

# Lisboa Summer School Course on Crossed Product C\*-Algebras

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## Comments on the notes

The handout contains much more material than I will give in the lectures.  
Please let me know of any misprints, mistakes, etc. found in the handout!

Sign up for the operator algebraist email directory, by emailing:  
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## Outline

A brief outline of the lectures:

- Introductory material and basic definitions
- Examples of group actions.
- Construction of the crossed product of an action by a discrete group.
- Examples of some elementary computations of crossed products.
- Simplicity of crossed products by minimal homeomorphisms.
- Toward the classification of crossed products by minimal homeomorphisms.

## Actions of groups on C\*-algebras

### Definition

Let  $G$  be a locally compact group, and let  $A$  be a C\*-algebra. An *action of  $G$  on  $A$*  is a homomorphism  $\alpha: G \rightarrow \text{Aut}(A)$ , usually written  $g \mapsto \alpha_g$ , such that, for every  $a \in A$ , the function  $g \mapsto \alpha_g(a)$ , from  $G$  to  $A$ , is norm continuous.

On a von Neumann algebra, substitute the  $\sigma$ -weak operator topology for the norm topology.

The continuity condition is the analog of requiring that a unitary representation of  $G$  on a Hilbert space be continuous in the strong operator topology. It is usually much too strong a condition to require that  $g \mapsto \alpha_g$  be a norm continuous map from  $G$  to the bounded operators on  $A$ .

Of course, if  $G$  is discrete, it doesn't matter. In this course, we will concentrate on discrete  $G$ .

## We will construct crossed products

Given  $\alpha: G \rightarrow \text{Aut}(A)$ , we will construct a crossed product  $C^*$ -algebra  $C^*(G, A, \alpha)$  and a reduced crossed product  $C^*$ -algebra  $C_r^*(G, A, \alpha)$ . (There are many other commonly used notations. See Remark 3.16 in the notes.)

If  $A$  is unital and  $G$  is discrete, it is a suitable completion of the algebraic skew group ring  $A[G]$ , with multiplication determined by  $gag^{-1} = \alpha_g(a)$  for  $g \in G$  and  $a \in A$ .

## Motivation for group actions on $C^*$ -algebras and their crossed products

Let  $G$  be a locally compact group obtained as a semidirect product  $G = N \rtimes H$ . The action of  $H$  on  $N$  gives actions of  $H$  on the full and reduced group  $C^*$ -algebras  $C^*(N)$  and  $C_r^*(N)$ , and one has  $C^*(G) \cong C^*(H, C^*(N))$  and  $C_r^*(G) \cong C_r^*(H, C^*(N))$ .

Probably the most important group action is time evolution: if a  $C^*$ -algebra  $A$  is supposed to represent the possible states of a physical system in some manner, then there should be an action  $\alpha: \mathbb{R} \rightarrow \text{Aut}(A)$  which describes the time evolution of the system. Actions of  $\mathbb{Z}$ , which are easier to study, can be thought of as “discrete time evolution”.

Crossed products are a common way of constructing simple  $C^*$ -algebras. We will see some examples later.

## Motivation for group actions on $C^*$ -algebras and their crossed products (continued)

If one has a homeomorphism  $h$  of a locally compact Hausdorff space  $X$ , the crossed product  $C^*(\mathbb{Z}, X, h)$  sometimes carries considerable information about the dynamics of  $h$ . The best known example is the result of Giordano, Putnam, and Skau on minimal homeomorphisms of the Cantor set: isomorphism of the transformation group  $C^*$ -algebras is equivalent to strong orbit equivalence of the homeomorphisms.

For compact groups, equivariant indices take values on the equivariant K-theory of a suitable  $C^*$ -algebra with an action of the group. When the group is not compact, one usually needs instead the K-theory of the crossed product  $C^*$ -algebra, or of the reduced crossed product  $C^*$ -algebra. (When the group is compact, this is the same thing.)

In other situations as well, the K-theory of the full or reduced crossed product is the appropriate substitute for equivariant K-theory.

## The commutative case

### Definition

A continuous action of a topological group  $G$  on a topological space  $X$  is a continuous function  $G \times X \rightarrow X$ , usually written  $(g, x) \mapsto g \cdot x$  or  $(g, x) \mapsto gx$ , such that  $(gh)x = g(hx)$  for all  $g, h \in G$  and  $x \in X$  and  $1 \cdot x = x$  for all  $x \in X$ .

