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The Schur–Horn theorem for unbounded operators with discrete spectrum

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Abstract

We characterize diagonals of unbounded self-adjoint operators on a Hilbert space $\mathcal{H}$ that have only discrete spectrum, that is, with empty essential spectrum. Our result extends the Schur–Horn theorem from a finite dimensional setting to an infinite dimensional Hilbert space, analogous to Kadison's theorem for orthogonal projections ['The Pythagorean theorem. I. The finite case' and 'The Pythagorean theorem. II. The infinite discrete case', *Proc. Natl. Acad. Sci. USA* 99 (2002) 4178–4184, 5217–5222], Kaftal and Weiss ['An infinite dimensional Schur–Horn theorem and majorization theory', *J. Funct. Anal.* 259 (2010) 3115–3162] results for positive compact operators, and Bownik and Jasper ['The Schur–Horn theorem for operators with finite spectrum', *Trans. Amer. Math. Soc.* 367 (2015) 5099–5140; 'Diagonals of self-adjoint operators with finite spectrum', *Bull. Pol. Acad. Sci. Math.* 63 (2015) 249–260; 'The Schur–Horn theorem for operators with three point spectrum', *J. Funct. Anal.* 265 (2013) 1494–1521] characterization for operators with finite spectrum. Furthermore, we show that if a symmetric unbounded operator E on $\mathcal{H}$ has a nondecreasing unbounded diagonal, then any sequence that weakly majorizes this diagonal is also a diagonal of E.

1. *Introduction*

The classical Schur–Horn theorem characterizes diagonals of Hermitian matrices in terms of their eigenvalues. An infinite dimensional extension of this result has been a subject of intensive study in recent years. This line of research was jumpstarted by the influential work of Kadison [**17**, **18**], who discovered a characterization of diagonals of orthogonal projections on separable Hilbert space, and by Arveson and Kadison [**6**], who extended the Schur–Horn theorem to positive trace class operators. This has been preceded by earlier work of Gohberg and Markus [**13**] and by Neumann [**25**]. The Schur–Horn theorem has been extended to compact positive operators by Kaftal and Weiss [**20**] and Loreaux and Weiss [**23**] in terms of majorization inequalities [**19**]. Lebesgue type majorization was used by Bownik and Jasper [**10**, **11**, **16**] to characterize diagonals of self-adjoint operators with finite spectrum operators. Other notable progress includes the work of Arveson [**5**] on diagonals of normal operators with finite spectrum and Antezana, Massey, Ruiz and Stojanoff's results [**1**]. Finally, there is a rapidly growing body of literature on the corresponding problems for von Neumann algebras [**2–4**, **8**, **12**, **21**, **26**].

The goal of this paper is to prove an infinite dimensional variant of the Schur–Horn theorem for unbounded self-adjoint operators with discrete spectrum. This represents a new direction in extending the Schur–Horn theorem to infinite dimensional setting since previous results dealt only with bounded operators.

Assume that an unbounded self-adjoint operator E on a separable Hilbert space $\mathcal{H}$ is bounded from below and has discrete spectrum. That is, the essential spectrum $\sigma_{ess}(E) = \emptyset$, and hence, every point $\lambda \in \sigma(E)$ is an isolated eigenvalue of finite multiplicity. Since E is bounded from below, its eigenvalues can be listed by a nondecreasing sequence $\boldsymbol{\lambda} = \{\lambda_i\}_{i\in\mathbb{N}}$ according to

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their multiplicities. Since $\sigma_{ess}(E) = \emptyset$, we must necessarily have $\lim_{i\to\infty} \lambda_i = \infty$, and thus E is unbounded from above.

Consequently, E is diagonalizable, that is, there exists an orthonormal basis $\{v_i\}_{i\in\mathbb{N}}$ of eigenvectors $Ev_i = \lambda_i v_i$ for all $i \in \mathbb{N}$, and the domain of E is given by

$$\mathcal{D} = \left\{ f \in \mathcal{H} : \sum_{i\in\mathbb{N}} |\lambda_i|^2 |\langle f, v_i\rangle|^2 < \infty \right\}. \tag{1.1}$$

In order to emphasize this point, we will use the notation $E = \operatorname{diag} \boldsymbol{\lambda}$ to denote the operator which has eigenvalues $\boldsymbol{\lambda}$ and domain (1.1) as above.

If $\{e_i\}_{i\in\mathbb{N}} \subset \mathcal{D}$ is any other orthonormal basis of $\mathcal{H}$, then the diagonal $d_i = \langle Ee_i, e_i\rangle$ of E with respect to $\{e_i\}$ satisfies

$$\sum_{i=1}^n \lambda_i \leqslant \sum_{i=1}^n d_i \quad \text{for all } n \in \mathbb{N}. \tag{1.2}$$

In particular, the same inequality holds true when $\{d_i\}_{i\in\mathbb{N}}$ is replaced by its nondecreasing rearrangement $\{d_i^\uparrow\}_{i\in\mathbb{N}}$. The necessity of condition (1.2) is often attributed to Schur [**27**]. Our main result says that (1.2) is also sufficient, thus generalizing Horn's theorem [**15**].

Theorem 1.1. *Suppose that $\boldsymbol{\lambda} = \{\lambda_i\}_{i\in\mathbb{N}}$ and $\{d_i\}_{i\in\mathbb{N}}$ are two nondecreasing and unbounded sequences. Let $E = \operatorname{diag} \boldsymbol{\lambda}$ be a self-adjoint operator with eigenvalues $\boldsymbol{\lambda}$ and eigenvectors $\{v_i\}_{i\in\mathbb{N}}$. If the majorization inequality (1.2) holds, then there exists an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$, which lies in the linear span of $\{v_i\}_{i\in\mathbb{N}}$, such that $d_i = \langle Ee_i, e_i\rangle$ for all $i \in \mathbb{N}$.*

The remarkable consequence of our main theorem is that majorization inequality (1.2) is the only condition that a sequence $\{d_i\}_{i\in\mathbb{N}}$ must satisfy in order to be diagonal of $\operatorname{diag} \boldsymbol{\lambda}$. Moreover, the required diagonal is achieved with respect to an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$, whose elements are finite linear combinations of eigenvectors $\{v_i\}_{i\in\mathbb{N}}$. In particular, it is possible that $\lambda_i = d_i$ for all but finitely many $i \in \mathbb{N}$, the trace condition is violated, that is, $\sum_{i=1}^\infty (d_i - \lambda_i) \neq 0$, but yet the conclusion of Theorem 1.1 still holds.

Despite the simplicity of the statement of Theorem 1.1, its proof is far from trivial as it needs to deal with two major cases. The majorization inequality (1.2) can be equivalently stated as

$$\delta_k = \sum_{i=1}^k (d_i - \lambda_i) \geqslant 0 \quad \text{for all } k \in \mathbb{N}.$$

After dealing with elementary reductions in Section 2, the first case deals with the conservation of mass scenario

$$\liminf_{k\to\infty} \delta_k = 0.$$

The second case deals with vanishing mass at infinity scenario

$$\alpha = \liminf_{k\to\infty} \delta_k > 0.$$

This further splits in two subcases: $\delta_k \geqslant \alpha$ for sufficiently large k, and $\delta_k < \alpha$ for infinitely many k, shown by Theorems 4.2 and 4.3, respectively.

The proofs of these cases require careful application of an infinite sequence of convex moves, also known as T-transforms [**20**], to guarantee that the limiting orthonormal sequence is a basis. In addition, we need to ensure that the constructed basis is contained in the dense domain $\mathcal{D}$. This constraint was not present in earlier work on bounded operators and requires new techniques of moving from a prescribed diagonal into a desired diagonal configuration. Our methods work not only for self-adjoint operators with discrete spectrum as in Theorem 1.1, but also for unbounded symmetric operators (possibly with continuous spectrum) as in Theorem 2.1. Indeed, 'eigenvalue to diagonal' Theorem 1.1 is an immediate consequence of a more general 'diagonal to diagonal' Theorem 2.1.

We end the paper by giving several examples illustrating Theorem 1.1 in Section 5. Laplacians, or more generally elliptic differential operators, provide a broad and interesting class of operators falling into the scope of this paper.

2. *Diagonal to diagonal elementary reductions*

In this section, we show several reductions that are employed in the proof of Theorem 1.1. To achieve this, we formulate a generalization of Theorem 1.1 for unbounded symmetric operators which are not necessarily diagonalizable. Recall that a linear operator E defined on a dense domain $\mathcal{D} \subset \mathcal{H}$ is *symmetric* if

$$\langle Ef, g\rangle = \langle f, Eg\rangle \quad \text{for all } f, g \in \mathcal{D}.$$

Theorem 1.1 is an immediate consequence of the following diagonal to diagonal theorem.

