Lisboa Summer School Course on Crossed Product C*-Algebras

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Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$

Fix $n \in \mathbb{Z}_{>0}$, and let $G = \mathbb{Z}/n\mathbb{Z}$ act on S^1 via the rotation by $2\pi/n$, that is, with generator the homeomorphism $h(\zeta) = e^{2\pi i/n}\zeta$ for $\zeta \in S^1$.

We describe what to expect. Every point in S^1 has a closed invariant neighborhood which is equivariantly homeomorphic to $G \times I$ for some closed interval $I \subset \mathbb{R}$, with the translation action on G and the trivial action on I. This leads to quotients of $C^*(G, S^1, h)$ isomorphic to $M_n \otimes C(I)$. Since S^1 itself is not such a product, one does not immediately get an isomorphism $C^*(G, S^1, h) \cong M_n \otimes C(Y)$ for any Y. Instead, one gets the section algebra of a locally trivial bundle over Y with fiber M_n. However, the appropriate Y is the orbit space $S^1/G \cong S^1$, and all locally trivial bundles over S^1 with fiber M_n are in fact trivial. Thus, one gets $C^*(G, S^1, h) \cong C(S^1, M_n)$ after all.

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

Here are the details. Let $\alpha \in \mathrm{Aut}(C(S^1))$ be the order n automorphism $\alpha(f) = f \circ h^{-1}$. Thus, $\alpha(f)(\zeta) = f(e^{-2\pi i/n}\zeta)$ for $\zeta \in S^1$. Let $s \in M_n$ be the shift unitary

$$s = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

The key computation is

$$s \,\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)\, s^* = \mathrm{diag}(\lambda_n, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}).$$

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

Set

$$B = \{f \in C([0,1], M_n) \colon f(0) = sf(1)s^*\}.$$

Define $\varphi_0 \colon C(S^1) \to B$ by sending $f \in C(S^1)$ to the continuously varying diagonal matrix

$$\varphi_0(f)(t) = \mathrm{diag}\big(f\big(e^{2\pi i t/n}\big),\, f\big(e^{2\pi i(t+1)/n}\big),\, \ldots,\, f\big(e^{2\pi i(t+n-1)/n}\big)\big).$$

(For fixed t, the diagonal entries are obtained by evaluating f at the points in the orbit of $e^{2\pi i t/n}$.) The diagonal entries of $f(0)$ are gotten from those of $f(1)$ by a forwards cyclic shift, so $\varphi_0(f)$ really is in B. For the same reason, we get

$$\begin{aligned} \varphi_0(\alpha(f))(t) &= \mathrm{diag}\big(f\big(e^{2\pi i(t-1)/n}\big),\, f\big(e^{2\pi i t/n}\big),\, \ldots,\, f\big(e^{2\pi i(t+n-2)/n}\big)\big) \\ &= s\varphi_0(f)(t)s^*. \end{aligned}$$

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

Now let $v \in C([0,1], M_n)$ be the constant function with value s. Note that $v \in B$. The calculation just done implies that

$$\varphi_0(\alpha^k(f)) = v^k \varphi_0(f) v^{-k}$$

for $0 \leq k \leq n-1$. Also clearly $v^n = 1$. We write the group elements as $0, 1, \ldots, n-1$, by abuse of notation treating them as integers when convenient. The universal property of the crossed product therefore implies that there is a homomorphism $\varphi \colon C^*(G, S^1, h) \to B$ such that $\varphi|_{C(S^1)} = \varphi_0$ and $\varphi(u_k) = v^k$ for $0 \leq k \leq n-1$.

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

We prove directly that φ is bijective. We can rewrite φ as the map $C(\mathbb{Z}/n\mathbb{Z} \times S^1) \to B$ given by

$$\varphi(f) = \sum_{k=0}^{n-1} \varphi_0(f(k, -)) v^k.$$

Injectivity now reduces to the easily verified fact that if $a_0, a_1, \ldots, a_{n-1} \in M_n$ are diagonal matrices, and $\sum_{k=0}^{n-1} a_k v^k = 0$, then $a_0 = a_1 = \cdots = a_{n-1} = 0$.

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

For surjectivity, let $a \in B$, and write

$$a(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) & \cdots & a_{1,n}(t) \\ a_{2,1}(t) & a_{2,2}(t) & \cdots & a_{2,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(t) & a_{n,2}(t) & \cdots & a_{n,n}(t) \end{pmatrix}$$

with $a_{j,k} \in C([0,1])$ for $1 \leq j, k \leq n$. The condition $a \in B$ implies that, taking the indices mod n in $\{1, 2, \ldots, n\}$, we have $a_{j,k}(1) = a_{j+1,\,k+1}(0)$ for all j and k. So we get a well defined element of $C(\mathbb{Z}/n\mathbb{Z} \times S^1)$ via

$$f\left(l,\, e^{2\pi i(t+j)/n}\right) = a_{l+j,\,j}(t)$$

for $t \in [0,1]$, $1 \leq j \leq n$, and $0 \leq l \leq n-1$, with $l+j$ taken mod n in $\{1, 2, \ldots, n\}$. One checks that $\varphi(f) = a$.

