

# OTTAWA SUMMER SCHOOL COURSE ON CROSSED PRODUCT C\*-ALGEBRAS (DRAFT)

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ABSTRACT. We present an introduction to the theory of crossed product C\*-algebras, with emphasis on the background needed for recent work on the classification of crossed products.

## 1. INTRODUCTION AND MOTIVATION

This is an expanded version of lectures on crossed product C\*-algebras given at the Ottawa Summer School in Operator Algebras, 20–24 Aug. 2007. These notes assume the basic theory of C\*-algebras. There will be occasional comments assuming other material, such as Morita equivalence, K-theory, multiplier algebras, and real and stable rank, but these are not essential to the development. Some things are given without proof: the proofs are either left as exercises or are beyond the scope of these notes. Although the definitions are presented for actions of general locally compact groups, most of the proofs and examples are restricted to the discrete case, which is often considerably easier. If the group is not discrete, one needs a theory of integration of Banach space valued functions, which we do not discuss. Some things go through in general, changing sums to integrals and using approximate identities, but some is unknown or fails. In the last section, we will use tracial rank zero; a definition will be given, but little will be said about its properties.

**Definition 1.1.** Let  $G$  be a locally compact group, and let  $A$  be a C\*-algebra. An *action of  $G$  on  $A$*  is a homomorphism  $\alpha: G \rightarrow \text{Aut}(A)$ , usually written  $g \mapsto \alpha_g$ , such that, for every  $a \in A$ , the function  $g \mapsto \alpha_g(a)$ , from  $G$  to  $A$ , is norm continuous.

The continuity condition is the analog of requiring that a unitary representation of  $G$  on a Hilbert space be continuous in the strong operator topology. It is usually much too strong a condition to require that  $g \mapsto \alpha_g$  be a norm continuous map from  $G$  to the bounded operators on  $A$ . Of course, if  $G$  is discrete, it doesn't matter.

Given  $\alpha: G \rightarrow \text{Aut}(A)$ , we will construct in Section 3 below a crossed product C\*-algebra  $C^*(G, A, \alpha)$  and a reduced crossed product C\*-algebra  $C_r^*(G, A, \alpha)$ . (There are many other commonly used notations. See Remark 3.16.) If  $A$  is unital and  $G$  is discrete, it is a suitable completion of the algebraic skew group ring  $A[G]$ , with multiplication determined by  $gag^{-1} = \alpha_g(a)$  for  $g \in G$  and  $a \in A$ .

A continuous action of a topological group  $G$  on a topological space  $X$  is a continuous function  $G \times X \rightarrow X$ , usually written  $(g, x) \mapsto g \cdot x$  or  $(g, x) \mapsto gx$ , such that  $(gh)x = g(hx)$  for all  $g, h \in G$  and  $x \in X$  and  $1 \cdot x = x$  for all  $x \in X$ . For a continuous action of a locally compact group  $G$  on a locally compact Hausdorff

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space  $X$ , there is a corresponding action  $\alpha: G \rightarrow \text{Aut}(C_0(X))$ , given by  $\alpha_g(f)(x) = f(g^{-1}x)$ . (The inverse appears for the same reason it does in the formula for the left regular representation of a group. If  $G$  is not abelian, the inverse is necessary to get  $\alpha_g \circ \alpha_h = \alpha_{gh}$  rather than  $\alpha_{hg}$ . Note, for example, that if  $K \subset X$  is a compact open set, so that its characteristic function  $\chi_K$  is in  $C_0(X)$ , then  $\alpha_g(\chi_K) = \chi_{gK}$ , not  $\chi_{g^{-1}K}$ .) We write  $C^*(G, X)$  for the crossed product  $C^*$ -algebra, and call it the transformation group  $C^*$ -algebra.

**Exercise 1.2.** Let  $G$  be a locally compact group, and let  $X$  be a locally compact Hausdorff space. Prove that the formulas given above determine a one to one correspondence between continuous actions of  $G$  on  $X$  and continuous actions of  $G$  on  $C_0(X)$ . (The main point is to show that an action on  $X$  is continuous if and only if the corresponding action on  $C_0(X)$  is continuous.)

For the special case  $G = \mathbb{Z}$ , the same notation is often used for the action and for the automorphism which generates it. Thus, if  $A$  is a  $C^*$ -algebra and  $\alpha \in \text{Aut}(A)$ , one often writes  $C^*(\mathbb{Z}, A, \alpha)$ . For a homeomorphism  $h$  of a locally compact Hausdorff space  $X$ , one gets an automorphism  $\alpha \in \text{Aut}(C_0(X))$ , and thus an action of  $\mathbb{Z}$  on  $C_0(X)$ . We abbreviate this crossed product to  $C^*(\mathbb{Z}, X, h)$ .

