \left|\widetilde{\varphi}_{j_1+\gamma,j_2+\gamma} * f(x)\right|^2 \\
&\quad \times \chi_P(x) \frac{|Q|^2}{[w(Q)]^2} \chi_Q(x) w(x)\,dx \\
&\lesssim \sup_{w(\Omega)<\infty} \frac{1}{[w(\ddot{\Omega})]^{\frac{2}{p}-1}} \int_{\ddot{\Omega}} \sum_{j_1,j_2\in\mathbb{Z}} \sum_{\substack{P\subset\ddot{\Omega}\\ \operatorname{scale}(P)=(-j_1-\gamma,-j_2-\gamma)}} \left|\widetilde{\varphi}_{j_1+\gamma,j_2+\gamma} * f(x)\right|^2 \\
&\quad \times \chi_P(x) \frac{|P|^2}{[w(P)]^2} \sum_{\substack{Q\subset\Omega\\ \operatorname{scale}(Q)=(-j_1,-j_2)}} \chi_Q(x) w(x)\,dx \lesssim \|f\|_{\mathcal{L}_{p,w}(\vec{A})}^2,
\end{aligned}$$

where

$$\bigcup_{\{P\in\mathcal{R}:\ Q\subset\Omega,\ P\cap Q\neq\emptyset,\ \operatorname{scale}(P)=\operatorname{scale}(Q)-(\gamma,\gamma)\}} P \subset \ddot{\Omega} := \left\{x \in \mathbb{R}^n \times \mathbb{R}^m:\ \mathcal{M}_s(\chi_\Omega)(x) > C_0\right\}$$

which is obtained by a proof similar to that of (3.43). Here $C_0 \in (0,1)$ is some positive constant independent of Ω. From the above estimate and the arbitrariness of Ω, we deduce that

$$\left\|\inf(f)\right\|_{\ell_{p,w}(\vec{A})} \lesssim \|f\|_{\mathcal{L}_{p,w}(\vec{A})}.$$

Obviously, for any $f \in \mathcal{L}_{p,w}(\mathbb{R}^n \times \mathbb{R}^m; \widetilde{\varphi})$, $\|f\|_{\mathcal{L}_{p,w}(\vec{A})} \leqslant \|\sup(f)\|_{\ell_{p,w}(\vec{A})}$. To finish the proof of Lemma 3.15, it still needs to prove $\|\sup(f)\|_{\ell_{p,w}(\vec{A})} \lesssim \|\inf(f)\|_{\ell_{p,w}(\vec{A})}$. Fix any $Q \in \mathcal{R}$ with $\operatorname{scale}(Q) = (-j_1, -j_2)$, $j_1, j_2 \in \mathbb{Z}$. Let $g(x) := (\widetilde{\varphi}_{j_1,j_2} * f)(A_1^{-j_1}x_1, A_2^{-j_2}x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m$. Then we have $\operatorname{supp}\widehat{g} \subset K := (\operatorname{supp}\widehat{\varphi^{(1)}} \times \operatorname{supp}\widehat{\varphi^{(2)}})$. Let

$\{a_Q\}_{Q\in\mathcal{R}}$ and $\{b_Q\}_{Q\in\mathcal{R}}$ be as in (3.39). A direct calculation shows that, for any fixed $Q\in\mathcal{R}$ with $\mathrm{scale}(Q)=(-j_1,-j_2)$,

$$a_{A_1^{j_1}Q_1\times A_2^{j_2}Q_2}:=|Q|^{-\frac{1}{2}}\sup_Q(f)\quad\text{and}\quad b_{A_1^{j_1}Q_1\times A_2^{j_2}Q_2}:=|Q|^{-\frac{1}{2}}\inf_Q(f),\quad Q\in\mathcal{R}.$$

Hence, applying Lemma 3.14 to the dilated rectangle $\widetilde{Q}:=A_1^{j_1}Q_1\times A_2^{j_2}Q_2$, we have

$$\bigl(\sup(f)^*_{2,\lambda}\bigr)_Q=|Q|^{\frac{1}{2}}\bigl(a^*_{2,\lambda}\bigr)_{\widetilde{Q}}\lesssim|Q|^{\frac{1}{2}}\bigl(b^*_{2,\lambda}\bigr)_{\widetilde{Q}}\sim\bigl(\inf(f)^*_{2,\lambda}\bigr)_Q. \tag{3.44}$$

Since $Q\in\mathcal{R}$ is arbitrary, letting $p\in(0,1]$ and $\lambda\in(2q_w/p+1,\infty)$ be as in Lemma 3.13, by Lemma 3.13(ii) and (3.44), we conclude that

$$\bigl\|\sup(f)\bigr\|_{\ell_{p,w}(\vec{A})}\lesssim\bigl\|\inf(f)\bigr\|_{\ell_{p,w}(\vec{A})},$$

which completes the proof of Lemma 3.15. □

Proof of Theorem 2.12. Let $w\in\mathcal{A}_\infty(\vec{A})$. By Lemmas 3.13 and 3.15 together with an argument similar to that used in the proofs of [5, Theorem 3.5] and [40, Theorem 1.4], we obtain the desired results for Theorem 2.12 on the spaces $\ddot{h}^p_w(\vec{A})$ and $\ddot{H}^p_w(\vec{A})$, the details being omitted by similarity.

In what follows, let us prove Theorem 2.12 on the spaces $\ell_{p,w}(\vec{A})$ and $\mathcal{L}_{p,w}(\vec{A})$ with $p\in(0,1]$. We first prove that T_ψ is bounded from $\ell_{p,w}(\vec{A})$ to $\mathcal{L}_{p,w}(\vec{A})$. Let

$$f:=T_\psi s=\sum_{Q\in\mathcal{R}}s_Q\psi_Q.$$

Then, by Lemma 3.5, we know that f is a well-defined element of $\mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, which implies that, for all $x\in\mathbb{R}^n\times\mathbb{R}^m$,

$$f*\varphi_{j_1,j_2}(x)=\sum_{\ell_1,\ell_2\in\mathbb{Z}}\sum_{\mathrm{scale}(Q)=(-\ell_1,-\ell_2)}s_Q\psi_Q*\varphi_{j_1,j_2}(x).$$