THEOREM 2.1. *Let E be a symmetric operator defined on a dense domain $\mathcal{D} \subset \mathcal{H}$. Let $\mathbf{d} = \{d_i\}_{i\in\mathbb{N}}$ and $\boldsymbol{\lambda} = \{\lambda_i\}_{i\in\mathbb{N}}$ be two nondecreasing unbounded sequences satisfying (1.2). If there exists an orthonormal sequence $\{f_i\}_{i\in\mathbb{N}} \subset \mathcal{D}$ such that*

$$\langle Ef_i, f_i\rangle = \lambda_i \quad \textit{for all } i \in \mathbb{N},$$

then there exists an orthonormal sequence $\{e_i\}_{i\in\mathbb{N}} \subset \operatorname{span}\{f_i\}_{i\in\mathbb{N}}$ such that $\overline{\operatorname{span}}\{e_i\}_{i\in\mathbb{N}} = \overline{\operatorname{span}}\{f_i\}_{i\in\mathbb{N}}$ and

$$\langle Ee_i, e_i\rangle = d_i \quad \textit{for all } i \in \mathbb{N}.$$

In the special case, when $\{f_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of eigenvectors with eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ of a self-adjoint operator $E = \operatorname{diag}\boldsymbol{\lambda}$, Theorem 2.1 immediately yields Theorem 1.1. To facilitate statements of reduction results, we shall make some formal definitions.

DEFINITION 2.2. Let $\boldsymbol{\lambda} = \{\lambda_i\}_{i\in I}$ and $\mathbf{d} = \{d_i\}_{i\in I}$ be two real sequences, where I is countable. Let E be unbounded (here it means not necessarily bounded) linear operator defined on a dense domain $\mathcal{D}$ of a Hilbert space $\mathcal{H}$. We say that an operator E has *diagonal* $\boldsymbol{\lambda}$ if there exists an orthonormal sequence $\{f_i\}_{i\in I}$ contained in $\mathcal{D}$ such that

$$\langle Ef_i, f_i\rangle = \lambda_i \quad \text{for all } i \in I.$$

We say that E has diagonal $\mathbf{d}$, which is *finitely derived* from diagonal $\boldsymbol{\lambda}$, if there exists an orthonormal sequence $\{e_i\}_{i\in I}$ in $\mathcal{D}$ satisfying $\langle Ee_i, e_i\rangle = d_i$ for all $i \in I$,

$$\overline{\operatorname{span}}\{e_i\}_{i\in\mathbb{N}} = \overline{\operatorname{span}}\{f_i\}_{i\in\mathbb{N}} \quad \text{and} \quad \forall k \in I \quad e_k \in \operatorname{span}\{f_i\}_{i\in I}. \tag{2.1}$$

For our purposes, it is more natural to define a majorization order using nondecreasing rearrangements instead of more classical nonincreasing rearrangements, see [**24**]. Suppose $\{\lambda_i\}_{i=1}^N$ and $\{d_i\}_{i=1}^N$ are two real sequences. Let $\{\lambda_i^{\uparrow}\}_{i=1}^N$ and $\{d_i^{\uparrow}\}_{i=1}^N$ be their nondecreasing rearrangements. We say that $\{d_i\} \preccurlyeq \{\lambda_i\}$ if and only if

$$\sum_{i=1}^N d_i^{\uparrow} = \sum_{i=1}^N \lambda_i^{\uparrow} \quad \text{and} \quad \sum_{i=1}^n \lambda_i^{\uparrow} \leqslant \sum_{i=1}^n d_i^{\uparrow} \quad \text{for all } 1 \leqslant n \leqslant N. \tag{2.2}$$

The classical Schur–Horn theorem [**15**, **27**] characterizes diagonals of self-adjoint (Hermitian) matrices with given eigenvalues. It can be stated as follows, where $\mathcal{H}_N$ is an N dimensional Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, that is, $\mathcal{H}_N = \mathbb{R}^N$ or $\mathbb{C}^N$.

THEOREM 2.3 (Schur–Horn theorem). *There exists a self-adjoint operator $E : \mathcal{H}_N \to \mathcal{H}_N$ with eigenvalues $\{\lambda_i\}_{i=1}^N$ and diagonal $\{d_i\}_{i=1}^N$ if and only if $\{d_i\} \preccurlyeq \{\lambda_i\}$.*

As a consequence of Theorem 2.3 we have the following block diagonal lemma.

LEMMA 2.4. *Let E be a symmetric operator defined on a dense domain $\mathcal{D} \subset \mathcal{H}$. Suppose that $\{d_i\}_{i\in I}$ and $\{\widetilde{d}_i\}_{i\in I}$ are two sequences of real numbers such that:*

(i) *there exists a collection of disjoint finite subsets $\{I_j\}_{j\in J}$ of the index set I,*
(ii) *$\{d_i\}_{i\in I_j} \preccurlyeq \{\widetilde{d}_i\}_{i\in I_j}$ for each $j \in J$,*
(iii) *$\widetilde{d}_i = d_i$ for all $i \in I \setminus \big(\bigcup_{j\in J} I_j\big)$.*

Suppose that E has diagonal $\{\widetilde{d}_i\}_{i\in I}$ with respect to an orthonormal sequence $\{f_i\}_{i\in I}$. Then, $\{d_i\}_{i\in I}$ is a finitely derived diagonal of E. That is, there exists an orthonormal sequence $\{e_i\}_{i\in I}$ satisfying (2.1) with respect to which E has diagonal $\{d_i\}_{i\in I}$.

Proof. Let P_j be the orthogonal projection of $\mathcal{H}$ onto finite dimensional block subspace $\mathcal{H}_j = \operatorname{span}\{f_i : i \in I_j\}$. Observe that a finite dimensional self-adjoint operator $E_j := (P_j E)|_{\mathcal{H}_j}$ has diagonal $\{\widetilde{d}_i\}_{i\in I_j}$ with respect to $\{f_i\}_{i\in I_j}$. By (ii) and the Schur–Horn theorem, there exists a unitary operator U_j on $\mathcal{H}_j$ such that $U_j E_j (U_j)^*$ has diagonal $\{d_i\}_{i\in I_j}$ with respect to $\{f_i\}_{i\in I_j}$. Define an orthonormal basis $\{e_i\}_{i\in I}$ by

$$e_i = \begin{cases} U_j f_j & i \in I_j, \\ f_i & i \in I \setminus \Big(\bigcup_{j\in J} I_j\Big). \end{cases}$$

For $i \in I_j$ we have

$$\langle E e_i, e_i\rangle = \langle E(U_j)^* f_i, (U_j)^* f_i\rangle = \langle P_j E(U_j)^* f_i, (U_j)^* f_i\rangle = \langle U_j E_j (U_j)^* f_i, f_i\rangle = d_i.$$

The same identity holds trivially for $i \notin \cup_{j\in J} I_j$, which shows that E has diagonal $\{d_i\}$ with respect to $\{e_i\}$. This completes the proof of the lemma. □

As an application of Lemma 2.4 we can show the special case of Theorem 2.1.

LEMMA 2.5. *Let E be a symmetric operator defined on a dense domain $\mathcal{D} \subset \mathcal{H}$. Let $\mathbf{d} = \{d_i\}_{i\in\mathbb{N}}$ and $\boldsymbol{\lambda} = \{\lambda_i\}_{i\in\mathbb{N}}$ be two nondecreasing sequences such that*

$$\delta_k := \sum_{i=1}^{k} (d_i - \lambda_i) \geqslant 0 \quad \text{for all } k \in \mathbb{N}. \tag{2.3}$$

Suppose that there are infinitely many $k \in \mathbb{N}$ such that $\delta_k = 0$. If $\boldsymbol{\lambda}$ is diagonal of E, then $\mathbf{d}$ is a finitely derived diagonal of E.