For a continuous action of a locally compact group  $G$  on a locally compact Hausdorff space  $X$ , there is a corresponding action  $\alpha: G \rightarrow \text{Aut}(C_0(X))$ , given by  $\alpha_g(f)(x) = f(g^{-1}x)$ .

(If  $G$  is not abelian, the inverse is necessary to get  $\alpha_g \circ \alpha_h = \alpha_{gh}$  rather than  $\alpha_{hg}$ .)

## The commutative case (continued)

### Exercise

Let  $G$  be a locally compact group, and let  $X$  be a locally compact Hausdorff space. Prove that the formulas given above determine a one to one correspondence between continuous actions of  $G$  on  $X$  and continuous actions of  $G$  on  $C_0(X)$ . (The main point is to show that an action on  $X$  is continuous if and only if the corresponding action on  $C_0(X)$  is continuous.)

## Examples of group actions on $C^*$ -algebras

We will give some examples of group actions on  $C^*$ -algebras. (Not all of them give interesting crossed products.)

We start with examples of group actions on locally compact spaces, which give rise to examples of group actions on commutative  $C^*$ -algebras.

We will discuss some of their crossed products later, but in some of the examples we state the results immediately. As one goes through the commutative examples, note that a closed orbit of the form  $Gx \cong G/H$  gives rise to a quotient of the crossed product isomorphic to  $K(L^2(G/H)) \otimes C^*(H)$

There are more examples in the notes.

## More about these lectures

There are many directions in the theory of crossed products. These lectures are biased towards the general problem of classifying crossed products by finite groups, by  $\mathbb{Z}$ , and by more complicated groups, in cases in which these crossed products are expected to be simple. (However, we will not get very far in that direction.)

I should at least mention some of the other directions:

- Coactions and actions of  $C^*$  Hopf algebras (“quantum groups”).
- Von Neumann algebra crossed products.
- Smooth crossed products.
- $C^*$ -algebras of groupoids.
- K-theory of crossed products: the Baum-Connes conjecture.
- The Connes spectrum and its generalizations.
- Ideal structure of crossed products.

For more, see the end of Section 1 of the notes.

## Example 1

### Example

The group  $G$  is arbitrary locally compact, the space  $X$  consists of just one point, and the action is trivial.

This gives the trivial action of  $G$  on the  $C^*$ -algebra  $\mathbb{C}$ . The full and reduced crossed products are the usual full and reduced group  $C^*$ -algebras  $C^*(G)$  and  $C_r^*(G)$ .

## Example 2

### Example

The group  $G$  is arbitrary locally compact,  $X = G$ , and the action is given by the group operation:  $g \cdot x = gx$ . (This action is called (left) translation.)

The full and reduced crossed products are both isomorphic to  $K(L^2(G))$ .

More generally, if  $H \subset G$  is a closed subgroup, then  $G$  acts continuously on  $G/H$  by translation. The trivial action above is the case  $H = G$ .

It turns out that  $C^*(G, G/H) \cong K(L^2(G/H)) \otimes C^*(H)$ . Note that there is no “twisting”.

## Example 4

### Example

Take  $X = \{0, 1\}^{\mathbb{Z}}$ , with elements being described as  $x = (x_n)_{n \in \mathbb{Z}}$ . Take  $G = \mathbb{Z}$ , with action generated by the *shift* homeomorphism  $h(x)_n = x_{n-1}$  for  $x \in X$  and  $n \in \mathbb{Z}$ .

Further examples (“subshifts”) can be gotten by restricting to invariant subsets of  $X$ . One can also replace  $\{0, 1\}$  by some other compact metric space  $S$ .

## Example 3

### Example

Take  $X = S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ . Taking  $G = S^1$ , acting by translation, gives a special case of a previous example. But we can also take  $G$  to be the finite subgroup (isomorphic to  $\mathbb{Z}/n\mathbb{Z}$ ) of  $S^1$  of order  $n$  generated by  $\exp(2\pi i/n)$ , still acting by translation (in this case, usually called rotation). Or we can fix  $\theta \in \mathbb{R}$ , and take  $G = \mathbb{Z}$ , with  $n \in \mathbb{Z}$  acting by  $\zeta \mapsto \exp(2\pi i n \theta) \zeta$ . These are *rational rotations* (for  $\theta \in \mathbb{Q}$ ) or *irrational rotations* (for  $\theta \notin \mathbb{Q}$ ).