Since $\mathrm{supp}\,\widehat{\varphi^{(i)}}$ and $\mathrm{supp}\,\widehat{\psi^{(i)}}$ are compact and bounded away from the origin, $i\in\{1,2\}$, then, for any $Q_i\in\mathcal{Q}_i$ with $\mathrm{scale}(Q_i)=-\ell_i$, $i\in\{1,2\}$, there exists a sufficiently large integer M such that, when $|j_i-\ell_i|>M$ and $\mathrm{scale}(Q_i)=-\ell_i$, $i\in\{1,2\}$, we have $\mathrm{supp}\,\widehat{\psi^{(i)}_{Q_i}}\cap\mathrm{supp}\,\widehat{\varphi^{(i)}_{j_i}}=\emptyset$ and hence, for any $\xi\in\mathbb{R}^n\times\mathbb{R}^m$, $|\ell_1-j_1|>M$ or $|\ell_2-j_2|>M$, we further have $(\psi_Q*\varphi_{j_1,j_2})\widehat{\ }(\xi)=0$. From this, it follows that, for any $x\in\mathbb{R}^n\times\mathbb{R}^m$, $\psi_Q*\varphi_{j_1,j_2}(x)=0$ when $|\ell_1-j_1|>M$ or $|\ell_2-j_2|>M$. Therefore, we conclude that, for all $x\in\mathbb{R}^n\times\mathbb{R}^m$,

$$f*\varphi_{j_1,j_2}(x)=\sum_{\substack{|\ell_i-j_i|\leqslant M\\ i\in\{1,2\}}}\sum_{\mathrm{scale}(Q)=(-\ell_1,-\ell_2)}s_Q\varphi_{j_1,j_2}*\psi_Q(x).$$

For any $p \in (0,1]$, we take $\lambda \in (2q_w/p+1,\infty)$. Since $\varphi_{j_1,j_2} * \psi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, it follows that, for all $x \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\begin{aligned}\varphi_{j_1,j_2} * \psi_Q(x) &= (\varphi_{j_1-\ell_1,j_2-\ell_2} * \psi)_Q(x)\\ &= |Q|^{-\frac{1}{2}}(\varphi_{j_1-\ell_1,j_2-\ell_2} * \psi)\big(A_1^{\ell_1}(x_1 - x_{Q_1}), A_2^{\ell_2}(x_2 - x_{Q_2})\big)\\ &\lesssim \frac{|Q|^{-\frac{1}{2}}}{\prod_{i=1}^2 [1+\rho_i(A_i^{\ell_i}x_i - x_{Q_i})]^{\lambda/2}}.\end{aligned}$$

Moreover, for any $x \in \mathbb{R}^n \times \mathbb{R}^m$ and $\ell_1, \ell_2 \in \mathbb{Z}$, there exists a unique $Q^x \in \mathcal{R}$ such that $x \in Q^x$ and $\mathrm{scale}(Q^x) = (-\ell_1,-\ell_2)$. Then, for any $Q \in \mathcal{R}$ with $\mathrm{scale}(Q) := (-\ell_1,-\ell_2)$, it is easy to show $1+\rho_i(A_i^{\ell_i}(x_{Q_i^x} - x_{Q_i})) \lesssim 1+\rho_i(A_i^{\ell_i}(x_i - x_{Q_i}))$, $i \in \{1,2\}$. From this, the above estimate, $p \in (0,1]$, $\lambda \in (2q_w/p+1,\infty)$ and Hölder's inequality, it follows that, for all $x \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\begin{aligned}\big|f * \varphi_{j_1,j_2}(x)\big| &\lesssim \sum_{\substack{|\ell_i - j_i| \leqslant M\\ i \in \{1,2\}}} \chi_{Q^x}(x)\big|Q^x\big|^{-\frac{1}{2}}\\ &\quad \times \sum_{\mathrm{scale}(Q)=(-\ell_1,-\ell_2)} \frac{|s_Q|}{\prod_{i=1}^2 [1+\rho_i(A_i^{\ell_i}(x_{Q_i^x} - x_{Q_i}))]^{\lambda/2}}\\ &\lesssim \sum_{\substack{|\ell_i - j_i| \leqslant M\\ i \in \{1,2\}}} \chi_{Q^x}(x)\big|Q^x\big|^{-\frac{1}{2}}\big(s_{2,\lambda}^*\big)_{Q^x}\\ &\lesssim \sum_{\substack{|\ell_i - j_i| \leqslant M\\ i \in \{1,2\}}} \sum_{\mathrm{scale}(Q)=(-\ell_1,-\ell_2)} \big(s_{2,\lambda}^*\big)_Q \chi_Q(x)|Q|^{-\frac{1}{2}}.\end{aligned}$$

Moreover, for any open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, using an argument similar to that used in the estimate for (3.43), there exists a positive constant $\eta_2 \in (0,1)$ such that

$$\bigcup_{\substack{|\ell_i - j_i| \leqslant M\\ i \in \{1,2\}}} \bigcup_{\substack{\mathrm{scale}(Q)=(-\ell_1,-\ell_2)\\ Q \subset \Omega}} Q \subset \check{\Omega} := \big\{x \in \mathbb{R}^n \times \mathbb{R}^m \colon\ \mathcal{M}_s(\chi_\Omega)(x) > \eta_2\big\}.$$

Consequently, for any $p \in (0,1]$, $\lambda \in (2q_w/p+1,\infty)$, $q \in (q_w,\infty)$ and open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ with $w(\Omega) < \infty$, by the last two estimates, (3.43), $w(\check{\Omega}) \lesssim w(\Omega)$ (see [8, Proposition 2.10(ii)]) and Lemma 3.15, we have

$$\begin{aligned}&\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \int_\Omega \sum_{j_1,j_2 \in \mathbb{Z}} \sum_{\substack{R \subset \Omega\\ \mathrm{scale}(R)=(-j_1,-j_2)}} \big|f * \varphi_{j_1,j_2}(x)\big|^2 \chi_R(x) \frac{|R|^2}{[w(R)]^2} w(x)\,dx\\ &\quad \lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \int_\Omega \sum_{j_1,j_2 \in \mathbb{Z}} \sum_{\substack{R \subset \Omega,\, R \in \mathcal{R}\\ \mathrm{scale}(R)=(-j_1,-j_2)}}\end{aligned}$$

$$\times \Bigg[\sum_{\substack{|\ell_i - j_i| \leqslant M \\ i\in\{1,2\}}} \sum_{\mathrm{scale}(Q)=(-\ell_1,-\ell_2)} \big(s^*_{2,\lambda}\big)_Q \chi_Q(x)|Q|^{-\frac{1}{2}} \Bigg]^2 \chi_R(x) \frac{|R|^2}{[w(R)]^2} w(x)\,dx$$

$$\lesssim \frac{1}{[w(\check{\Omega})]^{\frac{2}{p}-1}} \int_{\check{\Omega}} \sum_{\ell_1,\ell_2\in\mathbb{Z}} \sum_{\substack{Q\subset\check{\Omega} \\ \mathrm{scale}(Q)=(-\ell_1,-\ell_2)}} \big(s^*_{2,\lambda}\big)_Q^2 \chi_Q(x) \frac{|Q|}{[w(Q)]^2}$$

$$\times \sum_{\substack{|\ell_i - j_i| \leqslant M \\ i\in\{1,2\}}} \sum_{\mathrm{scale}(R)=(-j_1,-j_2)} \chi_R(x) w(x)\,dx$$

$$\lesssim \big\|s^*_{2,\lambda}\big\|^2_{\ell_{p,w}(\vec{A})} \lesssim \|s\|^2_{\ell_{p,w}(\vec{A})},$$

which implies that $\|T_\psi(s)\|_{\mathcal{L}_{p,w}(\vec{A})} \lesssim \|s\|_{\ell_{p,w}(\vec{A})}$.

