ANISOTROPIC CLASSES OF HOMOGENEOUS PSEUDODIFFERENTIAL SYMBOLS

ÁRPÁD BÉNYI AND MARCIN BOWNIK

ABSTRACT. We define homogeneous classes of x-dependent anisotropic symbols $\dot{S}_{\gamma,\delta}^m(A)$ in the framework determined by an expansive dilation A, thus extending the existing theory for diagonal dilations. We revisit anisotropic analogues of Hörmander-Mihlin multipliers introduced by Rivière [22] and provide direct proofs of their boundedness on Lebesgue and Hardy spaces by making use of the well-established Calderón-Zygmund theory on spaces of homogeneous type. We then show that x-dependent symbols in $\dot{S}_{1,1}^0(A)$ exhibit Calderón-Zygmund kernels, yet their L^2 boundedness fails. Finally, we prove boundedness results of the class $\dot{S}_{1,1}^m(A)$ on weighted anisotropic Besov and Triebel-Lizorkin spaces extending isotropic results of Grafakos and Torres [15].

1. INTRODUCTION: DEFINITIONS, EXAMPLES, NOTATION

Multiplier operators, and more generally pseudodifferential operators, continue to attract lots of attention due to their wide applications in the study of partial differential equations and signal analysis. Several classes of isotropic pseudodifferential symbols attached to such operators, both in linear and multilinear setting, are nowadays well understood. Among them we highlight the prominent role played by the classical Hörmander-Mihlin multipliers [16], [21], their space dependent counterparts—the Coifman-Meyer symbols [8], or more generally the so-called classical classes of symbols $S_{\gamma,\delta}^m$ or their homogeneous counterparts $\dot{S}_{\gamma,\delta}^m$.

We start by recalling the definition of the isotropic classes of homogeneous symbols $\dot{S}_{\gamma,\delta}^m$; see for example the work of Grafakos and Torres [15]. We say that a symbol σ belongs to the class $\dot{S}_{\gamma,\delta}^m$ if

$$(1.1) \qquad |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha\beta} |\xi|^{m+\delta|\alpha|-\gamma|\beta|},$$

for all multi-indices α, β, all $\xi \in \mathbb{R}^n$, and some positive constants $C_{\alpha\beta}$. In particular, if the symbol σ is x-independent, we refer to it as a multiplier.

A multiplier $\sigma(\xi)$ belongs to the class $\dot{S}^m_{\gamma,0}$, or simply $\dot{S}^m_\gamma$, if

$$(1.2) \qquad |\partial^\beta_\xi \sigma(\xi)| \leq C_\beta |\xi|^{m-\gamma|\beta|},$$

for all multi-indices β, all $\xi \in \mathbb{R}^n$, and some positive constants C_β. The non-homogeneous version of these classes is obtained by replacing the quantity $|\xi|$ with $1 + |\xi|$. For the remainder of this paper, the absence of the dot will refer to the nonhomogeneous version of a given class of symbols.

In the early 1970s Rivière extended the theory of singular integrals to operators with kernels that satisfy a homogeneity given by a one-parameter group of transformations. His work [22] anticipated future developments surrounding what is nowadays known as the Calderón-Zygmund theory on spaces of homogeneous type. Some of the motivation for the study of such spaces and operators acting on them comes from partial differential equations where several differential operators, such as the heat operator, satisfy an anisotropic homogeneity. Of particular interest was therefore the study of the boundedness properties of homogeneous multiplier operators; see, for example, [22], and the works of Madych and Rivière [20] and Seeger [23]. In the context of operators with x-dependent nonhomogeneous anisotropic symbols, several boundedness results are known, for instance, in the woks of Garello [13], Lascar [18], Leopold [19], and Yamazaki [29, 30]. As we will indicate below, the setting used by the latter authors involves diagonal dilations. However, the study of pseudodifferential operators with x-dependent anisotropic symbols associated with more general expansive dilations has not been previously explored.

In this paper we introduce and investigate the appropriate notion of *anisotropic class of multipliers* $\dot{S}^m_\gamma(A)$, and more generally of *anisotropic class of symbols* $\dot{S}^m_{\gamma,\delta}(A)$, associated to an expansive matrix A. We search for a definition analogous to the isotropic one stated above. We need to set up first some of the standard notation, which we borrow from Bownik's monograph [3]; see also [4], [6]. Given an expansive matrix A, that is a matrix for which all its eigenvalues λ satisfy $|\lambda| > 1$, we can first define a canonical quasi-norm ρ_A associated to it. Specifically, if we let P be some non-degenerate $n \times n$ matrix, and $|\cdot|$ the standard norm of $\mathbb{R}^n$, there exists an *ellipsoid* $\Delta = \{x \in \mathbb{R}^n : |Px| < 1\}$ such that $|\Delta| = 1$ and for some $r > 1$, $\Delta \subset r\Delta \subset A\Delta$. Then, we can define a family of dilated balls around the origin $B_k = A^k\Delta, k \in \mathbb{Z}$, that satisfy

$$B_k \subset rB_k \subset B_{k+1} \text{ and } |B_k| = b^k,$$

where $b = |\det A|$. The *step homogeneous quasi-norm* induced by A is defined by

$$\rho(x) = b^j \text{ for } x \in B_{j+1} \setminus B_j, \text{ and } \rho(0) = 0.$$

It is straightforward to verify that ρ satisfies a triangle inequality up to a constant and the homogeneity condition $\rho(Ax) = b\rho(x), x \in \mathbb{R}^n$. It is known that any two homogeneous quasi-norms associated to a dilation A are equivalent; therefore, we can talk about a *canonical quasi-norm associated to* A, which we denote by ρ_A. Moreover, endowed with the quasi-norm ρ_A and the Lebesgue measure, $\mathbb{R}^n$ becomes a space of homogeneous type. Similarly we shall consider a family of dilated balls B_k^*, $k \in \mathbb{Z}$, and a canonical quasi-norm ρ_{A^*} associated with the transposed dilation A^*.

Definition 1.1. We say that a bounded symbol $\sigma(x, \xi)$ belongs to the *homogeneous anisotropic class* $\dot{S}_{\gamma,\delta}^m(A)$ if it satisfies the estimates

(1.3) $\qquad |\partial_x^\alpha \partial_\xi^\beta [\sigma(A^{-k_1} \cdot, (A^*)^{k_2} \cdot)](A^{k_1} x, (A^*)^{-k_2} \xi)| \leq C_{\alpha,\beta} \rho_{A^*}(\xi)^m,$

for all multi-indices α, β and $(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Here, $k_1, k_2 \in \mathbb{Z}$ are given by

(1.4) $\qquad\qquad\qquad k_1 = \lfloor k\delta \rfloor, \qquad k_2 = \lfloor k\gamma \rfloor,$

where $k \in \mathbb{Z}$ is such that $\rho_{A^*}(\xi) \sim |\det A|^k$.

The derivatives above should be interpreted as

$$\partial_x^\alpha \partial_\xi^\beta \tilde{\sigma}(A^{k_1} x, (A^*)^{-k_2} \xi),$$

where

$$\tilde{\sigma}(x, \xi) = \sigma(A^{-k_1} x, (A^*)^{k_2} \xi),$$

and $k_1, k_2 \in \mathbb{Z}$ are as in the previous definition.

The notation $\sim$ has the following interpretation: we pick k to be the unique integer such that the frequency variable ξ belongs to the annulus $B_{k+1}^* \setminus B_k^*$. We would like to point out that, for some general expansive matrix A, we need to require estimates that hold uniformly after re-scaling to scale zero. This is intuitively clear, due to the definition of the quasi-norm induced by the adjoint matrix. As we shall soon see, however, this apparently small detail will translate into certain technical difficulties in our proofs.

When our symbol is x-independent, we will again refer to it as multiplier and simply write $\dot{S}_\gamma^m(A)$ for the corresponding class. At a first glance, Definition 1.1 might seem rather obscure. Nevertheless, it is not hard to see that in the isotropic case, that is when $A = 2I_n$ (I_n is the $n \times n$ identity

matrix), Definition 1.1 recovers the isotropic class $\dot{S}^m_{\gamma,\delta}$. Indeed, in this case we have $\rho_{A^*}(\xi) = |\xi|^n$ and we simply need to observe that our uniform estimates (1.3) are those in (1.1) written for $|\xi| \sim 2^k$. That is, we have that the isotropic class $\dot{S}^m_{\gamma,\delta}$ coincides with $\dot{S}^{m/n}_{\gamma,\delta}(2I_n)$. Note that the rescaling of parameter m by a factor of $1/n$ is an artifact of our definition of a quasi-norm. To be consistent with the isotropic definition of this class of symbols, one must require that the quasi-norm associated to A^* satisfies the homogeneity condition $\rho_{A^*}(A^*\xi) = |\det A|^{1/n}\rho_{A^*}(\xi)$ instead. When $A = 2I_n$ this leads to the quasi-norm $\rho_{A^*}(\xi) = |\xi|$ and the Definition 1.1 yields the isotropic class $\dot{S}^m_{\gamma,\delta}$.

More generally, suppose that the dilation A is diagonal

$$A = \begin{pmatrix} \lambda^{a_1} & 0 & \cdots & 0 \\ 0 & \lambda^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^{a_n} \end{pmatrix},$$

where $\lambda > 1$, $a_1, \ldots, a_n > 0$ and $a_1 + \cdots + a_n = na$. Consider

$$\rho_A(x_1, \ldots, x_n) = (|x_1|^{2/a_1} + \cdots + |x_n|^{2/a_n})^{a/2}.$$

It is easy to check that ρ_A is a quasi-norm associated with the dilation A. In particular, we have the homogeneity condition

$$\rho_A(Ax) = \rho_A(\lambda^{a_1}x_1, \ldots, \lambda^{a_n}x_n) = \lambda^a \rho_A(x) = |\det A|^{1/n}\rho_A(x).$$

Alternately, we could have chosen

$$\rho_A(x_1, \ldots, x_n) = \max_{1 \le j \le n} |x_j|^{a/a_j}.$$

Pick now $\xi \in B^*_{k+1} \setminus B^*_k$, for some $k \in \mathbb{Z}$, that is $\rho_{A^*}(\xi) \sim \lambda^{ak}$. Then

$$|\partial^\alpha_x \partial^\beta_\xi [\sigma(A^{-k_1}\cdot, (A^*)^{k_2}\cdot)](A^{k_1}x, (A^*)^{-k_2}\xi)|$$
$$= \lambda^{-k_1 \sum_{j=1}^n a_j\alpha_j + k_2 \sum_{j=1}^n a_j\beta_j} |\partial^\alpha_x \partial^\beta_\xi \sigma(x, \xi)|.$$

Therefore, using (1.4), we see that estimates (1.3) take the more familiar form

(1.5) $$|\partial^\alpha_x \partial^\beta_\xi \sigma(x, \xi)| \lesssim C_{\alpha,\beta}[\rho_{A^*}(\xi)]^{m+\delta\|\alpha\|-\gamma\|\beta\|},$$

where we denoted

$$\|\alpha\| = \frac{1}{a}\sum_{j=1}^n a_j\alpha_j, \qquad \|\beta\| = \frac{1}{a}\sum_{j=1}^n a_j\beta_j.$$

Estimates (1.5) define the so-called *homogeneous class* $\dot{S}^m_{a;\gamma,\delta}$; the corresponding nonhomogeneous version of this class, $S^m_{a;\gamma,\delta}$, was previously investigated in [13, 18, 19, 29, 30]. Our definition has the following advantage: for a general matrix A, say one that has some non-trivial Jordan blocks, the action of A on $\mathbb{R}^n$ could be rather complex, and the diagonal version employed by these authors does not capture the anisotropy of all directions. We also recover the nonhomogeneous class introduced by these authors with a straightforward adaptation of the previous definition. At least in the diagonal case, this definition is powerful enough to recover known properties of boundedness, symbolic calculus and microlocal analysis of classical Hörmander classes of symbols; see again [13, 18, 19, 29, 30] and the references therein. The relevant boundedness properties of certain nonhomogeneous classes of symbols associated to a general expansive matrix A are investigated in our complementary work [1]. The anisotropic nonhomogeneous class of symbols is simply the smoothed out version at $\xi = 0$ of the homogeneous one.