The crossed product for the action of  $\mathbb{Z}/n\mathbb{Z}$  turns out to be isomorphic to  $C(S^1, M_n)$ . (Note that there is no “twisting”.)

The crossed products for the actions of  $\mathbb{Z}$  are the well known (rational or irrational) rotation algebras. (This will be essentially immediate from the definitions.)

## Example 5

### Example

Fix a prime  $p$ , and let  $X = \mathbb{Z}_p$ , the group of  $p$ -adic integers. (This group can be defined as the completion of  $\mathbb{Z}$  in the metric  $d(m, n) = p^{-d}$  when  $p^d$  is the largest power of  $p$  which divides  $n - m$ . Alternatively, it is  $\varprojlim \mathbb{Z}/p^d \mathbb{Z}$ .) It is a compact topological group, and as a metric space it is homeomorphic to the Cantor set. Let  $h: X \rightarrow X$  be the homeomorphism defined on the dense subset  $\mathbb{Z}$  by  $h(n) = n + 1$ .

Many generalizations are possible in the inverse limit version of the construction. One need not use a prime, nor even the same number at each stage of the inverse limit.

## Example 6

### Example

Take  $X = S^n = \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\}$ . Then the homeomorphism  $x \mapsto -x$  has order 2, and so gives an action of  $\mathbb{Z}/2\mathbb{Z}$  on  $S^n$ .

The crossed product turns out to be isomorphic to the section algebra of a locally trivial but nontrivial bundle over the real projective space  $\mathbb{R}P^n = S^n/(\mathbb{Z}/2\mathbb{Z})$  with fiber  $M_2$ .

## Example 8

### Example

The group  $SL_2(\mathbb{Z})$  acts on  $S^1 \times S^1$  as follows. For

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix} \in SL_2(\mathbb{Z}),$$

let  $n$  act on  $\mathbb{R}^2$  via the usual matrix multiplication. Since  $n$  has integer entries, one gets  $n\mathbb{Z}^2 \subset \mathbb{Z}^2$ , and thus the action is well defined on  $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$ .

## Example 7

### Example

Take  $X = S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ , and consider the order 2 homeomorphism  $\zeta \mapsto \bar{\zeta}$ . We get an action of  $\mathbb{Z}/2\mathbb{Z}$  on  $S^1$ .

## Example 9

Notation: If  $A$  is a unital  $C^*$ -algebra and  $u \in A$  is unitary, then  $Ad(u)$  is the automorphism  $a \mapsto uau^*$  of  $A$ .

### Example

Let  $G$  be a locally compact group, let  $A$  be a unital  $C^*$ -algebra, and let  $g \mapsto z_g$  be a norm continuous group homomorphism from  $G$  to the unitary group  $U(A)$  of  $A$ . Then the formula

$$\alpha_g(a) = Ad(z_g),$$

for  $g \in G$  and  $a \in A$ , defines an action of  $G$  on  $A$ .

Actions obtained this way are called *inner actions*.

The crossed product turns out to be isomorphic to the crossed product by the trivial action.

## Example 10

An action via inner automorphisms is not necessarily an inner action. There are no counterexamples with  $G = \mathbb{Z}$  (trivial) or  $G$  finite cyclic and  $A$  is simple (easy; see exercise below). Here is the smallest counterexample.

### Example

Let  $A = M_2$ , let  $G = (\mathbb{Z}/2\mathbb{Z})^2$  with generators  $g_1$  and  $g_2$ , and set

$$\alpha_1 = \text{id}_A, \quad \alpha_{g_1} = \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_{g_2} = \text{Ad} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \alpha_{g_1 g_2} = \text{Ad} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These define an action  $\alpha: G \rightarrow \text{Aut}(A)$  such that  $\alpha_g$  is inner for all  $g \in G$ , but such that there is no homomorphism  $g \mapsto z_g \in U(A)$  such that  $\alpha_g = \text{Ad}(z_g)$  for all  $g \in G$ . The point is that the implementing unitaries for  $\alpha_{g_1}$  and  $\alpha_{g_2}$  commute up to a scalar, but can't be appropriately modified to commute exactly.