Suppose that $f \in \mathcal{L}_{p,w}(\vec{A})$ and $Q := A_1^{-j_1}([0,1)^n + k_1) \times A_2^{-j_2}([0,1)^m + k_2)$, $j_1, j_2 \in \mathbb{Z}$, $k_1 \in \mathbb{Z}^n$, $k_2 \in \mathbb{Z}^m$. Then,

$$\big|(S_\varphi f)_Q\big| = \big|\langle f, \varphi_Q\rangle\big| = |Q|^{\frac{1}{2}} \big|(\widetilde{\varphi}_{j_1,j_2} * f)(x_Q)\big| \leqslant \sup\nolimits_Q(f).$$

Therefore, by Lemma 3.15, we obtain the boundedness of S_φ from $\ell_{p,w}(\vec{A})$ to $\mathcal{L}_{p,w}(\vec{A};\widetilde{\varphi})$.

Finally, if (ψ,φ) is an admissible pair of frame wavelets as in Definition 2.2, then, by Lemma 2.3(ii), we know that $\mathcal{L}_{p,w}(\vec{A};\widetilde{\varphi}) \hookrightarrow \mathcal{L}_{p,w}(\vec{A};\varphi)$ is a bounded inclusion. Hence, by reversing the roles of φ and $\widetilde{\varphi}$, we find that

$$\mathcal{L}_{p,w}(\vec{A};\widetilde{\varphi}) = \mathcal{L}_{p,w}(\vec{A};\varphi),$$

which completes the proof of Theorem 2.12. $\square$

To prove Theorem 2.14, we need two technical lemmas first. By [8, Lemma 5.5] and a basic fact that $\psi_j * \varphi_k := (\psi_{j-k} * \varphi)_k$, $j,k \in \mathbb{Z}$, we obtain the following lemma, the details being omitted.

Lemma 3.16. *For $i \in \{1,2\}$, let $M_i \in (0,\infty)$, $N_i \in \mathbb{Z}_+$, $\varphi^{(i)}, \psi^{(i)} \in \mathcal{S}_{N_i}(\mathbb{R}^{n_i})$, $\varphi := \varphi^{(1)} \otimes \varphi^{(2)}$ and $\psi := \psi^{(1)} \otimes \psi^{(2)}$. Then there exists a positive constant C, depending on M_i and N_i with $i \in \{1,2\}$, such that, for all $j_i, k_i \in \mathbb{Z}$ with $j_i \geqslant k_i$, $i \in \{1,2\}$ and for all $x \in \mathbb{R}^n \times \mathbb{R}^m$,*

$$\big|\varphi_{j_1,j_2} * \psi_{k_1,k_2}(x)\big| \leqslant C \prod_{i=1}^{2} b_i^{k_i + (k_i - j_i)(N_i+1)\zeta_{i,-}} \big[1 + \rho_i\big(A_i^{k_i} x_i\big)\big]^{-M_i}.$$

We skip the proof of the following Lemma 3.17 since it is similar to those of [5, Theorem 4.1] and [40, Theorem 2.1].

Lemma 3.17. *Let $p \in (0,1]$ and $w \in \mathcal{A}_\infty(\vec{A})$ with q_w as in (3.14). An operator $\mathcal{A}$ is said to be almost diagonal, if its associated matrix $\{a_{R,Q}\}_{R,Q\in\mathcal{R}}$, where $a_{R,Q} := (\mathcal{A}e^Q)_R$, satisfies that there exists some positive constant ϵ such that*

$$\sup_{R,Q\in\mathcal{R}} |a_{R,Q}|/\kappa_{R,Q}(\epsilon) < \infty,$$

where

$$\kappa_{R,Q}(\epsilon) := \prod_{i=1}^{2}\left[1 + \frac{\rho_i(x_{R_i} - x_{Q_i})}{|R_i| \vee |Q_i|}\right]^{-\frac{q_w}{p}-\epsilon}\left[\left(\frac{|R_i|}{|Q_i|}\right)^{\frac{1+\epsilon}{2}} \wedge \left(\frac{|Q_i|}{|R_i|}\right)^{\frac{q_w}{p}+\frac{\epsilon-1}{2}}\right]. \quad (3.45)$$

Then the almost diagonal operator $\mathcal{A}$ is bounded on $\ddot{h}^p_w(\vec{A})$.

Proof of Theorem 2.14. Let $p \in (0,1]$ and $w \in \mathcal{A}_\infty(\vec{A})$. We first show that, for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, $\|f\|_{\widetilde{H}^p_w(\vec{A})} \lesssim \|f\|_{\ddot{H}^p_w(\vec{A})}$. For any $k_1, k_2 \in \mathbb{Z}$, $x \in \mathbb{R}^n \times \mathbb{R}^m$ and $\Phi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, we have

$$\begin{aligned} &b_1^{k_1} b_2^{k_2} \int_{\prod_{i=1}^2 B_{\rho_i}(x_i, b_i^{-k_i})} \left|f * \Phi_{k_1,k_2}(y)\right|^2 dy \\ &\quad = b_1^{k_1} b_2^{k_2} \int_{\prod_{i=1}^2 B_{\rho_i}(x_i, b_i^{-k_i})} \sum_{\substack{R\in\mathcal{R} \\ \mathrm{scale}(R)=(-k_1,-k_2)}} \left|f * \Phi_{k_1,k_2}(y)\right|^2 \chi_R(y)\, dy. \end{aligned} \quad (3.46)$$