Definition 1.2. We say that a bounded symbol $\sigma(x,\xi)$ belongs to the *nonhomogeneous anisotropic class* $S^m_{\gamma,\delta}(A)$ if it satisfies the estimates

$$(1.6) \qquad |\partial_x^\alpha \partial_\xi^\beta [\sigma(A^{-k_1}\cdot, (A^*)^{k_2}\cdot)](A^{k_1}x, (A^*)^{-k_2}\xi)| \leq C_{\alpha,\beta}(1 + \rho_{A^*}(\xi))^m,$$

for all multi-indices α, β and $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Here, $k_1, k_2 \in \mathbb{Z}$ are given by (1.4), where $k \in \mathbb{N}$ is such that $1 + \rho_{A^*}(\xi) \sim |\det A|^k$.

Associated to any symbol $\sigma(x,\xi)$ we have a pseudodifferential operator

$$(1.7) \qquad (\sigma(x,D)f)(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi;$$

here, $\widehat{f} = \mathcal{F}f$ denotes the Fourier transform of f. When the symbol is x-independent, we simply write $\sigma(D)$ and refer to it as a multiplier operator. Because of the $\xi = 0$ singularity of the symbol, it is natural to consider the operator $\sigma(x,D)$ initially defined on the subspace $\mathcal{S}_0(\mathbb{R}^n)$ of the space of Schwartz function $S(\mathbb{R}^n)$, consisting of all functions whose Fourier transform vanishes to infinite order at zero. Moreover, we can show that for any $\sigma \in \dot{S}^m_{\gamma,\delta}(A)$, $\sigma(x,D)$ maps $\mathcal{S}_0$ continuously to $\mathcal{S}$. We postpone the proof of this fact, which requires some additional notation specific to the anisotropic setting, until Section 4; see Lemma 4.9. It is also well known that $\mathcal{S}$ is dense in $L^p, 1 \leq p < \infty$, and $\mathcal{S} \cap H^p$ is dense in H^p, $0 < p \leq 1$; a similar statement holds for $\mathcal{S}_0$. For the appropriate definitions of these spaces and further properties, we refer again to the monograph [3].

The remainder of this paper will be concerned with *homogeneous* multipliers or pseudodifferential symbols, therefore allowing for a singularity at $\xi = 0$; we reiterate that the definition we provided is appropriate for any expansive matrix, not just for a diagonal one. We investigate the relevant properties of certain homogeneous classes of multipliers or symbols in this anisotropic setting, with the main goal of extending the classical isotropic results.

Our paper is organized as follows. In Sections 2 and 3 we revisit the anisotropic Mihlin and Hörmander multipliers, and provide alternate proofs of their boundedness on Lebesgue and Hardy spaces. Our approach is simpler than the one in [22], mainly because we can appeal now to the well established Calderón-Zygmund theory on spaces of homogeneous type. In particular, the continuity results for these multipliers follow immediately once we show that their Schwartz kernels satisfy anisotropic Calderón-Zygmund estimates. In Section 4, we prove that pseudodifferential operators with symbols in $\dot{S}^0_{1,1}(A)$ have anisotropic Calderón-Zygmund kernels. Then, by making use of wavelets techniques, we show that operators with symbols in the exotic classes $\dot{S}^m_{1,1}(A)$ are bounded on weighted anisotropic Besov and Triebel-Lizorkin spaces. This extends the corresponding isotropic results of Grafakos and Torres [15].

2. Anisotropic Mihlin multipliers

In analogy with its isotropic counterpart, we will define the *anisotropic Mihlin class* $\dot{S}^0_1(A)$. A multiplier $\sigma \in \dot{S}^0_1(A)$ satisfies

$$(2.1) \qquad |\partial^\beta_\xi[\sigma((A^*)^k \cdot)]((A^*)^{-k}\xi)| \leq C_\beta,$$

for all multi-indices β, for all $\xi \in \mathbb{R}^n \setminus \{0\}$, and $k \in \mathbb{Z}$ such that $\rho_{A^*}(\xi) \sim b^k = |\det A|^k$. In particular, (2.1) implies that σ is a bounded function. Note that in the isotropic case $A = 2I_n$, the condition (2.1) takes the familiar form

$$|\partial^\beta_\xi \sigma(\xi)| \leq C_\beta |\xi|^{-|\beta|} \qquad \text{for all } \beta.$$

Example 2.1. Consider the following simple partial differential equation in the variable function $u(x_1, x_2)$ on $\mathbb{R}^2$:

$$P(\partial)u = Q(\partial)f,$$

where $f(x_1, x_2)$ is some given Schwartz function on $\mathbb{R}^2$, and

$$P(\partial) = \partial^6_{x_1} + \partial^2_{x_2} + \partial^6_{x_2}, Q(\partial) = \partial^6_{x_1} + \partial^6_{x_2}.$$

By taking the Fourier transform on both sides of this equation, we obtain

$$(\xi_1^6 + \xi_2^2 + \xi_2^6)\widehat{u} = (\xi_1^6 + \xi_2^6)\widehat{f}$$

which gives (by taking now the inverse Fourier transform)

$$u = \mathcal{F}^{-1}(\sigma\widehat{f}).$$

Here,

$$\sigma(\xi_1, \xi_2) = \frac{\xi_1^6 + \xi_2^6}{\xi_1^6 + \xi_2^2 + \xi_2^6}$$

is a typical example of homogeneous multiplier to which the anisotropic setting seems to be more appropriate than the isotropic one. This is despite the fact that σ does not satisfy any obvious scaling property.

Indeed, a straightforward exercise verifies that $\sigma \in \dot{S}_1^0(A)$, where

$$A = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{pmatrix}.$$

More precisely, we can check that estimates (1.5) hold with $a_1 = 1/2, a_2 = 3/2, a = 1, n = 2$, i.e.,

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\sigma(\xi_1, \xi_2)| \lesssim \rho_{A^*}(\xi_1, \xi_2)^{-\|(\alpha_1, \alpha_2)\|},$$

where

$$\rho_{A^*}(\xi_1, \xi_2) = \max_{i=1,2}(|\xi_1|^2, |\xi_2|^{2/3}).$$

For example,

$$|\partial_{\xi_1}\sigma(\xi_1, \xi_2)| = \frac{6|\xi_1|^5|\xi_2|^2}{(\xi_1^6 + \xi_2^2 + \xi_2^6)^2} \lesssim \min_{i=1,2}(|\xi_1|^{-1}, |\xi_2|^{-1/3}) \lesssim \rho_{A^*}(\xi_1, \xi_2)^{-\|(1,0)\|}.$$

Let us briefly indicate how the estimate above was obtained. If $\rho_{A^*}(\xi_1, \xi_2) = |\xi_1|^2$, i.e., $|\xi_1|^3 \geq |\xi_2|$, then

$$\frac{|\xi_1|^5|\xi_2|^2}{(\xi_1^6 + \xi_2^2 + \xi_2^6)^2} \leq \frac{|\xi_1|^6|\xi_2|^2|\xi_1|^{-1}}{|\xi_1|^{12}} \leq \left(\frac{|\xi_2|}{|\xi_1|^3}\right)^2 |\xi_1|^{-1} \leq |\xi_1|^{-1}.$$

If $\rho_{A^*}(\xi_1, \xi_2) = |\xi_2|^{2/3}$, i.e., $|\xi_1|^3 \leq |\xi_2|$, then

$$\frac{|\xi_1|^5|\xi_2|^2}{(\xi_1^6 + \xi_2^2 + \xi_2^6)^2} \leq \frac{|\xi_2|^{5/3}|\xi_2|^2}{|\xi_2|^4} \leq |\xi_2|^{-1/3}.$$

Similar estimates hold for all multi-indices $|\alpha| \leq 2$. The results of Sections 2 and 3 will show that the L^p boundedness of the given function f is propagated to the solution u.

Consider then a multiplier operator $\sigma(D)$ with anisotropic Mihlin symbol $\sigma(\xi)$, initially defined on $\mathcal{S}_0$. The main result of this section is the following.

Theorem 2.2. *If $\sigma \in \dot{S}_1^0(A)$, then $\sigma(D)$ extends as a bounded operator*

    (i) $\sigma(D) : L^p \to L^p, p > 1$,

(ii) $\sigma(D) : L^1 \to L^{1,\infty}$,

(iii) $\sigma(D) : H^p \to H^p, 0 < p \le 1$,

(iv) $\sigma(D) : H^p \to L^p, 0 < p \le 1$.

In particular, in the isotropic case $A = 2I_n$, we recover the well known result about the Mihlin class that if $\sigma \in \dot{S}_1^0$, then $\sigma(D)$ is a bounded operator on all spaces $L^p, p > 1$.

Our proof will follow the classical approach. We refine the nice argument in Grafakos' book [14, Chapter 5], and show first that $\sigma(D)$ is a Calderón-Zygmund operator with respect to the dilation A and the canonical quasi-norm ρ_A.

Proposition 2.3. *Suppose that $\sigma \in \dot{S}_1^0(A)$. Then, the inverse Fourier transform $K = \mathcal{F}^{-1}\sigma$ is a Calderón-Zygmund kernel, that is*

$$(2.2) \qquad |\partial^\alpha[K(A^k\cdot)](A^{-k}x)| \le \frac{C_\alpha}{\rho_A(x)}$$

for some $C_\alpha > 0$, all multi-indices α, and all $x \in \mathbb{R}^n \setminus \{0\}$ such that $\rho_A(x) \sim |\det A|^k = b^k$.