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

It remains to prove that $B \cong C(S^1, M_n)$. Since $U(M_n)$ is connected, there is a unitary path $t \mapsto s_t$, for $t \in [0,1]$, such that $s_0 = 1$ and $s_1 = s$. Define $\psi \colon C(S^1, M_n) \to B$ by $\psi(f)(t) = s_t^* f(e^{2\pi i t}) s_t$. For $f \in C(S^1, M_n)$, we have

$$\psi(f)(1) = s^* f(1) s = s^* f(0) s = s^* \psi(f)(0) s,$$

so $\psi(f)$ really is in B. It is easily checked that ψ is bijective.

Example: $x \mapsto -x$ on S^n

Example

Let $X = S^n = \{x \in \mathbb{R}^{n+1} \colon \|x\|_2 = 1\}$, and let $\mathbb{Z}/2\mathbb{Z}$ act by sending the nontrivial group element to the order 2 homeomorphism $x \mapsto -x$.

The "local structure" of the crossed product $C^*(\mathbb{Z}/2\mathbb{Z}, X)$ is the same as in the previous example. However, for $n \geq 2$ the resulting bundle is no longer trivial. The crossed product is isomorphic to the section algebra of a locally trivial but nontrivial bundle over the real projective space $\mathbb{R}P^n = S^n/(\mathbb{Z}/2\mathbb{Z})$ with fiber M_2. See Proposition 4.15 of Williams' book.

Example: Complex conjugation on S^1

Example

Take $X = S^1 = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\}$, and let $\mathbb{Z}/2\mathbb{Z}$ act by sending the nontrivial group element to the order 2 homeomorphism $\zeta \mapsto \overline{\zeta}$. Let $\alpha \in \mathrm{Aut}(C(S^1))$ be the corresponding automorphism.

We compute the crossed product, but we first describe what to expect. We should expect that the points 1 and -1 contribute quotients isomorphic to $\mathbb{C} \oplus \mathbb{C}$, and that for $\zeta \neq \pm 1$, the pair of points $(\zeta, \overline{\zeta})$ contributes a quotient isomorphic to M_2. We will in fact show that $C^*(\mathbb{Z}/2\mathbb{Z}, X)$ is isomorphic to the C*-algebra

$$B = \{f \in C([-1, 1], M_2) \colon f(1) \text{ and } f(-1) \text{ are diagonal matrices}\}.$$

Example: Complex conjugation on S^1 (continued)

Here are the details. First, let $C_0 \subset M_2$ be the subalgebra consisting of all matrices of the form $\left(\begin{smallmatrix} \lambda & \mu \\ \mu & \lambda \end{smallmatrix}\right)$ with $\lambda, \mu \in \mathbb{C}$. Then define

$$C = \{f \colon [-1, 1] \to M_2 \colon f \text{ is continuous and } f(1), \, f(-1) \in C_0\}.$$

Let $v \in C$ be the constant function $v(t) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for all $t \in [-1, 1]$. Define $\varphi_0 \colon C(S^1) \to C$ by

$$\varphi_0(f)(t) = \begin{pmatrix} f\big(t + i\sqrt{1 - t^2}\big) & 0 \\ 0 & f\big(t - i\sqrt{1 - t^2}\big) \end{pmatrix}$$

for $f \in C(S^1)$ and $t \in [-1, 1]$. One checks that the conditions at ± 1 for membership in C are satisfied. Moreover, $v^2 = 1$ and $v\varphi_0(f)v^* = \varphi_0(\alpha(f))$ for $f \in C(S^1)$.

Example: Complex conjugation on S^1 (continued)

Therefore there is a homomorphism $\varphi \colon C^*(\mathbb{Z}/2\mathbb{Z}, X) \to C$ such that $\varphi|_{C(S^1)} = \varphi_0$ and φ sends the standard unitary u in $C^*(\mathbb{Z}/2\mathbb{Z}, X)$ to v. It is given by the formula

$$\varphi(f_0 + f_1 u)(t) = \begin{pmatrix} f_0\big(t + i\sqrt{1 - t^2}\big) & f_1\big(t + i\sqrt{1 - t^2}\big) \\ f_1\big(t - i\sqrt{1 - t^2}\big) & f_0\big(t - i\sqrt{1 - t^2}\big) \end{pmatrix}$$

for $f_1, f_2 \in C(S^1)$ and $t \in [-1, 1]$.