Let (φ, ψ) be the admissible pair of frame wavelets as in Definition 2.2. For any $f \in \ddot{H}^p_w(\vec{A})$, by Lemma 2.3(ii), we know that $f = \sum_{Q\in\mathcal{R}} \langle f, \varphi_Q\rangle \psi_Q$ in $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, which, together with $\Phi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, implies that

$$f * \Phi_{k_1,k_2} = \sum_{Q\in\mathcal{R}} \langle f, \varphi_Q\rangle \psi_Q * \Phi_{k_1,k_2} \quad (3.47)$$

holds true pointwise. Moreover, since $\psi := \psi^{(1)} \otimes \psi^{(2)}$ and $\Phi := \Phi^{(1)} \otimes \Phi^{(2)}$ with $\psi^{(i)}, \Phi^{(i)} \in \mathcal{S}_\infty(\mathbb{R}^{n_i})$, $i \in \{1,2\}$, then, for any $y \in R$ with $\mathrm{scale}(R) = (-k_1, -k_2)$, $Q \in \mathcal{R}$ with $\mathrm{scale}(Q) = (-j_1, -j_2)$, $M_i \in (0,\infty)$ and $N_i \in \mathbb{Z}_+$ to be fixed later, by Lemma 3.16, we conclude that

$$\begin{aligned} \left|\psi_Q * \Phi_{k_1,k_2}(y)\right| &\lesssim \prod_{i=1}^2 b_i^{-j_i/2 - |j_i - k_i|(N_i+1)\zeta_{i,-} + j_i\wedge k_i}\left[1 + b_i^{j_i\wedge k_i}\rho_i(y_i - x_{Q_i})\right]^{-M_i} \\ &\lesssim |R|^{-\frac{1}{2}} \prod_{i=1}^2 b_i^{-|j_i-k_i|[(N_i+1)\zeta_{i,-}+\frac{1}{2}]}\left[1 + b_i^{j_i\wedge k_i}\rho_i(x_{R_i} - x_{Q_i})\right]^{-M_i}. \end{aligned} \quad (3.48)$$

Since $p \in (0,1]$ and q_w is as in (3.14), if we let

$$a_{R,Q} := \prod_{i=1}^{2} b_i^{-|j_i-k_i|[(N_i+1)\zeta_{i,-}+\frac{1}{2}]}\left[1 + b_i^{j_i\wedge k_i}\rho_i(x_{R_i} - x_{Q_i})\right]^{-M_i},$$

$N_i > (q_w/p - 1)\zeta_{i,-}^{-1} - 1$, $M_i > q_w/p$, $N_i \in \mathbb{Z}_+$ and $\epsilon \in \mathbb{R}_+$ such that

$$\epsilon/2 < \min_{i\in\{1,2\}}\left\{(N_i+1)\zeta_{i,-} + 1 - q_w/p, M_i - q_w/p, (N_i+1)\zeta_{i,-}\right\},$$

then it is easy to show that, for any $R, Q \in \mathcal{R}$, $a_{R,Q} \lesssim \kappa_{R,Q}(\epsilon)$ uniformly, where $\kappa_{R,Q}(\epsilon)$ satisfies (3.45), which implies that $\{a_{R,Q}\}_{R,Q\in\mathcal{R}}$ induces an almost diagonal operator. Therefore, for any $f \in \ddot{H}_w^p(\vec{A})$ and $s := \{s_Q\}_{Q\in\mathcal{R}}$ with $s_Q := \langle f, \varphi_Q\rangle$, by (3.46), (3.47), (3.48), Lemma 3.17 and Theorem 2.12 with the inverse φ-transform $S_\varphi(f)$, we have

$$\begin{aligned}\|f\|_{\widetilde{H}_w^p(\vec{A})} &= \left\|\vec{S}_\Phi(f)\right\|_{L_w^p(\mathbb{R}^n\times\mathbb{R}^m)} \lesssim \left\|\left\{\sum_{Q\in\mathcal{R}} a_{R,Q}s_Q\right\}_{R\in\mathcal{R}}\right\|_{\ddot{h}_w^p(\vec{A})}\\ &\lesssim \left\|S_\varphi(f)\right\|_{\ddot{h}_w^p(\vec{A})} \lesssim \|f\|_{\ddot{H}_w^p(\vec{A})},\end{aligned}$$

which is desired.

Finally, we show that, for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, $\|f\|_{\ddot{H}_w^p(\vec{A})} \lesssim \|f\|_{\widetilde{H}_w^p(\vec{A})}$. To this end, letting $\gamma \in \mathbb{N}$ be as in Lemma 3.15, for any $Q \in \mathcal{R}$ with $\operatorname{scale}(Q) = (-k_1, -k_2)$, $k_1, k_2 \in \mathbb{Z}$, and any $x \in Q$, by [3, Lemma 2.9(b)], there exists some constant $k_0 \in \mathbb{N}$, independent of the choice of Q, such that

$$\bigcup_{\{P\in\mathcal{R}:\ P\cap Q\neq\emptyset,\ \operatorname{scale}(P_i)=(-k_1-\gamma,-k_2-\gamma)\}} P \subset B_{\rho_1}\left(x_1, b_1^{-k_1+k_0}\right)\times B_{\rho_2}\left(x_2, b_2^{-k_2+k_0}\right).$$

Thus, for any $x \in Q$ and $\Phi \in \mathscr{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, we have

$$\begin{aligned}&\sum_{\substack{P\cap Q\neq\emptyset\\ \operatorname{scale}(P)=(-k_1-\gamma,-k_2-\gamma)}} \inf_{y\in P}\left|f*\Phi_{k_1-k_0,k_2-k_0}(y)\right|^2\chi_Q(x)\\ &\quad\leqslant b_1^{k_1-k_0}b_2^{k_2-k_0}\int_{B_{\rho_1}(x_1,b_1^{-k_1+k_0})\times B_{\rho_2}(x_2,b_2^{-k_2+k_0})}\left|f*\Phi_{k_1-k_0,k_2-k_0}(y)\right|^2\,dy. \qquad (3.49)\end{aligned}$$

Let

$$\begin{aligned}&\inf\nolimits_Q(f)\\ &\quad:= |Q|^{\frac{1}{2}}\sup\left\{\inf_{y\in P}\left|f*\widetilde{\Phi}_{k_1-k_0,k_2-k_0}(y)\right|:\ \operatorname{scale}(P) = (-k_1-\gamma,-k_2-\gamma),\ P\cap Q\neq\emptyset\right\}.\end{aligned}$$