Proof. Set $k_1 = 0$ and let $\{k_j\}_{j=2}^\infty$ be a strictly increasing sequence in $\mathbb{N}$ such that $\delta_{k_j} = 0$ for all $j \geqslant 2$. For each $j \in \mathbb{N}$ set $I_j = \{k_j + 1, \dots, k_{j+1}\}$. For each $j \in \mathbb{N}$ and $k \in I_j$

$$\sum_{i=k_j+1}^{k} (d_i - \lambda_i) = \delta_k - \delta_{k_j} = \delta_k \geqslant 0.$$

Since $\delta_{k_{j+1}} = 0$ we have $\{d_i\}_{i\in I_j} \preccurlyeq \{\lambda_i\}_{i\in I_j}$. By our assumption, E has diagonal $\boldsymbol{\lambda}$ with respect to some orthonormal sequence $\{f_i\}_{i\in\mathbb{N}}$. By Lemma 2.4 there is an orthonormal sequence $\{e_i\}_{i\in\mathbb{N}}$ satisfying (2.1) with respect to which E has diagonal $\mathbf{d}$. ☐

In the proof of Theorem 2.1 it is convenient to make the reducing assumption (2.4) about nondecreasing sequences $\boldsymbol{\lambda}$ and $\mathbf{d}$.

THEOREM 2.6. *If Theorem 2.1 holds under an additional assumption*

$$\delta_k = \sum_{i=1}^{k}(d_i - \lambda_i) > 0 \quad \text{for all } k \in \mathbb{N}, \tag{2.4}$$

then it holds in a full generality.

Proof. Suppose that E has diagonal $\boldsymbol{\lambda}$ with respect to orthonormal sequence $\{f_i\}_{i\in\mathbb{N}}$. The case when $\delta_k = 0$ for infinitely many $k \in \mathbb{N}$ is covered by Lemma 2.5. Hence, we can assume that there are finitely many $k \in \mathbb{N}$ such that $\delta_k = 0$. Let $N \in \mathbb{N}$ be the largest such integer. Define the spaces

$$\mathcal{H}_0 = \overline{\operatorname{span}}\{f_i\}_{i=1}^N \quad \text{and} \quad \mathcal{H}_1 = \overline{\operatorname{span}}\{f_i\}_{i=N+1}^\infty.$$

Applying Theorem 2.1 to the sequences $\{d_i\}_{i=N+1}^\infty$ and $\{\lambda_i\}_{i=N+1}^\infty$, and noting that for $k \geqslant N+1$

$$\sum_{i=N+1}^{k}(d_i - \lambda_i) = \delta_k - \delta_N = \delta_k > 0$$

we obtain an orthonormal basis $\{e_i\}_{i=N+1}^\infty$ of $\mathcal{H}_1$ such that $\langle Ee_i, e_i\rangle = d_i$ for all $i \geqslant N+1$. Then the operator E has diagonal

$$\lambda_1, \dots, \lambda_N, d_{N+1}, d_{N+2}, \dots$$

with respect to orthonormal basis $\{f_1, \dots, f_N, e_{N+1}, e_{N+2}, \dots\}$ of $\mathcal{H}_0 \oplus \mathcal{H}_1$. Since $\delta_N = 0$ we have $\{d_i\}_{i=1}^N \preccurlyeq \{\lambda_i\}_{i=1}^N$. Applying Lemma 2.4 we obtain an orthonormal sequence $\{e_i\}_{i=1}^\infty$ with respect to which E has diagonal $\mathbf{d}$ and (2.1) holds. ☐

We end this section with a basic linear algebra lemma about convex moves of 2×2 Hermitian matrices. Lemma 2.7 generalizes the corresponding well-known result for matrices with zero off-diagonal entries.

LEMMA 2.7. *Let E be a symmetric operator on $\mathcal{D} \subset \mathcal{H}$. Assume that real numbers d_1, d_2, $\widetilde{d}_1$, $\widetilde{d}_2$ satisfy*

$$\widetilde{d}_1 \leqslant d_1, d_2 \leqslant \widetilde{d}_2, \quad \widetilde{d}_1 \neq \widetilde{d}_2, \quad \text{and} \quad \widetilde{d}_1 + \widetilde{d}_2 = d_1 + d_2. \tag{2.5}$$

If there exists an orthonormal set $\{f_1, f_2\} \subset \mathcal{D}$ such that $\langle Ef_i, f_i\rangle = \widetilde{d}_i$ for $i = 1, 2$, then there exists

$$\frac{\widetilde{d}_2 - d_1}{\widetilde{d}_2 - \widetilde{d}_1} \leqslant \alpha \leqslant 1$$

and $\theta \in [0, 2\pi)$ such that $\langle Ee_i, e_i\rangle = d_i$ for $i = 1, 2$, where

$$e_1 = \sqrt{\alpha} f_1 + \sqrt{1-\alpha}\, e^{i\theta} f_2 \quad \text{and} \quad e_2 = \sqrt{1-\alpha} f_1 - \sqrt{\alpha}\, e^{i\theta} f_2. \tag{2.6}$$

Moreover, if $\mathcal{H}$ is a real Hilbert space, then $e^{i\theta} = \pm 1$. If the inequalities in (2.5) are strict, then $\alpha < 1$.

Proof. Set

$$\beta := \langle Ef_1, f_2\rangle.$$

Choose $\theta \in [0, 2\pi)$ such that $e^{-i\theta}\beta \leqslant 0$. For $x \in [0,1]$ define

$$e_1^x = \sqrt{x} f_1 + \sqrt{1-x} e^{i\theta} f_2 \quad \text{and} \quad e_2^x = \sqrt{1-x} f_1 - \sqrt{x} e^{i\theta} f_2.$$

We calculate

$$\langle Ee_1^x, e_1^x\rangle = x\widetilde{d}_1 + (1-x)\widetilde{d}_2 + 2e^{-i\theta}\beta\sqrt{x(1-x)}$$

so that

$$\langle Ee_1^1, e_1^1\rangle = \widetilde{d}_1 \geqslant d_1$$

and for $\alpha_0 = (d_1 - \widetilde{d}_2)/(\widetilde{d}_1 - \widetilde{d}_2)$, since $e^{-i\theta}\beta \leqslant 0$ we have

$$\begin{aligned}\langle Ee_1^{\alpha_0}, e_1^{\alpha_0}\rangle &= \alpha_0(\widetilde{d}_1 - \widetilde{d}_2) + \widetilde{d}_2 + 2e^{-i\theta}\beta\sqrt{\alpha_0(1-\alpha_0)} \\ &= d_1 - \widetilde{d}_2 + \widetilde{d}_2 + 2e^{-i\theta}\beta\sqrt{\alpha_0(1-\alpha_0)} \leqslant d_1.\end{aligned}$$

Since $x \mapsto \langle Ee_1^x, e_1^x\rangle$ is continuous on $[\alpha_0, 1]$ there is some $\alpha \geqslant \alpha_0$ such that $\langle Ee_1^\alpha, e_1^\alpha\rangle = d_1$. Finally, using the assumption that $\widetilde{d}_1 + \widetilde{d}_2 = d_1 + d_2$, we have

$$\begin{aligned}\langle Ee_2^\alpha, e_2^\alpha\rangle &= (1-\alpha)\widetilde{d}_1 + \alpha\widetilde{d}_2 - 2e^{-i\theta}\beta\sqrt{\alpha(1-\alpha)} \\ &= \widetilde{d}_1 + \widetilde{d}_2 - \left(\alpha\widetilde{d}_1 + (1-\alpha)\widetilde{d}_2 + 2e^{-i\theta}\beta\sqrt{\alpha(1-\alpha)}\right) \\ &= \widetilde{d}_1 + \widetilde{d}_2 - \langle Ee_1^\alpha, e_1^\alpha\rangle = \widetilde{d}_1 + \widetilde{d}_2 - d_1 = d_2.\end{aligned}$$

This completes the proof of the lemma. □

3. *Conservation of mass scenario*

In this section, we will establish Theorem 2.1 under additional conservation of mass assumption

$$\liminf_{k\to\infty} \delta_k = 0, \quad \text{where } \delta_k = \sum_{i=1}^k (d_i - \lambda_i). \tag{3.1}$$

It is remarkable that we achieve this goal without assuming that the sequence $\{\lambda_i\}$ is unbounded. This requires a careful application of an infinite sequence of convex moves, also known as T-transforms [**20**], to the original orthonormal basis of eigenvectors $\{f_i\}_{i\in\mathbb{N}}$. The key Lemma 3.1 guarantees that the limiting orthonormal sequence is complete.