We claim that φ is an isomorphism. Since

$$C^*(\mathbb{Z}/2\mathbb{Z}, X) = \{f_0 + f_1 u \colon f_1, f_2 \in C(S^1)\},$$

it is easy to check injectivity.

Example: Complex conjugation on S^1 (continued)

For surjectivity, let

$$a(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) \\ a_{2,1}(t) & a_{2,2}(t) \end{pmatrix}$$

define an element $a \in C$. Then

$$a_{1,1}(-1) = a_{2,2}(-1) \quad \text{and} \quad a_{2,1}(-1) = a_{1,2}(-1), \tag{1}$$

and

$$a_{1,1}(1) = a_{2,2}(1) \quad \text{and} \quad a_{2,1}(1) = a_{1,2}(1). \tag{2}$$

Now set

$$f_0(\zeta) = \begin{cases} a_{1,1}(\mathrm{Re}(\zeta)) & \mathrm{Im}(\zeta) \geq 0 \\ a_{2,2}(\mathrm{Re}(\zeta)) & \mathrm{Im}(\zeta) \leq 0 \end{cases}$$

and

$$f_1(\zeta) = \begin{cases} a_{1,2}(\mathrm{Re}(\zeta)) & \mathrm{Im}(\zeta) \geq 0 \\ a_{2,1}(\mathrm{Re}(\zeta)) & \mathrm{Im}(\zeta) \leq 0 \end{cases}$$

for $\zeta \in S^1$. The relations (1) and (2) ensure that f_0 and f_1 are well defined at ± 1, and are continuous. One easily checks that $\varphi(f_0 + f_1 u) = a$. This proves surjectivity.

Example: Complex conjugation on S^1 (continued)

The algebra C is not quite what was promised. Set

$$w = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

which is a unitary in M_2. Then the isomorphism $\psi \colon C^*(\mathbb{Z}/2\mathbb{Z}, X) \to B$ is given by $\psi(a)(t) = w\varphi(a)(t)w^*$. (Check this!)

In this example, one choice of matrix units in M_2 was convenient for the free orbits, while another choice was convenient for the fixed points. It seemed better to compute everything in terms of the choice convenient for the free orbits, and convert afterwards.

Example: $x \mapsto -x$ on $[-1, 1]$

Exercise

Let $\mathbb{Z}/2\mathbb{Z}$ act on $[-1, 1]$ via $x \mapsto -x$. Compute the crossed product.

Example: $(x_1, x_2, \ldots, x_n, x_{n+1}) \mapsto (x_1, x_2, \ldots, x_n, -x_{n+1})$ on S^n

Exercise

Let $\mathbb{Z}/2\mathbb{Z}$ act on

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \colon x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

via $(x_1, x_2, \ldots, x_n, x_{n+1}) \mapsto (x_1, x_2, \ldots, x_n, -x_{n+1})$. Compute the crossed product.

Example: $\mathbb{Z}$ acting on $\mathbb{Z}/n\mathbb{Z}$ by translation

Example

Let $X = \mathbb{Z}/n\mathbb{Z}$, and let $\mathbb{Z}$ act on X by translation. We show that $C^*(\mathbb{Z}, X) \cong M_n \otimes C(S^1)$.

This is a special case of G acting on G/H by translation. In the general case, it turns out that $C^*(G,\ G/H) \cong K(L^2(G/H)) \otimes C^*(H)$. Note that there is no twisting.

We will be sketchy. See the notes for details.

Identify $\mathbb{Z}/n\mathbb{Z}$ with $\{1, 2, \ldots, n\}$. (We start at 1 instead of 0 to be consistent with common matrix unit notation.) Let $\alpha \in \mathrm{Aut}(C(\mathbb{Z}/n\mathbb{Z}))$ be $\alpha(f)(k) = f(k-1)$, with indices taken mod n in $\{1, 2, \ldots, n\}$. (Equivalently, $\alpha(\chi_{\{k\}}) = \chi_{\{k+1\}}$, with $k+1$ taken to be 1 when $k = n$.)

Example: $\mathbb{Z}$ acting on $\mathbb{Z}/n\mathbb{Z}$ by translation (continued)

In $C(S^1)$ let z be the function $z(\zeta) = \zeta$ for all ζ. In $M_n(C(S^1)) \cong M_n \otimes C(S^1)$, abbreviate $e_{j,k} \otimes 1$ to $e_{j,k}$, and let v be the unitary

$$v = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & z \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

(This unitary differs from the unitary s used before only in that here the upper right corner entry is z instead of 1.)

Example: $\mathbb{Z}$ acting on $\mathbb{Z}/n\mathbb{Z}$ by translation (continued)

Define $\varphi_0 \colon C(\mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ by $\varphi_0(\chi_{\{k\}}) = e_{k,k}$. Then one checks that $v\varphi_0(f)v^* = \varphi_0(\alpha(f))$ for all $f \in C(\mathbb{Z}/n\mathbb{Z})$. Therefore there is a homomorphism $\varphi \colon C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ such that $\varphi|_{C(\mathbb{Z}/n\mathbb{Z})} = \varphi_0$ and $\varphi(u) = v$. We claim that φ is an isomorphism.