Then, for any $f \in \widetilde{H}_w^p(\vec{A})$ with $p \in (0,1]$ and $w \in \mathcal{A}_\infty(\vec{A})$, by Corollary 2.3, Lemma 3.15 and (3.49), we conclude that

$$\begin{aligned}\|f\|_{\ddot{H}^p_w(\vec{A})} &\sim \left\| \left\{ \sum_{k_1,k_2\in\mathbb{Z}} |f * \widetilde{\Phi}_{k_1-k_0,k_2-k_0}|^2 \right\}^{\frac{1}{2}} \right\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} \\ &\lesssim \left\| \left\{ \sum_{k_1,k_2\in\mathbb{Z}} \sum_{\substack{Q\in\mathcal{R} \\ \mathrm{scale}(Q)=(-k_1,-k_2)}} \left[\inf_Q(f)\right]^2 |Q|^{-1}\chi_Q \right\}^{\frac{1}{2}} \right\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} \\ &\lesssim \left\|\vec{S}_\Phi(f)\right\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} \sim \|f\|_{\widetilde{H}^p_w(\mathbb{R}^n\times\mathbb{R}^m)},\end{aligned}$$

which completes the proof of Theorem 2.14. □

Proof of Proposition 2.15. Let $p \in (0,1]$ and $w \in \mathcal{A}_\infty(\vec{A})$ with q_w as in (3.14). We first prove that, for any $t := \{t_R\}_{R\in\mathcal{R}} \in \ell_{p,w}(\vec{A})$, its induced map L_t, defined by

$$L_t(s) := \sum_{R\in\mathcal{R}} s_R\bar{t}_R$$

for any $s \in \ddot{h}^p_w(\vec{A})$, belongs to $(\ddot{h}^p_w(\vec{A}))^*$. We show this by using some ideas from the proof of [40, Theorem 3.5]. For any $x \in \mathbb{R}^n \times \mathbb{R}^m$, $k \in \mathbb{Z}$ and $R \in \mathcal{R}$, let

$$\begin{aligned} G(x) &:= \left\{ \sum_{R\in\mathcal{R}} |s_R|^2|R|^{-1}\chi_R(x) \right\}^{\frac{1}{2}}, \\ \Omega_k &:= \left\{x \in \mathbb{R}^n \times \mathbb{R}^m \colon G(x) > 2^k\right\}, \\ \mathcal{R}_k &:= \left\{R \in \mathcal{R}\colon |R\cap\Omega_k| > |R|/2,\ |R\cap\Omega_{k+1}| \leqslant |R|/2\right\} \end{aligned}$$

and $E_{R,k} := R \cap (\Omega_{k+1})^\complement$. Then, for any $k \in \mathbb{Z}$ and $R \in \mathcal{R}_k$, by Lemma 3.3 with $w \in \mathcal{A}_q(\vec{A})$ and $q \in (q_w,\infty)$, we obtain

$$\frac{1}{2^q} \leqslant \frac{|E_{R,k}|^q}{|R|^q} \lesssim \frac{w(E_{R,k})}{w(R)}. \tag{3.50}$$

We choose a positive integer $c_0 > 2$ such that $b_1^{-c_0u_1}b_2^{-c_0u_2} < b_1^{-2u_1}b_2^{-2u_2}/2$ and, for all $k \in \mathbb{Z}$, let

$$\widetilde{\Omega}_k := \left\{x \in \mathbb{R}^n \times \mathbb{R}^m \colon \mathcal{M}_s(\chi_{\Omega_k})(x) > b_1^{-c_0u_1}b_2^{-c_0u_2}\right\}.$$

Then, for all $R \in \mathcal{R}_k$ and all $x \in R$, by Lemma 3.7(iv), we see that

$$\begin{aligned}\mathcal{M}_s(\chi_{\Omega_k})(x) &\geqslant \frac{1}{b_1^{v_1\ell(\dot{R}_1)+u_1}b_2^{v_2\ell(\dot{R}_2)+u_2}} \int_{x_R+B^{(1)}_{v_1\ell(\dot{R}_1)+u_1}\times B^{(2)}_{v_2\ell(\dot{R}_2)R+u_2}} \chi_{\Omega_k}(y)\,dy \\ &\geqslant b_1^{-2u_1}b_2^{-2u_2}\frac{|\Omega_k\cap R|}{|R|} > b_1^{-c_0u_1}b_2^{-c_0u_2},\end{aligned}$$

which implies that

$$\bigcup_{R\in\mathcal{R}_k} R \subset \widetilde{\Omega}_k. \tag{3.51}$$

Moreover, for any $w \in \mathcal{A}_q(\vec{A})$ with $q \in (q_w, \infty)$, by the $L^q_w(\mathbb{R}^n \times \mathbb{R}^m)$-boundedness of $\mathcal{M}_s$ (see [8, Proposition 2.10(ii)]), we obtain $w(\widetilde{\Omega}_k) \lesssim w(\Omega_k)$ for all $k \in \mathbb{Z}$.

Therefore, for all $s \in \ddot{h}^p_w(\vec{A})$, by (3.50), (3.51), Hölder's inequality and $w(\widetilde{\Omega}_k) \lesssim w(\Omega_k)$, we have

$$\begin{aligned}
\left|L_t(s)\right| &\leqslant \sum_{k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_k} |t_R||s_R| \\
&\lesssim \sum_{k\in\mathbb{Z}} \int_{\widetilde{\Omega}_k} \sum_{R\in\mathcal{R}_k} |t_R| \frac{|R|^{\frac{1}{2}}}{w(R)} \chi_R(x)|s_R||R|^{-\frac{1}{2}} \chi_{E_{R,k}}(x) w(x)\, dx \\
&\lesssim \sum_{k\in\mathbb{Z}} \left\{ \int_{\widetilde{\Omega}_k} \sum_{R\in\mathcal{R}_k} |t_R|^2 \frac{|R|}{[w(R)]^2} \chi_R(x) w(x)\, dx \right\}^{\frac{1}{2}} \\
&\quad \times \left\{ \int_{\widetilde{\Omega}_k} \sum_{R\in\mathcal{R}_k} |s_R|^2 |R|^{-1} \chi_{E_{R,k}}(x) w(x)\, dx \right\}^{\frac{1}{2}} \\
&\lesssim \|t\|_{\ell_{p,w}(\vec{A})} \sum_{k\in\mathbb{Z}} \left[w(\widetilde{\Omega}_k)\right]^{(\frac{2}{p}-1)\frac{1}{2}} \left\{ \int_{\widetilde{\Omega}_k\setminus\Omega_{k+1}} \left[G(x)\right]^2 w(x)\, dx \right\}^{\frac{1}{2}} \\
&\lesssim \|t\|_{\ell_{p,w}(\vec{A})} \sum_{k\in\mathbb{Z}} 2^k \left[w(\Omega_k)\right]^{\frac{1}{p}} \lesssim \|t\|_{\ell_{p,w}(\vec{A})} \|s\|_{\ddot{h}^p_w(\vec{A})},
\end{aligned} \tag{3.52}$$

which implies that the induced fractional $L_t \in (\ddot{h}^p_w(\vec{A}))^*$ and $|L_t| \lesssim \|t\|_{\ell_{p,w}(\vec{A})}$.