The crossed product turns out to be isomorphic to  $M_4$ .

## Example 12

### Example

For  $\theta \in \mathbb{R}$ , let  $A_\theta$  be the rotation algebra, which is the universal  $C^*$ -algebra generated by two unitaries  $u$  and  $v$  satisfying the commutation relation  $vu = \exp(2\pi i\theta)uv$ . (The convention  $e^{2\pi i\theta}$  instead of  $e^{i\theta}$  has become so standard that it can't be changed.) (If  $\theta \notin \mathbb{Q}$ , then  $A_\theta$  is known to be simple. Thus, one may take any  $C^*$ -algebra generated by two unitaries satisfying the appropriate commutation relation.) The group  $\text{SL}_2(\mathbb{Z})$  acts on  $A_\theta$  by sending the matrix

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$$

to the automorphism determined by

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}} \quad \text{and} \quad \alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$

## Example 11 (continued)

### Exercise

Let  $A$  be a simple unital  $C^*$ -algebra, and let  $\alpha: \mathbb{Z}/n\mathbb{Z} \rightarrow \text{Aut}(A)$  be an action such that each automorphism  $\alpha_g$ , for  $g \in \mathbb{Z}/n\mathbb{Z}$ , is an inner automorphism. Prove that  $\alpha$  is an inner action in the sense above.

### Problem

Find a counterexample when  $A$  is not assumed simple. (I presume that a counterexample exists, but I do not know of one.)

### Exercise

Prove the statements made in the example on the previous slide.

## Example 12 (continued)

The algebra  $A_\theta$  is often considered to be a noncommutative analog of the torus  $S^1 \times S^1$  (more accurately, of  $A_0 \cong C(S^1 \times S^1)$ ), and this action is the analog of the action of  $\text{SL}_2(\mathbb{Z})$  on  $S^1 \times S^1$  above.

The group  $\text{SL}_2(\mathbb{Z})$  has finite subgroups of orders 2, 3, 4, and 6. Restriction of the action gives actions of these groups on rotation algebras. The crossed products by these actions have been intensively studied. Recently, it has been proved that for  $\theta \notin \mathbb{Q}$  they are all AF.

## Example 13

### Example

Let  $A_\theta$  be generated by unitaries  $u$  and  $v$  as in the previous example. For  $\zeta_1, \zeta_2 \in S^1$ , the unitaries  $\zeta_1 u$  and  $\zeta_2 v$  satisfy the same commutation relation. Therefore there is an action  $\alpha: S^1 \times S^1 \rightarrow \text{Aut}(A_\theta)$  determined by

$$\alpha_{(\zeta_1, \zeta_2)}(u) = \zeta_1 u \quad \text{and} \quad \alpha_{(\zeta_1, \zeta_2)}(v) = \zeta_2 v.$$

Checking continuity of the action requires a  $3\epsilon$  argument.

If we fix  $\zeta_1, \zeta_2 \in S^1$ , then  $\alpha_{(\zeta_1, \zeta_2)}$  generates an action of  $\mathbb{Z}$ . If both have finite order, we get an action of a finite cyclic group. For example, there is an action of  $\mathbb{Z}/n\mathbb{Z}$  generated by the automorphism which sends  $u$  to  $\exp(2\pi i/n)u$  and  $v$  to  $v$ .

## Example 14 (continued)

### Exercise

Verify that the formula above does in fact define a continuous action of  $U(M_n)$  on  $\mathcal{O}_n$ .

(Check that the elements  $\zeta_j s_j$  satisfy the required relations. Use a  $3\epsilon$  argument to prove continuity.)

## Example 14

### Example

Recall that the Cuntz algebra  $\mathcal{O}_n$  is the universal unital  $C^*$ -algebra on generators  $s_1, s_2, \dots, s_n$ , subject to the relations  $s_j^* s_j = 1$  for  $1 \leq j \leq n$  and  $\sum_{j=1}^n s_j s_j^* = 1$ . (It is in fact simple.)

There is an action of  $(S^1)^n$  on  $\mathcal{O}_n$  such that  $\alpha_{(\zeta_1, \zeta_2, \dots, \zeta_n)}(s_j) = \zeta_j s_j$  for  $1 \leq j \leq n$ .