LEMMA 3.1. *Let $\{f_i\}_{i\in\mathbb{N}}$ be an orthonormal set, and let $\{\alpha_i\}_{i\in\mathbb{N}}$ be a sequence in $[0,1]$. Set $\widetilde{e}_1 = f_1$ and inductively define for $i \in \mathbb{N}$,*

$$e_i = \sqrt{\alpha_i}\,\widetilde{e}_i + \sqrt{1-\alpha_i} f_{i+1} \quad \text{and} \quad \widetilde{e}_{i+1} = \sqrt{1-\alpha_i}\widetilde{e}_i - \sqrt{\alpha_i} f_{i+1}. \tag{3.2}$$

If for each $n \in \mathbb{N}$

$$\prod_{i=n}^{\infty}(1-\alpha_i) = 0, \tag{3.3}$$

then $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis for $\overline{\operatorname{span}}\{f_i\}_{i\in\mathbb{N}}$ and (2.1) holds. In particular, if $\alpha_i < 1$ for all i and $\sum_{i=1}^\infty \alpha_i/(1-\alpha_i) = \infty$, then $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis for $\overline{\operatorname{span}}\{f_i\}_{i\in\mathbb{N}}$.

Proof. By induction, we see that for each $i \in \mathbb{N}$,

$$\{e_1, e_2, \dots, e_{i-1}, \widetilde{e}_i, f_{i+1}, f_{i+2}, \dots\}$$

is an orthonormal sequence. Hence, $\{e_i\}_{i\in\mathbb{N}}$ is orthonormal and it is enough to show that $f_j \in \overline{\operatorname{span}}\{e_i\}_{i\in\mathbb{N}}$ for all $j \in \mathbb{N}$. Note that $e_i \in \operatorname{span}\{f_j\}_{j=1}^{i+1}$ and $\widetilde{e}_i \in \operatorname{span}\{f_j\}_{j=1}^{i}$. Thus, $\langle f_j, e_i\rangle = 0$ for $i \leqslant j-2$ and $\langle f_j, \widetilde{e}_i\rangle = 0$ for $i \leqslant j-1$. Also note that for each $n \in \mathbb{N}$ the sequence $\{e_1, e_2, \dots, e_n, \widetilde{e}_{n+1}\}$ is an orthonormal basis for $\operatorname{span}\{f_i\}_{i=1}^{n+1}$. Thus, for $n \geqslant j-1$ we have

$$1 - |\langle f_j, \widetilde{e}_{n+1}\rangle|^2 = \sum_{i=1}^{n} |\langle f_j, e_i\rangle|^2. \tag{3.4}$$

If we set $\alpha_0 = 1$, then

$$\langle f_j, \widetilde{e}_j\rangle = -\sqrt{\alpha_{j-1}}$$

for all $j \in \mathbb{N}$. For $n \geqslant 0$ we have

$$\langle f_j, \widetilde{e}_{j+n}\rangle = \sqrt{1-\alpha_{j+n-1}}\langle f_j, \widetilde{e}_{j+n-1}\rangle,$$

so that, by induction for $n \geqslant 0$ we have

$$\langle f_j, \widetilde{e}_{j+n}\rangle = -\left(\alpha_{j-1}\prod_{k=j}^{j+n-1}(1-\alpha_{j+k})\right)^{1/2}. \tag{3.5}$$

Letting $n \to \infty$ in (3.5), we see from (3.3) that $\lim_{n\to\infty}\langle f_j, \widetilde{e}_n\rangle = 0$. Hence, (3.4) implies that for each $j \in \mathbb{N}$

$$\sum_{i=1}^{\infty} |\langle f_j, e_i\rangle|^2 = 1,$$

That is, $f_j \in \overline{\operatorname{span}}\{e_i\}_{i\in\mathbb{N}}$, which completes the proof.

Finally, consider the case that $\alpha_i < 1$ for all $i \in \mathbb{N}$, and $\sum_{i=1}^{\infty} \alpha_i/(1-\alpha_i) = \infty$. In this case, we have

$$\sum_{i=n}^{k} \frac{\alpha_i}{1-\alpha_i} \leqslant \prod_{i=n}^{k}\left(1 + \frac{\alpha_i}{1-\alpha_i}\right) = \frac{1}{\prod_{i=n}^{k}(1-\alpha_i)}.$$

Letting $k \to \infty$ we obtain (3.3). □

LEMMA 3.2. *If $\{t_n\}$ is a positive nonincreasing sequence with limit zero, then*

$$\sum_{n=1}^{\infty} \frac{t_n - t_{n+1}}{t_{n+1}} = \infty.$$

Proof. Since $(t_n - t_{n+1})/t_{n+1} = t_n/t_{n+1} - 1$, we may assume $t_{n+1}/t_n \to 1$ as $n \to \infty$. Since $t_n/t_{n+1} \geqslant 1$ we have

$$\sum_{n=1}^{k} \frac{t_n - t_{n+1}}{t_{n+1}} = \sum_{n=1}^{k}\left(\frac{t_n}{t_{n+1}} - 1\right) \geqslant \sum_{n=1}^{k} \log\left(\frac{t_n}{t_{n+1}}\right) = \log(t_1) - \log(t_{k+1}) \to \infty \quad \text{as } k \to \infty.$$

□

Next, we prove the first preliminary version of Theorem 2.1 under the additional assumption that $\{\delta_k\}$ is strictly decreasing to 0. However, we do not assume in Lemma 3.3 that $\{d_i\}$ is arranged in nondecreasing order. Also in all subsequent results in Section 3 we do not assume that $\{\lambda_i\}$ is an unbounded sequence.

LEMMA 3.3. *Let $\boldsymbol{\lambda} = \{\lambda_i\}_{i\in\mathbb{N}}$ be a nondecreasing sequence. Let E be a symmetric operator with diagonal $\boldsymbol{\lambda}$ as in Definition 2.2. If $\mathbf{d} = \{d_i\}_{i\in\mathbb{N}}$ is a sequence such that the following two properties hold:*

$$\lambda_1 \leqslant d_n < \lambda_n \quad \text{for all } n \geqslant 2, \tag{3.6}$$

$$d_1 = \lambda_1 + \sum_{i=2}^{\infty}(\lambda_i - d_i) < \lambda_2, \tag{3.7}$$

then E has diagonal $\mathbf{d}$, which is finitely derived from $\boldsymbol{\lambda}$.

Proof. For each $n \in \mathbb{N}$ set

$$t_n = \sum_{i=n}^{\infty}(\lambda_i - d_i).$$

Note that $t_1 = 0$, and $\{t_i\}_{i=2}^{\infty}$ is a positive, nonincreasing sequence with limit zero since

$$\sum_{i=1}^{\infty}(\lambda_i - d_i) = 0.$$

For each $n \in \mathbb{N}$ set

$$\widetilde{\lambda}_n := d_n - t_{n+1} = \lambda_n - t_n.$$

From (3.6) for each $n \geqslant 2$, we have

$$\widetilde{\lambda}_n < d_n < \lambda_n \leqslant \lambda_{n+1}.$$

From (3.7) we have

$$\widetilde{\lambda}_1 = \lambda_1 < d_1 < \lambda_2.$$

Thus, for all $n \in \mathbb{N}$ we have

$$\widetilde{\lambda}_n < d_n, \widetilde{\lambda}_{n+1} < \lambda_{n+1} \quad \text{and} \quad \widetilde{\lambda}_n + \lambda_{n+1} = d_n + \widetilde{\lambda}_{n+1}. \tag{3.8}$$

We conclude that for all $n \in \mathbb{N}$

$$\widetilde{\alpha}_n := \frac{\lambda_{n+1} - d_n}{\lambda_{n+1} - \widetilde{\lambda}_n} = \frac{\lambda_{n+1} - d_n}{\lambda_{n+1} - d_n + t_{n+1}} \in (0,1).$$

By Lemma 3.2 we have

$$\sum_{n=1}^{\infty}\frac{\widetilde{\alpha}_n}{1-\widetilde{\alpha}_n} = \sum_{n=1}^{\infty}\frac{\lambda_{n+1} - d_n}{t_{n+1}} \geqslant \sum_{n=1}^{\infty}\frac{\lambda_n - d_n}{t_{n+1}} = \sum_{n=1}^{\infty}\frac{t_n - t_{n+1}}{t_{n+1}} = \infty. \tag{3.9}$$

Let $\{f_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence with respect to which E has diagonal $\boldsymbol{\lambda}$. We shall now define an orthonormal sequence $\{e_n\}_{n\in\mathbb{N}}$ as in Lemma 3.1 for an appropriate choice of the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$.

