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Wavelets for Non-expanding Dilations and the Lattice Counting Estimate

Marcin Bownik[1,2,*] **and Jakob Lemvig**[3]

[1]Department of Mathematics, University of Oregon, Eugene, OR 97403–1222, USA, [2]Institute of Mathematics, Polish Academy of Sciences, ul. Wita Stwosza 57, 80–952 Gdańsk, Poland, and [3]Department of Applied Mathematics and Computer Science, Technical University of Denmark, Matematiktorvet 303, 2800 Kongens Lyngby, Denmark

[*]*Correspondence to be sent to: e-mail: mbownik@uoregon.edu*

We show that problems of existence and characterization of wavelets for non-expanding dilations are intimately connected with the geometry of numbers; more specifically, with a bound on the number of lattice points in balls dilated by the powers of a dilation matrix $A \in \mathrm{GL}(n,\mathbb{R})$. This connection is not visible for the well-studied class of expanding dilations since the desired lattice counting estimate holds automatically. We show that the lattice counting estimate holds for all dilations A with $|\det A| \neq 1$ and for almost every lattice Γ with respect to the invariant probability measure on the set of lattices. As a consequence, we deduce the existence of minimally supported frequency (MSF) wavelets associated with such dilations for almost every choice of a lattice. Likewise, we show that MSF wavelets exist for all lattices and almost every choice of a dilation A with respect to the Haar measure on $\mathrm{GL}(n,\mathbb{R})$.

1 Introduction

A wavelet system is a collection of dilates and translates of a function $\psi \in L^2(\mathbb{R}^n)$ given by $\{|\det A|^{j/2}\,\psi(A^j\cdot-\gamma)\}_{j\in\mathbb{Z},\gamma\in\Gamma}$, where A is an invertible $n\times n$ real matrix and Γ is a full

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rank lattice in $\mathbb{R}^n$. The study of wavelets in higher dimensions is generally restricted to the class of expanding dilations A, often additionally assumed to preserve the integer lattice, $A\mathbb{Z}^n \subset \mathbb{Z}^n$. Recall that a real $n \times n$ matrix is expanding, or expansive, if all of its eigenvalues λ satisfy $|\lambda| > 1$. This is due to the fact that many classical results initially established for dyadic dilations $A = 2\,\mathrm{Id}$, first in dimension $n = 1$, and then in higher dimensions, often extend to the setting of expanding dilations. This includes existence of several classes of wavelets: well-localized wavelets in time and frequency, minimally supported frequency (MSF) wavelets, Haar-type wavelets, and Parseval wavelet frames. For example, Dai *et al.* [8, 9] have shown the existence of MSF wavelets for all expanding dilations with real coefficients. In addition, wavelet expansions associated with expanding dilations characterize many classical function spaces such as: Lebesgue, Hardy, Lipschitz, Sobolev, Besov, and Triebel–Lizorkin spaces.

In contrast, much less attention has been devoted to the study of wavelets associated with general invertible dilations. Speegle in his thesis raised the problem of existence of MSF wavelets for non-expanding dilations, and the first example of a wavelet of this kind appeared in [3]. Laugesen [17] and Hernández *et al.* [11] then initiated a systematic study of wavelets $\psi \in L^2(\mathbb{R}^n)$ in two distinct settings: amplifying dilations for ψ and dilations expanding on a subspace, respectively. In particular, Hernández *et al.* [11] introduced an important concept, known as the local integrability condition (LIC), that yields characterization results for Parseval wavelet frames for non-expanding dilations, see [10, 15]. Soon after, Speegle [22] achieved breakthrough results giving necessary and sufficient conditions for the existence of MSF wavelets for non-expanding dilations. Based on Speegle's work, Ionascu and Wang [12] proved a beautiful result that gives a complete characterization of dilations admitting MSF wavelet in the dimension $n = 2$. The corresponding problem in higher dimensions $n \geq 3$ remains open.

In this paper we show that problems of existence and characterization of wavelets for non-expanding dilations are intimately connected with the geometry of numbers and, more specifically, with the problem of bounding the number of lattice points lying inside balls dilated by powers of a dilation matrix A. The existence of a link between wavelets for non-expanding dilations and diophantine approximation was already manifested in the papers of Speegle [22] and Ionascu and Wang [12]. However, this link is completely invisible in the standard setting of expanding dilations, where the desired *lattice counting estimate*, see Definition 1.1, holds automatically.

Definition 1.1. Suppose A is an $n \times n$ invertible matrix such that $|\det A| > 1$. We say that a pair (A, Γ) satisfies the *lattice counting estimate* if

$$\#\left|\Gamma \cap A^j(\mathbf{B}(0,r))\right| \le C \max(1, |\det A|^j) \qquad \text{for all } j \in \mathbb{Z}, \tag{1.1}$$

where $\mathbf{B}(0,r)$ denotes the open ball of radius $r > 0$ centered at 0. □

We remark that if (1.1) holds for some $r = r_0 > 0$, then it holds for all $r > 0$. This can be deduced from Lemma 4.5, which guarantees the existence of large arithmetic progressions in the intersection of a lattice with a symmetric convex body. Of course, the constant C in (1.1) will depend on r.

In the context of wavelets we shall consider the lattice counting estimate in Fourier domain for the transpose dilation $B = A^T$ and the dual lattice

$$\Gamma^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \in \mathbb{Z} \quad \text{for all } y \in \Gamma\}$$

that takes the form

$$\#\left|\Gamma^* \cap B^j(\mathbf{B}(0,r))\right| \le C \max(1, |\det B|^j) \qquad \text{for all } j \in \mathbb{Z}. \tag{1.2}$$

It is well-known [7, 11, 17] that an orthonormal wavelet $\psi \in L^2(\mathbb{R}^n)$, or more generally a Parseval frame wavelet, associated with an expanding dilation A satisfies the Calderón formula

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(B^{-j}\xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{1.3}$$

This equation, together with the off-diagonal equations (5.1) for $\alpha \neq 0$, constitutes characterizing equations of Parseval wavelets. As an immediate consequence of our results, we show that the same characterization of Parseval frame wavelets is true under the lattice counting estimate (1.2). Moreover, we show that the lattice counting estimate is essential for establishing wavelet characterizing equations. More precisely, (1.2) characterizes the pairs of dilations and lattices (B, Γ^*) for which a rather technical LIC is actually equivalent with the much less technical integrability of the Calderón sum $\sum_{j\in\mathbb{Z}} |\hat{\psi}(B^{-j}\xi)|^2$, that is known to plays a key role in characterization of frame wavelets.

We also show that the lattice counting estimate holds not only for expanding dilations, including those expanding on a subspace, it is even ubiquitous in a probabilistic sense. That is, for any dilation A with $|\det A| > 1$, almost any choice of lattice Γ yields the lattice counting estimate. It also holds for any fixed lattice Γ and almost every choice of a dilation A. These results are shown using techniques introduced by Skriganov [20] in his study of the logarithmically small errors in the lattice problem for polyhedra. In particular, our arguments rely on diophantine characteristic of a lattice, introduced by Skriganov [20], and on several result in the geometry of numbers on intersection of convex bodies with lattices.

An interesting consequence of our ubiquity results is the existence of MSF wavelets for almost all random choices of dilations and lattices (A, Γ). That is, for any fixed lattice Γ, which by standard arguments reduces to the key case $\Gamma = \mathbb{Z}^n$, there exists an MSF wavelet for almost every choice of a dilation $A \in \mathrm{GL}(n, \mathbb{R})$. Likewise, for any choice of a dilation $A \in \mathrm{GL}(n, \mathbb{R})$, outside the exceptional case $|\det A| = 1$ for which MSF wavelets do not exist by the work of Larson *et al.* [16], almost every (with respect to appropriate invariant measure on the set of all lattice) choice of a lattice Γ yields an MSF wavelet. Hence, MSF wavelets exists not only for all expanding dilations as was shown in [8], but also for all invertible dilations A and a generic choice of a lattice Γ. Consequently, the pairs (A, Γ) that do not admit MSF wavelets form a thin and rather pathological exceptional set which is challenging to characterize beyond the known case [12] of the dimension $n = 2$.

2 Dilations expanding on a subspace

In this section we investigate the properties of the class of dilations that are *expanding on a subspace*, that were introduced by Hernández *et al.* [11]. For this class of dilations wavelet characterization results, such as the characterization of Parseval wavelet frames, are known to hold, see [11, Theorem 5.3] and [10, Theorem 1.1]. Guo and Labate [10] corrected an error in the proof of the characterization result from [11] by redefining the class of dilation matrices expanding on a subspace. We give an explicit characterization of dilations that are expanding on a subspace. We also show that they correspond exactly to those dilations A that satisfy the lattice counting estimate (1.1) for all possible choices of a lattice Γ.

Following Guo and Labate [10] we adopt the following definition.