With this fact in our hands, the proof of our theorem is immediate.

Proof of Theorem 2.2. Parts (i) and (ii) follow from the general Calderón-Zygmund theory on spaces of homogeneous type as explained, for example, in Stein's book [24], Chapter 1. This is due to the fact that condition (2.2) implies Hörmander condition (3.2); see Proposition 3.3.

Part (iii) is implied by [3, Theorem 9.8]. We only need to note that the multiplier operator is a Calderón-Zygmund singular integral of convolution type

$$\sigma(D)f = K * f$$

which is L^2 bounded, because $K \in L^\infty$. Moreover, as a convolution operator $\sigma(D)$ preserves vanishing moments, i.e., $(\sigma(D))^*(x^\alpha) = 0$ for all α. Finally, part (iv) is a consequence of [3, Theorem 9.9]. $\qquad\square$

To prove the proposition we will need the following elementary lemma, see [3, 4, 6].

Lemma 2.4. *Suppose A is an expansive matrix, and λ_- and λ_+ are any positive real numbers such that $1 < \lambda_- < \min_{\lambda \in \sigma(A)} |\lambda|$ and $\max_{\lambda \in \sigma(A)} |\lambda| < \lambda_+ < b = |\det A|$. Then, there exists $c > 0$ such that*

$$(2.3) \qquad (1/c)(\lambda_-)^j|x| \le |A^j x| \le c(\lambda_+)^j|x| \qquad \text{for } j \ge 0,$$

$$(2.4) \qquad (1/c)(\lambda_+)^j|x| \le |A^j x| \le c(\lambda_-)^j|x| \qquad \text{for } j \le 0.$$

Furthermore, if A is diagonalizable over $\mathbb{C}$, then we may take $\lambda_- = \min_{\lambda \in \sigma(A)} |\lambda|$ and $\lambda_+ = \max_{\lambda \in \sigma(A)} |\lambda|$.

We are ready to prove our proposition.

Proof of Proposition 2.3. We start by considering the Littlewood-Paley decomposition

$$\sum_{j \in \mathbb{Z}} \varphi((A^*)^j \xi) = 1, \qquad \xi \neq 0,$$

where φ is a C^∞ function compactly supported away from the origin. Define

$$\sigma_j(\xi) = \sigma(\xi)\varphi((A^*)^j \xi),$$

and note that $\sigma_j(\xi) \neq 0$ if and only $\rho_{A^*}(\xi) \sim b^{-j}$. Clearly $\sum_{j \in \mathbb{Z}} \sigma_j(\xi)$ converges boundedly to $\sigma(\xi)$ for $\xi \neq 0$; see, for example, Frazier, Jawerth and Weiss' monograph [12]. If we let

$$K_j(x) = \int_{\mathbb{R}^n} \sigma_j(\xi) e^{ix \cdot \xi} \, d\xi,$$

then $\sum_{j \in \mathbb{Z}} K_j$ converges to K in $\mathcal{S}'$.

Claim 1. *For all multi-indices α and for all $x \in \mathbb{R}^n$ such that $\rho_A(x) \sim 1$, we have*

$$(2.5) \qquad \sum_{j \in \mathbb{Z}} |\partial^\alpha K_j(x)| \leq C_\alpha,$$

for some positive constant C_α.

Notice again that we continue with our paradigm of "rescaling to scale zero". It turns out that it suffices to have the estimate (2.5) on the kernel K, since, as we will see later, we can perform a "rescaling back to scale k" argument and therefore obtain estimate (2.2).

Let us first prove Claim 1. We denote by $D_{A^k} f(\cdot) = f(A^k \cdot)$, the dilation by A^k for all $k \in \mathbb{Z}$ of some function f. We can write

$$K_j = \mathcal{F}^{-1}\sigma_j = \mathcal{F}^{-1}(D_{A^{*j}} D_{(A^*)^{-j}} \sigma_j) = b^{-j} D_{A^{-j}} \mathcal{F}^{-1}(D_{(A^*)^{-j}} \sigma_j).$$

Let

$$f_j = D_{(A^*)^{-j}} \sigma_j \text{ and } g_j = \mathcal{F}^{-1} f_j.$$

It is easy to check that $\{f_j\}_{j \in \mathbb{Z}}$ is a subset of the *normal class*

$$\mathcal{N}_R = \{\phi \in C^\infty : \operatorname{supp}(\phi) \subset \{\xi : R^{-1} < |\xi| < R\}, \|\partial^\alpha \phi\|_{L^\infty} \leq c_\alpha\},$$

where $R, c_\alpha > 0$ is some fixed collection of parameters depending only on φ. Consequently, $\{g_j\}_{j \in \mathbb{Z}}$ is a subset of

$$\mathcal{M} = \{\Phi \in C^\infty : \|\Phi\|_m = \sup_{|\alpha| \leq m} (1 + |x|)^m |\partial^\alpha \Phi(x)| \leq c_m\},$$

where $(c_m)_{m \in \mathbb{N}}$ is some fixed sequence depending on R and c_α.

Therefore, we have

$$(2.6) \qquad K = \sum_{j \in \mathbb{Z}} K_j = \sum_{j \in \mathbb{Z}} b^{-j} D_{A^{-j}}(g_j),$$

where $\{g_j\}_{j \in \mathbb{Z}} \subset \mathcal{M}$. Recall that we are trying to achieve the scale zero estimate (2.5) of K. We fix $s \in \mathbb{N}$ and try to find summable estimates on $|\partial^\alpha D_{A^{-j}}(g_j)(x)|, |\alpha| = s$, for $\rho_A(x) \sim 1$. We distinguish two cases.

Case 1. *"Estimates of flat functions": $j \geq 0$*

We use the chain rule and the fact that $\|g_j\|_s \leq c_s$ to conclude that

$$|\partial^\alpha D_{A^{-j}}(g_j)(x)| \leq C_s \|A^{-j}\|^s \sup_{|\beta|=s} |\partial^\beta g_j(A^{-j}x)| \leq C_s \|A^{-j}\|^s.$$

Clearly, the series $\displaystyle\sum_{j \geq 0} b^{-j} \|A^{-j}\|^s$ converges as a geometric series with ratio strictly less than 1.

Case 2. *"Decay estimates at infinity": $j < 0$*

We use again the chain rule and Lemma 2.4. For some convenient $N \in \mathbb{N}$ (to be chosen later), we can write

$$|\partial^\alpha D_{A^{-j}}(g_j)(x)| \leq C_{N,s} \|A^{-j}\|^s |A^{-j}x|^{-N} \leq C_{N,s} \lambda_+^{-js} \lambda_-^{jN}.$$

Note that the second inequality implicitly assumes that we work at scale zero, that is $\rho_A(x) \sim 1$. We also used the fact that $g_j \in \mathcal{M}$. If we now recall that $1 < \lambda_- < \lambda_+ < b$, we can choose N such that

$$b^{-1} \lambda_+^{-s} \lambda_-^N < 1,$$

which guarantees the convergence of the geometric series $\displaystyle\sum_{j < 0} b^{-j} \lambda_+^{-js} \lambda_-^{jN}$.

Hence, by combining the two summable estimates, we conclude that the scale-zero estimate (2.5) holds:

$$\sum_{j \in \mathbb{Z}} |\partial^\alpha K_j(x)| \leq C_s, \qquad |\alpha| = s.$$

In order to finish the proof, we only need to prove the following

Claim 2. *If K satisfies (2.5), then K also satisfies (2.2).*

We use a rather natural rescaling argument. For a fixed $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$ such that $\rho_A(x) \sim b^k$, we need to estimate

$$\partial^\alpha [K(A^k \cdot)](A^{-k}x) = \partial^\alpha (D_{A^k} K)(A^{-k}x).$$

Note that

$$D_{A^k} K = \sum_{j \in \mathbb{Z}} b^{-j} D_{A^{k-j}}(g_j) = b^{-k} \sum_{j \in \mathbb{Z}} b^{-j} D_{A^{-j}}(\tilde{g}_j),$$

where $\tilde{g}_j = g_{j+k} \in \mathcal{M}$. Consequently, since $\rho_A(A^{-k}x) \sim 1$ (i.e, we are back at scale zero!), we can repeat the same argument following (2.6) and conclude that

$$|\partial^\alpha [K(A^k \cdot)](A^{-k}x)| \leq C_\alpha b^{-k} \lesssim \frac{C_\alpha}{\rho_A(x)}.$$

This shows (2.2) and the proof is complete. $\qquad\square$

3. ANISOTROPIC HÖRMANDER MULTIPLIERS

In this section we revisit the class of anisotropic Hörmander multipliers introduced by Rivière [22]. We show that in the setting of expansive dilations this class corresponds to anisotropic convolution-type Calderón-Zygmund operators. As a consequence, we obtain an alternative proof of L^p boundedness of anisotropic Hörmander multipliers to the original approach of Rivière.

Definition 3.1. Let σ be a bounded function on $\mathbb{R}^n \setminus \{0\}$. We say that σ is an *anisotropic Hörmander multiplier of order M* (associated to the matrix A) if it is M-times differentiable and there exists $C_\alpha > 0$ such that

$$(3.1) \qquad \fint_{B_1^* \setminus B_0^*} |\partial_\xi^\alpha (\sigma(A^{*k} \cdot))(\xi)|^2 \, d\xi \leq C_\alpha,$$

for all multi-indices α, $|\alpha| \leq M$, and $k \in \mathbb{Z}$ such that $\rho_{A^*}(\xi) \sim b^k = |\det A|^k$.

Note that in the isotropic case $A = 2I_n$, the condition (3.1) takes the more familiar form

$$\sup_{k \in \mathbb{Z}} 2^{k(-n+2|\alpha|)} \int_{2^k < |\xi| < 2^{k+1}} |\partial_\xi^\alpha \sigma(\xi)|^2 d\xi \leq C_\alpha \qquad \text{for all } |\alpha| \leq M.$$

We remark immediately that any Mihlin multiplier is a Hörmander multiplier. Indeed, if $\sigma \in \dot{S}_1^0(A)$, then

$$\int_{B_1^* \setminus B_0^*} |\partial^\alpha (\sigma(A^{*k} \cdot))(\xi)|^2 \, d\xi = \int_{B_{k+1}^* \setminus B_k^*} |\partial^\alpha (\sigma(A^{*k} \cdot))((A^*)^{-k}\xi)|^2 b^{-k} \, d\xi$$

$$\leq \int_{B_{k+1}^* \setminus B_k^*} C_\alpha b^{-k} \, d\xi \leq C_\alpha.$$

The constant C_α is of course the one appearing in the estimates (2.1) that define the class $\dot{S}_1^0(A)$.