We have $\langle Ef_1, f_1\rangle = \lambda_1 = \widetilde{\lambda}_1$, and $\langle Ef_2, f_2\rangle = \lambda_2$. By Lemma 2.7 there exist $\alpha_1 \in [\widetilde{\alpha}_1, 1)$ and $\theta_2 \in [0, 2\pi)$ such that vectors

$$e_1 = \sqrt{\alpha_1} f_1 + \sqrt{1-\alpha_1}e^{i\theta_2} f_2 \quad \text{and} \quad \widetilde{e}_2 = \sqrt{1-\alpha_1} f_1 - \sqrt{\alpha_1}e^{i\theta_2} f_2$$

form an orthonormal basis for $\operatorname{span}\{f_1, f_2\}$ and

$$\langle Ee_1, e_1\rangle = d_1 \quad \text{and} \quad \langle E\widetilde{e}_2, \widetilde{e}_2\rangle = \widetilde{\lambda}_2.$$

Now, we may inductively assume that for some $n \geqslant 2$ we have an orthonormal basis $\{e_1, \dots, e_{n-1}, \widetilde{e}_n\}$ for $\operatorname{span}\{f_j\}_{j=1}^n$ such that

$$\langle Ee_j, e_j\rangle = d_j \quad \text{for } j \leqslant n-1 \quad \text{and} \quad \langle E\widetilde{e}_n, \widetilde{e}_n\rangle = \widetilde{\lambda}_n.$$

Using (3.8), by Lemma 2.7 there exist $\alpha_n \in [\widetilde{\alpha}_n, 1)$ and $\theta_{n+1} \in [0, 2\pi)$ such that the vectors

$$e_n = \sqrt{\alpha_n}\widetilde{e}_n + \sqrt{1-\alpha_n}e^{i\theta_n} f_{n+1} \quad \text{and} \quad \widetilde{e}_{n+1} = \sqrt{1-\alpha_n}\widetilde{e}_n - \sqrt{\alpha_n}e^{i\theta_{n+1}} f_{n+1}$$

form an orthonormal basis for $\operatorname{span}\{\widetilde{e}_n, f_{n+1}\}$ and

$$\langle Ee_n, e_n\rangle = d_n \quad \text{and} \quad \langle E\widetilde{e}_{n+1}, \widetilde{e}_{n+1}\rangle = \widetilde{\lambda}_{n+1}.$$

The fact that $\alpha_n < 1$ for all $n \in \mathbb{N}$ is a consequence of strict inequalities in (3.8).

Observe that the above procedure yields an orthonormal sequence $\{e_n\}_{n=1}^\infty$ that is obtained by applying Lemma 3.1 to $\{e^{i\theta_n} f_n\}_{n\in\mathbb{N}}$ with $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ as already defined and $\theta_1 = 0$. Since for all $n \in \mathbb{N}$, $\alpha_n \in [\widetilde{\alpha}_n, 1)$, by (3.9) we have

$$\sum_{n=1}^\infty \frac{\alpha_n}{1-\alpha_n} \geqslant \sum_{n=1}^\infty \frac{\widetilde{\alpha}_n}{1-\widetilde{\alpha}_n} = \infty.$$

Hence, by Lemma 3.1 $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $\overline{\operatorname{span}}\{f_n\}_{n\in\mathbb{N}}$. By (3.2) each vector e_n is a linear combination $f_1, \dots, f_{n+1}$. Therefore, E has diagonal $\mathbf{d}$, which is finitely derived from $\boldsymbol{\lambda}$. $\square$

The following is the second preliminary version of the main result of this section. The final result of this section, which is Theorem 3.6, will be identical with the exception of the two extra assumptions: $d_1 < \lambda_2$ and $\delta_k > 0$ for all $k \in \mathbb{N}$.

Lemma 3.4. *Let $\mathbf{d} = \{d_i\}_{i\in\mathbb{N}}$ and $\boldsymbol{\lambda} = \{\lambda_i\}_{i\in\mathbb{N}}$ be nondecreasing sequences such that (2.3) and (3.1) hold. Let E be a symmetric operator with diagonal $\boldsymbol{\lambda}$. If $\lambda_2 > d_1$ and $\delta_k > 0$ for all $k \in \mathbb{N}$, then E has diagonal $\mathbf{d}$, which is finitely derived from $\boldsymbol{\lambda}$.*

Proof. Inductively define the sequence $\{m_j\}$ as follows. Set $m_1 = 1$ and for $j \geqslant 2$ set $m_j = \min\{n > m_{j-1} : \delta_n < \delta_{m_{j-1}}\}$.

For each $j \in \mathbb{N}$ and $i = m_j + 1, \dots, m_{j+1}$ set

$$\widetilde{d}_i = \frac{\delta_{m_{j+1}} - \delta_{m_j}}{m_{j+1} - m_j} + \lambda_i.$$

Also set $\widetilde{d}_1 = d_1$ and define

$$\widetilde{\delta}_k := \sum_{i=1}^k (\widetilde{d}_i - \lambda_i).$$

By induction, for each $j \in \mathbb{N}$ and $k = m_j + 1, \dots, m_{j+1}$ we have

$$\widetilde{\delta}_k = \delta_{m_j} + \frac{\delta_{m_{j+1}} - \delta_{m_j}}{m_{j+1} - m_j}(k - m_j). \tag{3.10}$$

In particular, we have

$$\widetilde{\delta}_{m_j} = \delta_{m_j} \quad \text{for all } j \in \mathbb{N}. \tag{3.11}$$

Since $\delta_k \geqslant \delta_{m_j} > \delta_{m_{j+1}}$ for all $m_j < k < m_{j+1}$, we have

$$\delta_{m_j} + \sum_{i=m_j+1}^k (\widetilde{d}_i - \lambda_i) = \widetilde{\delta}_k = \delta_{m_j} + \frac{\delta_{m_{j+1}} - \delta_{m_j}}{m_{j+1} - m_j}(k - m_j) \leqslant \delta_{m_j} \leqslant \delta_k = \delta_{m_j} + \sum_{i=m_j+1}^k (d_i - \lambda_i).$$

Combining this with (3.11) shows that $\{d_i\}_{i=m_j+1}^{m_{j+1}} \preccurlyeq \{\widetilde{d}_i\}_{i=m_j+1}^{m_{j+1}}$ for each $j \in \mathbb{N}$.

Using $\delta_{m_{j+1}} - \delta_{m_j} < 0$, (3.10), and (3.11) we deduce that the sequence $\{\widetilde{\delta}_k\}_{k\in\mathbb{N}}$ is decreasing and $\lim_{k\to\infty}\widetilde{\delta}_k = 0$. Moreover, we have $\widetilde{d}_1 = d_1 < \lambda_2$ and $\lambda_1 \leqslant \widetilde{d}_n$ for all $n \geqslant 2$. Applying Lemma 3.3 to the sequences $\boldsymbol{\lambda}$ and $\widetilde{\mathbf{d}} := \{\widetilde{d}_i\}_{i\in\mathbb{N}}$ shows that E has diagonal $\widetilde{\mathbf{d}}$, which is finitely derived from $\boldsymbol{\lambda}$.

Finally, since the sets $I_j = \{m_j+1,\ldots,m_{j+1}\}$ are disjoint, and $\{d_i\}_{i\in I_j} \preccurlyeq \{\widetilde{d}_i\}_{i\in I_j}$, Lemma 2.4 shows that E has diagonal $\mathbf{d}$, which is finitely derived from $\widetilde{\mathbf{d}}$, and hence finitely derived from $\boldsymbol{\lambda}$. □

LEMMA 3.5. *Let $\mathbf{d} = \{d_i\}_{i\in\mathbb{N}}$ and $\boldsymbol{\lambda} = \{\lambda_i\}_{i\in\mathbb{N}}$ be nondecreasing sequences such that such that (2.3) and (3.1) hold. Let E be a symmetric operator defined on a dense domain $\mathcal{D}$. If the following two conditions hold:*

(i) *there exists $N \in \mathbb{N}$ such that $\delta_N \leqslant \delta_k$ for all $k \leqslant N$,*
(ii) *E has diagonal $\widetilde{\mathbf{d}} := \{\widetilde{d}_i\}_{i\in\mathbb{N}}$, where*

$$\widetilde{d}_i := \begin{cases} \lambda_1 + \delta_N & i = 1, \\ \lambda_i & i = 2,\ldots,N, \\ d_i & i > N, \end{cases} \tag{3.12}$$

then E has diagonal $\mathbf{d}$, which is finitely derived from $\widetilde{\mathbf{d}}$.