Definition 2.1. Given $A \in \mathrm{GL}(n, \mathbb{R})$ and a non-zero linear subspace $F \subset \mathbb{R}^n$, we say that A is expanding on F if there exists a complementary (not necessarily orthogonal) linear subspace E of $\mathbb{R}^n$ with the following properties:

(i) $\mathbb{R}^n = F + E$ and $F \cap E = \{0\}$,

(ii) F and E are invariant under A, that is, $A(F) = F$ and $A(E) = E$,

(iii) $\exists c \geq 1 \, \exists \gamma > 1 \, \forall j \geq 0 : \left|A^j x\right| \geq (1/c)\gamma^j \left|x\right|$ for all $x \in F$,

(iv) $\exists k > 0 \, \forall j \geq 0 : \left|A^j x\right| \geq k \left|x\right|$ for all $x \in E$. □

Remark 1. If A expanding on a subspace, then all eigenvalues satisfy $|\lambda| \geq 1$. Indeed, eigenvalues λ of $A|_F$ must satisfy $|\lambda| > 1$, whereas eigenvalues λ of $A|_E$ satisfy $|\lambda| \geq 1$. Hence, we can take E to be the (real) eigenspace associated with eigenvalues of modulus one; here we take the real and imaginary parts of eigenvectors associated with a complex conjugate pair of eigenvalues. □

Since E is invariant under A, condition (iv) in Definition 2.1 is equivalent to the existence of $k > 0$ such that, for all $j \geq 0$, we have $|x| \geq k \left|A^{-j}x\right|$ for all $x \in E$. This is equivalent to saying that the discrete time mapping $x \mapsto A^{-1}x, E \to E$, has a Lyapunov stable (sometimes called a marginally stable) fixed point at $x = 0$. It is well-known that this discrete time mapping is Lyapunov stable if and only if all eigenvalues of A^{-1} are no greater than one, and eigenvalues of modulus one have Jordan blocks of order one, that is, the algebraic and geometric multiplicity agree. We thereby obtain a simple characterization of the class of dilation matrices that are expansive on a subspace.

Proposition 2.2. Let $A \in \mathrm{GL}(n, \mathbb{R})$ be given. Then A is expanding on a subspace if and only if

(i) all eigenvalues of A have modulus greater than or equal to 1, and

(ii) at least one eigenvalue has modulus strictly greater than 1, and

(iii) all eigenvalues of modulus equal to 1 have Jordan blocks of order one. □

Proof. To prove the "only if"-direction, assume towards a contradiction that the eigenvalue λ of A^{-1}, $|\lambda| = 1$, has an algebraic multiplicity strictly greater than its geometric multiplicity. Let v_1 be an eigenvector and v_2 a generalized eigenvector of A^{-1} associated with λ such that

$$A^{-1}v_1 = \lambda v_1 \qquad \text{and} \qquad A^{-1}v_2 = \lambda v_2 + v_1.$$

Assume that λ is non-real; the case $\lambda = \pm 1$ can be handled similarly. Then $\overline{\lambda}$ is also an eigenvalue with eigenvector $\overline{v_1}$ and generalized eigenvector $\overline{v_2}$. Take $x = v_2 + \overline{v_2} = 2\Re v_2 \in \mathbb{R}^n$. By Remark 1, we can take E to be the span of the basis vectors associated with eigenvalues of modulus one from the real Jordan form of A^{-1}. Then $x \in E$ and

$$A^{-j}x = 2j\Re(\lambda^{j-1}v_1) + 2\Re(\lambda^j v_2).$$

We see that the orbit $\{A^{-j}x\}_{j=1}^{\infty} \subset E$ is unbounded, hence (iv) in Definition 2.1 cannot hold. Thus, all eigenvalues λ of A with $|\lambda| = 1$ have Jordan blocks of order one. Moreover, at least one eigenvalue λ of A satisfies $|\lambda| > 1$ in light of (iii). The proof of the "if"-direction is a simple verification of properties (i)–(iv) in Definition 2.1. ■

It was shown in [10] that eigenvalues alone do not give the complete picture of when the wavelet characterization results hold. The interaction of a dilation A and a lattice Γ has to be taken into account to get the more optimal result. In fact, the following result combining [10, Lemma 3.2] and [10, Lemma 3.3] motivates the definition of lattice counting estimate (1.1), see also [1, Lemma 2.8].

Lemma 2.3. Let $A \in \mathrm{GL}(n,\mathbb{R})$ be expanding on a subspace of $\mathbb{R}^n$, and let $r > 0$. Then (A,Γ) satisfies the lattice counting estimate (1.1) for any full-rank lattice $\Gamma \subset \mathbb{R}^n$. □

We finish this section by showing that the converse of Lemma 2.3 holds. Hence, the class of dilations expanding on a subspace consists precisely of those dilations for which the lattice counting estimate holds for every choice of a lattice.

Theorem 2.4. Suppose that $A \in \mathrm{GL}(n,\mathbb{R})$ and $|\det A| > 1$. Then, (A,Γ) satisfies the lattice counting estimate (1.1) for *all* full-rank lattices $\Gamma \subset \mathbb{R}^n$ if and only if A is expanding on a subspace. □

Proof. Lemma 2.3 shows the "if"-implication. To show the converse implication, assume that (A,Γ) satisfies (1.1) for all lattices Γ. We will show that the properties (i)–(iii) in Proposition 2.2 hold.

On the contrary, suppose that (i) fails, that is, there exists an eigenvalue λ of A such that $|\lambda| < 1$. Then, there exists one-dimensional eigenspace V if λ is real, or two-dimensional invariant space V corresponding to a pair of complex conjugate eigenvalues λ and $\overline{\lambda}$. In either case, we have $|Av| = |\lambda||v|$ for all $v \in V$. Choose a full rank lattice Γ in V and extend it to a full rank lattice in $\mathbb{R}^n$. Then,

$$\#|\Gamma \cap A^j(\mathbf{B}(0,r))| \geq \#|\Gamma \cap A^j(V \cap \mathbf{B}(0,r))| = \#|\Gamma \cap V \cap \mathbf{B}(0,|\lambda|^j r)| \to \infty \qquad \text{as } j \to -\infty.$$

This contradicts (1.1). Hence, (i) holds and so does (ii) since $|\det A| > 1$.

Finally, suppose that (iii) fails. That is, there exists a Jordan block of order ≥ 2 corresponding to an eigenvalue $|\lambda| = 1$. If λ is real, then $\lambda = \pm 1$ and there exists a two-dimensional invariant subspace V such that $A|_V$ has a matrix representation $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

For simplicity assume $\lambda = 1$. The case $\lambda = -1$ is similar. Then,

$$(A|_V)^j = \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix}. \tag{2.1}$$

Choose $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and define a lattice Γ in V of the form $\Gamma = \mathbb{Z}(0,1)+\mathbb{Z}(1,\alpha)$. Then, for any $N \in \mathbb{N}$, we can find $\gamma = (\gamma_1, \gamma_2) \in \Gamma$ such that $\gamma_1 < 0$ and $0 < \gamma_2 < 1/N$. Let $j = \lfloor \gamma_1/\gamma_2 \rfloor < 0$. Since the image of the unit square $[0,1]^2$ under (2.1) is a parallelogram with vertices $(0,0)$, $(1,0)$, $(j,1)$, and $(j+1,1)$, it contains line segments going through the origin with slopes m such that $1/j \le m \le 1/(j+1)$. In particular, the slope m of the line $\mathbb{R}(\gamma_1,\gamma_2)$ lies in this range. Since $0 < \gamma_2 < 1/N$, at least N points of the lattice Γ lie in the above parallelogram. Thus,

$$\sup_{j<0} \#|\Gamma \cap (A|_V)^j(\mathbf{B}(0,r))| = \infty. \tag{2.2}$$

Extending the rank 2 lattice Γ to a full rank lattice yields a pair (A,Γ) that fails the lattice counting estimate (1.1) for $j < 0$, which is a contradiction.

If $\lambda = \mathrm{e}^{i\theta}$ is not real, then there exists a four-dimensional invariant subspace V such that $A|_V$ has a matrix representation

$$\begin{bmatrix} R(\theta) & \mathbf{I} \\ \mathbf{0} & R(\theta) \end{bmatrix}, \qquad \text{where } R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Here, $\mathbf{I}$ and $\mathbf{0}$ are the 2×2 identity matrix and the 2×2 zero matrix, respectively. Hence, by conjugating we can find a basis of V in which

$$A|_V = \begin{bmatrix} R(\theta) & R(\theta) \\ \mathbf{0} & R(\theta) \end{bmatrix}.$$

Observe that

$$(A|_V)^j = \begin{bmatrix} R(\theta j) & jR(\theta j) \\ \mathbf{0} & R(\theta j) \end{bmatrix}. \tag{2.3}$$

Let $\mathbf{B}_2$ be the unit ball in $\mathbb{R}^2$. Since $R(\theta)$ is a rotation matrix, the image of $\mathbf{B}_2 \times \mathbf{B}_2$ under the matrix (2.3) is the same as the image of the same set under the matrix

$$\begin{bmatrix} \mathbf{I} & j\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

By permuting the basis elements, the above matrix consists of two blocks of the form (2.1). By the same argument as in the real case we can find a lattice Γ satisfying (2.2). Again this contradicts (1.1) and completes the proof of Theorem 2.4. ■

3 The LIC

In this section we investigate the LIC that was originally introduced by Hernández *et al.* [11]. While this condition can be studied in full generality of generalized shift-invariant (GSI) system, we restrict our attention to wavelet systems. We show that the integrability of the Calderón formula implies the LIC precisely for pairs (B, Γ^*) satisfying the lattice counting estimate (1.2). As a consequence, in Section 5 we shall extend characterization results for Parseval and dual frames to this more general (than expanding on a subspace) setting.