Now let us prove the converse by borrowing some ideas from the proof of [21, Theorem 5.9]. For any $N \in \mathbb{N}$, $w \in \mathcal{A}_\infty(\vec{A})$ and $R \in \mathcal{R}$, let $B_N := B_{\rho_1}(0, b_1^N) \times B_{\rho_2}(0, b_2^N)$,

$$I_N := \left\{R \in \mathcal{R}\colon\ R \subset B_N,\ \left|\mathrm{scale}(R_i)\right| \leqslant N,\ i \in \{1,2\}\right\},$$

and $\ell^2(B_N)$ be the *set of all* $s^{(N)} := \{s_R^{(N)}\}_{R\in I_N}$ *satisfying*

$$\left\|s^{(N)}\right\|_{\ell^2(B_N)} := \left\{ \sum_{R\in I_N} \left|s_R^{(N)}\right|^2 \right\}^{1/2} < \infty.$$

For any $t \in \ddot{h}^p_w(\vec{A})$ and $N \in \mathbb{N}$, let $t^{(N)} := \{t_R^{(N)}\}_{R\in I_N}$ with $t_R^{(N)} := t_R$ if $R \in I_N$. We denote by $\ddot{h}^p_w(\vec{A}; B_N)$ the *set of all such* $t^{(N)}$. Obviously, $\ddot{h}^p_w(\vec{A}; B_N)$ endowed with the norm $\|\cdot\|_{\ddot{h}^p_w(\vec{A})}$ is a subspace of $\ddot{h}^p_w(\vec{A})$.

Notice that, for any $s^{(N)} \in \ell^2(B_N)$, $p \in (0,1]$ and $w \in \mathcal{A}_\infty(\vec{A})$, by Hölder's inequality, we know that

$$\|s^{(N)}\|_{\ddot{h}^p_w(\vec{A})} \leqslant \|s^{(N)}\|_{\ell^2(B_N)}\big[w(B_N)\big]^{1-\frac{p}{2}},$$

which implies that $\ell^2(B_N) \subset \ddot{h}^p_w(\vec{A};B_N)$ and hence $(\ddot{h}^p_w(\vec{A};B_N))^* \subset (\ell^2(B_N))^* = \ell^2(B_N)$. Then, for any $L \in (\ddot{h}^p_w(\vec{A}))^*$, by the above estimate and $(\ddot{h}^p_w(\vec{A}))^* \subset (\ddot{h}^p_w(\vec{A};B_N))^*$, there exists some $t^{(N)} \in \ell^2(B_N)$ such that, for all $s^{(N)} \in \ddot{h}^p_w(\vec{A};B_N)$,

$$L\big(s^{(N)}\big) = \sum_{R\in I_N} s^{(N)}_R \overline{t^{(N)}_R}. \tag{3.53}$$

For $N+1$, repeating the above process, there exists some $t^{(N+1)} \in \ell^2(B_N)$ such that, for all $s^{(N+1)} \in \ddot{h}^p_w(\vec{A};B_{N+1})$,

$$L\big(s^{(N+1)}\big) = \sum_{R\in I_{N+1}} s^{(N+1)}_R \overline{t^{(N+1)}_R},$$

and $t^{(N+1)}|_{\ddot{h}^p_w(\vec{A};B_N)} = t^{(N)}$. By this extension, we obtain a sequence $t^* := \{t^*_R\}_{R\in\mathcal{R}}$, where $t^*_R := t^{(N)}_R$ if $R \in I_N$ for all $N \in \mathbb{N}$.

We now show that $t^* \in \ell_{p,w}(\vec{A})$. To this end, let Ω be any open set with $w(\Omega) < \infty$ and ϑ the measure on $\mathcal{R}$ such that, for any $R \in \mathcal{R}$, $\vartheta(R) := [w(\Omega)]^{1-2/p}|R|[w(R)]^{-1}$ if $R \subset \Omega$, or else $\vartheta(R) := 0$. Define $\ell^2(\Omega;\vartheta)$ to be the *set of all complex-valued sequences* $s := \{s_R\}_{R\in\mathcal{R},\, R\subset\Omega}$ *such that*

$$\|s\|_{\ell^2(\Omega;\vartheta)} := \bigg\{\sum_{R\subset\Omega} |s_R|^2\big[w(\Omega)\big]^{1-\frac{2}{p}}|R|\big[w(R)\big]^{-1}\bigg\}^{\frac{1}{2}} < \infty.$$

Then, by (3.53) and $(\ell^2(\Omega;\vartheta))^* = \ell^2(\Omega;\vartheta)$, we have

$$\begin{aligned}
&\bigg\{\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{R\subset\Omega} \chi_{I_N}(R)\big|t^*_R\big|^2|R|\big[w(R)\big]^{-1}\bigg\}^{\frac{1}{2}} \\
&\quad = \big\|\chi_{I_N}t^*\big\|_{\ell^2(\Omega,\vartheta)} = \sup_{\|s\|_{\ell^2(\Omega;\vartheta)}\leqslant 1}\bigg|\sum_{R\in I_N,\, R\subset\Omega} t^{(N)}_R\overline{s_R}\big[w(\Omega)\big]^{1-\frac{2}{p}}|R|\big[w(R)\big]^{-1}\bigg| \\
&\quad \leqslant \|L\|_{(\ddot{h}^p_w(\vec{A}))^*}\sup_{\|s\|_{\ell^2(\Omega;\vartheta)}\leqslant 1}\Big\|\Big\{s_R\big[w(\Omega)\big]^{1-\frac{2}{p}}|R|\big[w(R)\big]^{-1}\Big\}_{R\subset\Omega}\Big\|_{\ddot{h}^p_w(\vec{A})} \leqslant \|L\|_{(\ddot{h}^p_w(\vec{A}))^*},
\end{aligned}$$

where, in the last step, we used the following inequality that

$$\begin{aligned}
&\big\|\big\{s_R\big[w(\Omega)\big]^{1-\frac{2}{p}}|R|\big[w(R)\big]^{-1}\big\}_{R\subset\Omega}\big\|_{\ddot{h}^p_w(\vec{A})}\\
&=\bigg\{\int_\Omega\bigg[\sum_{R\subset\Omega}\big|s_R\big[w(\Omega)\big]^{1-\frac{2}{p}}\big|^2|R|\big[w(R)\big]^{-2}\chi_R(x)\bigg]^{p/2}w(x)\,dx\bigg\}^{1/p}\\
&\leqslant\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\bigg\{\int_\Omega\sum_{R\subset\Omega}|s_R|^2|R|\big[w(R)\big]^{-2}\chi_R(x)w(x)\,dx\bigg\}^{\frac{1}{2}}\big[w(\Omega)\big]^{\frac{1}{p}-\frac{1}{2}}\\
&=\|s\|_{\ell^2(\Omega,\vartheta)}\leqslant 1.
\end{aligned}$$