Before going into details, we give some motivation for studying group actions on  $C^*$ -algebras and their crossed products.

- (1) Let  $G$  be a locally compact group obtained as a semidirect product  $G = N \rtimes H$ . The action of  $H$  on  $N$  gives actions of  $H$  on the full and reduced group  $C^*$ -algebras  $C^*(N)$  and  $C_r^*(N)$ , and one has  $C^*(G) \cong C^*(H, C^*(N))$  and  $C_r^*(G) \cong C_r^*(H, C^*(N))$ .
- (2) Probably the most important group action is time evolution: if a  $C^*$ -algebra  $A$  is supposed to represent the possible states of a physical system in some manner, then there should be an action  $\alpha: \mathbb{R} \rightarrow \text{Aut}(A)$  which describes the time evolution of the system. Actions of  $\mathbb{Z}$ , which are easier to study, can be thought of as “discrete time evolution”.
- (3) Crossed products are a common way of constructing simple  $C^*$ -algebras. We will see some examples later.
- (4) If one has a homeomorphism  $h$  of a locally compact Hausdorff space  $X$ , the crossed product  $C^*(\mathbb{Z}, X, h)$  sometimes carries considerable information about the dynamics of  $h$ . The best known example is the result of [9] on minimal homeomorphisms of the Cantor set: isomorphism of the transformation group  $C^*$ -algebras is equivalent to strong orbit equivalence of the homeomorphisms.

There are many directions in the theory of crossed products. These notes are biased towards the general problem of classifying crossed products by finite groups, by  $\mathbb{Z}$ , and by more complicated groups, in cases in which these crossed products are expected to be simple. (However, we will not get very far in that direction.) I should at least mention some of the other directions:

- The structure of crossed products of continuous trace  $C^*$ -algebras by actions for which the induced action on the primitive ideal space is proper.
- Extensions of the notion of crossed product to coactions and actions of  $C^*$  Hopf algebras (“quantum groups”), and the associated duality theory.
- Crossed products twisted by cocycles.
- Von Neumann algebra crossed products.

- Smooth crossed products.
- C\*-algebras of groupoids, and crossed products by actions of groupoids on C\*-algebras.
- Computation of the K-theory of crossed products, from the Pimsner-Voiculescu exact sequence and the Connes isomorphism through the Baum-Connes conjecture.
- The ideal structure of crossed products, without assuming analogs of freeness or properness.
- The Connes spectrum and its generalizations.
- Crossed products by endomorphisms and by semigroups.

The textbook references on crossed products that I know are: Chapters 7 and 8 of [25] (very condensed; the primary emphasis is on properties of group actions rather than of crossed products), [38] (quite detailed; the primary emphasis is on ideal structure of general crossed products), and Chapter 8 of [4] (the primary emphasis is on crossed products, especially by  $\mathbb{Z}$ , as a means of constructing interesting examples of C\*-algebras). There are no textbooks with primary emphasis on classification of crossed products or on crossed products by minimal homeomorphisms.

The “further reading” section in the introduction of [38] gives a number of references for various directions in the theory of crossed products which are treated neither in [38] nor here.

## 2. EXAMPLES OF GROUP ACTIONS ON C\*-ALGEBRAS

In this section, we give a number of examples of actions of locally compact groups on C\*-algebras. We start with examples of group actions on locally compact spaces, which give rise to examples of group actions on commutative C\*-algebras. We will compute the crossed products for some of these examples in Section 4, but there are many more examples here than there are computations of examples in Section 4.

**Example 2.1.** The group  $G$  is arbitrary locally compact, the space  $X$  consists of just one point, and the action is trivial. This gives the trivial action of  $G$  on the C\*-algebra  $\mathbb{C}$ . The full and reduced crossed products are the usual full and reduced group C\*-algebras  $C^*(G)$  and  $C_r^*(G)$ . As we will see, this is essentially immediate by comparing definitions. See Example 4.7 below.

**Example 2.2.** The group  $G$  is arbitrary locally compact,  $X = G$ , and the action is given by the group operation:  $g \cdot x = gx$ . (This action is called (left) translation.) The full and reduced crossed products are both isomorphic to  $K(L^2(G))$ . We will prove this for the discrete case in Example 4.13 below.

