Lecture 4: Crossed Products by Actions with the Rokhlin Property

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University of Oregon

18 July 2016

The Second Summer School on Operator Algebras and Noncommutative Geometry 2016

East China Normal University, Shanghai

11–29 July 2016

- Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
- Lecture 2 (13 July 2016): Introduction to Crossed Products and More Examples of Actions.
- Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
- Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
- Lecture 6 (20 July 2016): Applications and Problems.

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- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
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$$\begin{aligned}\|abe_g - e_g ab\| &= \big\|a(be_g - e_g b) + (e_g a - ae_g)b\big\| \\ &\leq \|a\| \cdot \|be_g - e_g b\| + \|e_g a - ae_g\| \cdot \|b\|.\end{aligned}$$

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Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C*-algebra A. Let $T \subset A$ generate A as a C*-algebra. Suppose that for every finite set $F \subset T$ and every $\varepsilon > 0$, there are projections $e_g \in A$ for $g \in G$ such that:

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3. $\sum_{g \in G} e_g = 1$.

Then α has the Rokhlin property.

Hint 1: The *-algebra generated by T is dense.

Hint 2: F only appears in condition (2). If, say, a and b approximately commute with e_g, then ab approximately commutes with e_g because

$$\begin{aligned} \| a b e_g - e_g a b \| &= \big\| a (b e_g - e_g b) + (e_g a - a e_g) b \big\| \\ &\leq \| a \| \cdot \| b e_g - e_g b \| + \| e_g a - a e_g \| \cdot \| b \|. \end{aligned}$$

Using a generating set

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The proof that $\beta(e_1) = e_0$ is the same. We are done.

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Exactly permuting the projections

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections e_g such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
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Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then α has the Rokhlin property if and only if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

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AF algebras

We give a very brief summary of AF algebras, restricted for convenience to the unital case, and refer to the lectures of Zhuang Niu for more.

Definition

Let A be a unital C*-algebra. Then A is an *AF algebra* if there is an increasing sequence

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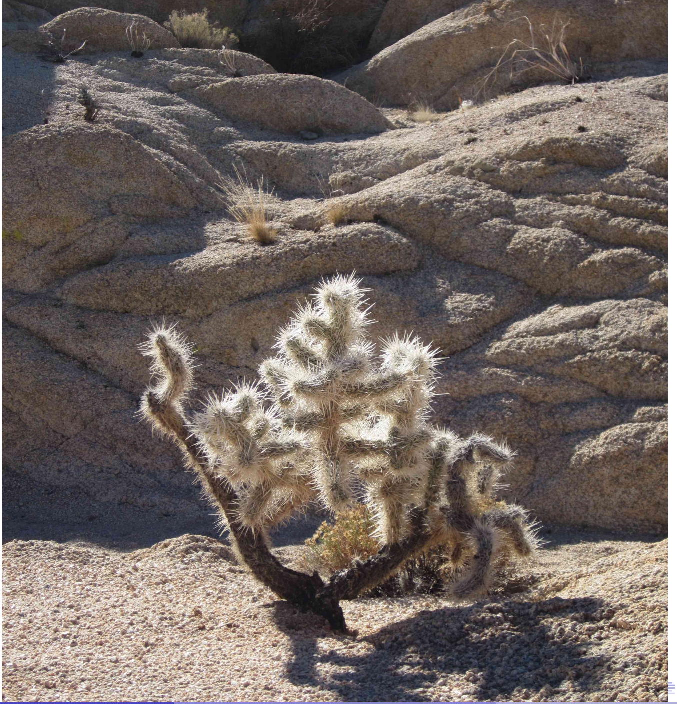
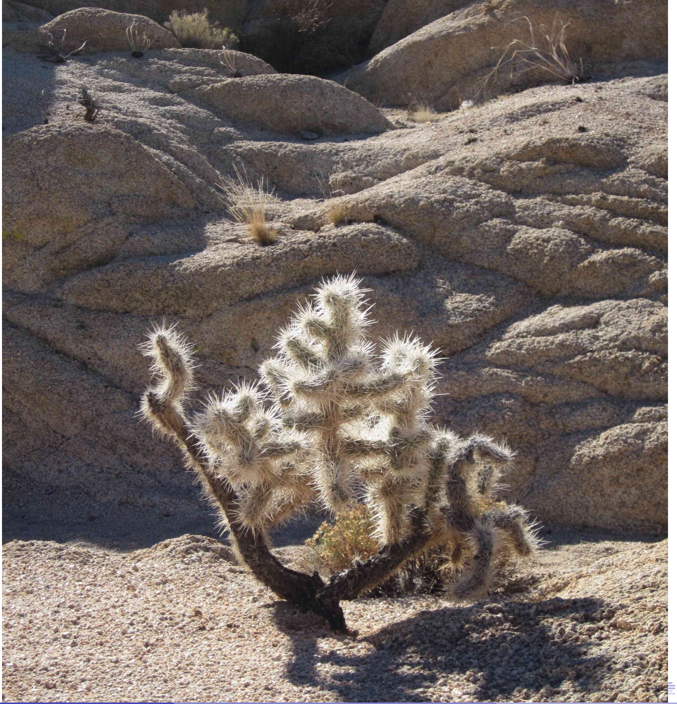
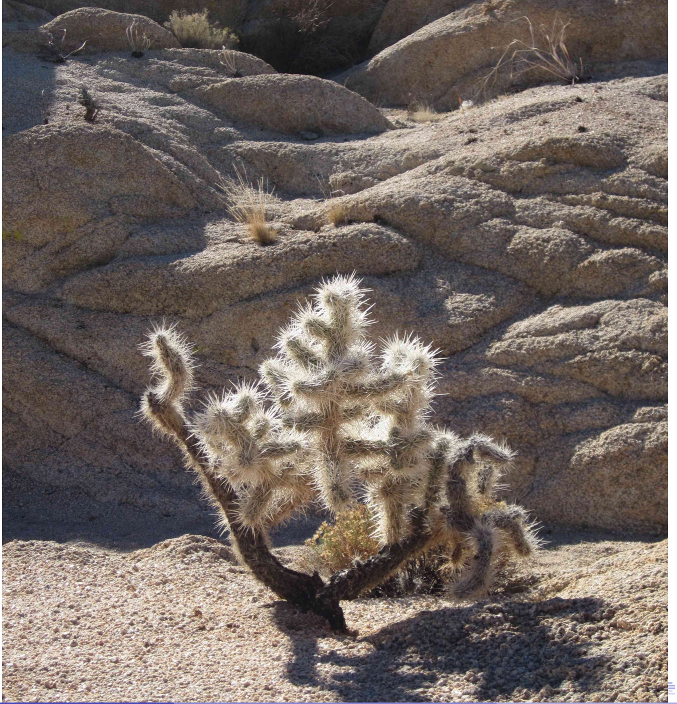
2. For every finite set $F \subset A$ and every $\varepsilon > 0$, there is a unital finite dimensional subalgebra $D \subset A$ such that $\operatorname{dist}(a, D) < \varepsilon$ for all $a \in F$.