Proof. Let $I_1 = \{1,\ldots,N\}$. In light of Lemma 2.4 it is enough to show that $\{d_i\}_{i\in I_1} \preccurlyeq \{\widetilde{d}_i\}_{i\in I_1}$. Let $\{\widetilde{d}_i^{\uparrow}\}_{i=1}^N$ denote the nondecreasing rearrangement of $\{\widetilde{d}_i\}_{i=1}^N$, then for $k = 1,\ldots,N$

$$\sum_{i=1}^k \widetilde{d}_i^{\uparrow} \leqslant \sum_{i=1}^k \widetilde{d}_i = \delta_N + \sum_{i=1}^k \lambda_i = \sum_{i=1}^k d_i + \delta_N - \delta_k \leqslant \sum_{i=1}^k d_i.$$

Together with the observation that both of the inequalities above become equality when $k = N$ demonstrates the desired majorization. □

We are now ready to show Theorem 1.1 under the additional hypothesis (3.1), but without the assumption that $\{\lambda_i\}$ is unbounded.

THEOREM 3.6. *Let $\mathbf{d} = \{d_i\}_{i\in\mathbb{N}}$ and $\boldsymbol{\lambda} = \{\lambda_i\}_{i\in\mathbb{N}}$ be nondecreasing sequences such that (2.3) and (3.1) hold. Let E be a symmetric operator with diagonal $\boldsymbol{\lambda}$ as in Definition 2.2. Then, E has diagonal $\mathbf{d}$, which is finitely derived from $\boldsymbol{\lambda}$.*

Proof. By Theorem 2.6 we may assume that $\delta_k > 0$ for all $k \in \mathbb{N}$. We also claim that $\boldsymbol{\lambda}$ is not a constant sequence. On the contrary, suppose $\lambda_i = L$ for all $i \in \mathbb{N}$. Since $\mathbf{d}$ is nondecreasing and $\liminf_{k\to\infty}\delta_k = 0$ we conclude that $d_i \nearrow L$ as $i \to \infty$. The assumption that $\delta_1 > 0$ implies $d_1 > L$, which is a contradiction.

Since $\boldsymbol{\lambda}$ is not constant, there is some $M \in \mathbb{N}$ such that $\lambda_1 < \lambda_M$. Choose $N > M$ such that

$$\delta_N \leqslant \delta_k \quad \text{for all } k \leqslant N \tag{3.13}$$

and $\delta_N < \lambda_M - \lambda_1$. Since $\lambda_M \leqslant \lambda_{N+1}$ we also have

$$\lambda_{N+1} > \delta_N + \lambda_1. \tag{3.14}$$

Define the sequence $\widetilde{\mathbf{d}} = \{\widetilde{d}_i\}_{i\in\mathbb{N}}$ as in (3.12). Define the sequences $\{c_i\}$ and $\{\mu_i\}$ by

$$c_i = \begin{cases} \widetilde{d}_1 & i = 1, \\ \widetilde{d}_{i+N-1} & i \geqslant 2, \end{cases} \quad \text{and} \quad \mu_i = \begin{cases} \lambda_1 & i = 1, \\ \lambda_{i+N-1} & i \geqslant 2. \end{cases}$$

Note that

$$\widetilde{\delta}_k := \sum_{i=1}^{k}(c_i - \mu_i) = \delta_{N+k-1} \quad \text{for all } k \in \mathbb{N}.$$

Hence, $\widetilde{\delta}_k > 0$ for all $k \in \mathbb{N}$ and by (3.14) we have $c_1 = \delta_N + \lambda_1 < \lambda_{N+1} = \mu_2$. By our hypothesis, E has diagonal $\{\mu_i\}$ with respect to orthonormal sequence $\{f_i\}_{i=1,i>N}$. Applying Lemma 3.4 yields an orthonormal basis $\{\widetilde{e}_i\}_{i=1,i>N}$ of $\overline{\operatorname{span}}\{f_i\}_{i=1,i>N}$ with respect to which E has diagonal $\{c_i\}$, which is finitely derived from $\{\mu_i\}$. Letting $\widetilde{e}_i = f_i$ for $2 \leqslant i \leqslant N$, yields an orthonormal sequence $\{\widetilde{e}_i\}_{i\in\mathbb{N}}$ with respect to which E has diagonal $\widetilde{\mathbf{d}}$. By (3.13) we can apply Lemma 3.5 to obtain a desired orthonormal sequence $\{e_i\}_{i\in\mathbb{N}}$, with respect to which E has diagonal $\mathbf{d}$. Moreover, $\mathbf{d}$ is finitely derived from $\widetilde{\mathbf{d}}$, and hence from $\boldsymbol{\lambda}$. □

4. *Mass vanishing at infinity scenario*

In this section, we will show Theorem 2.1 under complementary assumption to (3.1). This involves a construction of an infinite sequence of convex moves continually transforming a diagonal sequence, where some of the mass must necessarily vanish at infinity. First, we handle the strong domination case $\lambda_k \leqslant d_k$ for every $k \in \mathbb{N}$. Equivalently, the sequence $\{\delta_k\}_{k\in\mathbb{N}}$ is assumed to be nondecreasing in Lemma 4.1.

LEMMA 4.1. *Let E be a symmetric operator defined on a dense domain $\mathcal{D}$. Let $\mathbf{d} = \{d_i\}_{i=1}^\infty$ and $\boldsymbol{\lambda} = \{\lambda_i\}_{i=1}^\infty$ be nondecreasing unbounded sequences with $d_i \geqslant \lambda_i$ for every i. If there exists an orthonormal sequence $\{f_i\}_{i\in\mathbb{N}} \subset \mathcal{D}$ such that*

$$\langle Ef_i, f_i\rangle = \lambda_i \quad \textit{for all } i \in \mathbb{N},$$

then there exists an orthonormal sequence $\{e_i\}_{i\in\mathbb{N}}$ satisfying (2.1) and

$$\langle Ee_i, e_i\rangle = d_i \quad \textit{for all } i \in \mathbb{N}.$$

Proof. Without loss of generality we can assume that sequences $\mathbf{d}$ and $\boldsymbol{\lambda}$ consist of positive terms. This can be seen by adding a positive multiple of the identity to E, which corresponds to translating these sequences by a positive constant. This process can be reversed by subtracting the same multiple of the identity.

Suppose that I is an infinite subset of $\mathbb{N}$. For any such subset we define inductively an increasing sequence $\{i_k\}_{k=1}^\infty$ in I by letting $i_1 = \min I$ and choosing $i_k \in I$ large enough to have

$$\lambda_{i_k} > 2d_{i_{k-1}} \quad k \geqslant 2. \tag{4.1}$$

In addition, we require that $I \setminus \{i_k : k \in \mathbb{N}\}$ is infinite. This is possible since the sequence $\{\lambda_i\}$ is not bounded. Now recursively define another sequence by $x_{i_1} = \lambda_{i_1}$ and

$$x_{i_{k+1}} = \lambda_{i_{k+1}} + x_{i_k} - d_{i_k} \quad k \geqslant 1.$$

Note that $x_{i_2} \leqslant \lambda_{i_2}$ and $x_{i_2} - d_{i_1} > 0$ (using condition (4.1)). By induction we get that for any $k \geqslant 1$

$$x_{i_k} \leqslant d_{i_k} < x_{i_{k+1}} \leqslant \lambda_{i_{k+1}}. \tag{4.2}$$

Furthermore,

$$\widetilde{\alpha}_k := \frac{\lambda_{i_{k+1}} - d_{i_k}}{\lambda_{i_{k+1}} - x_{i_k}} > \frac{\lambda_{i_{k+1}}/2}{\lambda_{i_{k+1}}} = \frac{1}{2}. \tag{4.3}$$

Now we are ready to start constructing an orthonormal sequence $\{e_{i_k}\}_{k=1}^\infty$.