In fact, regarding  $(S^1)^n$  as the diagonal unitary matrices, this action extends to an action of the unitary group  $U(M_n)$  on  $\mathcal{O}_n$ , defined as follows. If  $u = (u_{j,k})_{j,k=1}^n \in M_n$  is unitary, then define an automorphism  $\alpha_u$  of  $\mathcal{O}_n$  by the following action on the generating isometries  $s_1, s_2, \dots, s_n$ :

$$\alpha_u(s_j) = \sum_{k=1}^n u_{k,j} s_k.$$

This determines a continuous action of the compact group  $U(M_n)$  on  $\mathcal{O}_n$ . Any individual automorphism from this action gives an action of  $\mathbb{Z}$  on  $\mathcal{O}_n$ .

## Example 15

### Example

Let  $k_1, k_2, \dots$  be integers with all  $k_n \geq 2$ . Consider the UHF algebra  $A$  of type  $\prod_{n=1}^{\infty} k_n$ . We construct it as  $\bigotimes_{n=1}^{\infty} M_{k_n}$ , or, in more detail, as  $\varinjlim A_n$  with  $A_n = M_{k_1} \otimes M_{k_2} \otimes \dots \otimes M_{k_n}$ . Note that  $A_n = A_{n-1} \otimes M_{k_n}$ , and the map  $\varphi_n: A_{n-1} \rightarrow A_n$  is given by  $a \mapsto a \otimes 1_{M_{k_n}}$ .

Let  $G$  be a locally compact group, and let  $\beta^{(n)}: G \rightarrow \text{Aut}(M_{k_n})$  be an action of  $G$  on  $M_{k_n}$ . Then define an action  $\alpha^{(n)}: G \rightarrow \text{Aut}(A_n)$  by

$$\alpha_g^{(n)}(a_1 \otimes a_2 \otimes \dots \otimes a_n) = \beta_g^{(1)}(a_1) \otimes \beta_g^{(2)}(a_2) \otimes \dots \otimes \beta_g^{(n)}(a_n).$$

One checks immediately that  $\varphi_n \circ \alpha_g^{(n-1)} = \alpha_g^{(n)} \circ \varphi_n$  for all  $n \in \mathbb{Z}_{>0}$  and  $g \in G$ , so there is a direct limit action  $g \mapsto \alpha_g$  of  $G$  on  $A = \varinjlim A_n$ . (One needs a  $3\epsilon$  argument to prove continuity.)

## Example 15 (continued)

The easiest way to get such an action is to choose a unitary representation  $g \mapsto u_n(g)$  on  $\mathbb{C}^{k_n}$ , and set  $\beta_g^{(n)}(a) = u_n(g)au_n(g)^*$  for  $g \in G$  and  $a \in M_{k_n}$ . In this case, the resulting action is called a *product type action*.

As a specific example, take  $G = \mathbb{Z}/2\mathbb{Z}$ , and for every  $n$  take  $k_n = 2$  and take  $\beta^{(n)}$  to be generated by  $\text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

### Exercise

Prove that the actions above really are continuous.

## Covariant representations

To define the crossed product, we need:

### Definition

Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a locally compact group  $G$  on a  $C^*$ -algebra  $A$ . A *covariant representation* of  $(G, A, \alpha)$  on a Hilbert space  $H$  is a pair  $(\nu, \pi)$  consisting of a unitary representation  $\nu: G \rightarrow U(H)$  (the unitary group of  $H$ ) and a representation  $\pi: A \rightarrow L(H)$  (the algebra of all bounded operators on  $H$ ), satisfying the *covariance condition*

$$\nu(g)\pi(a)\nu(g)^* = \pi(\alpha_g(a))$$

for all  $g \in G$  and  $a \in A$ . It is called *nondegenerate* if  $\pi$  is nondegenerate.

By convention, unitary representations are strong operator continuous. Representations of  $C^*$ -algebras, and of other  $*$ -algebras are  $*$ -representations (and, similarly, homomorphisms are  $*$ -homomorphisms).

## Example 16

### Example

Let  $A$  be a unital  $C^*$ -algebra. The *tensor flip* is the automorphism  $\varphi \in \text{Aut}(A \otimes_{\max} A)$  of order 2 determined by the formula  $\varphi(a \otimes b) = b \otimes a$  for  $a, b \in A$ . (Use the universal property of  $A \otimes_{\max} A$ .) This gives an action of  $\mathbb{Z}/2\mathbb{Z}$  on  $A \otimes_{\max} A$ .