Definition 3.1. Let E be a proper subspace of $\mathbb{R}^n$. Consider the following dense subspace of $L^2(\mathbb{R}^n)$,

$$\mathcal{D}_E = \left\{ f \in L^2(\mathbb{R}^n) : \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \overline{\operatorname{supp} \hat{f}} \subset \mathbb{R}^n \setminus E \text{ is compact} \right\}. \tag{3.1}$$

Let $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R}^n)$, $A \in \mathrm{GL}(n, \mathbb{R})$ and Γ is a full-rank lattice. The corresponding wavelet system associated with the pair (A, Γ) is defined as

$$\mathcal{A}(\Psi, A, \Gamma) = \{D_{A^j} T_\gamma \psi_\ell : j \in \mathbb{Z}, \gamma \in \Gamma, \ell = 1, \dots, L\},$$

where $D_A f(x) = |\det A|^{1/2} f(Ax)$ is the dilation operator and $T_\gamma f(x) = f(x - \gamma)$ is the translation operator. We say $\mathcal{A}(\Psi, A, \Gamma)$ satisfies the LIC if

$$\begin{aligned} L(f) &= \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \Gamma^*} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\xi + B^j k)|^2 \, |\det A|^j \left| \mathcal{F} D_{A^j} \psi_l(\xi) \right|^2 \, \mathrm{d}\xi \\ &= \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \Gamma^*} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\xi + B^j k)|^2 \, |\hat{\psi}_l(B^{-j} \xi)|^2 \, \mathrm{d}\xi < \infty \quad \text{for all } f \in \mathcal{D}_E. \end{aligned} \tag{3.2}$$

Here, the Fourier transform is defined for $f \in L^1(\mathbb{R}^n)$ by

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-2\pi i \langle \xi, x \rangle} \mathrm{d}x$$

with the usual extension to $L^2(\mathbb{R}^n)$. □

3.1 The Calderón condition

The following fact shows that the LIC (3.2) for the wavelet system $\mathcal{A}(\Psi, A, \Gamma)$ implies the local integrability of the Calderón sum (3.3).

Lemma 3.2. Let $A \in \mathrm{GL}(n, \mathbb{R})$ and let E be a proper subspace of $\mathbb{R}^n$. Suppose $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R}^n)$ satisfies the LIC (3.2) for $f \in \mathcal{D}_E$. Then

$$\sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}_l(B^{-j}\xi) \right|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus E). \tag{3.3}$$

□

Proof. Suppose that $L(f) < \infty$ for all $f \in \mathcal{D}_E$. Then, in particular by choosing $\hat{f} = \chi_S$ for a compact set $S \subset \mathbb{R}^n \setminus E$, we have

$$\int_S \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}_l(B^{-j}\xi) \right|^2 \, \mathrm{d}\xi = \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \int_S \left| \hat{\psi}_l(B^{-j}\xi) \right|^2 \, \mathrm{d}\xi \leq L(f) < \infty.$$

Since the set S was arbitrarily chosen, the validity of (3.3) follows. ■

Definition 3.3. We say that a Lebesgue measurable set $S \subset \mathbb{R}^n$ is a *multiplicative tiling set* under $A \in \mathrm{GL}(n, \mathbb{R})$ if

(a) $\bigcup_{j \in \mathbb{Z}} A^j(S) = \mathbb{R}^n$,
(b) $A^j(S) \cap A^i(S) = \emptyset$ whenever $j \neq i \in \mathbb{Z}$,

where each equality is up to sets of measure zero. □

Larson *et al.* [16, Theorem 4] have shown the following interesting result about multiplicative tilings of $\mathbb{R}^n$.

Theorem 3.4. Let $A \in \mathrm{GL}(n, \mathbb{R})$.

(i) There exists a multiplicative tiling set if and only if A is not orthogonal.
(ii) There exists a multiplicative tiling set of finite measure if and only if $|\det A| \neq 1$.
(iii) There exists a bounded multiplicative tiling set if and only if all eigenvalues of A, in modulus, are either strictly greater or strictly smaller than 1. □

The following fact is a consequence of the main result of Laugesen *et al.* in [18]. It can also be deduced from Theorem 3.4 as we see below.

Lemma 3.5. Let $A \in \mathrm{GL}(n,\mathbb{R})$. Then $|\det A| \neq 1$ if and only if there exists a function $\psi \in L^2(\mathbb{R}^n)$ such that

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(B^{-j}\xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{3.4}$$

□

Proof. If $|\det A| \neq 1$, then we simply take $\hat{\psi} = \chi_S$, where S is a multiplicative tiling set of finite measure for $B = A^T$. Conversely, assume that (3.4) holds. There are two cases to consider. Suppose that A is an orthogonal matrix. Then, by the change of variables formula for any $R > 1$, we have

$$|\{\xi \in \mathbb{R}^n : 1/R < |\xi| < R\}| = \int_{1/R<|\xi|<R} \sum_{j\in\mathbb{Z}} |\hat{\psi}(B^{-j}\xi)|^2 \mathrm{d}\xi = \sum_{j\in\mathbb{Z}} \int_{1/R<|\xi|<R} |\hat{\psi}(\xi)|^2 \mathrm{d}\xi.$$

The left-hand side of this equation is finite and positive, while the term of the far right is either zero or infinite, which is a contradiction. Suppose next that A is not orthogonal and $|\det A| = 1$. Let S be a multiplicative tiling set for B. Since $|\det B| = 1$, by the change of variables formula, we have

$$|S| = \int_S \sum_{j\in\mathbb{Z}} |\hat{\psi}(B^{-j}\xi)|^2 \mathrm{d}\xi = \sum_{j\in\mathbb{Z}} \int_{B^{-j}S} |\hat{\psi}(\xi)|^2 \mathrm{d}\xi = ||\hat{\psi}||^2 = ||\psi||^2.$$

This implies that S has finite measure. Then, Theorem 3.4(ii) yields $|\det B| \neq 1$, which is a contradiction. Consequently, we have $|\det A| \neq 1$. ■

3.2 The LIC and the lattice counting estimate

The main result of this section shows a link between the lattice counting estimate, the LIC, and Calderón's formula. We start with a necessary definition of sets that appear in the proof of Theorem 3.8.

Definition 3.6. For a given $A \in \mathrm{GL}(n,\mathbb{R})$ we consider $B = A^T$ as a linear map acting on $\mathbb{C}^n$. Let $E^c \subset \mathbb{C}^n$ and $F^c \subset \mathbb{C}^n$ be the span of eigenspaces corresponding to eigenvalues λ of B satisfying $|\lambda| \leq 1$ and $|\lambda| > 1$, respectively. Define $E = E^c \cap \mathbb{R}^n$ and $F = F^c \cap \mathbb{R}^n$. For $p, q, s > 0$, define

$$Q(p,q,s) = \{x = x_E + x_F : x_E \in E, x_F \in F, |x_E| < p, s < |x_F| < q\}.$$

□

Since complex eigenvalues of B come in conjugate pairs, the spaces E^c and F^c are complexifications of the real spaces E and F, respectively.

Lemma 3.7. Let $B \in \mathrm{GL}(n, \mathbb{R})$ with $|\det B| > 1$ be given. For any $\varepsilon, s > 0$, there exists a multiplicative tiling set S for the dilation B such that for some $p, q > 0$ we have

$$|S \setminus Q(p, q, s)| < \varepsilon\, |S|\,. \tag{3.5}$$

□

That is, for a given $\varepsilon > 0$ and $s > 0$, we can always find a multiplicative tiling set for the dilation B that, up to a relative error ε, lies inside $Q(p, q, s)$ for sufficiently large $p, q > 0$.

Proof. By Theorem 3.4, there exists a multiplicative tiling set S_0 for B of finite measure. Since

$$\mathbb{R}^n \setminus E = \bigcup_{\delta > 0} Q(\infty, \infty, \delta),$$

we can find $\delta > 0$ such that

$$|S_0 \setminus Q(\infty, \infty, \delta)| < \frac{\varepsilon}{2}\, |S_0|\,. \tag{3.6}$$

For any $j \in \mathbb{N}$, define

$$S_j = B^j (S_0 \cap Q(\infty, \infty, \delta)) \cup (S_0 \setminus Q(\infty, \infty, \delta)).$$

Clearly, S_j is a multiplicative tiling set for B. Since the dilation B is expanding in the direction of the space F, there exists $j \in \mathbb{N}$ such that

$$B^j (Q(\infty, \infty, \delta)) \subset Q(\infty, \infty, s). \tag{3.7}$$

Combining (3.6) and (3.7) yields

$$|S_j \setminus Q(\infty, \infty, s)| < \frac{\varepsilon}{2} |S_0| \le \frac{\varepsilon}{2} |S_j|.$$

Hence, by choosing sufficiently large $p, q > 0$ we have

$$|S_j \setminus Q(p, q, s)| < \varepsilon |S_j|,$$

which shows (3.5). ■

The following result characterizes the LIC.