Conversely, suppose that σ is a Hörmander multiplier of sufficiently large order M. Then, by the Sobolev imbedding theorem, $\partial^\beta (\sigma(A^{*k} \cdot))(\xi)$ are bounded on $B_1^* \setminus B_0^*$ for all multi-indices $|\beta| < M - n/2$. Thus, σ satisfies Mihlin estimates (2.1) up to that order. Thus, one could easily deduce an analogue of Theorem 2.2 for Hörmander multipliers with sufficiently large

order M. Instead, we will show the following more concrete result general-izing the isotropic Hörmander Multiplier Theorem, see [14, Theorem 5.2.7], which can also be deduced from [22, Theorem II.1.2].

Theorem 3.2. *If σ is an anisotropic Hörmander multiplier of order $\lfloor n/2 \rfloor + 1$, then $\sigma(D)$ extends as a bounded operator*
   (i) $\sigma(D) : L^p \to L^p, p > 1$,
   (ii) $\sigma(D) : L^1 \to L^{1,\infty}$.

We follow the strategy outlined in the previous section. The Calderón-Zygmund theory for spaces of homogeneous type immediately yields Theorem 3.2, once we prove that $\sigma(D)$ has a Calderón-Zygmund kernel.

Proposition 3.3. *Let σ be an anisotropic Hörmander multiplier of order $\lfloor n/2 \rfloor + 1$. Then the inverse Fourier transform $K = \mathcal{F}^{-1}\sigma$ satisfies the anisotropic Hörmander condition*

$$(3.2) \qquad \sup_{y \neq 0} \int_{\rho_A(x) \geq 2c\rho_A(y)} |K(x-y) - K(x)|\, dx \leq C,$$

for some constants $c > 1$ and $C > 0$.

Proof. We start by observing that the annulus $B_1^* \setminus B_0^*$ can be replaced by any other annulus $B_{i+1}^* \setminus B_i^*$ in Definition 3.1. Indeed,

$$\int_{B_{i+1}^* \setminus B_i^*} |\partial^\alpha(\sigma(A^{*k}\cdot))(\xi)|^2\, d\xi = b^i \int_{B_1^* \setminus B_0^*} |\partial^\alpha(\sigma(A^{*k}\cdot))(A^{*i}\xi)|^2\, d\xi$$

$$(3.3) \qquad \leq Cb^i \|(A^*)^{-i}\|^{|\alpha|} \sum_{|\beta|=|\alpha|} \int_{B_1^* \setminus B_0^*} |\partial^\beta(\sigma((A^*)^{k+i}\cdot))(\xi)|^2\, d\xi$$

$$\leq Cb^i \|(A^*)^{-i}\|^{|\alpha|} \sum_{|\beta|=|\alpha|} C_\beta.$$

With the notation in the proof of Proposition 2.3, we have

$$\sigma_j(\xi) = \sigma(\xi)\varphi(A^{*j}\xi),$$

so that $\sigma_j(\xi) \neq 0$ if and only if $\rho_{A^*}(\xi) \sim b^j = |\det A|^j$. Let

$$K_j = b^{-j} D_{A^{-j}} \mathcal{F}^{-1}(D_{(A^*)^{-j}}\sigma_j).$$

By the product rule and the fact that $\operatorname{supp}\hat{\varphi} \subset B_R^* \setminus B_{-R}^*$ for some $R \in \mathbb{N}$, we have

$$\int_{\mathbb{R}^n} |\partial^\alpha D_{(A^*)^{-j}}\sigma_j(\xi)|^2\, d\xi \lesssim \int_{B_R^* \setminus B_{-R}^*} |\partial^\alpha D_{(A^*)^{-j}}\sigma(\xi)|^2\, d\xi$$

$$(3.4)$$
$$= \sum_{i=-R}^{R-1} \int_{B_{i+1}^* \setminus B_i^*} |\partial^\alpha D_{(A^*)^{-j}}\sigma(\xi)|^2\, d\xi \lesssim \sum_{|\beta|=|\alpha|} C_\beta,$$

for all $|\alpha| \leq M := \lfloor n/2 \rfloor + 1$, where in the last step we used (3.3). From (3.4) we infer that

$$\int_{\mathbb{R}^n} |\mathcal{F}^{-1}(D_{(A^*)^{-j}}\sigma_j)(x)(1+|x|)^M|^2 \, dx \leq C_M,$$

which is equivalent to

$$(3.5) \qquad \int_{\mathbb{R}^n} |b^j K_j(A^j x)(1+|x|)^M|^2 \, dx \leq C_M.$$

Fix now $0 < \epsilon < M - n/2$. From the Cauchy-Schwarz inequality and (3.5) we see that, for all $j \in \mathbb{Z}$, we have

$$(3.6)$$
$$\int_{\mathbb{R}^n} |K_j(x)|(1+|A^{-j}x|)^\varepsilon \, dx = b^j \int_{\mathbb{R}^n} |K_j(A^j x)|(1+|x|)^\epsilon \, dx$$
$$\leq \left(\int_{\mathbb{R}^n} |b^j K_j(A^j x)(1+|x|)^M|^2 \, dx \right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|x|)^{2\epsilon-2M} \, dx \right)^{1/2} \leq C_\epsilon.$$

Likewise, the estimate

$$\int_{\mathbb{R}^n} |\xi^\beta \partial^\alpha (D_{(A^*)^{-j}}\sigma_j(\xi))|^2 \, d\xi \leq C_{\alpha,\beta},$$

yields

$$b^j \int_{\mathbb{R}^n} |\partial^\beta (D_{A^j} K_j)(x)(1+|x|)^M|^2 \, dx \leq C_{M,\beta}$$

and a similar argument to the one proving estimate (3.6) yields the following general estimates on the derivatives of the localized kernels:

$$(3.7) \qquad b^j \int_{\mathbb{R}^n} |\partial^\beta (D_{A^j} K_j)(x)|(1+|x|)^\epsilon \, dx \leq C_{\beta,\epsilon},$$

for all $j \in \mathbb{Z}$ and all multi-indices β. Using the estimates above, we see that $\sum_{j \in \mathbb{Z}} K_j(x)$ converges to some function $K(x)$ for all $x \neq 0$; this being the case, the function $K(x)$ coincides with the distribution $\mathcal{F}^{-1}\sigma$ for $x \neq 0$. Indeed, since for $j \geq 0$ we have $|K_j(x)| \leq Cb^{-j}$, we immediately conclude that $\sum_{j \geq 0} K_j(x)$ is convergent. On the other hand, (3.6) implies that for any $\delta > 0$ we have

$$\int_{|x| \geq \delta} \sum_{j < 0} |K_j(x)| \, dx < \infty,$$

and this implies that the function $\sum_{j < 0} K_j(x)$ is finite almost everywhere away from the origin.

To prove now that K satisfies the anisotropic Hörmander condition (3.2), it is sufficient to show that for all $y \neq 0$, we have

$$\sum_{j \in \mathbb{Z}} \int_{\rho_A(x) \geq 2c\rho_A(y)} |K_j(x-y) - K_j(x)| \leq C.$$

For a fixed $y \neq 0$, let $k \in \mathbb{Z}$ (fixed) be such that $\rho_A(y) \sim b^k$. We will break down our summation into two sums, one over $j > k$ and one over $j \leq k$.

By appropriately choosing $c > 1$ (this choice being determined by the triangle inequality satisfied by the quasi-norm ρ_A) and using again (3.6), we can write

$$
\begin{aligned}
\sum_{j>k} \int_{\rho_A(x) \geq 2c\rho_A(y)} |K_j(x-y) - K_j(x)|\, dx &\leq \sum_{j>k} \int_{\rho_A(x) \geq \rho_A(y)} 2|K_j(x)| \\
&= 2 \sum_{j>k} \int_{\rho_A(x) \geq \rho_A(y)} |K_j(x)| \frac{(1 + \rho_A(A^{-j}x))^\epsilon}{(1 + \rho_A(A^{-j}x))^\epsilon}\, dx \\
&\leq C_\epsilon \sum_{j>k} \sup_{x:\rho_A(x) \geq \rho_A(y)} (1 + \rho_A(A^{-j}x))^{-\epsilon} \\
&\leq C_\epsilon \sum_{j>k} (1 + \rho_A(A^{-j}y))^{-\epsilon} \leq C_\epsilon \sum_{j>k} b^{(j-k)\epsilon} \lesssim 1.
\end{aligned}
$$

(3.8)

To estimate the summation over the indices $j \leq k$, we will use the Mean Value Theorem and estimate (3.7). We have

$$
\begin{aligned}
\int_{\rho_A(x) \geq 2c\rho_A(y)} &|K_j(x-y) - K_j(x)|\, dx \\
&= b^j \int_{\rho_A(x) \geq 2cb^{k-j}} |D_{A^j} K_j(x - A^{-j}y) - D_{A^j} K_j(x)|\, dx \\
&\leq b^j |A^{-j}y| \int_{\rho_A(x) \geq 2cb^{k-j}} \int_0^1 |\nabla D_{A^j} K_j(x - \theta A^{-j}y)|\, dx d\theta \\
&\leq b^j |A^{-j}y| \int_0^1 \int_{\rho_A(x) \geq 2cb^{k-j}} |\nabla D_{A^j} K_j(x - \theta A^{-j}y)| \frac{(1 + |x - \theta A^{-j}y|)^\epsilon}{(1 + |x - \theta A^{-j}y|)^\epsilon}\, dx d\theta \\
&\leq C_{1,\epsilon} \frac{|A^{-j}y|}{(1 + |A^{-j}y|)^{-\epsilon}}.
\end{aligned}
$$

Summing over $j \leq k$ and taking $\epsilon < 1$, we conclude that

(3.9) $$ \sum_{j \leq k} \int_{\rho_A(x) \geq 2c\rho_A(y)} |K_j(x-y) - K_j(x)|\, dx \lesssim C_{1,\epsilon} \sum_{j \leq k} b^{(j-k)(1-\epsilon)} \lesssim 1. $$

By combining estimates (3.8) and (3.9), we obtain (3.2). The proof is complete. $\qquad\square$

4. THE CLASS $\dot{S}_{1,1}^0(A)$

It is well known [24, p. 267] that in contrast with the isotropic Mihlin class of multipliers $\dot{S}_1^0$, the isotropic class of x-dependent symbols $\sigma(x, \xi) \in \dot{S}_{1,0}^0$ does not yield L^p-bounded pseudo-differential operators

$$ (\sigma(x, D)f)(x) = \int_{\mathbb{R}^n} \sigma(x, \xi) \widehat{f}(\xi) e^{ix \cdot \xi}\, d\xi. $$

Recall that a symbol $\sigma \in \dot{S}^0_{1,0}(A)$ satisfies

$$|\partial_x^\alpha \partial_\xi^\beta [\sigma(\cdot, A^{*k}\cdot)](x, (A^*)^{-k}\xi)| \leq C_{\alpha,\beta},$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ and $k \in \mathbb{Z}$ is such that $\rho_{A^*}(\xi) \sim b^k$. The L^p-unboundedness is propagated to the more general anisotropic class $\dot{S}^0_{1,0}(A)$. In fact, for each expansive matrix A, one can construct an appropriate symbol σ for which $\sigma(x,D)$ is unbounded; our examples are simple extensions of the isotropic ones appearing, for example, in [15].