We use the following description of $M_n \otimes C(S^1)$: it is the universal unital C*-algebra generated by a system $(e_{j,k})_{1 \le j,k \le n}$ of matrix units such that $\sum_{j=1}^{n} e_{j,j} = 1$ and a central unitary y. The $e_{j,k}$ are the matrix units we have already used, and the central unitary is $1 \otimes z$. (Proof: Exercise.)

Example: $\mathbb{Z}$ acting on $\mathbb{Z}/n\mathbb{Z}$ by translation (continued)

To prove that φ is surjective, it suffices to prove that its image contains the generators above. This is easy; see the notes.

To prove injectivity, it suffices to prove that whenever A is a unital C*-algebra, $\psi_0 \colon C(\mathbb{Z}/n\mathbb{Z}) \to A$ is a unital homomorphism, and $w \in A$ is a unitary such that $w\psi_0(f)w^* = \psi_0(\alpha(f))$ for all $f \in C(\mathbb{Z}/n\mathbb{Z})$, then there is a homomorphism $\gamma \colon M_n \otimes C(S^1) \to A$ such that $\gamma \circ \varphi_0 = \psi_0$ and $\gamma(v) = w$. That is, we are showing that $M_n \otimes C(S^1)$ satisfies the universal property of the crossed product. If φ were not injective, taking ψ_0 and w to come from $\mathrm{id}_{C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})}$ would yield a contradiction. It suffices to define γ on the generators above.

The rest of the details are omitted; see the notes. The main point is to use the description of $M_n \otimes C(S^1)$ as the universal C*-algebra on the generators and relations above.

Where ideals in crossed products come from

We have implicitly seen two sources of ideals in a reduced crossed product $C^*_{\mathrm{r}}(G, A, \alpha)$: invariant ideals in A, and group elements which act trivially on A. There is a theorem due to Gootman and Rosenberg which gives a description of the primitive ideals of any crossed product $C^*(G, A)$ with G amenable, and which, very roughly, says that they all come from some combination of these two sources. (One does not even need to restrict to discrete groups.) There is a bit more discussion, with references, in the notes.

Example: A product type action

Example

We consider the action of $\mathbb{Z}/2\mathbb{Z}$ on the 2^∞ UHF algebra A generated by $\bigotimes_{n=1}^{\infty} \mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Again, we will be sketchy. See the notes for more details.

Write $A = \varinjlim M_{2^n}$, with maps $\varphi_n \colon M_{2^n} \to M_{2^{n+1}}$ given by $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$. Define unitaries $z_n \in M_{2^n}$ inductively by $z_0 = 1$ and $z_{n+1} = \left(\begin{smallmatrix} z_n & 0 \\ 0 & -z_n \end{smallmatrix}\right)$. In tensor product notation, and with an appropriate choice of isomorphism $M_{2^n} \otimes M_2 \to M_{2^{n+1}}$, these are

$$\varphi_n(a) = a \otimes 1_{M_2} \quad \text{and} \quad z_{n+1} = z_n \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

Let

$$\overline{\varphi}_n \colon C^*(\mathbb{Z}/2\mathbb{Z},\, M_{2^n},\, \mathrm{Ad}(z_n)) \to C^*(\mathbb{Z}/2\mathbb{Z},\, M_{2^{n+1}},\, \mathrm{Ad}(z_{n+1}))$$

be the corresponding map on the crossed products.

Example: A product type action (continued)

From what we did with inner actions, we get isomorphisms

$$\sigma_n \colon C^*(\mathbb{Z}/2\mathbb{Z},\, M_{2^n},\, \mathrm{Ad}(z_n)) \to M_{2^n} \oplus M_{2^n}$$

given by $a + bu_n \mapsto (a + bz_n,\, a - bz_n)$. We now need a map

$$\psi_n \colon M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$$

which makes the following diagram commute:

$$\begin{array}{ccc} C^*(\mathbb{Z}/2\mathbb{Z},\, M_{2^n},\, \mathrm{Ad}(z_n)) & \xrightarrow{\sigma_n} & M_{2^n} \oplus M_{2^n} \\ {\scriptstyle \overline{\varphi}_n}\big\downarrow & & \big\downarrow{\scriptstyle \psi_n} \\ C^*(\mathbb{Z}/2\mathbb{Z},\, M_{2^{n+1}},\, \mathrm{Ad}(z_{n+1})) & \xrightarrow{\sigma_{n+1}} & M_{2^{n+1}} \oplus M_{2^{n+1}} \end{array}$$

Example: A product type action (continued)

We need a map

$$\psi_n \colon M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$$

which makes the diagram on the previous slide commute. That is, ψ_n sends

$$\sigma_n(a + bu_n) = (a + bz_n,\, a - bz_n)$$

to

$$\sigma_{n+1}(\varphi_n(a) + \varphi_n(b)u_{n+1}).$$

We have enough information to write an explicit formula for this expression (see the notes), and it turns out that we can take

$$\psi_n(b, c) = \left(\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix} \right).$$

Example: A product type action (continued)

Those familiar with Bratteli diagrams will now be able to write down the Bratteli diagram for the crossed product. There is a direct computation in the notes. It turns out that the crossed product is isomorphic to A.

