Lisboa Summer School Course on Crossed Product C*-Algebras

N. Christopher Phillips

15 June 2009

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Sign up for the operator algebraist email directory, by emailing: `ncp@uoregon.edu`.

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- Construction of the crossed product of an action by a discrete group.
- Examples of some elementary computations of crossed products.
- Simplicity of crossed products by minimal homeomorphisms.
- Toward the classification of crossed products by minimal homeomorphisms.

Actions of groups on C*-algebras

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The continuity condition is the analog of requiring that a unitary representation of G on a Hilbert space be continuous in the strong operator topology. It is usually much too strong a condition to require that $g \mapsto \alpha_g$ be a norm continuous map from G to the bounded operators on A.

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Of course, if G is discrete, it doesn't matter. In this course, we will concentrate on discrete G.

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If A is unital and G is discrete, it is a suitable completion of the algebraic skew group ring $A[G]$, with multiplication determined by $gag^{-1} = \alpha_g(a)$ for $g \in G$ and $a \in A$.

Motivation for group actions on C*-algebras and their crossed products

Let G be a locally compact group obtained as a semidirect product $G = N \rtimes H$. The action of H on N gives actions of H on the full and reduced group C*-algebras $C^*(N)$ and $C^*_{\mathrm{r}}(N)$, and one has $C^*(G) \cong C^*(H, C^*(N))$ and $C^*_{\mathrm{r}}(G) \cong C^*_{\mathrm{r}}(H, C^*(N))$.

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Probably the most important group action is time evolution: if a C*-algebra A is supposed to represent the possible states of a physical system in some manner, then there should be an action $\alpha \colon \mathbb{R} \to \mathrm{Aut}(A)$ which describes the time evolution of the system. Actions of $\mathbb{Z}$, which are easier to study, can be though of as "discrete time evolution".

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Crossed products are a common way of constructing simple C*-algebras. We will see some examples later.

Motivation for group actions on C*-algebras and their crossed products (continued)

If one has a homeomorphism h of a locally compact Hausdorff space X, the crossed product $C^*(\mathbb{Z}, X, h)$ sometimes carries considerable information about the dynamics of h. The best known example is the result of Giordano, Putnam, and Skau on minimal homeomorphisms of the Cantor set: isomorphism of the transformation group C*-algebras is equivalent to strong orbit equivalence of the homeomorphisms.

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For compact groups, equivariant indices take values on the equivariant K-theory of a suitable C*-algebra with an action of the group. When the group is not compact, one usually needs instead the K-theory of the crossed product C*-algebra, or of the reduced crossed product C*-algebra. (When the group is compact, this is the same thing.)

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In other situations as well, the K-theory of the full or reduced crossed product is the appropriate substitute for equivariant K-theory.

The commutative case

Definition

A continuous action of a topological group G on a topological space X is a continuous function $G \times X \to X$, usually written $(g, x) \mapsto g \cdot x$ or $(g, x) \mapsto gx$, such that $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$ and $1 \cdot x = x$ for all $x \in X$.

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For a continuous action of a locally compact group G on a locally compact Hausdorff space X, there is a corresponding action $\alpha \colon G \to \mathrm{Aut}(C_0(X))$, given by $\alpha_g(f)(x) = f(g^{-1}x)$.

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(If G is not abelian, the inverse is necessary to get $\alpha_g \circ \alpha_h = \alpha_{gh}$ rather than α_{hg}.)

The commutative case (continued)

Exercise

Let G be a locally compact group, and let X be a locally compact Hausdorff space. Prove that the formulas given above determine a one to one correspondence between continuous actions of G on X and continuous actions of G on $C_0(X)$.

The commutative case (continued)

Exercise

Let G be a locally compact group, and let X be a locally compact Hausdorff space. Prove that the formulas given above determine a one to one correspondence between continuous actions of G on X and continuous actions of G on $C_0(X)$. (The main point is to show that an action on X is continuous if and only if the corresponding action on $C_0(X)$ is continuous.)

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For more, see the end of Section 1 of the notes.

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There are more examples in the notes.

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This gives the trivial action of G on the C*-algebra $\mathbb{C}$. The full and reduced crossed products are the usual full and reduced group C*-algebras $C^*(G)$ and $C^*_{\mathrm{r}}(G)$.

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It turns out that $C^*(G, G/H) \cong K(L^2(G/H)) \otimes C^*(H)$. Note that there is no "twisting".

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The crossed products for the actions of $\mathbb{Z}$ are the well known (rational or irrational) rotation algebras. (This will be essentially immediate from the definitions.)

Example 4

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Take $X = \{0, 1\}^{\mathbb{Z}}$, with elements being described as $x = (x_n)_{n \in \mathbb{Z}}$. Take $G = \mathbb{Z}$, with action generated by the *shift* homeomorphism $h(x)_n = x_{n-1}$ for $x \in X$ and $n \in \mathbb{Z}$.

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Further examples ("subshifts") can be gotten by restricting to invariant subsets of X. One can also replace $\{0,1\}$ by some other compact metric space S.