Theorem 3.8. Let $A \in \mathrm{GL}(n,\mathbb{R})$ with $|\det A| > 1$ be given, and let $\Gamma \subset \mathbb{R}^n$ be a full-rank lattice. The following assertions are equivalent:

(i) (B,Γ^*) satisfies the lattice counting estimate (1.2),

(ii) For any $\Psi = \{\psi_1,\dots,\psi_L\} \subset L^2(\mathbb{R}^n)$, the LIC (3.2) holds for $\mathcal{A}(\Psi,A,\Gamma)$ if and only if Ψ satisfies the Calderón integrability condition (3.3). □

Proof. Let E, F, and $Q(p,q,s)$, be as in Definition 3.6.

(i) $\Rightarrow$ (ii): Let $\Psi = \{\psi_1,\dots,\psi_L\} \subset L^2(\mathbb{R}^n)$. Suppose that the lattice counting estimate (1.2) holds. If $\mathcal{A}(\Psi,A,\Gamma)$ satisfies the LIC, then, by Lemma 3.2, Ψ satisfies the Calderón integrability condition.

Assume on the other hand that the Calderón integrability condition (3.3) holds. For simplicity assume that $L=1$. Let $f \in \mathcal{D}_E$. Then $T := \operatorname{supp}\hat{f} \subset Q(p,q,s) \subset \mathbb{R}^n \setminus E$ for some $s,p,q > 0$. By the lattice counting estimate (1.2), we have

$$\#\left|B^j\Gamma^* \cap (\operatorname{supp}\hat{f} - \operatorname{supp}\hat{f})\right| \le C\max\left(1,|\det B|^{-j}\right).$$

Hence,

$$\begin{aligned} L(f) &\le \sum_{j\in\mathbb{Z}} \|\hat{f}\|_\infty^2 C\max\left(1,|\det B|^{-j}\right) \int_{\operatorname{supp}\hat{f}} \left|\hat{\psi}(B^{-j}\xi)\right|^2 \,\mathrm{d}\xi \\ &= \|\hat{f}\|_\infty^2 C \sum_{j\ge 0} \int_T \left|\hat{\psi}(B^{-j}\xi)\right|^2 \,\mathrm{d}\xi + \|\hat{f}\|_\infty^2 C \sum_{j<0} \int_{B^{-j}(T)} \left|\hat{\psi}(\xi)\right|^2 \,\mathrm{d}\xi. \end{aligned}$$

Since the matrix B is expansive on F, there exists a constant $K \in \mathbb{N}$ such that each trajectory $\{B^j\xi\}_{j\in\mathbb{Z}}$ hits $Q(p,q,s)$ at most K times. Thus,

$$\#\left|\left\{j\in\mathbb{Z} : \xi \in B^{-j}(T)\right\}\right| \le K.$$

Thereby, we can continue the above estimate:

$$L(f) \le \|\hat{f}\|_\infty^2 C \int_T \sum_{j\ge 0} \left|\hat{\psi}(B^{-j}\xi)\right|^2 \,\mathrm{d}\xi + \|\hat{f}\|_\infty^2 CK \int_{\mathbb{R}^n} \left|\hat{\psi}(\xi)\right|^2 \,\mathrm{d}\xi < \infty,$$

where the last inequality is a consequence of (3.3) and $\psi \in L^2(\mathbb{R}^n)$.

(ii) $\Rightarrow$ (i): We prove the contrapositive assertion. So suppose that (B,Γ^*) does not satisfy the lattice counting estimate (1.2) for some $r > 0$. Since the lattice counting estimate fails for either $j \in -\mathbb{N}$ or $j \in \mathbb{N}$, we have two cases:

(a) $\sup_{j<0} w_j = \infty$, where $w_j := \#|B^j\Gamma^* \cap \mathbf{B}(0,r)|\, |\det B|^j$,

(b) $\sup_{j\ge 0} v_j = \infty$, where $v_j := \#|B^j\Gamma^* \cap \mathbf{B}(0,r)|$.

Suppose case (a) holds. Choose a subsequence $\{w_{j_i}\}_{i=1}^{\infty}$ of $\{w_{-1}, w_{-2}, \dots\}$ such that $0 > j_1 > j_2 > \cdots$ and

$$\sum_{i=1}^{\infty} \frac{1}{w_{j_i}} < \infty.$$

Let $P : \mathbb{R}^n \to \mathbb{R}^n$ be a projection (not necessarily orthogonal) such that $\ker P = E$ and $P(\mathbb{R}^n) = F$. Let $\varepsilon > 0$ and pick $s > ||P||r$. Let S be a multiplicative tiling set for B as in Lemma 3.7. Define $\psi : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{\psi}(\xi) = \sum_{i=1}^{\infty} \frac{1}{\sqrt{v_{j_i}}} \chi_{B^{-j_i}(S)}(\xi).$$

We have $\psi \in L^2(\mathbb{R}^n)$ since

$$\int_{\mathbb{R}^n} |\hat{\psi}(\xi)|^2 \mathrm{d}\xi = \sum_{i=1}^{\infty} \frac{1}{v_{j_i}} \int_{\mathbb{R}^n} \chi_{B^{-j_i}(S)} \mathrm{d}\xi = |S| \sum_{i=1}^{\infty} \frac{|\det B|^{-j_i}}{v_{j_i}} = |S| \sum_{i=1}^{\infty} \frac{1}{w_{j_i}} < \infty.$$

Since $\xi \mapsto \sum_{j\in\mathbb{Z}} |\hat{\psi}(B^{-j}\xi)|^2$ is B-dilative periodic, we see that

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(B^{-j}\xi)|^2 = \sum_{i=1}^{\infty} \frac{1}{v_{j_i}} < \infty \qquad \text{for a.e. } \xi \in S,$$

also holds for a.e. $\xi \in \mathbb{R}^n$. Hence, the Calderón integrability condition (3.3) holds.

We now show that $\mathcal{A}(\Psi, A, \Gamma)$ does not satisfy the LIC. Define $T = (S \cap Q(p,q,s)) + \mathbf{B}(0,r)$. Since $s > ||P||r$, we claim that $\overline{T} \subset \mathbb{R}^n \setminus E$. Indeed, take any $x \in T$ and write it as

$$x = x_1 + x_2, \qquad x_1 \in Q(p,q,s),\ x_2 \in \mathbf{B}(0,r).$$

Then,

$$||Px|| \geq ||Px_1|| - ||Px_2|| \geq s - ||P||r > 0.$$

Hence, $\overline{T} \subset \mathbb{R}^n \setminus E$ is compact.

Let $\hat{f} = \chi_T$. Then $f \in \mathcal{D}_E$, and by definition of T and v_j, we have for $j \in \mathbb{Z}$,

$$\#\left|\{k \in \Gamma^* : \hat{f}(\xi + B^j k) = 1 \text{ for } \xi \in S \cap Q(p,q,s)\}\right| \geq v_j.$$

From this and $S \cap Q(p,q,s) \subset T = \operatorname{supp} \hat{f}$, it follows that

$$L(f) \geq \sum_{j<0} \sum_{k\in\Gamma^*} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + B^j k)|^2 \, |\hat{\psi}(B^{-j}\xi)|^2 \, \mathrm{d}\xi \geq \sum_{j<0} v_j \int_{S\cap Q(p,q,s)} |\hat{\psi}(B^{-j}\xi)|^2 \, \mathrm{d}\xi.$$

By a change of variables and Lemma 3.7

$$L(f) \geq \sum_{i=1}^{\infty} v_{j_i} \int_{B^{-j_i}(S \cap Q(p,q,s))} |\hat{\psi}(\xi)|^2 \, |\det B|^{j_i} \; d\xi$$
$$= \sum_{i=1}^{\infty} |S \cap Q(p,q,s)| \geq \sum_{i=1}^{\infty} (1-\varepsilon)\, |S| = \infty.$$

Suppose now case (b) holds. Choose a subsequence $\{v_{j_i}\}_{i=1}^{\infty}$ of $\{v_0, v_1, \ldots\}$ such that $0 \leq j_1 < j_2 < \cdots$ and

$$\sum_{i=1}^{\infty} \frac{1}{v_{j_i}} < \infty,$$

and define $\psi : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{\psi}(\xi) = \sum_{i=1}^{\infty} \frac{1}{\sqrt{w_{j_i}}} \chi_{B^{j_i}(S)}(\xi).$$

The rest of the proof is dealt in a similar way as in the case (a) and is left to the reader. ■

4 Ubiquity of the lattice counting estimate

In this section we will show that the lattice counting estimate holds almost surely for generic choices of dilations and lattices. By Theorem 2.4, for any dilation A that is not expanding on a subspace, one can find a full-rank lattice Γ for which the lattice counting estimate (1.1) fails. On the other hand, we shall show in this section that the lattice counting estimate (1.1) holds for any dilation A with $|\det A| > 1$ for almost every choice of a lattice Γ. In Section 5 we shall establish similar results on the existence of MSF wavelets.