Remarks on the product type example

The fact that we got the same algebra back is somewhat special, but the general principle of the computation is much more generally applicable.

Since the crossed product is simple, the action is not inner.

The theorem of Gootman and Rosenberg described above gives no information here.

Exercises 4.23 and 4.24 in the notes combine direct limit methods with computations of the sort done above.

Example: The irrational rotation algebras

Let $\theta \in \mathbb{R}$. Recall that the rotation algebra A_θ is the universal C*-algebra generated by unitaries u and v satisfying $vu = e^{2\pi i\theta} uv$.

Let $h_\theta \colon S^1 \to S^1$ be the homeomorphism $h_\theta(\zeta) = e^{2\pi i\theta}\zeta$. We claim that there is an isomorphism $\varphi \colon A_\theta \to C^*(\mathbb{Z}, S^1, h_\theta)$ which sends u to the standard unitary u_1 in the crossed product, and sends v to the function $z \in C(S^1)$ defined by $z(\zeta) = \zeta$ for all $\zeta \in S^1$.

Example: The irrational rotation algebras (continued)

The proof of the claim is by comparison of universal properties. First, one checks that $zu_1 = e^{2\pi i\theta} u_1 z$, so at least there is a homomorphism φ as claimed.

Next, define $\psi_0 \colon C(S^1) \to A_\theta$ by $\psi_0(f) = f(v)$ (continuous functional calculus) for $f \in C(S^1)$. For $n \in \mathbb{Z}$, we have

$$u\psi_0(z^n)u^* = (uvu^*)^n = e^{2\pi i n\theta} v^n = \psi_0\big(e^{2\pi i n\theta} z^n\big) = \psi_0\big(z^n \circ h_\theta^{-1}\big).$$

Since the functions z^n span a dense subspace of $C(S^1)$, it follows that $u\psi_0(f)u^* = \psi_0\big(f \circ h_\theta^{-1}\big)$ for all $f \in C(S^1)$. By the universal property of the crossed product, there is a homomorphism $\psi \colon C^*(\mathbb{Z}, S^1, h_\theta) \to A_\theta$ such that $\psi|_{C(S^1)} = \psi_0$ and $\psi(u_1) = u$.

We have $(\psi \circ \varphi)(u) = u$ and $(\psi \circ \varphi)(v) = v$. Since u and v generate A_θ, we conclude that $\psi \circ \varphi = \mathrm{id}_{A_\theta}$. Similarly, one proves $\varphi \circ \psi = \mathrm{id}_{C^*(\mathbb{Z}, S^1, h_\theta)}$.

We will see below that for $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the algebra $C^*(\mathbb{Z}, S^1, h_\theta)$ is simple.

Minimal actions

We will prove that reduced crossed products by free minimal actions of countable discrete groups on compact metric spaces are simple.

Definition

Let a locally compact group G act continuously on a locally compact space X. The action is called *minimal* if whenever $T \subset X$ is a closed subset such that $gT \subset T$ for all $g \in G$, then $T = \varnothing$ or $T = X$.

In short, there are no nontrivial invariant closed subsets. For those knowing some ergodic theory, this is the topological analog of an ergodic action on a measure space.

If the action of G on X is not minimal, then there is a nontrivial invariant closed subset $T \subset X$, and $C^*(G, X \setminus T)$ turns out to be a nontrivial ideal in $C^*(G,X)$. Thus $C^*(G,X)$ is not simple.

Minimal homeomorphisms

For the case $G = \mathbb{Z}$, the conventional terminology is a bit different.

Definition

Let X be a locally compact Hausdorff space, and let $h\colon X \to X$ be a homeomorphism. Then h is called *minimal* if whenever $T \subset X$ is a closed subset such that $h(T) = T$, then $T = \varnothing$ or $T = X$.

Almost all work on minimal homeomorphisms has been on compact spaces. For these, we have the following equivalent conditions.