Example 4.1. Assuming that φ is a smooth bump supported away from the origin consider a sequence of smooth functions $(m_j(x))_{j \in \mathbb{Z}}$ defined on $\mathbb{R}^n$ such that

(4.1) $$\|\partial^\alpha [m_j(A^{-j}\cdot)]\|_\infty \leq C_\alpha \qquad \text{for all } \alpha.$$

Then, a straightforward calculation using Definition 1.1 shows that the symbol

(4.2) $$\sigma(x,\xi) = \sum_{j \in \mathbb{Z}} m_j(x)\varphi((A^*)^{-j}\xi)$$

belongs to the class $\dot{S}^0_{1,1}(A)$. Indeed, recall that a symbol $\sigma \in \dot{S}^0_{1,1}(A)$ if

(4.3) $$|\partial_x^\alpha \partial_\xi^\beta [\sigma(A^{-k}\cdot, A^{*k}\cdot)](A^k x, (A^*)^{-k}\xi)| \leq C_{\alpha,\beta},$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ and $k \in \mathbb{Z}$ is such that $\rho_{A^*}(\xi) \sim b^k$. The constraint that $\rho_{A^*}(\xi) \sim b^k$ reduces the series (4.2) to a finite sum over $|j - k| \leq R$, where R depends on the size of $\operatorname{supp}\varphi$. Then, (4.3) follows by the chain rule.

Example 4.2. Assume that φ is as in Example 2 and let $(v_j)_{j \in \mathbb{Z}}$ be a bounded sequence in $\mathbb{R}^n$. Define the symbol

$$\sigma_1(x,\xi) = \sum_{j \in \mathbb{Z}} e^{ix \cdot (A^*)^j v_j} \varphi((A^*)^{-j}\xi).$$

Since the functions $m_j(x) = e^{ix \cdot (A^*)^j v_j}$ satisfy (4.1) we have that $\sigma_1 \in \dot{S}^0_{1,1}(A)$. A similar argument shows that the symbol

$$\sigma_0(x,\xi) = \sum_{j=0}^\infty e^{ix \cdot (A^*)^j v_j} \varphi((A^*)^{-j}\xi)$$

is in the nonhomogeneous class $S^0_{1,1}(A)$. However, one can show that the operator $\sigma_0(x,D)$ is unbounded on L^2, see [1]. Thus, the L^2-boundedness of operators with symbols in $\dot{S}^0_{1,1}(A)$ also fails in general.

4.1. **Calderón-Zygmund estimates.** A natural approach for proving continuity results of pseudodifferential operators, when applicable, is via their singular integral realization. It turns out that the anisotropic homogeneous forbidden symbols $\sigma \in \dot{S}_{1,1}^0(A)$ enjoy a remarkable property: the Schwartz kernels associated to $\sigma(x, D)$ satisfy Calderón-Zygmund estimates. The isotropic analogue of this fact is due to Coifman and Meyer [9]. Combined with the failure of L^2 bounds, this shows that, in general, this class yields unbounded operators on all L^p spaces.

The x-dependence of the symbol $\sigma(x, \xi)$ implies that the operator $\sigma(x, D)$ is of non-convolution type. For $f \in \mathcal{S}$, we can write

$$(\sigma(x, D)f)(x) = \int_{\mathbb{R}^n} \tilde{K}(x, y) f(y) \, dy,$$

where $\tilde{K}(x, y) = K(x, x - y)$, and (in a distribution sense)

$$K(x, z) = \int_{\mathbb{R}^n} \sigma(x, \xi) e^{iz \cdot \xi} \, d\xi = \mathcal{F}_2^{-1} \sigma(x, z).$$

Theorem 4.3. *If $\sigma \in \dot{S}_{1,1}^0(A)$, then the corresponding kernel K satisfies the Calderón-Zygmund estimates*

(4.4) $$|\partial_x^\alpha \partial_z^\beta [K(A^k \cdot, A^k \cdot)](A^{-k} x, A^{-k} z)| \leq \frac{C_{\alpha,\beta}}{\rho_A(z)},$$

for $z \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ such that $\rho_A(z) \sim b^k$, for all multi-indices α, β, and some constants $C_{\alpha,\beta} > 0$.

As an immediate corollary of this result we obtain that the Schwartz kernel $\tilde{K}$ satisfies the *symmetric Calderón-Zygmund condition*

$$|\partial_x^\alpha \partial_y^\beta [\tilde{K}(A^k \cdot, A^k \cdot)](A^{-k} x, A^{-k} y)| \leq \frac{C_{\alpha,\beta}}{\rho_A(x - y)},$$

for $x, y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ such that $\rho_A(x - y) \sim b^k$.

Proof of Theorem 4.3. Our proof is a fine tune-up of the one given for Proposition 2.3. We start again by considering the Littlewood-Paley decomposition

$$\sum_{j \in \mathbb{Z}} \varphi((A^*)^j \xi) = 1, \qquad \xi \neq 0,$$

where φ is a C^∞ function compactly supported away from the origin. Define the symbols

$$\sigma_j(x, \xi) = \sigma(x, \xi) \varphi((A^*)^j \xi),$$

and note that $\sigma_j(x,\xi) \neq 0$ if and only $\rho_{A^*}(\xi) \sim b^{-j}$. Clearly $\sum\limits_{j \in \mathbb{Z}} \sigma_j(x,\xi)$ converges boundedly to $\sigma(x,\xi)$ for $\xi \neq 0$. Let

$$K_j(x,z) = \int_{\mathbb{R}^n} \sigma_j(x,\xi)e^{iz \cdot \xi}\, d\xi.$$

We have the following

Claim 1. *For all multi-indices α, β and for all $z \in \mathbb{R}^n$ such that $\rho_A(z) \sim 1$, we have*

(4.5) $$\sum_{j \in \mathbb{Z}} |\partial_x^\alpha \partial_z^\beta K_j(x,z)| \leq C_{\alpha,\beta}$$

for some positive constant $C_{\alpha,\beta}$.

The proof of Claim 1 follows now a familiar path. For a function of two variables $f(x,y)$, we denote by $D_{A,B}f(\cdot,\cdot) = f(A\cdot, B\cdot)$, the dilation by A in the first variable and by B in the second one. Thus, we can write

$$K_j = b^{-j} D_{A^{-j},A^{-j}} \mathcal{F}^{-1}(D_{A^j,(A^*)^{-j}} \sigma_j).$$

Let

$$f_j = D_{A^j,(A^*)^{-j}} \sigma_j \text{ and } g_j = \mathcal{F}^{-1} f_j.$$

Then

(4.6) $$K_j = b^{-j} D_{A^{-j},A^{-j}} g_j.$$

It is easy to check that $\{f_j\}_{j \in \mathbb{Z}}$ is a subset of the *normal class* $\mathcal{N}_R$ consisting of functions $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\operatorname{supp}(\phi) \subset \mathbb{R}^n \times \{\xi : R^{-1} < |\xi| < R\} \text{ and } |\partial_x^\alpha \partial_\xi^\beta \phi(x,\xi)| \leq c_{\alpha,\beta}, \forall x, \xi,$$

where $R, c_{\alpha,\beta} > 0$ is some fixed collection of parameters depending only on φ. Consequently, $\{g_j\}_{j \in \mathbb{Z}}$ is a subset of $\mathcal{M}$ which consists of functions $\Phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\|\Phi\|_{m,\alpha} = \sup_{|\beta| \leq m} \sup_{z \in \mathbb{R}^n} (1 + |z|)^m |\partial_x^\alpha \partial_z^\beta \Phi(x,z)| \leq c_{m,\alpha},$$

where $(c_{m,\alpha})$ is some fixed sequence depending on R and the constants $c_{\alpha,\beta}$.

Our stated goal is to achieve estimates (4.5). Based on the equality (4.6), if we fix $s_1 = |\alpha|, s_2 = |\beta| \in \mathbb{N}$, it suffices to prove summable estimates on $|\partial_x^\alpha \partial_z^\beta D_{A^{-j},A^{-j}} g_j(x,z)|$ for $\rho_A(z) \sim 1$. We split our analysis into two cases.

Case 1. *"Estimates of flat functions": $j \geq 0$*

We use the chain rule and the fact that $\|g_j\|_{s,\alpha} \le c_{s,\alpha}$ to conclude that:

$$|\partial_x^\alpha \partial_z^\beta g_j(A^{-j}\cdot, A^{-j}\cdot)(x,z)|$$
$$\le C_{s_1,s_2}\|A^{-j}\|^{s_1}\|A^{-j}\|^{s_2} \sup_{|\alpha_1|=s_1} \sup_{|\alpha_2|=s_2} |\partial_x^{\alpha_1}\partial_z^{\alpha_2} g_j(A^{-j}x, A^{-j}z)|$$
$$\le C_{s_1,s_2}\|A^{-j}\|^{s_1+s_2}.$$

Clearly, the series $\sum_{j\ge 0} b^{-j}\|A^{-j}\|^{s_1+s_2}$ converges as a geometric series with ratio strictly less than 1.

Case 2. *"Decay estimates at infinity": $j < 0$*

We use again the chain rule, Lemma 2.4, and the fact that $g_j \in \mathcal{M}$, to write for some sufficiently large N

$$|\partial_x^\alpha \partial_z^\beta g_j(A^{-j}\cdot, A^{-j}\cdot)(x,z)| \le C_{N,s_1,s_2}\|A^{-j}\|^{s_1+s_2}|A^{-j}z|^{-N}$$
$$\le C_{N,s_1,s_2}\lambda_+^{-j(s_1+s_2)}\lambda_-^{jN}.$$

Note that the second inequality implicitly assumes that we work at scale zero, that is $\rho_A(z) \sim 1$. Recall also that $1 < \lambda_- < \lambda_+ < b$. Therefore, we can choose N such that

$$b^{-1}\lambda_+^{-s_1-s_2}\lambda_-^N < 1,$$

which again guarantees the convergence of the series $\sum_{j<0} b^{-j}\lambda_+^{-j(s_1+s_2)}\lambda_-^{jN}$.