Our techniques rely on the work of Skriganov [20, 21] on the logarithmically small errors in the lattice point problem for polyhedra. These results initially involve averaging over the $\mathrm{SO}(n)$ group. Using the Iwasawa decomposition we deduce similar results by averaging over the $\mathrm{SL}(n,\mathbb{R})$ group. Similar lattice point results using Fourier analysis and averaging over the orthogonal group $\mathrm{O}(n)$ can be found in [4, 5, 13].

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define the norm form $\mathrm{Nm}\, x = x_1 x_2 \cdots x_n$. Let

$$\mathrm{Nm}\, \Lambda = \inf\{|\mathrm{Nm}\, x| : x \in \Lambda \setminus \{0\}\}.$$

A lattice Λ is said to be *admissible* if $\mathrm{Nm}\, \Lambda > 0$. For such lattices Skriganov [19] has established the following asymptotic bound on the number of lattice points inside a dilation of a parallelepiped $\Pi \subset \mathbb{R}^n$ with edges parallel to the coordinates axes:

$$\#(\Lambda \cap t\Pi) = t^n\,|\Pi| + O((\log t)^{n-1}) \qquad \text{as } t \to \infty. \tag{4.1}$$

Let $\mathcal{L}_n$ be the set of unimodular lattices (with volume = 1) which can be identified with

$$\mathcal{L}_n = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z}).$$

Even though the subset of admissible lattices is dense in $\mathcal{L}_n$, it has zero measure with respect to the invariant (probability) measure $\mu_{\mathcal{L}}$ on $\mathcal{L}_n$, see [20]. Hence, admissible lattices are rare, that is, $\mu_{\mathcal{L}}$-almost surely we have $\mathrm{Nm}\,\Lambda = 0$.

Skriganov [20] introduced a diophantine characteristic of a lattice Λ, which measures the rate at which $\mathrm{Nm}\,\Lambda = 0$ is achieved, defined by

$$\nu(\Lambda,\rho) = \min\{|\mathrm{Nm}\,\gamma| : \gamma \in \Lambda, 0 < |\gamma| < \rho\}, \qquad \rho > ||\Lambda|| := \min\{|\gamma| : \gamma \in \Lambda \setminus \{0\}\}. \tag{4.2}$$

The following result, Lemma 4.1, plays a key role in showing the main result of [20] which says that the bound (4.1) holds when Π is replaced by any compact polyhedron for almost every choice of Λ, albeit with a slightly worse exponent $(\log t)^{n-1+\varepsilon}$, $\varepsilon > 0$.

Lemma 4.1 is a slight generalization of [20, Lemma 4.3] due to presence of a matrix $P \in \mathrm{GL}(n,\mathbb{R})$. This change corresponds to a more general norm form $x \mapsto \mathrm{Nm}(Px)$.

Lemma 4.1. Let $\Lambda \in \mathcal{L}_n$ be an arbitrary lattice and let $P \in \mathrm{GL}(n,\mathbb{R})$. Then for almost all orthogonal matrices $U \in \mathrm{SO}(n)$ (in the sense of the Haar measure on $\mathrm{SO}(n)$) we have

$$\nu(PU\Lambda,\rho) > (\log\rho)^{1-n-\varepsilon} \qquad \text{as } \rho \to \infty, \tag{4.3}$$

where $\varepsilon > 0$ is arbitrary. □

Proof. We shall follow the proof of [20, Lemma 4.2] with some necessary modifications. Suppose that $\omega : [||\Lambda||,\infty) \to (0,\infty)$ is an arbitrary monotone decreasing function satisfying

$$\sum_{\gamma \in \Lambda\setminus\{0\}} |\gamma|^{-n}\left(\log\frac{|\gamma|^n}{\omega(|\gamma|)}\right)^{n-2}\omega(|\gamma|) < \infty. \tag{4.4}$$

Let σ be the unique $\mathrm{SO}(n)$-invariant measure on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ that is normalized such that $\sigma(S^{n-1}) = 1$. In particular, for any $x \in S^{n-1}$ we have

$$\sigma(V) = \mu_{\mathrm{SO}}(\{U \in \mathrm{SO}(n) : Ux \in V\}) \qquad \text{for any open set } V \subset S^{n-1}. \tag{4.5}$$

Given $P \in \mathrm{GL}(n, \mathbb{R})$, define

$$s_P(\theta) = \sigma(\{x \in S^{n-1} : |\operatorname{Nm}(Px)| < \theta\}).$$

Let $\mathbf{I}$ be $n \times n$ identity matrix. By the estimate (4.21) in the proof of [20, Lemma 4.2] we have

$$s_{\mathrm{I}}(x) < c(n)\theta\left(1 + \log\frac{1}{\theta}\right)^{n-2} \qquad \text{for } 0 < \theta < \frac{1}{\sqrt{n}}, \tag{4.6}$$

where a positive constant $c(n)$ depends only on the dimension n.

The mapping $\phi : S^{n-1} \to S^{n-1}$ given by $\phi(x) = Px/|Px|$ is a smooth diffeomorphism. Since S^{n-1} is compact, the Jacobian of ϕ is bounded from above and below by positive constants. Thus, by the change of variables formula, there exists a constant $c = c(P) > 0$ depending on P such that

$$\frac{1}{c}\sigma(V) \le \sigma(\phi^{-1}(V)) \le c\sigma(V) \qquad \text{for any open set } V \subset S^{n-1}.$$

Consequently, we have

$$\begin{aligned} s_P(\theta) &\le \sigma(\{x \in S^{n-1} : |\operatorname{Nm}(Px)| < \theta||P^{-1}||^n |Px|^n\}) \\ &= \sigma(\{x \in S^{n-1} : |\operatorname{Nm}(Px/|Px|)| < \theta||P^{-1}||^n\}) \le c s_{\mathrm{I}}(\theta||P^{-1}||^n). \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7) yields

$$\begin{aligned} s_P(\theta) &< c(P)c(n)||P^{-1}||^n\theta\left(1 + \log\frac{1}{\theta||P^{-1}||^n}\right)^{n-2} \\ &< c(n,P)\theta\left(1 + \log\frac{1}{\theta}\right)^{n-2} \qquad \text{for } 0 < \theta < \frac{1}{\sqrt{n}||P^{-1}||^n}, \end{aligned} \tag{4.8}$$

where the positive constant $c(n, P)$ depends on n and P.

For any $\gamma \in \Lambda \setminus \{0\}$ and $\theta > 0$, let

$$m_\gamma(\theta) = \mu_{\mathrm{SO}}(\{U \in \mathrm{SO}(n) : |\operatorname{Nm}(PU\gamma)| < \theta\}).$$

By (4.5) and (4.8), we have

$$m_\gamma(\theta) = s_P\left(\frac{\theta}{|\gamma|^n}\right) < c(n,P)|\gamma|^{-n}\theta\left(1+\log\frac{|\gamma|^n}{\theta}\right)^{n-2} \qquad \text{for } 0<\theta<\frac{||\Lambda||^n}{\sqrt{n}||P^{-1}||^n}. \tag{4.9}$$

Using (4.4) and (4.9), the Borel–Cantelli Lemma implies that for almost all $U \in \mathrm{SO}(n)$

$$|\operatorname{Nm}(PU\gamma)| \geq \omega(|\gamma|) \qquad \text{for all but finitely many } \gamma \in \Lambda \setminus \{0\}. \tag{4.10}$$

Observe also that for every $\gamma \in \Lambda \setminus \{0\}$ we have

$$|\operatorname{Nm}(PU\gamma)| > 0 \qquad \text{for almost all } U \in \mathrm{SO}(n). \tag{4.11}$$

Note that

$$\begin{aligned} \nu(PU\Lambda,\rho) &= \min\{|\operatorname{Nm}(PU\gamma)| : \gamma \in \Lambda, 0 < |PU\gamma| < \rho\} \\ &\leq \min\{|\operatorname{Nm}(PU\gamma)| : \gamma \in \Lambda, 0 < |\gamma| < ||P^{-1}||\rho\}. \end{aligned} \tag{4.12}$$

Let $\{\gamma_1,\ldots,\gamma_q\} \subset \Lambda$ be the exceptional set, which depends on U, where (4.10) fails. Combining (4.10)–(4.12) yields $\rho_0 > 0$ such that

$$\begin{aligned} \nu(PU\Lambda,\rho) &\geq \min\{\omega(||P^{-1}||\rho), |\operatorname{Nm}(PU\gamma_1)|,\ldots,|\operatorname{Nm}(PU\gamma_q)|\} \\ &= \omega(||P^{-1}||\rho) \qquad \text{for } \rho > \rho_0. \end{aligned}$$

Since the function $\omega(\rho) = (\log\rho)^{1-n-\varepsilon}$ with $\varepsilon > 0$ satisfies (4.4), we obtain the bound (4.3). This completes the proof of Lemma 4.1. ■

Skriganov's Lemma 4.1 plays a key role in the proof of the following main result of this section.

Theorem 4.2. Let B be any matrix in $\mathrm{GL}(n,\mathbb{R})$ with $|\det B| > 1$. Then for any lattice $\Lambda \in \mathcal{L}_n$, the pair $(B, U\Lambda)$ satisfies the lattice counting estimate (1.1) for almost all (in the sense of Haar measure) $U \in \mathrm{SO}(n)$. □

To prove Theorem 4.2 we need two lemmas about intersection of lattices with convex symmetric bodies. The first result is the volume packing lemma which can be found in the book of Tao and Vu [23, Lemmas 3.24 and 3.26].