Minimal homeomorphisms (continued)

Lemma

Let X be a compact Hausdorff space, and let $h\colon X \to X$ be a homeomorphism. Then the following are equivalent:

1. h is minimal.
2. Whenever $T \subset X$ is a closed subset such that $h(T) \subset T$, then $T = \varnothing$ or $T = X$.
3. Whenever $U \subset X$ is an open subset such that $h(U) = U$, then $U = \varnothing$ or $U = X$.
4. Whenever $U \subset X$ is an open subset such that $h(U) \subset U$, then $U = \varnothing$ or $U = X$.
5. For every $x \in X$, the orbit $\{h^n(x)\colon n \in \mathbb{Z}\}$ is dense in X.
6. For every $x \in X$, the forward orbit $\{h^n(x)\colon n \geq 0\}$ is dense in X.

Minimal homeomorphisms (continued)

Conditions (1), (3), and (5) are equivalent even when X is only locally compact, and in fact there is an analog for actions of arbitrary groups. Minimality does not imply the other three conditions without compactness, as can be seen by considering the homeomorphism $n \mapsto n+1$ of $\mathbb{Z}$. Also, even for compact X, it isn't good enough to merely have the existence of some dense orbit, as can be seen by considering the homeomorphism $n \mapsto n+1$ on the two point compactification $\mathbb{Z} \cup \{\pm\infty\}$ of $\mathbb{Z}$.

Exercise

Prove the lemma.

Examples of minimal actions

Example

Let G be a locally compact group, let $H \subset G$ be a closed subgroup, and let G act on G/H be translation. This action is minimal: there are no nontrivial invariant subsets, closed or not.

This is a "trivial" example of a minimal action. Here are several more interesting ones.

Example

Irrational rotations of the circle are minimal.

Minimality of irrational rotations

Minimality of irrational rotations follows from the following lemma.

Lemma

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then $\{ e^{2\pi i n \theta} \colon n \in \mathbb{Z} \}$ is dense in S^1.

Proof.

It suffices to prove that $\mathbb{Z} + \theta\mathbb{Z}$ is dense in $\mathbb{R}$. Suppose not. Because we are dealing with groups, there is an open set $U \subset \mathbb{R}$ such that $U \cap (\mathbb{Z} + \theta\mathbb{Z}) = \{0\}$. Let $t = \inf\left(\left\{ x \in \overline{\mathbb{Z} + \theta\mathbb{Z}} \colon x > 0 \right\} \right)$. Then $t > 0$. Since $\overline{\mathbb{Z} + \theta\mathbb{Z}}$ is a group, one checks that $\overline{\mathbb{Z} + \theta\mathbb{Z}} = \mathbb{Z}t$. Then we must have $\mathbb{Z} + \theta\mathbb{Z} = \mathbb{Z}t$. It follows that both 1 and θ are integer multiples of t, so that $\theta \in \mathbb{Q}$. $\square$

There are other proofs of minimality.

More examples of minimal homeomorphisms

Example

The homeomorphism $x \mapsto x + 1$ on the p-adic integers is minimal. The orbit of 0 is $\mathbb{Z}$, which is dense, essentially by definition. Every other orbit is a translate of this one, so also dense.

Example

The shift homeomorphism on $\{0, 1\}^{\mathbb{Z}}$ and the action of $\mathrm{SL}_2(\mathbb{Z})$ on $S^1 \times S^1$ are not minimal. In fact, they have fixed points.

Other examples of minimal homeomorphisms include Furstenberg transformations, restrictions of Denjoy homeomorphisms of the circle to their minimal sets, and certain irrational time maps of suspension flows.

Minimal actions are plentiful

Minimal actions are plentiful: a Zorn's Lemma argument shows that every nonempty compact G-space X contains a nonempty invariant closed subset on which the restricted action is minimal.

Fres actions

The transformation group C*-algebra of a minimal action need not be simple. Consider, for example, the trivial action of a group G (particularly an abelian group) on a one point space, for which the transformation group C*-algebra is $C^*(G)$.

Definition

Let a locally compact group G act continuously on a locally compact space X. The action is called *free* if whenever $g \in G \setminus \{1\}$ and $x \in X$, then $gx \neq x$. The action is called *essentially free* if whenever $g \in G \setminus \{1\}$, the set $\{x \in X : gx = x\}$ has empty interior.

Remarks on freeness

Let X be an infinite compact Hausdorff space, and let $h: X \to X$ be a minimal homeomorphism. Then one easily checks that the corresponding action of $\mathbb{Z}$ on X is free.

An essentially free minimal action of an abelian group is free. This is because the fixed point set of any group element is invariant under the action.

Simplicity of the transformation group C*-algebra

Theorem (Archbold-Spielberg)

Let a discrete group G act minimally and essentially freely on a locally compact space X. Then $C^*_{\mathrm{r}}(G, X)$ is simple.

Archbold and Spielberg actually proved something stronger, involving actions on not necessarily commutative C*-algebras. See the notes, and also see the notes for discussions of other proofs and related results.