Similarly, the same formula defines a tensor flip action of  $\mathbb{Z}/2\mathbb{Z}$  on  $A \otimes_{\min} A$ . (Choose an injective representation  $\pi: A \rightarrow L(H)$ , and consider  $\pi \otimes \pi$  as a representation of  $A \otimes_{\min} A$  on  $H \otimes H$ .)

In a similar manner, the symmetric group  $S_n$  acts on the  $n$ -fold maximal and minimal tensor products of  $A$  with itself.

There is also a “tensor shift”, a noncommutative analog, defined on  $\bigotimes_{n \in \mathbb{Z}} A$ , of the shift on  $S^{\mathbb{Z}}$ .

## Remarks on Banach space valued integration

The crossed product  $C^*$ -algebra  $C^*(G, A, \alpha)$  is the universal  $C^*$ -algebra for covariant representations of  $(G, A, \alpha)$ , in essentially the same way that the (full) group  $C^*$ -algebra  $C^*(G)$  is the universal  $C^*$ -algebra for unitary representations of  $G$ . We construct it in a similar way to the group  $C^*$ -algebra. We start with the analog of  $L^1(G)$ .

For a general locally compact group, one needs an appropriate notion of integration of Banach space valued functions. One must prove that twisted convolution below is well defined, associative, distributive, and satisfies  $(ab)^* = b^*a^*$ . Once one has the appropriate notion of integration, the proofs are similar to the proofs of the corresponding facts about convolution in  $L^1(G)$ . Integration of continuous functions with compact support is much easier than integration of  $L^1$  functions, but the “right” way to do this is to define measurable Banach space valued functions and their integrals. This has been done; one reference is Appendix B of the book of Williams. Things simplify considerably if  $G$  is second countable and  $A$  is separable, but neither of these conditions is necessary.



## Twisted convolution

### Definition

Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a locally compact group  $G$  on a  $C^*$ -algebra  $A$ . We let  $C_c(G, A, \alpha)$  be the  $*$ -algebra of continuous functions  $a: G \rightarrow A$ , with pointwise addition and scalar multiplication. Using Haar measure in the integral, we define multiplication by the following “twisted convolution”:

$$(ab)(g) = \int_G a(h)\alpha_h(b(h^{-1}g)) dh.$$

Let  $\Delta$  be the modular function of  $G$ . We define the adjoint by

$$a^*(g) = \Delta(g)^{-1}\alpha_g(a(g^{-1})^*).$$

We define a norm  $\|\cdot\|_1$  on  $C_c(G, A, \alpha)$  by  $\|a\|_1 = \int_G \|a(g)\| dg$ . One checks that  $\|ab\|_1 \leq \|a\|_1\|b\|_1$  and  $\|a^*\|_1 = \|a\|_1$ . Then  $L^1(G, A, \alpha)$  is the Banach  $*$ -algebra obtained by completing  $C_c(G, A, \alpha)$  in  $\|\cdot\|_1$ .

## Twisted convolution (continued)

### Exercise

Assuming suitable versions of Fubini's Theorem for Banach space valued integrals, check that that multiplication in  $C_c(G, A, \alpha)$  is associative. Further check for  $a, b \in C_c(G, A, \alpha)$  that  $\|ab\|_1 \leq \|a\|_1\|b\|_1$ , that  $(ab)^* = b^*a^*$ , and that  $\|a^*\|_1 = \|a\|_1$ .

## Twisted convolution (continued)

### Exercise

Suppose  $A = C_0(X)$ , and  $\alpha$  comes from an action of  $G$  on  $X$ . Since we complete in a suitable norm later on, it suffices to use only the dense subalgebra  $C_c(X)$  in place of  $C_0(X)$ . There is an obvious identification of  $C_c(G, C_c(X))$  with  $C_c(G \times X)$ . Check that, on  $C_c(G \times X)$ , the formulas for multiplication and adjoint become

$$(f_1 f_2)(g, x) = \int_G f_1(h, x) f_2(h^{-1}g, h^{-1}x) dh$$

and

$$f^*(g, x) = \Delta(g)^{-1} \overline{f(g^{-1}, g^{-1}x)}.$$