Lemma 4.3. Let $\Gamma \subset \mathbb{R}^n$ be a full rank lattice, and let Ω be a symmetric convex body in $\mathbb{R}^n$. Then,

$$\frac{|\Omega|}{2^n\,|\mathbb{R}^n/\,\Gamma|} \le \#|\Omega \cap \Gamma|. \tag{4.13}$$

In, addition if the vectors $\Omega \cap \Gamma$ linearly span $\mathbb{R}^n$, then

$$\#|\Omega \cap \Gamma| \le \frac{3^n n!\,|\Omega|}{2^n\,|\mathbb{R}^n/\,\Gamma|}. \tag{4.14}$$

□

The following lemma, which is a consequence of Minkowski's second theorem [23, Theorem 3.30], shows the existence of large proper arithmetic progressions inside $\Omega \cap \Gamma$, see [23, Lemma 3.33].

Definition 4.4. We say that $S \subset \mathbb{R}^n$ is a *symmetric arithmetic progression of rank s*, if there exist $(v_1, \ldots, v_s) \in (\mathbb{R}^n)^s$ and $(N_1, \ldots, N_s) \in \mathbb{N}^s$ such that

$$S = \{n_1 v_1 + \ldots n_s v_s : n_j \in \mathbb{Z}, \left|n_j\right| \le N_j \text{ for all } 1 \le j \le s\}.$$

We say that such S is *proper* if elements of S are uniquely represented, or equivalently if the cardinality of S equals $(2N_1 + 1) \cdots (2N_s + 1)$. □

Lemma 4.5. Let $\Gamma \subset \mathbb{R}^n$ be a lattice (not necessarily of full rank), and let Ω be a symmetric convex body in $\mathbb{R}^n$. Then there exists a proper symmetric arithmetic progression S in $\Omega \cap \Gamma$ of rank $s \le \dim \operatorname{span}(\Omega \cap \Gamma)$ such that

$$\#|S| \ge c_n \#|\Omega \cap \Gamma|,$$

where $c_n > 0$ is a universal constant which depends only on dimension n. □

Finally, we shall need an elementary lemma on the behavior of the norm form $\mathrm{Nm}(x)$ under dilations.

Lemma 4.6. Let B be any matrix in $\mathrm{GL}(n, \mathbb{R})$ with $|\det B| > 1$. Let $P \in \mathrm{GL}(n, \mathbb{R})$ be such that $P^{-1}BP$ is the real Jordan form of B. Then for any $\varepsilon > 0$ and $r > 0$, there exists $C = C(\varepsilon)$ such that

$$|\,\mathrm{Nm}(P^{-1}x)| \le C|\det B|^{j+|j|\varepsilon} \qquad \text{for all } x \in B^j(\mathbf{B}(0,r)), j \in \mathbb{Z}. \tag{4.15}$$

□

Proof. Let J be a Jordan block of order k corresponding to a complex eigenvalue $\lambda = a + ib$. That is, J is $(2k) \times (2k)$ matrix of the form

$$J = \begin{bmatrix} R_\lambda & \mathbf{I}_2 & & \\ & R_\lambda & \mathbf{I}_2 & \\ & & \ddots & \ddots \\ & & & R_\lambda \end{bmatrix}, \qquad \text{where } R_\lambda = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, an elementary calculation shows that there exists $C > 0$ such that for all $j \in \mathbb{Z} \setminus \{0\}$,

$$|J^j y| \le C|j|^k |\lambda|^j |y| \qquad \text{for } y \in \mathbb{R}^{2k}. \tag{4.16}$$

Thus, we have

$$|\operatorname{Nm}(J^j y)| \le C^{2k} |j|^{2k^2} |\lambda|^{2kj} |y|^{2k} = C^{2k} |j|^{2k^2} |\det J|^j |y|^{2k} \qquad \text{for } j \ne 0, \ y \in \mathbb{R}^{2k}.$$

A similar estimate holds when J is a Jordan block of order k corresponding to a real eigenvalue λ, that is,

$$|\operatorname{Nm}(J^j y)| \le C^k |j|^{k^2} |\det J|^j |y|^k \qquad \text{for } j \ne 0, \ y \in \mathbb{R}^k.$$

Since $P^{-1} B^j P$ is a block diagonal matrix consisting of such Jordan blocks, we can find a constant $C > 0$ such that

$$|\operatorname{Nm}(P^{-1} B^j P y)| \le C |j|^{n^2} |\det B|^j |y|^n \qquad \text{for } j \ne 0, \ y \in \mathbb{R}^n.$$

Now, take any $x \in B^j(\mathbf{B}(0,r))$ and write it as $x = B^j y$, where $|y| < r$. Then, for any $j \ne 0$,

$$|\operatorname{Nm}(P^{-1} x)| = |\operatorname{Nm}(P^{-1} B^j y)| \le C |j|^{n^2} |\det B|^j |P^{-1} y|^n \le C ||P^{-1}||^n r^n |j|^{n^2} |\det B|^j.$$

For any $\varepsilon > 0$, there exists j_0 such that $|j|^{n^2} \le |\det B|^{|j|\varepsilon}$ for $|j| > j_0$. This shows (4.15) and completes the proof of the lemma. ■

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. First, we shall show that for almost every $U \in SO(n)$,

$$\#|U\Gamma \cap B^j(\mathbf{B}(0,r))| \le C\,|\det B|^j \qquad \text{for } j \ge 0. \tag{4.17}$$

Let $j \geq 0$. By Lemma 4.3, it suffices to show that the vectors $U\Gamma \cap B^j(\mathbf{B}(0,r))$ linearly span $\mathbb{R}^n$. On the contrary, suppose they do not. By Lemma 4.5 there exists a proper symmetric arithmetic progression S of rank $s < n$ in $U\Gamma \cap B^j(\mathbf{B}(0,r))$, see Definition 4.4, such that

$$\#|S| = (2N_1+1)\cdots(2N_s+1) \geq c_n \#|U\Gamma \cap B^j(\mathbf{B}(0,r))| \geq \frac{c_n\,|\mathbf{B}(0,r)|}{2^n\,|\mathbb{R}^n/\Gamma|}\,|\det B|^j\,.$$

Thus, there exists $1 \leq k \leq s$ such that $N_k \geq C\,|\det B|^{j/s}$. Since $N_k v_k \in B^j(\mathbf{B}(0,r))$, it follows from Lemma 4.6 that for any $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that

$$\left|\mathrm{Nm}(P^{-1}N_k v_k)\right| \leq C\,|\det B|^{j(1+\varepsilon)}\,.$$

Thus,

$$\left|\mathrm{Nm}(P^{-1}v_k)\right| \leq C\,|\det B|^{j(1+\varepsilon)}/(N_k)^n \leq C\,|\det B|^{j(1+\varepsilon-n/s)}\,.$$

By choosing $\varepsilon > 0$ small enough we therefore have

$$\left|\mathrm{Nm}(P^{-1}v_k)\right| \leq C\,|\det B|^{-j\eta}\,, \qquad \text{where } \eta = n/s - 1 - \varepsilon > 0. \tag{4.18}$$

Since $v_k \in B^j(\mathbf{B}(0,r))$, we have $|P^{-1}v_k| \leq C'\,\|B\|^j$, where $C' = ||P^{-1}||r$. Hence, by (4.2) and (4.18), we have

$$\nu(P^{-1}U\Gamma, C'\,\|B\|^j) \leq C\,|\det B|^{-j\eta} \tag{4.19}$$

since $v_k \in U\Gamma$. On the other hand, Lemma 4.1 implies that for almost every $U \in \mathrm{SO}(n)$ we have

$$\nu(P^{-1}U\Gamma, C'\|B\|^j) \geq \left(\log(C'\|B\|^j)\right)^{1-n-\varepsilon} \geq cj^{1-n-\varepsilon} \qquad \text{as } j \to \infty. \tag{4.20}$$

Combining (4.19) and (4.20) yields a contradiction for sufficiently large $j > j_0$. Therefore, the vectors $U\Gamma \cap B^j((\mathbf{B}(0,r))$ must linearly span $\mathbb{R}^n$ for all $j > j_0$. Applying Lemma 4.3 shows (4.17) for $j > j_0$. By increasing the constant C (if necessary), we obtain (4.17) for the remaining values $0 \leq j \leq j_0$.