The usual proof for $G = \mathbb{Z}$ depends on Rokhlin type arguments. See the proof of Lemma VIII.3.7 of Davidson's book. We have avoided such arguments here, but will use them later. To obtain more information about simple transformation group C*-algebras, such arguments are necessary, at least with the current state of knowledge. Examples show that, in the absence of some form of the Rokhlin property, stronger structural properties of crossed products of noncommutative C*-algebras need not hold, even when they are simple.

First lemma

The proof of the theorem on simplicity needs several lemmas, which are special cases of the corresponding lemmas in the paper of Archbold and Spielberg.

Lemma

Let A be a C*-algebra, let $B \subset A$ be a subalgebra, and let ω be a state on A such that $\omega|_B$ is multiplicative. Then for all $a \in A$ and $b \in B$, we have $\omega(ab) = \omega(a)\omega(b)$ and $\omega(ba) = \omega(b)\omega(a)$.

This is also a special case of a result of Choi.

Proof of the first lemma

We prove $\omega(ab) = \omega(a)\omega(b)$. The other equation will follow by using adjoints and the relation $\omega(c^*) = \overline{\omega(c)}$.

If A is not unital, then ω extends to a state on the unitization A^+. Thus, we may assume that A is unital. Also, if ω is multiplicative on B, one easily checks that ω is multiplicative on $B + \mathbb{C} \cdot 1$. Thus, we may assume that $1 \in B$.

Proof of the first lemma (continued)

We recall from the Cauchy-Schwarz inequality that $|\omega(x^*y)|^2 \leq \omega(y^*y)\omega(x^*x)$. Replacing x by x^*, we get $|\omega(xy)|^2 \leq \omega(y^*y)\omega(xx^*)$. Now let $a \in A$ and $b \in B$. Then

$$\begin{aligned} |\omega(ab) - \omega(a)\omega(b)|^2 &= \big|\omega\big(a(b - \omega(b) \cdot 1\big)\big|^2 \\ &\leq \omega\big((b - \omega(b) \cdot 1)^*(b - \omega(b) \cdot 1)\big)\omega(aa^*). \end{aligned}$$

Since ω is multiplicative on B, we have

$$\omega\big((b - \omega(b) \cdot 1)^*(b - \omega(b) \cdot 1)\big) = \omega\big((b - \omega(b) \cdot 1)^*\big)\omega\big(b - \omega(b) \cdot 1\big) = 0.$$

So $|\omega(ab) - \omega(a)\omega(b)|^2 = 0$. This completes the proof.

Second lemma

Lemma

Let a discrete group G act on a locally compact space X. Let $x \in X$, let $g \in G$, and assume that $gx \neq x$. Let $\mathrm{ev}_x \colon C_0(X) \to \mathbb{C}$ be the evaluation map $\mathrm{ev}_x(f) = f(x)$ for all $f \in C_0(X)$, and let ω be a state on $C^*_{\mathrm{r}}(G, X)$ which extends ev_x. Then $\omega(fu_g) = 0$ for all $f \in C_0(X)$.

For the proof, let $\alpha \colon G \to \mathrm{Aut}(C_0(X))$ be $\alpha_g(f)(x) = f(g^{-1}x)$ for $f \in C_0(X)$, $g \in G$, and $x \in X$. Choose $f_0 \in C_0(X)$ such that $f_0(x) = 1$ and $f_0(gx) = 0$. Applying the first lemma to ω at the second and fourth steps, and using $\omega(f_0) = 1$ at the first step, we have

$$\begin{aligned} \omega(fu_g) &= \omega(f_0)\omega(fu_g) = \omega(f_0 f u_g) = \omega\big(fu_g \alpha_g^{-1}(f_0)\big) \\ &= \omega(fu_g)\omega(\alpha_g^{-1}(f_0)) = \omega(fu_g)f_0(gx) = 0. \end{aligned}$$

This completes the proof.

Reminder: The conditional expectation

Recall the conditional expectation from the previous lecture:
Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Then the conditional expectation $E \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ does the following: if $a = \sum_{g \in G} a_g u_g \in C^*_{\mathrm{r}}(G, A, \alpha)$ with $a_g = 0$ for all but finitely many g, then $E(a) = a_1$. Moreover, E has the following properties:

1. $E(E(b)) = E(b)$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
2. If $b \geq 0$ then $E(b) \geq 0$.
3. $\|E(b)\| \leq \|b\|$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
4. If $a \in A$ and $b \in C^*_{\mathrm{r}}(G, A, \alpha)$, then $E(ab) = aE(b)$ and $E(ba) = E(b)a$.
5. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $E(a^*a) = 0$, then $a = 0$.

Proof of the simplicity theorem

Let $I \subset C^*_{\mathrm{r}}(G, X)$ be a nonzero closed ideal.