We have $\langle Ef_{i_1}, f_{i_1}\rangle = x_{i_1}$, and $\langle Ef_{i_2}, f_{i_2}\rangle = \lambda_{i_2}$. By Lemma 2.7 there exist $\alpha_1 \in [\widetilde{\alpha}_1, 1]$ and $\theta_2 \in [0, 2\pi)$ such that vectors

$$e_{i_1} = \sqrt{\alpha_1} f_{i_1} + \sqrt{1-\alpha_1} e^{i\theta_2} f_{i_2} \quad \text{and} \quad \widetilde{e}_{i_2} = \sqrt{1-\alpha_1} f_{i_1} - \sqrt{\alpha_1} e^{i\theta_2} f_{i_2}$$

form an orthonormal basis for $\operatorname{span}\{f_{i_1}, f_{i_2}\}$ and

$$\langle Ee_{i_1}, e_{i_1}\rangle = d_{i_1} \quad \text{and} \quad \langle E\widetilde{e}_{i_2}, \widetilde{e}_{i_2}\rangle = x_{i_2}.$$

Now, we may inductively assume that for some $k \geqslant 2$ we have an orthonormal basis $\{e_{i_1}, \ldots, e_{i_{k-1}}, \widetilde{e}_{i_k}\}$ for $\operatorname{span}\{f_{i_j}\}_{j=1}^k$ such that

$$\langle Ee_{i_j}, e_{i_j}\rangle = d_{i_j} \quad \text{for } j \leqslant k-1 \quad \text{and} \quad \langle E\widetilde{e}_{i_k}, \widetilde{e}_{i_k}\rangle = x_{i_k}.$$

Using (4.2), by Lemma 2.7 there exist $\alpha_k \in [\widetilde{\alpha}_k, 1]$ and $\theta_{k+1} \in [0, 2\pi)$ such that the vectors

$$e_{i_k} = \sqrt{\alpha_k}\widetilde{e}_{i_k} + \sqrt{1-\alpha_k} e^{i\theta_k} f_{i_{k+1}} \quad \text{and} \quad \widetilde{e}_{i_{k+1}} = \sqrt{1-\alpha_k}\widetilde{e}_{i_k} - \sqrt{\alpha_k} e^{i\theta_{k+1}} f_{i_{k+1}}$$

form an orthonormal basis for $\operatorname{span}\{\widetilde{e}_{i_k}, f_{i_{k+1}}\}$ and

$$\langle Ee_{i_k}, e_{i_k}\rangle = d_{i_k} \quad \text{and} \quad \langle E\widetilde{e}_{i_{k+1}}, \widetilde{e}_{i_{k+1}}\rangle = x_{i_{k+1}}.$$

This completes the inductive step, and thus we have an orthonormal sequence $\{e_{i_k}\}_{k=1}^\infty$.

Observe that this is exactly the orthonormal sequence obtained by applying Lemma 3.1 to $\{e^{i\theta_k} f_{i_k}\}_{k\in\mathbb{N}}$ with $\{\alpha_k\}_{k\in\mathbb{N}}$ and $\{\theta_k\}_{k\in\mathbb{N}}$ as already defined with $\theta_1 = 0$. By (4.3) we have $\alpha_k > 1/2$ for all $k \in \mathbb{N}$. Hence, by Lemma 3.1 $\{e_i\}_{i\in I_1}$ is an orthonormal basis for $\mathcal{H}_1 = \overline{\operatorname{span}}\{f_i\}_{i\in I_1}$, with respect to which E has diagonal $\{d_i\}_{i\in I_1}$, where $I_1 = \{i_k : k \in \mathbb{N}\}$. Moreover, diagonal $\{d_i\}_{i\in I_1}$ is finitely derived from $\{\lambda_i\}_{i\in I_1}$.

In the initial step, we run the above construction starting with the full index set $I = \mathbb{N}$ to obtain the required diagonal subsequence indexed by I_1. Then, we repeat the above construction inductively with respect to the unused index set $I = \mathbb{N} \setminus (I_1 \cup \ldots \cup I_{k-1})$, $k \geqslant 2$, to obtain the required diagonal subsequence indexed by I_k. Since we always include the smallest unused element in I and we leave out infinitely many unused indices, the family $\{I_k\}_{k\in\mathbb{N}}$ is a partition of $\mathbb{N}$. Thus, we obtain an orthogonal decomposition

$$\overline{\operatorname{span}}\{f_i\}_{i\in\mathbb{N}} = \bigoplus_{k=1}^\infty \mathcal{H}_k, \quad \text{where } \mathcal{H}_k = \overline{\operatorname{span}}\{f_i\}_{i\in I_k}.$$

For each subspace $\mathcal{H}_k$ we have constructed an orthonormal basis $\{e_i\}_{i\in I_k}$, with respect to which E has diagonal $\{d_i\}_{i\in I_k}$, that is finitely derived from $\{\lambda_i\}_{i\in I_k}$. This defines the required orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of $\overline{\operatorname{span}}\{f_i\}_{i\in\mathbb{N}}$ with respect to which E has diagonal $\mathbf{d}$. $\square$

We are now ready to show Theorem 2.1 in the case when sequence $\{\delta_k\}_{k\in\mathbb{N}}$ as in (2.3), eventually stays above its $\liminf_{k\to\infty} \delta_k$.

THEOREM 4.2. *Let E be a symmetric operator defined on a dense domain $\mathcal{D}$. Let $\mathbf{d} = \{d_i\}_{i=1}^\infty$ and $\boldsymbol{\lambda} = \{\lambda_i\}_{i=1}^\infty$ be nondecreasing unbounded sequences such that (2.3) holds. Assume that there exists $M \geqslant 0$ such that*

$$\delta_k \geqslant \alpha := \liminf_{i\to\infty} \delta_i \quad \textit{for all } k \geqslant M.$$

If $\boldsymbol{\lambda}$ is a diagonal of E, then $\mathbf{d}$ is a finitely derived diagonal of E.

Proof. By Lemma 2.6 we may assume $\delta_k > 0$ for all $k \in \mathbb{N}$. Fix $N \in \mathbb{N}$ such that $N > \max_{k \leqslant M-1}\{k\alpha/\delta_k, M\}$. Hence,

$$\delta_k \geqslant \frac{k\alpha}{N} \quad \text{for } k \leqslant M-1.$$

Define

$$\widetilde{d}_i = \begin{cases} d_i - \dfrac{\alpha}{N} & i = 1, \dots, N, \\ d_i & i \geqslant N+1. \end{cases}$$

Observe that

$$\sum_{i=1}^{k}(\widetilde{d}_i - \lambda_i) = \begin{cases} \delta_k - \dfrac{k\alpha}{N} \geqslant 0 & k \leqslant M-1, \\ \delta_k - \dfrac{k\alpha}{N} \geqslant \alpha - \dfrac{k\alpha}{N} \geqslant 0 & M \leqslant k \leqslant N, \\ \delta_k - \alpha \geqslant 0 & k \geqslant N+1. \end{cases}$$

The last equation implies that $\liminf_{k\to\infty} \sum_{i=1}^{k}(\widetilde{d}_i - \lambda_i) = 0$. We may apply Theorem 3.6 to deduce that E has diagonal $\{\widetilde{d}_i\}_{i\in\mathbb{N}}$, which is finitely derived from $\boldsymbol{\lambda}$. Since $d_i \geqslant \widetilde{d}_i$ for all $i \in \mathbb{N}$, Lemma 4.1 yields the desired diagonal $\{d_i\}_{i\in\mathbb{N}}$. □

Finally, we are left we the case when the sequence $\{\delta_k\}_{k\in\mathbb{N}}$ dips infinitely many times below its $\liminf_{k\to\infty} \delta_k$.

THEOREM 4.3. *Let E be a symmetric operator defined on a dense domain $\mathcal{D}$. Let $\mathbf{d} = \{d_i\}_{i\in\mathbb{N}}$ and $\boldsymbol{\lambda} = \{\lambda_i\}_{i=1\in\mathbb{N}}$ be nondecreasing unbounded sequences such that (2.3) holds. Assume that*

$$\delta_k < \alpha := \liminf_{i\to\infty} \delta_i \quad \textit{for infinitely many } k.$$

If $\boldsymbol{\lambda}$ is a diagonal of E, then $\mathbf{d}$ is a finitely derived diagonal of E.

Proof. We define inductively the index sequence $\{m_j\}_{j=0}^{\infty}$ as follows. Let $m_0 = 0$. For $j \geqslant 1$ set

$$m_j = \min\{n > m_{j-1} : \forall k \geqslant n \quad \delta_n \leqslant \delta_k\}.$$

That is, the sequence $\{m_j\}$ records consecutive global minima of the tail $\{\delta_n\}_{n>m_{j-1}}$. In particular, using the convention that $\delta_0 = 0$, we have

$$\delta_{m_{j-1}} \leqslant \delta_{m_j} \leqslant \delta_k \quad \text{for all } m_{j-1} < k \leqslant m_j,\ j \geqslant 1. \tag{4.4}$$

Define the sequence $\{\widetilde{d}_i\}_{i\in\mathbb{N}}$ by

$$\widetilde{d}_i = \begin{cases} \lambda_i + (\delta_{m_j} - \delta_{m_{j-1}}) & \text{for } i = m_j,\ j \geqslant 1, \\ \lambda_i & \text{otherwise.} \end{cases} \tag{4.5}$$

Set

$$\widetilde{\delta}_k = \sum_{i=1}^{k}(\widetilde{d}_i - \lambda_i).$$

For $j \geqslant 1$, set $I_j = \{m_{j-1}+1, \dots, m_j\}$. By (4.4) and (4.5), for any $k \in I_j$ we have

$$\delta_{m_{j-1}} + \sum_{i=m_{j-1}+1}^{k}(d_i - \lambda_i) = \delta_k \geqslant \delta_{m_j} \geqslant \delta_{m_{j-1}} + \sum_{i=m_{j-1}+1}^{k}(\widetilde{d}_i - \lambda_i)$$

with equalities when $k = m_j$. This shows that $\{d_i\}_{i\in I_j} \preccurlyeq \{\widetilde{d}_i\}_{i\in I_j}$. Since the sets $\{I_j\}_{j\in\mathbb{N}}$ form a partition of $\mathbb{N}$, we can apply Lemma 2.4 to reduce the problem to showing that E has diagonal $\{\widetilde{d}_i\}_{i\in\mathbb{N}}$. This case is already covered by Lemma 4.1 since the sequence $\{\widetilde{\delta}_i\}_{i\in\mathbb{N}}$ is nondecreasing. □

Theorem 2.1 now follows immediately by combining Theorems 4.2 and 4.3.