Next, we shall show that for almost every $U \in \mathrm{SO}(n)$, there exists $C > 0$ such that

$$\#|U\Gamma \cap B^j(\mathbf{B}(0,r))| \leq C \qquad \text{for } j < 0. \tag{4.21}$$

Take any $0 \neq v \in U\Gamma \cap B^j(\mathbf{B}(0,r))$, where $j < 0$. By Lemma 4.6 we have $\left|\mathrm{Nm}(P^{-1}v)\right| \leq C\,|\det B|^{j(1-\varepsilon)}$, where $\varepsilon > 0$ and $C = C(\varepsilon)$. Since $\left|P^{-1}v\right| \leq \left\|P^{-1}\right\| r\,\|B^j\| \leq C'\|B^{-1}\|^{|j|}$, by (4.2), we have

$$\nu(P^{-1}U\Gamma, C'\|B^{-1}\|^{|j|}) \leq C|\det B|^{j(1-\varepsilon)}.$$

On other hand, Lemma 4.1 implies that for almost every $U \in \mathrm{SO}(n)$,

$$\nu(P^{-1}U\Gamma, C'\|B^{-1}\|^{|j|}) \geq \left(\log(C\|B^{-1}\|^{-j})\right)^{1-n-\varepsilon} \geq c|j|^{1-n-\varepsilon} \qquad \text{as } j \to -\infty.$$

Combining the last two estimates implies that $j \geq -j_0$ for some sufficiently large $j_0 > 0$. Therefore, the intersection

$$U\Gamma \cap B^j(\mathbf{B}(0,r)) = \{0\} \qquad \text{for all } j < -j_0. \tag{4.22}$$

By increasing constant C (if necessary) we obtain (4.21). This completes the proof of Theorem 4.2. ■

As a consequence of Theorem 4.2 and the properties of the invariant measures $\mu_{\mathcal{L}}$ from [20, Appendix 1], we have the following corollary.

Corollary 4.7. The following statements are true.

(i) Let B be any matrix in $\mathrm{GL}(n,\mathbb{R})$ with $|\det B| > 1$. Then the pair (B,Γ) satisfies lattice counting estimate (1.1) for almost all lattices $\Gamma \in \mathcal{L}_n$ in the sense of the invariant measure $\mu_{\mathcal{L}}$.

(ii) Let $\Gamma \subset \mathbb{R}^n$ be any full rank lattice. Then the pair (B,Γ) satisfies lattice counting estimate (1.1) for almost every $B \in \mathrm{GL}(n,\mathbb{R})$ with $|\det B| > 1$. □

To deduce Corollary 4.7 from Theorem 4.2 we shall use the following lemma that is implicitly contained Skriganov's paper [20].

Lemma 4.8. Suppose that for any lattice $\Lambda \in \mathcal{L}_n$, a certain property holds for lattices of the form $U\Lambda$ for almost all $U \in \mathrm{SO}(n)$ in the sense of Haar measure μ_{SO}. Then, the same property holds for almost all lattices $\Lambda \in \mathcal{L}_n$ in the sense of the invariant measure $\mu_{\mathcal{L}}$. □

Proof. The proof follows along the lines of the argument by Skriganov in [20, Lemma 4.5] using the fact the measure μ_Λ on Λ_n can be identified with a product measure

$$\mu_\Lambda = \mu_{\mathcal{F}} \times \mu_{\mathrm{SO}}.$$

More precisely, following [20, Appendix 1] consider the quotient spaces

$$\mathcal{H}_n = \mathrm{SO}(n)\backslash \mathrm{SL}(n,\mathbb{R}),$$
$$\mathcal{F}_n = \mathcal{H}_n/\mathrm{SL}(n,\mathbb{Z}) = \mathrm{SO}(n)\backslash \mathcal{L}_n.$$

We regard $\mathcal{H}_n$ as a homogeneous space of the group $\mathrm{SL}(n,\mathbb{R})$ and $\mathcal{F}_n \subset \mathcal{H}_n$ as a fundamental set of the discrete subgroup $\mathrm{SL}(n,\mathbb{Z}) \subset \mathrm{SL}(n,\mathbb{R})$. Then, $\mathcal{H}_n$ admits the unique $\mathrm{SL}(n,\mathbb{R})$-invariant measure $\mu_{\mathcal{F}}$ normalized so that $\mu_{\mathcal{F}}(\mathcal{F}_n) = 1$. Moreover, the space $\mathcal{H}_n$ can be identified as a submanifold in $\mathcal{K}$

$$\mathcal{H}_n = \{A \in \mathcal{K} : \det A = 1\},$$

where $\mathcal{K}$ is the open cone of all $n \times n$ real symmetric matrices.

For any lattice $\Lambda = P\mathbb{Z}^n \in \mathcal{L}_n$, the polar decomposition of $P \in \mathrm{SL}(n,\mathbb{R})$ yields

$$P = VA^{1/2}, \qquad \text{where } V \in \mathrm{SO}(n),\ A = P^T P \in \mathcal{F}_n.$$

By [20, (13.14)], we have the following product formula for $\psi \in L^1(\mathcal{L}_n, \mu_{\mathcal{L}})$

$$\int_{\mathcal{L}_n} \psi(\Lambda)\mathrm{d}\mu_{\mathcal{L}}(\Lambda) = \int_{\mathcal{F}_n}\int_{\mathrm{SO}(n)} \psi(VA^{1/2}\mathbb{Z}^n)\mathrm{d}\mu_{\mathrm{SO}}(V)\mathrm{d}\mu_{\mathcal{F}}(A), \tag{4.23}$$

where μ_{SO} is the normalized Haar measure on $\mathrm{SO}(n)$.

Define a function $\psi(\Lambda) = 1$ when a certain property holds for $\Lambda \in \mathcal{L}_n$, and $\psi(\Lambda) = 0$ otherwise. By our hypothesis for all symmetric positive matrices $A \in \mathcal{F}_n$ we have

$$\psi(UA^{1/2}\mathbb{Z}^n) = 1 \qquad \text{for all } U \in \mathrm{SO}(n) \setminus \mathcal{E}_A,$$

where the exceptional set $\mathcal{E}_A \subset \mathrm{SO}(n)$ has measure $\mu_{\mathrm{SO}}(\mathcal{E}_A) = 0$. Define the exceptional set as

$$\mathcal{E} = \{\Lambda \in \mathcal{L}_n : \psi(\Lambda) = 0\} = \{\Lambda = UA^{1/2}\mathbb{Z}^n \in \mathcal{L}_n : A \in \mathcal{F}_n,\ U \in \mathcal{E}_A\}.$$

Then, by (4.23)

$$\mu_{\mathcal{L}}(\mathcal{E}) = \int_{\mathcal{L}_n} \psi(\Lambda)\mathrm{d}\mu_{\mathcal{L}}(\Lambda) = \int_{\mathcal{F}_n} \mu_{\mathrm{SO}}(\mathcal{E}_A)\mathrm{d}\mu_{\mathcal{F}}(A) = 0.$$

This completes the proof of Lemma 4.8. ■

Proof of Corollary 4.7. Part (i) is an immediate consequence of Theorem 4.2 and Lemma 4.8. To show part (ii) we consider the exceptional set

$$\mathcal{E} = \{(B,\Gamma) \in \mathrm{GL}(n,\mathbb{R}) \times \mathcal{L}_n : (1.1) \text{ fails for } (B,\Gamma),\ |\det B| > 1\}.$$

By part (i) each section

$$\mathcal{E}_B = \{\Gamma \in \mathcal{L}_n : (B,\Gamma) \in \mathcal{E}\}$$

has measure $\mu_{\mathcal{L}}(\mathcal{E}_B) = 0$. Thus, by Fubini's Theorem $(\mu_{\mathrm{GL}} \times \mu_{\mathcal{L}})(\mathcal{E}) = 0$, where μ_{GL} is the Haar measure on $\mathrm{GL}(n,\mathbb{R})$. Consequently, for almost every lattice $\Lambda \in \mathcal{L}_n$ we have

$$\mu_{\mathrm{GL}}(\mathcal{E}^{\Lambda}) = 0, \qquad \text{where } \mathcal{E}^{\Lambda} = \{B \in \mathrm{GL}(n,\mathbb{R}) : (B,\Lambda) \in \mathcal{E}\}. \tag{4.24}$$

Observe that the lattice counting estimate (1.1) holds for (B,Γ) if and only if it holds for $(P^{-1}BP, P^{-1}\Gamma)$ for any $P \in \mathrm{GL}(n,\mathbb{R})$. Given any $\Lambda \in \mathcal{L}_n$, take $P \in \mathrm{SL}(n,\mathbb{R})$ such that $\Lambda = P^{-1}\Gamma$. Since $\mathcal{E}^{\Gamma} = P\mathcal{E}^{\Lambda}P^{-1}$ and the Haar measure on $\mathrm{GL}(n,\mathbb{R})$ is unimodular, we have $\mu_{\mathrm{GL}}(\mathcal{E}^{\Gamma}) = \mu_{\mathrm{GL}}(\mathcal{E}^{\Lambda})$. Choosing $\Lambda \in \mathcal{L}_n$ such that (4.24) holds, yields $\mu_{\mathrm{GL}}(\mathcal{E}^{\Gamma}) = 0$. This completes the proof of the corollary. ■

5 Applications to wavelets

In this section we apply results of Sections 3 and 4 to show characterization and existence results for wavelets associated with non-expanding dilations.

5.1 MSF wavelets

An MSF wavelet $\psi \in L^2(\mathbb{R}^n)$ associated with (A,Γ) is a function of the form $\hat{\psi} = |W|^{-1/2}\mathbf{1}_W$, where W is a measurable subset of $\mathbb{R}^n$, often called a wavelet set. A wavelet set W tiles $\mathbb{R}^d$ translationally by Γ^* and simultaneously W tiles $\mathbb{R}^d$ multiplicatively by $B = A^T$. As a corollary of Theorem 4.2 and [12, Theorem 2.5] we deduce the ubiquity of MSF wavelets with respect to random choices of dilations and lattices.