First suppose $I \cap C_0(X) = 0$. Choose $a \in I$ with $a \neq 0$. Let

$$E\colon C^*_{\mathrm{r}}(G, X) \to C_0(X)$$

be the standard conditional expectation. Then $E(a^*a) \neq 0$ because E is faithful. Choose $b \in C_{\mathrm{c}}(G,\, C_0(X),\, \alpha)$ such that $\|b - a^*a\| < \frac{1}{4}\|E(a^*a)\|$. We can write $b = \sum_{g \in S} b_g u_g$ for some finite set $S \subset G$ and with $b_g \in C_0(X)$ for $g \in S$. Without loss of generality $1 \in S$. Since $E(a^*a)$ is a positive element of $C_0(X)$, there is $x_0 \in X$ such that $E(a^*a)(x_0) = \|E(a^*a)\|$. Essential freeness implies that

$$\{x \in X \colon gx \neq x \text{ for all } g \in S \setminus \{1\}\}$$

is dense in X. In particular, there is $x \in X$ so close to x_0 that $E(a^*a)(x) > \frac{3}{4}\|E(a^*a)\|$, and also satisfying $gx \neq x$ for all $g \in S$.

Proof of the simplicity theorem (continued)

The set $C_0(X) + I$ is a C*-subalgebra of $C^*_{\mathrm{r}}(G, X)$. Let $\omega_0 \colon C_0(X) + I \to \mathbb{C}$ be the following composition:

$$C_0(X) + I \longrightarrow (C_0(X) + I)/I \xrightarrow{\cong} C_0(X)/(C_0(X) \cap I) = C_0(X) \xrightarrow{\mathrm{ev}_x} \mathbb{C}.$$

Then ω_0 is a homomorphism. Use the Hahn-Banach Theorem in the usual way to get a state $\omega \colon C^*_{\mathrm{r}}(G, X) \to \mathbb{C}$ which extends ω. Note that $\omega(a^*a) = 0$.

We now have, using the second lemma at the fifth step,

$$\begin{aligned} \tfrac{1}{4}\|E(a^*a)\| > \|b - a^*a\| &\geq |\omega(b - a^*a)| = |\omega(b)| = \left|\sum\nolimits_{g \in S} \omega(b_g u_g)\right| \\ &= |\omega(b_1)| = |\omega_0(b_1)| = |b_1(x)| \geq E(a^*a)(x) - \|E(a^*a) - b_1\| \\ &\geq E(a^*a)(x) - \|a^*a - b\| > \tfrac{3}{4}\|E(a^*a)\| - \tfrac{1}{4}\|E(a^*a)\| = \tfrac{1}{2}\|E(a^*a)\|. \end{aligned}$$

This contradiction shows that $I \cap C_0(X) \neq 0$.

Proof of the simplicity theorem (continued)

Since $I \cap C_0(X)$ is an ideal in $C_0(X)$, it has the form $C_0(U)$ for some nonempty open set $U \subset X$. We claim that U is G-invariant. Let $g \in G$ and let $f \in C_0(U)$. Let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for $C_0(X)$. Then the elements $e_\lambda u_g$ are in $C^*_{\mathrm{r}}(G, X)$, and we have $(e_\lambda u_g) f (e_\lambda u_g)^* = e_\lambda \alpha_g(f) e_\lambda$, which converges to $\alpha_g(f)$. We also have $(e_\lambda u_g) f (e_\lambda u_g)^* \in I \cap C_0(X)$, since I is an ideal. So $\alpha_g(C_0(U)) \subset C_0(U)$ for all $g \in G$, and the claim follows.

Since U is open, invariant, and nonempty, we have $U = X$. One easily checks that an approximate identity for $C_0(X)$ is also an approximate identity for $C^*_{\mathrm{r}}(G, X)$, so $I = C^*_{\mathrm{r}}(G, X)$, as desired. This completes the proof.

Kishimoto's result

The following theorem, originally due to Kishimoto, also follows from the full Archbold-Spielberg theorem.

Theorem

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a simple C*-algebra A. Suppose that α_g is outer for every $g \in G \setminus \{1\}$. Then $C^*_{\mathrm{r}}(G, A, \alpha)$ is simple.

What to do in the last lecture?

There are two possibilities:

Theorem

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a simple unital C*-algebra A. Suppose that A has tracial rank zero and α has the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has tracial rank zero.

Tracial rank zero is important because it is a hypothesis in classification theorems.

(This is not in the notes.)

What to do in the last lecture? (continued)

Towards the proof of:

Theorem

Let X be an infinite compact metric space with finite covering dimension, and let $h\colon X \to X$ be a minimal homeomorphism. Suppose that $\rho(K_0(C^*(\mathbb{Z}, X, h)))$ is dense in $\mathrm{Aff}(T(C^*(\mathbb{Z}, X, h)))$. Then $C^*(\mathbb{Z}, X, h)$ is a simple unital C*-algebra with tracial rank zero.