Hence, both cases yield summable estimates and we conclude that

$$\sum_{j\in\mathbb{Z}} |\partial_x^\alpha \partial_z^\beta K_j(x,z)| \le C_{s_1,s_2}, \qquad |\alpha| = s_1, |\beta| = s_2.$$

This, in turn, clearly implies the zero-scale estimates

(4.7) $$|\partial_x^\alpha \partial_z^\beta K(x,z)| \le C_{\alpha,\beta},$$

for all multi-indices α, β and all pairs (x,z) such that $\rho_A(z) \sim 1$.

Finally, we now use the standard re-scaling argument to prove the following claim.

Claim 2. *If K satisfies (4.7), then K also satisfies (4.4).*

If $\rho_A(z) \sim b^k$, for some $k \in \mathbb{Z}$ fixed, we need to estimate

$$\partial_x^\alpha \partial_z^\beta [K(A^k\cdot, A^k\cdot)](A^{-k}x, A^{-k}z) = \partial^\alpha (D_{A^k,A^k}K)(A^{-k}x, A^{-k}z).$$

Note that

$$D_{A^k,A^k}K = \sum_{j\in\mathbb{Z}} b^{-j}D_{A^{k-j},A^{k-j}}(g_j) = b^{-k}\sum_{j\in\mathbb{Z}} b^{-j}D_{A^{-j},A^{-j}}(\tilde{g}_j),$$

where $\tilde{g}_j = g_{j+k} \in \mathcal{M}$. Consequently, since $\rho_A(A^{-k}x) \sim 1$ (that is, we are back at scale zero), $b^k D_{A^k,A^k}K$ satisfies (4.7). Thus,

$$b^k|\partial_x^\alpha \partial_z^\beta [K(A^k\cdot, A^k\cdot)](A^{-k}x, A^{-k}z)| \le C_{\alpha,\beta}.$$

The proof is complete. $\qquad\square$

4.2. Anisotropic homogeneous Triebel-Lizorkin and Besov spaces.

The study of pseudodifferential operators with exotic isotropic symbols $\dot{S}_{1,1}^0$ has received lots of attention. For example, using wavelet techniques, Torres [25, 26] and Grafakos-Torres [15] have studied the properties of this class of symbols on general spaces of smooth functions. These works suggest that the right setting for studying the boundedness of pseudodifferential operators with symbols in the forbidden class $\dot{S}_{1,1}^0(A)$ is provided by the anisotropic homogeneous Triebel-Lizorkin spaces $\dot{\mathbf{F}}_p^{\alpha,q} = \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ or Besov spaces $\dot{\mathbf{B}}_p^{\alpha,q} = \dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$. Before stating our result, we recall the definitions and the molecular characterizations of these spaces. For an extensive treatment of these spaces, the reader is referred to [4, 5, 6]; see also [2].

Definition 4.4. Let μ be ρ_A-*doubling* measure on $\mathbb{R}^n$, i.e., there exists $\beta = \beta(\mu) > 0$ such that

$$(4.8) \qquad \mu(x + B_{k+1}) \leq |\det A|^\beta \mu(x + B_k) \qquad \text{for all } x \in \mathbb{R}^n, \ k \in \mathbb{Z}.$$

The smallest such β is called a doubling constant of μ.

For $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, we define the *anisotropic Triebel-Lizorkin space* $\dot{\mathbf{F}}_p^{\alpha,q} = \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ as the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$(4.9) \qquad \|f\|_{\dot{\mathbf{F}}_p^{\alpha,q}} = \left\| \left(\sum_{j \in \mathbb{Z}} (|\det A|^{j\alpha} |f * \varphi_j|)^q \right)^{1/q} \right\|_{L^p(\mu)} < \infty,$$

where $\varphi_j(x) = |\det A|^j \varphi(A^j x)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (4.10), (4.11)

$$(4.10) \qquad \operatorname{supp} \hat{\varphi} := \overline{\{\xi \in \mathbb{R}^n : \hat{\varphi}(\xi) \neq 0\}} \subset [-\pi, \pi]^n \setminus \{0\},$$

$$(4.11) \qquad \sup_{j \in \mathbb{Z}} |\hat{\varphi}((A^*)^j \xi)| > 0 \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

Likewise, we define the *anisotropic Besov space* $\dot{\mathbf{B}}_p^{\alpha,q} = \dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ as the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$(4.12) \qquad \|f\|_{\dot{\mathbf{B}}_p^{\alpha,q}} = \left(\sum_{j \in \mathbb{Z}} (|\det A|^{j\alpha} \|f * \varphi_j\|_{L^p(\mu)})^q \right)^{1/q} < \infty.$$

Recall that $\mathcal{S}'/\mathcal{P}$ can be identified with the space of all continuous functionals on $\mathcal{S}_0$. In [4, 5, 6] it is proved that the inclusion maps $\dot{\mathbf{F}}_p^{\alpha,q} \hookrightarrow \mathcal{S}'/\mathcal{P}$, $\dot{\mathbf{B}}_p^{\alpha,q} \hookrightarrow \mathcal{S}'/\mathcal{P}$ are continuous, and therefore these spaces are quasi-Banach. Moreover, the definitions of anisotropic $\dot{\mathbf{F}}_p^{\alpha,q}$ and $\dot{\mathbf{B}}_p^{\alpha,q}$ spaces are independent of φ. Let $\mathcal{Q}$ be the collection of all *dilated cubes*

$$\mathcal{Q} = \{A^{-j}([0,1]^n + k) : j \in \mathbb{Z}, \ k \in \mathbb{Z}^n\}$$

adapted to the action of a dilation A. For $Q = Q_{j,k} = A^{-j}([0,1]^n + k)$ define scale$(Q) = -j$ and its "lower-left" corner $x_Q = A^{-j}k$. If φ is a function on $\mathbb{R}^n$, we define its wavelet system as

$$(4.13) \qquad \varphi_Q(x) = |Q|^{1/2}\varphi(A^j x - k), \qquad Q = A^{-j}([0,1]^n + k) \in \mathcal{Q}.$$

Obviously, if $A = 2I_n$ we obtain the usual collection of *dyadic cubes*.

Definition 4.5. The discrete Triebel-Lizorkin sequence space $\dot{\mathbf{f}}_p^{\alpha,q}(A, \mu)$ is defined as the collection of all complex-valued sequences $s = \{s_Q\}_{Q \in \mathcal{Q}}$ such that

$$(4.14) \qquad \|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}} = \left\| \left(\sum_{Q \in \mathcal{Q}} (|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q)^q \right)^{1/q} \right\|_{L^p(\mu)} < \infty,$$

where $\tilde{\chi}_Q = |Q|^{-1/2}\chi_Q$ is the L^2-normalized characteristic function of the dilated cube Q. Likewise, the discrete Besov sequence space $\dot{\mathbf{b}}_p^{\alpha,q} = \dot{\mathbf{b}}_p^{\alpha,q}(A, \mu)$ is the collection of all complex-valued sequences $s = \{s_Q\}_{Q \in \mathcal{Q}}$ such that

$$\|s\|_{\dot{\mathbf{b}}_p^{\alpha,q}} = \left(\sum_{j \in \mathbb{Z}} \left\| \sum_{Q \in \mathcal{Q},\ \text{scale}(Q)=j} |Q|^{-\alpha}|s_Q|\tilde{\chi}_Q \right\|_{L^p(\mu)}^q \right)^{1/q} < \infty.$$

We will also need a definition of anisotropic molecules introduced in [4, 6] generalizing isotropic molecules of Frazier and Jawerth [11, 12]. These molecules come in two flavors depending whether they are used in the analysis or synthesis transforms.

Definition 4.6. Suppose $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is a ρ_A-doubling measure with a doubling constant $\beta \ge 1$. Let $0 < \zeta_- \le 1/n \le \zeta_+ < 1$ be the parameters measuring the eccentricity of the dilation A:

$$\zeta_+ := \frac{\ln \lambda_+}{\ln |\det A|}, \qquad \zeta_- := \frac{\ln \lambda_-}{\ln |\det A|},$$

where λ_- and λ_+ are any positive real numbers such that

$$1 < \lambda_- < \min_{\lambda \in \sigma(A)} |\lambda| \le \max_{\lambda \in \sigma(A)} |\lambda| < \lambda_+ < |\det A|.$$

Let

$$(4.15) \qquad J = \begin{cases} \beta \max(1, 1/p, 1/q), & \text{for } \dot{\mathbf{F}}_p^{\alpha,q} \text{ spaces,} \\ \beta/p + \max(0, 1 - 1/p), & \text{for } \dot{\mathbf{B}}_p^{\alpha,q} \text{ spaces,} \end{cases}$$
$$N = \max(\lfloor (J - \alpha - 1)/\zeta_- \rfloor, -1).$$

We say that $\Psi_Q(x)$ is a *smooth synthesis molecule* for $\dot{\mathbf{F}}_p^{\alpha,q}$ (or $\dot{\mathbf{B}}_p^{\alpha,q}$) supported near $Q \in \mathcal{Q}$ with scale$(Q) = -j$ and $j \in \mathbb{Z}$, if there exist $M > J$

such that
(4.16)
$$|\partial^\gamma[\Psi_Q(A^{-j}\cdot)](x)| \leq \frac{|\det A|^{j/2}}{(1+\rho_A(x-A^j x_Q))^M} \qquad \text{for } |\gamma| \leq \lfloor \alpha/\zeta_- \rfloor + 1,$$

(4.17)
$$|\Psi_Q(x)| \leq \frac{|\det A|^{j/2}}{(1+\rho_A(A^j(x-x_Q)))^{\max(M,(M-\alpha)\zeta_+/\zeta_-)}},$$

(4.18)
$$\int x^\gamma \Psi_Q(x)dx = 0 \qquad \text{for } |\gamma| \leq N.$$

We say that $\Phi_Q(x)$ is a *smooth analysis molecule* for $\dot{\mathbf{F}}_p^{\alpha,q}$ (or $\dot{\mathbf{B}}_p^{\alpha,q}$) supported near $Q \in \mathcal{Q}$ with scale(Q) $= -j$ and $j \in \mathbb{Z}$, if there exists $M > J$ such that

(4.19)
$$|\partial^\gamma[\Phi_Q(A^{-j}\cdot)](x)| \leq \frac{|\det A|^{j/2}}{(1+\rho_A(x-A^j x_Q))^M} \qquad \text{for } |\gamma| \leq N+1,$$

(4.20)
$$|\Phi_Q(x)| \leq \frac{|\det A|^{j/2}}{(1+\rho_A(A^j(x-x_Q)))^{\max(M,1+\alpha\zeta_+/\zeta_-+M-J)}},$$

(4.21)
$$\int x^\gamma \Phi_Q(x)dx = 0 \qquad \text{for } |\gamma| \leq \lfloor \alpha/\zeta_- \rfloor.$$

We say that $\{\Phi_Q\}_{Q\in\mathcal{Q}}$ is a *family of smooth synthesis (analysis) molecules*, if each Φ_Q is a smooth synthesis (analysis) molecule supported near Q.