Theorem 5.1. The following statements are true.

(i) Let A be any matrix in $\mathrm{GL}(n,\mathbb{R})$ with $|\det A| > 1$ and let $\Lambda \subset \mathbb{R}^n$ be any full rank lattice. Then there exists an MSF wavelet associated with $(A, U\Lambda)$ for almost every (in the sense of Haar measure) $U \in \mathrm{SO}(n)$.

(ii) Let A be any matrix in $\mathrm{GL}(n,\mathbb{R})$ with $|\det A| > 1$. Then there exists an MSF wavelet associated with (A,Γ) for almost all unimodular lattices $\Gamma \in \mathcal{L}_n$ in the sense of the invariant measure $\mu_{\mathcal{L}}$.

(iii) Let $\Gamma \subset \mathbb{R}^n$ be any full rank lattice. Then there exists an MSF wavelet associated with (A,Γ) for almost every $A \in \mathrm{GL}(n,\mathbb{R})$. □

Proof. To prove part (i) let $\Gamma = \Lambda^*$. The proof of Theorem 4.2 shows that for some sufficiently large $j_0 = j_0(U,\Gamma,r) > 0$, the trivial intersection property (4.22) holds for all $r > 0$ and for a.e. $U \in \mathrm{SO}(n)$. In particular, for any $r > 0$, there are infinitely many $j \in \mathbb{N}$ such that $B^{-j}(\mathbf{B}(0,r/2))$ packs translationally by $U\Lambda^*$. By [12, Theorem 2.5], there exists a set $W \subset \mathbb{R}^n$ such that W tiles $\mathbb{R}^d$ multiplicatively by B and translationally by $U\Lambda^*$. In other words, W is a wavelet set associated with the dilation B and the lattice $U\Lambda^*$. Thus, $\psi \in L^2(\mathbb{R}^n)$, defined by $\hat{\psi} = |W|^{-1/2}\mathbf{1}_W$, is an MSF wavelet associated with $(A, U\Lambda)$, where $B = A^T$ and $(U\Lambda)^* = U\Lambda^*$. This shows part (i).

Part (ii) follows then from Lemma 4.8. To show part (iii) observe that

$$\mu_{\mathrm{GL}}(\{A \in \mathrm{GL}(n,\mathbb{R}) : |\det A| = 1\}) = 0,$$

so it is enough to show the existence of MSF wavelets for almost every $A \in \mathrm{GL}(n,\mathbb{R})$ with $|\det A| > 1$. Then (iii) is deduced from (ii) along the same lines as the proof of Corollary 4.7(ii) using the observation that there exists an MSF wavelets associated with (A,Γ) if and only if it exists for $(P^{-1}AP, P^{-1}\Gamma)$ for any $P \in \mathrm{GL}(n,\mathbb{R})$. ■

5.2 Characterizing equations

The papers [10] and [11] establish wavelet characterizing equations for dilations that are expanding on a subspace. In light of Theorem 2.4, these are optimal results unless extra information about a lattice is also taken into account. Here we shall also show the characterizing equations for pairs of dilations and lattices (B,Γ^*) satisfying the lattice counting estimate (1.2).

The following result generalizes [11, Theorem 6.6], see also [10, Theorem 1.1]. For the definitions of (Parseval) frames, dual frames and Bessel sequences, we refer the reader to the book [6].

Theorem 5.2. Let $A \in \mathrm{GL}(n,\mathbb{R})$, $|\det A| > 1$, and $\Gamma \subset \mathbb{R}^n$ is a full-rank lattice. Suppose that (B,Γ^*) satisfies the lattice counting estimate (1.2). Then, the wavelet system $\mathcal{A}(\Psi, A, \Gamma)$ generated by $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R}^n)$ is a Parseval frame if and only if for all $\alpha \in \Gamma^*$ we have

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}:B^{-j}\alpha\in\Gamma^*} \hat{\psi}_l(B^{-j}\xi)\overline{\hat{\psi}_l(B^{-j}(\xi+\alpha))} = \delta_{\alpha,0} \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{5.1}$$

□

Proof. It is well-known that if a wavelet system is a Bessel sequence with bound $C > 0$, then the Calderón formula is bounded by C, that is,

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}_l(B^{-j}\xi)|^2 \leq C \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{5.2}$$

This result holds without any a priori assumptions on the dilation A and the lattice Γ, as it is a consequence of a more general result that holds for GSI systems, see [11, Proposition 4.1]. Thus, if Ψ is a Parseval frame, then (5.2) holds for $C = 1$. Likewise, if (5.1) holds, then by setting $\alpha = 0$, we also have (5.2) for $C = 1$. In either case, the LIC holds for $\mathcal{A}(\Psi, A, \Gamma)$ in light of Theorem 3.8. Consequently, the general machinery of Hernández *et al.* [11] applies. By [11, Theorem 4.2], the wavelet system $\mathcal{A}(\Psi, A, \Gamma)$ is a Parseval frame if and only if (5.1) holds. ■

If the wavelet system generated by $\Psi = \{\psi_1, \ldots, \psi_L\}$ is a Parseval frame, it is an orthonormal basis precisely when $\|\psi_\ell\| = 1$ for each $\ell = 1, \ldots, L$. Hence, Theorem 5.2 also characterizes orthonormal wavelets. Moreover, Theorem 5.2 generalizes to dual wavelet frames. Indeed, using [11, Theorem 9.1], one can easily show the generalization of [11, Theorem 9.6] from the setting of dilations expanding on a subspace to the lattice counting estimate (1.2).

Theorem 5.3. Suppose that (B, Γ^*) satisfies the lattice counting estimate (1.2). Suppose that the wavelet systems $\mathcal{A}(\Psi, A, \Gamma)$ and $\mathcal{A}(\Phi, A, \Gamma)$ generated by $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R}^n)$ and $\Phi = \{\phi_1, \ldots, \phi_L\} \subset L^2(\mathbb{R}^n)$, respectively, are Bessel sequences. Then they are dual frames if and only if for all $\alpha \in \Gamma^*$ we have

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}:B^{-j}\alpha\in\Gamma^*} \hat{\psi}_l(B^{-j}\xi)\overline{\hat{\phi}_l(B^{-j}(\xi+\alpha))} = \delta_{\alpha,0} \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{5.3}$$

□

Theorem 3.8 shows that the lattice counting estimate (1.2) is the optimal hypothesis under which one should expect the characterizing equations (5.1) to hold. Indeed, if (B, Γ^*) does not satisfy (1.2), then the LIC must fail for some choice of Ψ and the known techniques collapse. However, this does not completely close the problem since the LIC is merely a convenient sufficient condition for showing characterization results. In particular, the following problem raised in [2, 22] remains open.

Conjecture 1. Suppose that wavelet system $\mathcal{A}(\Psi, A, \Gamma)$ is an orthonormal basis, or more generally a Parseval frame for $L^2(\mathbb{R}^n)$. Then the Calderón sum formula holds

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}_l(B^{-j}\xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{5.4}$$

□

Theorem 5.2 implies that (5.4) is true when the lattice counting estimate holds; in particular, the conjecture is true in one dimension. Moreover, this conjecture is also valid for continuous wavelets where translates along a fixed lattice Γ are replaced by translates along $\mathbb{R}^n$, see [18, Theorem 1.1] and [16, Proposition 1]. In particular, by Lemma 3.5, we must necessarily have $|\det A| \neq 1$. For continuous wavelet systems $\{T_\gamma D_{A^j}\psi\}_{\gamma\in\mathbb{R}^n, j\in\mathbb{Z}, \psi\in\Psi}$ with respect to discrete group of dilations $\{A^j : j \in \mathbb{Z}\}$ and translations along $\mathbb{R}^n$, the Calderón sum formula (5.4) is a necessary and sufficient condition for $\{T_\gamma D_{A^j}\psi\}_{\gamma\in\mathbb{R}^n, j\in\mathbb{Z}, \psi\in\Psi}$ to be a continuous Parseval frame, see [16, Theorem 2]. Actually, this is true for any continuous translation invariant system $\{T_\gamma g_p\}_{\gamma\in\mathbb{R}^n, p\in P}$, where (P, μ_P) is a σ-finite measure space and $\{g_p\}_{p\in P} \subset L^2(\mathbb{R}^n)$, see [14]. In this case, the (generalized) Calderón formula is $\int_P |\hat{g}_p(\xi)|^2 \, \mathrm{d}\mu_P(p) = 1$ for a.e. $\xi \in \mathbb{R}^n$.

In contrast to the continuous case, Conjecture 1 remains a surprisingly intractable problem. In particular, it is not even known whether the existence of discrete orthonormal (or Parseval) wavelet Ψ for the pair (A, Γ) implies $|\det A| \neq 1$. Finally, Conjecture 1 is a special case of the stronger conjecture that if the wavelet system $\mathcal{A}(\Psi, A, \Gamma)$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1 and C_2, then $C_1 \leq \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}_l(B^{-j}\xi)|^2 \leq C_2$ for a.e. $\xi \in \mathbb{R}^n$.

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