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The algebraic version of this is true. That is, if a complex *-algebra A can be written as $A = \bigcup_{n=0}^{\infty} A_n$ for finite dimensional C*-algebras $A_0 \subset A_1 \subset \cdots$, and if $\alpha \colon G \to \mathrm{Aut}(A)$ is an action of a finite group G on A,

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A structure theorem for crossed products by actions with the Rokhlin property:

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The basic idea (details later). Set $n = \mathrm{card}(G)$. Recall that

$$C^*(G, A, \alpha) = A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g \colon c_g \in A \text{ for } g \in G \right\}.$$

and $(a \cdot u_g)(b \cdot u_h) = (a\alpha_g(b)) \cdot u_{gh}$.

Let $e_g \in A$, for $g \in G$, be Rokhlin projections, with $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$. Then $\mathrm{span}\big(\{e_g \colon g \in G\}\big)$ is a G-invariant subalgebra isomorphic to $C(G)$ with the action from translation of G on G. Let $u_g \in C^*(G, A, \alpha)$ be the canonical unitary implementing the automorphism α_g. Then $v_{g,h} = e_g u_{gh^{-1}}$ defines a system of matrix units in $C^*(G, A, \alpha)$. (This is essentially the same formula as was used in the proof that $C^*(G, \, C(G)) \cong M_n$.) Using the homomorphism $M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ given by $v_{g,h} \otimes d \mapsto v_{g,1} d v_{1,h}$, one can approximate $C^*(G, A, \alpha)$ by matrix algebras over corners of A.

Idea of the proof

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Apply the Rokhlin property to the finite set F. Use the version in which the projections are exactly permuted by the group. Thus, we get projections $e_g \in A$ for $g \in G$ such that:

1. $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
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Recall the conclusion of the theorem: $C^*(G, A, \alpha)$ is AF.

To prove the theorem, we prove that for every finite set $S \subset C^*(G, A, \alpha)$ and every $\varepsilon > 0$, there is an AF subalgebra $D \subset C^*(G, A, \alpha)$ such that every element of S is within ε of an element of D.

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Apply the Rokhlin property to α with F as given and with δ in place of ε, obtaining projections $e_g \in A$ for $g \in G$ such that $\alpha_g(e_h) = e_{gh}$ for $g, h \in G$, $\|e_g a - a e_g\| < \delta$ for $g \in G$ and $a \in F$, and $\sum_{g \in G} e_g = 1$.

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Approximating elements of F

We have to approximate elements of $S = F \cup \{u_g : g \in G\}$, with $F \subset A$ finite, by elements of $D = \varphi(M_n \otimes e_1 A e_1)$. Recall that $v_{g,h} = e_g u_{gh^{-1}}$, and that $\varphi \colon M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ is defined by $\varphi(w_{g,h} \otimes d) = v_{g,1} d v_{1,h}$ for $g, h \in G$ and $d \in e_1 A e_1$. We already took care of u_g.

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1. Show that $\left\| a - \sum_{g \in G} e_g a e_g \right\| < \varepsilon$.

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Once we have these, we are done: we have $\operatorname{dist}(a, D) < \varepsilon$.

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