From this and the Fatou lemma, it follows that

$$\bigg\{\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{R\subset\Omega}\big|t^*_R\big|^2\frac{|R|}{w(R)}\bigg\}^{\frac{1}{2}}\leqslant\|L\|_{(\ddot{h}^p_w(\vec{A}))^*},$$

which, together with the arbitrariness of Ω, implies that $t^*\in\ell_{p,w}(\vec{A})$ and $\|t^*\|_{\ell_{p,w}(\vec{A})}\leqslant\|L\|_{(\ddot{h}^p_w(\vec{A}))^*}$.

By (3.52), for all $s\in\ddot{h}^p_w(\vec{A})$, we have

$$\sum_{R\in\mathcal{R}}\big|t^*_R s_R\big|\lesssim\big\|t^*\big\|_{\ell_{p,w}(\vec{A})}\|s\|_{\ddot{h}^p_w(\vec{A})}\lesssim\|L\|_{(\ddot{h}^p_w(\vec{A}))^*}\|s\|_{\ddot{h}^p_w(\vec{A})},$$

which, together with the Lebesgue dominated convergence theorem on series and (3.53), yields that, for all $s\in\ddot{h}^p_w(\vec{A})$,

$$L(s)=\lim_{N\to\infty}L\big(s^{(N)}\big)=\lim_{N\to\infty}\sum_{R\in I_N}s^{(N)}_R\overline{t^*_R}=\sum_{R\in\mathcal{R}}s_R\overline{t^*_R}.$$

This finishes the proof of Proposition 2.15. □

From Theorem 2.12 and Proposition 2.15, the proof of Theorem 2.16 follows by a straightforward adaption of methods by Frazier and Jawerth [21, Theorem 5.13].

Proof of Theorem 2.16. Let $p\in(0,1]$ and $w\in\mathcal{A}_\infty(\vec{A})$. Let (φ,ψ) be an admissible pair of frame wavelets as in Definition 2.2 such that $\varphi=\psi$. In other words, φ is an admissible Parseval wavelet. Using Theorem 2.12, we conclude that $T_\varphi\circ S_\varphi$ is also an identity on $\ddot{H}^p_w(\vec{A})$.

For $s:=\{s_R\}_{R\in\mathcal{R}}$ and $t:=\{t_R\}_{R\in\mathcal{R}}$, let $\langle s,t\rangle:=\sum_{R\in\mathcal{R}}s_R\overline{t_R}$. Then, for any $f\in\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, the φ-transform S_φ and the inverse φ-transform T_φ, we have

$$\big\langle S_\varphi(f),t\big\rangle=\sum_{R\in\mathcal{R}}\langle f,\varphi_R\rangle\overline{t_R}=\big\langle f,T_\varphi(t)\big\rangle.\tag{3.54}$$

For any $g\in\mathcal{L}_{p,w}(\vec{A})$, define a linear functional $\widetilde{L}_g$ by $\widetilde{L}_g(f):=\langle g,f\rangle$ for any $f\in\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$.

Then, for any $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, by Theorem 2.12, (3.54) and Proposition 2.15, we find that

$$\begin{aligned} \bigl|\widetilde{L}_g(f)\bigr| &= \bigl|\langle g, f\rangle\bigr| = \bigl|\bigl\langle T_\varphi\bigl(S_\varphi(g)\bigr), f\bigr\rangle\bigr| = \bigl|\bigl\langle S_\varphi(g), S_\varphi(f)\bigr\rangle\bigr| \\ &\lesssim \bigl\|S_\varphi(g)\bigr\|_{\ell_{p,w}(\vec{A})}\bigl\|S_\varphi(f)\bigr\|_{\ddot{h}^p_w(\vec{A})} \lesssim \|g\|_{\mathcal{L}_{p,w}(\vec{A})}\|f\|_{\ddot{H}^p_w(\vec{A})}, \end{aligned}$$

which implies that $\|\widetilde{L}_g\|_{(\ddot{H}^p_w(\vec{A}))^*} \lesssim \|g\|_{\mathcal{L}_{p,w}(\vec{A})}$ and hence $\widetilde{L}_g$ defines a continuous linear functional on $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Moreover, since $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ is a dense subspace of $\ddot{H}^p_w(\vec{A})$, using Theorem 2.12, we conclude that $\widetilde{L}_g$ is uniquely extended to a continuous linear functional L_g on $\ddot{H}^p_w(\vec{A})$.

Conversely, for any $L \in (\ddot{H}^p_w(\vec{A}))^*$ and the inverse φ-transform T_φ, by Theorem 2.12, we have $\ell_1 := L \circ T_\varphi \in (\ddot{h}^p_w(\vec{A}))^*$. Then, by Proposition 2.15 and Theorem 2.12, there exists $t = \{t_R\}_{R\in\mathcal{R}} \in \ell_{p,w}(\vec{A})$ such that $\ell_1(s) = \sum_{R\in\mathcal{R}} s_R\overline{t_R}$ for any $s := \{s_R\}_{R\in\mathcal{R}} \in \ddot{h}^p_w(\vec{A})$ and

$$\|t\|_{\ell_{p,w}(\vec{A})} \sim \|\ell_1\|_{(\ddot{h}^p_w(\vec{A}))^*} \lesssim \|L\|_{(\ddot{H}^p_w(\vec{A}))^*}.$$

Hence, for any $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and $g := T_\varphi(t) = \sum_{R\in\mathcal{R}} t_R\varphi_R$, by $\ell_1 \circ S_\varphi = L \circ T_\varphi \circ S_\varphi = L$, $T_\varphi \circ S_\varphi = \mathrm{Id}$ on $\ddot{H}^p_w(\vec{A})$ and (3.54), we know that

$$L(f) = \ell_1 \circ \bigl(S_\varphi(f)\bigr) = \bigl\langle S_\varphi(f), t\bigr\rangle = \langle f, g\rangle = L_g(f),$$

which implies that $L = L_g$. Moreover, by Theorem 2.12, we conclude that

$$\|g\|_{\mathcal{L}_{p,w}(\vec{A})} \lesssim \|t\|_{\ell_{p,w}(\vec{A})} \lesssim \|L_g\|_{(\ddot{H}^p_w(\vec{A}))^*},$$

which completes the proof of Theorem 2.16. $\square$

References

[1] M. Bownik, Anisotropic Hardy spaces and wavelets, Mem. Amer. Math. Soc. 164 (781) (2003), vi+122 pp.