5. *Remarks and examples*

5.1. *Diagonals and eigenvalues of inverse operators*

It is worth observing how our main result, Theorem 1.1, is related to the result of Kaftal and Weiss [**20**] who characterized the diagonals of positive compact operators. The earlier result of Arveson and Kadison [**6**] characterized diagonals of positive trace class operators. In the case of positive compact operators that are not trace class, the trace condition is not present both in [**20**] and in Theorem 1.1. Hence, one might attempt to deduce Theorem 1.1 from [**20**].

For simplicity assume that the first eigenvalue of E is $\lambda_1 > 0$. Then, the inverse E^{-1} is a compact positive operator with eigenvalues $1/\lambda_1 \geqslant 1/\lambda_2 \geqslant \ldots \searrow 0$. Conversely, the inverse of positive self-adjoint operator with trivial kernel is unbounded with discrete spectrum. However, the diagonal does not behave in such controlled way when taking inverses. Thus, Theorem 1.1 does not follow from [**20**] in any obvious way. For the converse direction, Theorem 3.6 implies a special case of [**20**] when $\liminf_{k\to\infty} \delta_k = 0$. Nevertheless, it is possible to deduce majorization for sums of inverses from the majorization of sums of eigenvalues as follows.

We say that a sequence $\{a_i\}_{i\in\mathbb{N}}$ is (weakly) majorized by a sequence $\{b_i\}_{i\in\mathbb{N}}$, and write $\{a_i\} \prec \{b_i\}$, if

$$\sum_{i=1}^{n} a_i \leqslant \sum_{i=1}^{n} b_i \quad \text{for all } n \in \mathbb{N}.$$

Note that unlike (strong) majorization order $\preccurlyeq$, we do not alter the order of elements of the sequences. Recall the classical Hardy–Littlewood–Pólya majorization theorem [**14**, § 3.17].

Theorem 5.1 (Hardy–Littlewood–Pólya majorization). *Assume that $\{a_i\}$ and $\{b_i\}$ are nondecreasing sequences of positive real numbers such that $\{a_i\} \prec \{b_i\}$. Then for any concave increasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ we have $\{\Phi(a_i)\} \prec \{\Phi(b_i)\}$. Similarly, when $\{a_i\}$ and $\{b_i\}$ are nonincreasing, then the result holds for convex increasing functions Φ.*

Let $a_i = \lambda_i$ and $b_i = d_i$ with the sequences coming from the unbounded operator E as in Theorem 1.1. Now choose $\Phi(x) = -1/x$ to get that $\{1/d_i\} \prec \{1/\lambda_i\}$. Therefore, whenever $\{d_i\}$ is a possible diagonal for E, the sequence of inverses is a valid diagonal for the compact operator E^{-1}. Interestingly, the inverse procedure does not work. Even if $\{\widetilde{d}_i\}$ is majorized by $\{1/\lambda_i\}$, the sequence of inverses $\{1/\widetilde{d}_i\}$ does not need to majorize $\{\lambda_i\}$, since $\Phi(x) = -1/x$ is not convex.

As another consequence of Hardy–Littlewood–Pólya majorization we get that whenever $\{d_i\}$ is a valid diagonal for E, the sequence of eigenvalues $\{e^{-\lambda_i t}\}$ of the heat operator e^{-tE} majorizes $\{e^{-d_i t}\}$. Therefore the heat operator associated with E admits diagonal $\{e^{-d_i t}\}$.

5.2. *Examples using Laplacians*

Elliptic differential operators provide a broad and interesting class of operators falling into the scope of this paper. In particular, Laplace operators on domains $\Omega \subset \mathbb{R}^d$ imposed with various boundary conditions can be closed in $L^2(\Omega)$ leading to essentially self-adjoint operators with discrete spectrum. This follows from classical considerations involving compactness of their inverses and compactness of the Sobolev embeddings. For more details, see Bandle [**7**] or Blanchard–Brüning [**9**].

To be more specific, consider two Laplace operators defined (weakly) on Sobolev spaces, via the corresponding quadratic forms:

- Neumann Laplacian Δ_N: domain $H^1(\Omega)$, quadratic form $\langle \Delta_N u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dA$;
- Dirichlet Laplacian Δ_D: domain $H_0^1(\Omega)$, quadratic form $\int_\Omega \nabla u \cdot \nabla v \, dA$.

It turns out that the eigenfunctions for these operators satisfy appropriate classical boundary conditions: Neumann $\partial_n u = 0$ on $\partial\Omega$, and Dirichlet $u = 0$ on $\partial\Omega$, respectively. See Chapters 5 and 6 of Laugesen [**22**] for a nice overview.

Let μ_j and λ_j denote the eigenvalues (in nondecreasing order) for the Neumann and Dirichlet Laplacians, respectively. It is easy to see (via operator domain inclusion) that for any j we have $\mu_j \leqslant \lambda_j$, see [**7**] or [**22**, Chapter 10]. Therefore we have two sequences exhibiting strong domination as in Lemma 4.1.

Interestingly, these operator are not self-adjoint, or even symmetric, according to the theory of unbounded operators. They are defined on a dense subspace $H^1(\Omega)$ of $L^2(\Omega)$; however, their adjoints have much smaller domain. One can however consider the same operators restricted to $H^2(\Omega)$. Assuming that Ω is somewhat smooth (locally Lipschitz boundary is enough), elliptic regularity theory implies that domain of the adjoint is now the same as for the operator. Hence we get self-adjoint operators on $H^2(\Omega)$ which agree with the weak formulation on their domains. See [**22**, Chapters 18 and 19] for a detailed exposition.

5.2.1. *Dirichlet eigenvalues and Neumann Laplacian.* We can ask for an orthonormal basis of $L^2(\Omega)$ such that the diagonal entries of the Neumann Laplacian equal the Dirichlet eigenvalues $\lambda_j \geqslant \mu_j$. Theorem 1.1 asserts that such a basis must exist.

In the simplest possible case of an interval, $\Omega = [0, \pi]$, the Dirichlet eigenfunctions equal $\{u_j = \sin(jx)\}_{j\geqslant 1}$ and they form an orthonormal basis of L^2. These functions certainly belong to $H^1(\Omega)$ (or even $H^2(\Omega)$), so we already have the required orthonormal basis for $L^2(\Omega)$ (Fourier sine series). However, we are acting on these functions using Neumann Laplacian. This is irrelevant for the quadratic form definition, but the pointwise action is not simply the second derivative. In order to compute the Neumann Laplacian of $\sin(jx)$ we must first find the Fourier cosine series expansion of that function, since $\{\cos(jx)\}_{j\geqslant 0}$ is the orthonormal basis formed by the eigenfunction of the Neumann Laplacian. Therefore our transformations amount to constructing a cosine series for sine functions.

5.2.2. *Domain monotonicity for Dirichlet Laplacian.* It is also easy to see that if $\Omega_1 \subset \Omega_2$, then $\lambda_j(\Omega_1) \geqslant \lambda_j(\Omega_2)$, simply because $H_0^1(\Omega_1) \subset H_0^1(\Omega_2)$ (by setting functions equal 0 outside). If Ω_1 is a relatively compact subset of Ω_2 then the eigenfunctions of the Dirichlet Laplacian on Ω_1 are concentrated on a compact subset of Ω_2, hence they cannot form an orthonormal basis for $L^2(\Omega_2)$. Theorem 1.1 still asserts that there is an orthonormal basis of $L^2(\Omega_2)$ such that the diagonal of the Dirichlet Laplacian on Ω_2 equals $\{\lambda_j(\Omega_1)\}$. However, it is not at all clear how to find such a basis.

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