We emphasize that in the context of smooth molecules, Ψ_Q and Φ_Q are understood as some functions indexed by $Q \in \mathcal{Q}$ which are not necessarily given by (4.13). Nevertheless, if $\varphi \in \mathcal{S}$ has sufficiently many vanishing moments, then the wavelet system $\{\varphi_Q\}_Q$ is a family of smooth molecules (both for synthesis and analysis).

The following result about smooth molecular analysis and synthesis transforms was established in the setting of anisotropic $\dot{\mathbf{B}}_p^{\alpha,q}$ spaces [4, Theorems 5.5 and 5.7] and anisotropic $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces [5, Theorem 5.4]. Theorem 4.7 is simply a generalization of the corresponding isotropic results of Frazier and Jawerth [10, 11].

Theorem 4.7. *Suppose that A is an expansive matrix, $\alpha \in \mathbb{R}$, $0 < p,q \leq \infty$, and μ is a ρ_A-doubling measure. Then there exists a constant $C > 0$, such that*

(i) *If $\{\Psi_Q\}_Q$ is a family of smooth synthesis molecules for $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$, then*
$$\left\| \sum_{Q\in\mathcal{Q}} s_Q \Psi_Q \right\|_{\dot{\mathbf{F}}_p^{\alpha,q}} \leq C \|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}} \qquad \text{for all } s = \{s_Q\}_Q \in \dot{\mathbf{f}}_p^{\alpha,q}(A,\mu).$$

(ii) *If $\{\Phi_Q\}_Q$ is a family of smooth analysis molecules, then*
$$\|\{\langle f, \Phi_Q\rangle\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha,q}} \leq C \|f\|_{\dot{\mathbf{F}}_p^{\alpha,q}} \qquad \text{for all } f \in \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu).$$

Furthermore, the same result holds for Besov spaces $\dot{\mathbf{B}}_p^{\alpha,q}$.

4.3. **Boundedness on anisotropic** $\dot{\mathbf{F}}_p^{\alpha,q}$ **and** $\dot{\mathbf{B}}_p^{\alpha,q}$ **spaces.** Finally, we are ready to prove our anisotropic boundedness result extending the isotropic result of Grafakos and Torres [15].

Theorem 4.8. *Let* $\sigma \in \dot{S}_{1,1}^m(A)$, $0 < p, q < \infty$, *and*

$$(4.22) \quad (\sigma(x,D))^*(x^\gamma) = 0 \qquad for \ |\gamma| \leq N = \max(\lfloor (J - \alpha - 1)/\zeta_- \rfloor, -1).$$

Note that if $\alpha > J - 1$, *then* (4.22) *is trivially satisfied. Then, the pseudo-differential operator* $\sigma(x,D)$ *(a priori defined on* $\mathcal{S}_0$*) extends as a bounded operator*

(i) $\sigma(x,D) : \dot{\mathbf{F}}_p^{\alpha+m,q}(\mathbb{R}^n, A, \mu) \to \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu),$

(ii) $\sigma(x,D) : \dot{\mathbf{B}}_p^{\alpha+m,q}(\mathbb{R}^n, A, \mu) \to \dot{\mathbf{B}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu).$

In the statement of the theorem, we need specify how T^* acts on polynomials, where $T = \sigma(x,D)$. Using Lemma 4.9 and a formal duality relation $\langle T^*(x^\gamma), f \rangle = \langle T(f), x^\gamma \rangle$, the condition (4.22) means that $\langle \sigma(x,D)(f), x^\gamma \rangle = 0$ for all $f \in \mathcal{S}_0$ and $|\gamma| \leq N$. To prove Theorem 4.8, we adapt carefully the argument in [15] to the anisotropic setting. It is sufficient to prove that $\sigma(x,D)$ maps a wavelet system $\{\varphi_Q\}_Q$ into a family of smooth molecules $\{\Psi_Q\}_Q$ (modulo some scalar rescaling). As we shall see shortly, this can be achieved by some relatively easy computations. We need the following technical lemma.

Lemma 4.9. *Let* $\sigma \in \dot{S}_{\gamma,\delta}^m(A)$. *Then,* $\sigma(x,D)$ *maps* $\mathcal{S}_0$ *continuously into* $\mathcal{S}$.

Proof. The homogeneous anisotropic symbol estimate (1.3) combined with the chain rule and Lemma 2.4 implies that

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{|\alpha|,|\beta|} \rho_{A^*}(\xi)^m \cdot \begin{cases} (\lambda_+)^{k\delta|\alpha|}(\lambda_-)^{-k\gamma|\beta|} & \text{if } k \geq 0, \\ (\lambda_-)^{k\delta|\alpha|}(\lambda_+)^{-k\gamma|\beta|} & \text{if } k < 0, \end{cases}$$

for all multi-indices α, β, and $\xi \in \mathbb{R}^n \setminus \{0\}$, where $k \in \mathbb{Z}$ is such that $\rho_{A^*}(\xi) \sim b^k$. Thus, we can rewrite the above as

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{|\alpha|,|\beta|} \cdot \begin{cases} \rho_{A^*}(\xi)^{m+(\zeta_+)\delta|\alpha|-(\zeta_-)\gamma|\beta|} & \text{if } \rho_{A^*}(\xi) \geq 1, \\ \rho_{A^*}(\xi)^{m+(\zeta_-)\delta|\alpha|-(\zeta_+)\gamma|\beta|} & \text{if } \rho_{A^*}(\xi) < 1. \end{cases}$$

Applying [3, Lemma 3.2] this implies polynomial growth/decay estimates as $\xi \to 0$ and $\xi \to \infty$,

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{|\alpha|,|\beta|} \cdot \begin{cases} |\xi|^{(m+(\zeta_+)\delta|\alpha|-(\zeta_-)\gamma|\beta|)/\zeta_\pm} & \text{if } |\xi| \geq 1, \\ |\xi|^{(m+(\zeta_-)\delta|\alpha|-(\zeta_+)\gamma|\beta|)/\zeta_\mp} & \text{if } |\xi| < 1. \end{cases}$$

Here $\zeta_\pm$ means ζ_+ or ζ_- if the exponent is negative or positive, respectively, and similarly for $\zeta_\mp$. However, for the rest of the argument it is unimportant what the exact exponents are. That is, we shall only use the fact that

$$(4.23) \qquad |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta} \min(1, |\xi|^{-d_1}) \max(1, |\xi|^{d_2})$$

for some exponents $d_1 = d_1(\alpha,\beta) > 0$ and $d_2 = d_2(\alpha,\beta) > 0$ depending on multi-indices α and β. In fact, the rest of the proof follows directly the standard argument as in [15, Lemma 2.1]. One should emphasize here that we must impose a stronger topology on $\mathcal{S}_0$ than the induced topology of $\mathcal{S}_0$ as a closed subspace of the Schwartz class $\mathcal{S}$.

Indeed, let $f \in \mathcal{S}_0$ and $N \in \mathbb{N}$. Let Δ_ξ be the Laplacian in ξ variable. Using the identity

$$(I - \Delta_\xi)^N (e^{ix\cdot\xi}) = (1 + |x|^2)^N e^{ix\cdot\xi}$$

and the bound (4.23), the integration by parts yields

$$(\sigma(x,D)f)(x) = \int e^{ix\cdot\xi} \frac{(I - \Delta_\xi)^N}{(1 + |x|^2)^N} (\sigma(x,\xi)\hat{f}(\xi)) d\xi.$$

This formula works since $f \in \mathcal{S}_0$ and thus $\hat{f}$ and all of its partial derivatives vanish to infinity order at the origin. Moreover, the differentiation under the integral is allowed resulting in

$$(\partial_x^\alpha \sigma(x,D)f)(x) = \int (I - \Delta_\xi)^N \left(\partial_x^\alpha \left(\frac{e^{ix\cdot\xi}}{(1 + |x|^2)^N} \sigma(x,\xi) \right) \hat{f}(\xi) \right) d\xi$$

for any multi-index α. Applying the product rule, the bound (4.23), and the fact that $f \in \mathcal{S}_0$ yields

$$|(\partial_x^\alpha \sigma(x,D)f)(x)| \le C \frac{||f||_M}{(1 + |x|^2)^{N-|\alpha|}},$$

for sufficiently large M. Here,

$$||f||_M = \sup_{|\beta| \le M} \sup_{\xi \in \mathbb{R}^n} |\partial^\beta \hat{f}(\xi)|(|\xi|^M + |\xi|^{-M}) < \infty$$

are seminorms defining the topology of a locally convex space $\mathcal{S}_0$. Since α and N are arbitrary, the above shows that $\sigma(x,D) : \mathcal{S}_0 \to \mathcal{S}$ is continuous. $\qquad\square$

Proof of Theorem 4.8. Let $\varphi \in \mathcal{S}$ be such that

$$(4.24) \qquad \operatorname{supp} \hat{\varphi} \subset [-\pi, \pi]^n \setminus \{0\},$$

$$\sum_{j \in \mathbb{Z}} |\hat{\varphi}((A^*)^j \xi)|^2 = 1 \qquad \text{for all } \xi \ne 0.$$

Hence, φ is a Parseval wavelet for $L^2(\mathbb{R}^n)$. By the combination of [7, Lemma 2.12] with [6, Lemma 2.8] we have

$$f = \sum_{Q \in \mathcal{Q}} \langle f, \varphi_Q \rangle \varphi_Q \qquad \text{for any } f \in \mathcal{S}_0,$$

with the unconditional convergence in $\mathcal{S}$. Therefore, by Lemma 4.9 the action of the pseudodifferential operator $\sigma(x, D)$ can be expressed as

$$(4.25) \qquad \sigma(x, D)f = \sum_{Q \in \mathcal{Q}} \langle f, \varphi_Q \rangle \sigma(x, D)\varphi_Q \qquad \text{for } f \in \mathcal{S}_0.$$

Assume for a moment that there exists a constant $C > 0$ such that

$$(4.26) \qquad \sigma(x, D)\varphi_Q = C|Q|^{-m}\Psi_Q \qquad \text{for all } Q \in \mathcal{Q}$$

where $\{\Psi_Q\}_Q$ is family of smooth synthesis molecules for $\dot{\mathbf{F}}_p^{\alpha,q}$ or $\dot{\mathbf{B}}_p^{\alpha,q}$. Then, by Theorem 4.7(i) the analysis transforms $f \mapsto (\langle f, \varphi_Q \rangle)_Q$ is bounded as $\dot{\mathbf{F}}_p^{\alpha+m,q} \to \dot{\mathbf{f}}_p^{\alpha+m,q}$. Clearly, the multiplication map $(s_Q)_Q \mapsto (|Q|^{-m}s_Q)_Q$ is an isometry as $\dot{\mathbf{f}}_p^{\alpha+m,q} \to \dot{\mathbf{f}}_p^{\alpha,q}$. Thus, by (4.25), (4.26), and Theorem 4.7(ii) we have that $||\sigma(x, D)f||_{\dot{\mathbf{F}}_p^{\alpha,q}} \lesssim ||f||_{\dot{\mathbf{F}}_p^{\alpha+m,q}}$ for all $f \in \mathcal{S}_0$. Since $\mathcal{S}_0$ is a dense subspace of $\dot{\mathbf{F}}_p^{\alpha,q}$ if $p, q < \infty$, this yields the required conclusion. The same argument works for $\dot{\mathbf{B}}_p^{\alpha,q}$ spaces.