[2] M. Bownik, Atomic and molecular decompositions of anisotropic Besov spaces, Math. Z. 250 (2005) 539–571.

[3] M. Bownik, Anisotropic Triebel–Lizorkin spaces with doubling measures, J. Geom. Anal. 17 (2007) 387–424.

[4] M. Bownik, Duality and interpolation of anisotropic Triebel–Lizorkin spaces, Math. Z. 259 (2008) 131–169.

[5] M. Bownik, K.-P. Ho, Atomic and molecular decompositions of anisotropic Triebel–Lizorkin spaces, Trans. Amer. Math. Soc. 358 (2006) 1469–1510.

[6] M. Bownik, J. Lemvig, Affine and quasi-affine frames for rational dilations, Trans. Amer. Math. Soc. 363 (2011) 1887–1924.

[7] M. Bownik, B. Li, D. Yang, Y. Zhou, Weighted anisotropic Hardy spaces and their applications in boundedness of sublinear operators, Indiana Univ. Math. J. 57 (2008) 3065–3100.

[8] M. Bownik, B. Li, D. Yang, Y. Zhou, Weighted anisotropic product Hardy spaces and boundedness of sublinear operators, Math. Nachr. 283 (2010) 392–442.

[9] A.-P. Calderón, An atomic decomposition of distributions in parabolic H^p spaces, Adv. Math. 25 (1977) 216–225.
[10] A.-P. Calderón, A. Torchinsky, Parabolic maximal functions associated with a distribution, Adv. Math. 16 (1975) 1–64.
[11] A.-P. Calderón, A. Torchinsky, Parabolic maximal functions associated with a distribution. II, Adv. Math. 24 (1977) 101–171.
[12] S.-Y.A. Chang, R. Fefferman, A continuous version of duality of H^1 with BMO on the bidisc, Ann. of Math. (2) 112 (1980) 179–201.
[13] S.-Y.A. Chang, R. Fefferman, The Calderón–Zygmund decomposition on product domains, Amer. J. Math. 104 (1982) 455–468.
[14] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, Colloq. Math. 60/61 (1990) 601–628.
[15] R.R. Coifman, G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc. 83 (1977) 569–645.
[16] W. Farkas, Atomic and subatomic decompositions in anisotropic function spaces, Math. Nachr. 209 (2000) 83–113.
[17] R. Fefferman, Singular integrals on product H^p spaces, Rev. Mat. Iberoam. 1 (1985) 25–31.
[18] R. Fefferman, Harmonic analysis on product spaces, Ann. of Math. (2) 126 (1987) 109–130.
[19] R. Fefferman, A_p weights and singular integrals, Amer. J. Math. 110 (1988) 975–987.
[20] G.B. Folland, E.M. Stein, Hardy Spaces on Homogeneous Group, Math. Notes, vol. 28, Princeton University Press, Princeton, NJ, 1982.
[21] M. Frazier, B. Jawerth, A discrete transform and decompositions of distribution spaces, J. Funct. Anal. 93 (1990) 34–170.
[22] M. Frazier, B. Jawerth, G. Weiss, Littlewood–Paley Theory and the Study of Function Spaces, CBMS Reg. Conf. Ser. Math., vol. 79, American Mathematical Society, Providence, RI, 1991, published for the Conference Board of the Mathematical Sciences, Washington, DC.
[23] R.F. Gundy, E.M. Stein, H^p theory for the poly-disc, Proc. Nat. Acad. Sci. 76 (1979) 1026–1029.
[24] D.D. Haroske, E. Tamási, Wavelet frames for distributions in anisotropic Besov spaces, Georgian Math. J. 12 (2005) 637–658.
[25] R. Johnson, C.J. Neugebauer, Homeomorphisms preserving A_p, Rev. Mat. Iberoam. 3 (1987) 249–273.
[26] D. Krug, A weighted version of the atomic decomposition for H^p (bi-halfspace), Indiana Univ. Math. J. 37 (1988) 277–300.
[27] D. Krug, A. Torchinsky, A weighted version of Journé's lemma, Rev. Mat. Iberoam. 10 (1994) 363–378.
[28] B. Li, M. Bownik, D. Yang, Y. Zhou, Anisotropic singular integrals in product spaces, Sci. China Math. 53 (2010) 3163–3178.
[29] S. Sato, Lusin functions on product spaces, Tohoku Math. J. (2) 39 (1987) 41–59.
[30] S. Sato, An atomic decomposition for parabolic H^p spaces on product domains, Proc. Amer. Math. Soc. 104 (1988) 185–192.
[31] H.-J. Schmeisser, H. Triebel, Topics in Fourier Analysis and Function Spaces, John Wiley & Sons Ltd., Chichester, 1987.
[32] A. Seeger, T. Tao, Sharp Lorentz space estimates for rough operators, Math. Ann. 320 (2001) 381–415.
[33] E.M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, NJ, 1993.
[34] E.M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton, NJ, 1971.
[35] H. Triebel, Theory of Function Spaces, Birkhäuser Verlag, Basel, 1983.
[36] H. Triebel, Theory of Function Spaces II, Birkhäuser Verlag, Basel, 1992.
[37] H. Triebel, Theory of Function Spaces III, Birkhäuser Verlag, Basel, 2006.
[38] J. Vybíral, Function spaces with dominating mixed smoothness, Dissertationes Math. (Rozprawy Mat.) 436 (2006) 1–73.
[39] J. Vybíral, W. Sickel, Traces of functions with a dominating mixed derivative in $\mathbb{R}^3$, Czechoslovak Math. J. 57 (132) (2007) 1239–1273.
[40] W. Wang, A discrete transform and Triebel–Lizorkin spaces on the bidisc, Trans. Amer. Math. Soc. 347 (1995) 1351–1364.
[41] D. Yang, W. Yuan, A new class of function spaces connecting Triebel–Lizorkin spaces and Q spaces, J. Funct. Anal. 255 (2008) 2760–2809.

[42] D. Yang, W. Yuan, New Besov-type spaces and Triebel–Lizorkin-type spaces including Q spaces, Math. Z. 265 (2010) 451–480.
[43] X. Zhu, Atomic decomposition for weighted H^p spaces on product domains, Sci. China Ser. A 35 (1992) 158–168.