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Many generalizations are possible in the inverse limit version of the construction. One need not use a prime, nor even the same number at each stage of the inverse limit.

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The crossed product turns out to be isomorphic to the section algebra of a locally trivial but nontrivial bundle over the real projective space $\mathbb{R}P^n = S^n/(\mathbb{Z}/2\mathbb{Z})$ with fiber M_2.

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Take $X = S^1 = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\}$, and consider the order 2 homeomorphism $\zeta \mapsto \overline{\zeta}$. We get an action of $\mathbb{Z}/2\mathbb{Z}$ on S^1.

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Notation: If A is a unital C*-algebra and $u \in A$ is unitary, then $\mathrm{Ad}(u)$ is the automorphism $a \mapsto uau^*$ of A.

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The crossed product turns out to be isomorphic to the crossed product by the trivial action.

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Example 11 (continued)

Exercise

Let A be a simple unital C*-algebra, and let $\alpha\colon \mathbb{Z}/n\mathbb{Z} \to \mathrm{Aut}(A)$ be an action such that each automorphism α_g, for $g \in \mathbb{Z}/n\mathbb{Z}$, is an inner automorphism.

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Prove the statements made in the example on the previous slide.

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Recall that the Cuntz algebra $\mathcal{O}_n$ is the universal unital C*-algebra on generators $s_1, s_2, \ldots, s_n$, subject to the relations $s_j^* s_j = 1$ for $1 \leq j \leq n$ and $\sum_{j=1}^{n} s_j s_j^* = 1$. (It is in fact simple.)

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There is an action of $(S^1)^n$ on $\mathcal{O}_n$ such that $\alpha_{(\zeta_1, \zeta_2, \ldots, \zeta_n)}(s_j) = \zeta_j s_j$ for $1 \leq j \leq n$.

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There is an action of $(S^1)^n$ on $\mathcal{O}_n$ such that $\alpha_{(\zeta_1, \zeta_2, \ldots, \zeta_n)}(s_j) = \zeta_j s_j$ for $1 \leq j \leq n$.

In fact, regarding $(S^1)^n$ as the diagonal unitary matrices, this action extends to an action of the unitary group $U(M_n)$ on $\mathcal{O}_n$, defined as follows. If $u = (u_{j,k})_{j,k=1}^{n} \in M_n$ is unitary, then define an automorphism α_u of $\mathcal{O}_n$ by the following action on the generating isometries $s_1, s_2, \ldots, s_n$:

$$\alpha_u(s_j) = \sum_{k=1}^{n} u_{k,j} s_k.$$

This determines a continuous action of the compact group $U(M_n)$ on $\mathcal{O}_n$. Any individual automorphism from this action gives an action of $\mathbb{Z}$ on $\mathcal{O}_n$.

Example 14 (continued)

Exercise

Verify that the formula above does in fact define a continuous action of $U(M_n)$ on $\mathcal{O}_n$.

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(Check that the elements $\zeta_j s_j$ satisfy the required relations. Use a 3ε argument to prove continuity.)

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The easiest way to get such an action is to choose a unitary representation $g \mapsto u_n(g)$ on $\mathbb{C}^{k_n}$, and set $\beta_g^{(n)}(a) = u_n(g) a u_n(g)^*$ for $g \in G$ and $a \in M_{k_n}$.

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Exercise

Prove that the actions above really are continuous.

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There is also a "tensor shift", a noncommutative analog, defined on $\bigotimes_{n \in \mathbb{Z}} A$, of the shift on $S^{\mathbb{Z}}$.

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By convention, unitary representations are strong operator continuous. Representations of C*-algebras, and of other *-algebras are *-representations (and, similarly, homomorphisms are *-homomorphisms).

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We define a norm $\|\cdot\|_1$ on $C_{\mathrm{c}}(G, A, \alpha)$ by $\|a\|_1 = \int_G \|a(g)\|\, dg$. One checks that $\|ab\|_1 \leq \|a\|_1\|b\|_1$ and $\|a^*\|_1 = \|a\|_1$. Then $L^1(G, A, \alpha)$ is the Banach *-algebra obtained by completing $C_{\mathrm{c}}(G, A, \alpha)$ in $\|\cdot\|_1$.

Twisted convolution (continued)

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Assuming suitable versions of Fubini's Theorem for Banach space valued integrals, check that that multiplication in $C_c(G, A, \alpha)$ is associative. Further check for $a, b \in C_c(G, A, \alpha)$ that $\|ab\|_1 \leq \|a\|_1 \|b\|_1$, that $(ab)^* = b^* a^*$, and that $\|a^*\|_1 = \|a\|_1$.

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$$(f_1 f_2)(g, x) = \int_G f_1(h, x) f_2(h^{-1} g, \, h^{-1} x) \, dh$$

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$$(f_1 f_2)(g, x) = \int_G f_1(h, x) f_2(h^{-1} g, \, h^{-1} x) \, dh$$

and

$$f^*(g, x) = \Delta(g)^{-1} \overline{f(g^{-1}, \, g^{-1} x)}.$$