Thus, it remains to establish (4.26) by computing the action of $\sigma(x, D)$ on a fixed wavelet φ_Q associated to a dilated cube $Q = Q_{j,k}$. For $x \in \mathbb{R}^n, x \neq 0$, let $\tilde{x} = A^j x - k$, and $b = |\det A|$. We have the following sequence of equalities:

$$\begin{aligned}(\sigma(x, D)\varphi_Q)(x) &= \int \sigma(x, \xi)\widehat{\varphi}_Q(\xi)e^{ix\cdot\xi}\,d\xi \\ &= \int \sigma(x, \xi)b^{-j/2}\widehat{\varphi}((A^*)^{-j}\xi)e^{-ik\cdot(A^*)^{-j}\xi}e^{ix\cdot\xi}\,d\xi \\ &= \int \sigma(x, A^{*j}\xi)b^{j/2}\widehat{\varphi}(\xi)e^{-ik\cdot\xi}e^{ix\cdot A^{*j}\xi}\,d\xi \\ &= b^{j/2}\int \sigma(A^{-j}(\tilde{x}+k), A^{*j}\xi)\widehat{\varphi}(\xi)e^{i\tilde{x}\cdot\xi}\,d\xi.\end{aligned}$$

Therefore,

$$(4.27) \qquad (\sigma(x, D)\varphi_Q)(x) = b^{j/2}(\sigma_Q(x, D)\varphi)(A^j x - k),$$

where we denoted

$$(\sigma_Q(x, D)f)(x) = \int \sigma(A^{-j}(x+k), A^{*j}\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi.$$

For a fixed multi-index γ, we can write
(4.28)
$$(\partial^\gamma \sigma_Q(x, D)\varphi)(x) = \int e^{ix\cdot\xi}\widehat{\varphi}(\xi)\sum_{\delta \le \gamma} C_\delta(i\xi)^\delta \partial_x^{\gamma-\delta}[\sigma(A^{-j}(\cdot+k), A^{*j}\cdot)](x, \xi)\,d\xi.$$

Fix now β. An integration by parts in (4.28) then gives

(4.29)
$$(\partial^\gamma \sigma_Q(x, D)\varphi)(x) = \sum_{\delta \leq \gamma} C_\delta \int e^{ix\cdot\xi}|x|^{-|\beta|}$$
$$\sum_{|\beta_1|+|\beta_2|=|\beta|} \partial_x^{\gamma-\delta}\partial_\xi^{\beta_1}[\sigma(A^{-j}(\cdot + k), A^{*j}\cdot)](x, \xi)\partial_\xi^{\beta_2}((i\xi)^\delta\widehat{\varphi}(\xi))\,d\xi.$$

By the support assumption (4.24), the above integral runs only over $\xi \in \mathbb{R}^n$ with $\rho_{A^*}(\xi) \sim 1$. Let $\xi = (A^*)^{-j}\xi'$ and $x = A^j x'$. Then, $\rho_{A^*}(\xi') \sim b^j$ and since $\sigma \in \dot{S}_{1,1}^m(A)$, we have

$$|\partial_x^{\gamma-\delta}\partial_\xi^{\beta_1}[\sigma(A^{-j}\cdot, A^{*j}\cdot)](A^j x', (A^*)^{-j}\xi')| \leq C_{\beta_1,\gamma,\delta}b^{jm}.$$

Therefore, by taking absolute values on both sides of (4.29) and using the triangle inequality, we get

$$|(\partial^\gamma\sigma_Q(x, D)\varphi)(x)| \leq C_{\beta,\gamma}|x|^{-|\beta|}b^{jm}.$$

Since the multi-index β is arbitrary, by the quasi-norm estimate

$$(1 + \rho_A(x))^{\zeta_-} \leq C(1 + |x|) \qquad \text{for all } x \in \mathbb{R}^n,$$

we can obtain the anisotropic version of this estimate

(4.30) $$|(\partial^\gamma\sigma_Q(x, D)\varphi)(x)| \leq C_{M,P}(1 + \rho_A(x))^{-M}b^{jm} \qquad \text{for } |\gamma| \leq P,$$

where $M, P > 0$ are some fixed integers. More precisely, we let $P = \lfloor \alpha/\zeta_- \rfloor + 1$ and $M > \max(J, (J - \alpha)\zeta_+/\zeta_-)$.

Recall now that $Q = A^{-j}((0, 1]^n + k)$ and define

(4.31) $$\Psi_Q(x) = (C_{M,P})^{-1}b^{j/2-jm}(\sigma_Q(x, D)\varphi)(A^j x - k),$$

so that by (4.27)

$$\sigma(x, D)(\varphi_Q) = C_{M,P}|Q|^{-m}\Psi_Q.$$

It remains to verify that Ψ_Q is a molecule. By (4.30), (4.31), and

$$\Psi_Q(A^{-j}\cdot) = (C_{M,P})^{-1}b^{j/2-jm}(\sigma_Q(x, D)\varphi)(\cdot - k)$$

we have that

$$|\partial^\gamma[\Psi_Q(A^{-j}\cdot)](x)| \leq b^{j/2}(1 + \rho_A(x - A^j x_Q))^{-M} \qquad \text{for } |\gamma| \leq P.$$

By our choice of M, this simultaneously yields (4.16) and (4.17). Finally, the condition (4.18) is a direct consequence of the vanishing moment hypothesis (4.22). Thus, we conclude that Ψ_Q is a smooth synthesis molecule, which finishes the proof. $\qquad\square$

References

[1] Á. Bényi and M. Bownik, *Anisotropic classes of nonhomogeneous pseudodifferential symbols,* preprint.

[2] L. Borup and M. Nielsen, *On anisotropic Triebel-Lizorkin type spaces, with applications to the study of pseudo-differential operators*, J. Funct. Spaces Appl. **6** (2008), 107–154.

[3] M. Bownik, *Anisotropic Hardy spaces and wavelets*, Mem. Amer. Math. Soc. **164** (2003), no. 781, 122pp.

[4] M. Bownik, *Atomic and molecular decompositions of anisotropic Besov spaces*, Math. Z. **250** (2005), 539–571.

[5] M. Bownik, *Anisotropic Triebel-Lizorkin spaces with doubling measures*, J. Geom. Anal. **17** (2007), 387–424.

[6] M. Bownik and K.-P. Ho, *Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces*, Trans. Amer. Math. Soc. **358** (2006), 1469–1510.

[7] M. Bownik, B. Li, D. Yang, and Y. Zhou, *Weighted anisotropic product Hardy spaces and boundedness of sublinear operators*, Math. Nachr. **283** (2010), 392–442.

[8] R. R. Coifman and Y. Meyer, *Au delá des opérateurs pseudo-différentiels,* Astérisque **57** (1978).

[9] R. R. Coifman and Y. Meyer, *Wavelets. Caldern-Zygmund and multilinear operators*, Cambridge Studies in Advanced Mathematics, 48. Cambridge University Press, Cambridge, 1997.

[10] M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. **34** (1985), 777–799.

[11] M. Frazier and B. Jawerth, *A Discrete transform and decomposition of distribution spaces*, J. Funct. Anal. **93**, (1990), 34–170.

[12] M. Frazier, B. Jawerth, and G. Weiss, *Littlewood-Paley theory and the study of function spaces*, CBMS Regional Conference Series **79**, Amer. Math. Soc. (1991).

[13] G. Garello, *Anisotropic pseudodifferential operators of type 1,1*, Ann. Mat. Pura Appl. **IV** (1998), 135–160.

[14] L. Grafakos, *Classical and modern Fourier analysis,* Prentice Hall (2003).

[15] L. Grafakos and R.H. Torres, *Pseudodifferential operators with homogeneous symbols,* Michigan Math. J. **46** (1999), 261–269.

[16] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140.

[17] L. Hörmander, *Continuity of pseudo-differential operators of type 1,1*, Comm. Partial Differential Equations **14** (1989), 231–243.

[18] R. Lascar, *Propagation des singularités des solutions d'équations pseudo-différentielles quasi homogènes*, Ann. Inst. Fourier **27** (1977), 79–123.

[19] H.-G. Leopold, *Boundedness of anisotropic pseudo-differential operators in function spaces of Besov-Hardy-Sobolev type*, Z. Anal. Anwendungen **5** (1986), 409–417.

[20] W. R. Madych and N. M. Rivière, *Multipliers of the Hölder classes,* J. Func. Anal. **21** (1976), 369–379.

[21] S. G. Mihlin, *On the multipliers of Fourier integrals*, Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 701–703.

[22] N. M. Rivière, *Singular integrals and multiplier operators,* Ark. Mat. **9** (1971), 243–278.

[23] A. Seeger, *Some inequalities for singular convolution operators in L^p-spaces*, Trans. Amer. Math. Soc. **308** (1988), 259–272.

[24] E.M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press (1993).

[25] R. H. Torres, *Continuity properties of pseudodifferential operators of type 1,1*, Comm. Partial Differential Equations **15** (1990), 1313–1328.

[26] R. H. Torres, *Boundedness results for operators with singular kernels on distribution spaces*, Mem. Amer. Math. Soc. **90** (1991), no. 442, viii+172 pp.

[27] H. Triebel, *Theory of function spaces*, Monographs in Math., #78, Birkhäuser (1983).

[28] H. Triebel, *Theory of function spaces II*, Monographs in Math., #84, Birkhäuser Verlag, Basel, (1992).

[29] M. Yamazaki, *A quasihomogeneous version of paradifferential operators. I. Boundedness on spaces of Besov type*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986), 131–174.

[30] M. Yamazaki, *A quasihomogeneous version of paradifferential operators. II. A symbol calculus*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986), 311–345.

ÁRPÁD BÉNYI, DEPARTMENT OF MATHEMATICS, 516 HIGH STREET, WESTERN WASHINGTON UNIVERSITY, BELLINGHAM, WA 98225-9063, USA
  E-mail address: arpad.benyi@wwu.edu

MARCIN BOWNIK, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR 97403-1222, USA
  Current address: Institute of Mathematics, Polish Academy of Sciences, ul. Abrahama 18, 81-825 Sopot, Poland
  E-mail address: mbownik@uoregon.edu