More generally, if  $H \subset G$  is a closed subgroup, then  $G$  acts continuously on  $G/H$  by translation. Example 2.1 is the case  $H = G$ . See Example 4.16 below for the computation when  $G = \mathbb{Z}$  and  $H = n\mathbb{Z}$ , and for the description of the crossed product in the general case.

**Example 2.3.** Take  $X = S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ . Taking  $G = S^1$ , acting by translation, gives a special case of Example 2.2. But we can also take  $G$  to be the finite subgroup of  $S^1$  of order  $n$  generated by  $\exp(2\pi i/n)$ , still acting by translation (in this case, usually called rotation). The computation for this case is in Example 4.14. Or we can fix  $\theta \in \mathbb{R}$ , and take  $G = \mathbb{Z}$ , with  $n \in \mathbb{Z}$  acting by  $\zeta \mapsto \exp(2\pi i n \theta) \zeta$ . These are *rational rotations* (for  $\theta \in \mathbb{Q}$ ) or *irrational rotations* (for  $\theta \notin \mathbb{Q}$ ). See Example 4.25.

**Example 2.4.** Take  $X = \{0, 1\}^{\mathbb{Z}}$ , with elements being described as  $x = (x_n)_{n \in \mathbb{Z}}$ . Take  $G = \mathbb{Z}$ , with action generated by the *shift* homeomorphism  $h(x)_n = x_{n-1}$  for  $x \in X$  and  $n \in \mathbb{Z}$ .

Further examples (“subshifts”) can be gotten by restricting to invariant subsets of  $X$ . One can also replace  $\{0, 1\}$  by some other compact metric space  $S$ .

**Example 2.5.** Fix a prime  $p$ , and let  $X = \mathbb{Z}_p$ , the group of  $p$ -adic integers. This group can be defined as the completion of  $\mathbb{Z}$  in the metric  $d(m, n) = p^{-d}$  when  $p^d$  is the largest power of  $p$  which divides  $n - m$ . Alternatively, it is  $\varprojlim \mathbb{Z}/p^d \mathbb{Z}$ . It is a compact topological group, and as a metric space it is homeomorphic to the Cantor set. Let  $h: X \rightarrow X$  be the homeomorphism defined on the dense subset  $\mathbb{Z}$  by  $h(n) = n + 1$ .

Many generalizations are possible in the inverse limit version of the construction. One need not use a prime, nor even the same number at each stage of the inverse limit; thus, one could consider  $X = \varprojlim \mathbb{Z}/k_d \mathbb{Z}$ , with  $k_1 | k_2 | \dots$  ( $k_1$  divides  $k_2$ ,  $k_2$  divides  $k_3$ , etc.) and  $\lim_{d \rightarrow \infty} k_d = \infty$ .

**Example 2.6.** Take  $X = S^n = \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\}$ . Then the homeomorphism  $x \mapsto -x$  has order 2, and so gives an action of  $\mathbb{Z}/2\mathbb{Z}$  on  $S^n$ . See Example 4.15 below for a description of the crossed product (without proof).

**Example 2.7.** Take  $X = S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ , and consider the order 2 homeomorphism  $\zeta \mapsto \bar{\zeta}$ . We get an action of  $\mathbb{Z}/2\mathbb{Z}$  on  $S^1$ . See Example 4.19 below for the computation of the crossed product.

**Example 2.8.** The group  $\mathrm{SL}_2(\mathbb{Z})$  acts on  $S^1 \times S^1$  as follows. For

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

let  $n$  act on  $\mathbb{R}^2$  via the usual matrix multiplication. Since  $n$  has integer entries, one gets  $n\mathbb{Z}^2 \subset \mathbb{Z}^2$ , and thus the action is well defined on  $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$ .

Similarly,  $\mathrm{SL}_d(\mathbb{Z})$  acts on  $(S^1)^d$ .

Now we turn to examples of group actions on noncommutative  $C^*$ -algebras. If  $A$  is a unital  $C^*$ -algebra and  $u \in A$  is unitary, we will sometimes use the notation  $\mathrm{Ad}(u)$  for the automorphism  $a \mapsto uau^*$  of  $A$ .

**Example 2.9.** Let  $G$  be a locally compact group, let  $A$  be a unital  $C^*$ -algebra, and let  $g \mapsto z_g$  be a norm continuous group homomorphism from  $G$  to the unitary group  $U(A)$  of  $A$ . Then the formula  $\alpha_g = \mathrm{Ad}(z_g)$ , for  $g \in G$  and  $a \in A$ , defines an action of  $G$  on  $A$ . Actions obtained this way are called *inner actions*.

If  $A$  is not unital, let  $M(A)$  be its multiplier algebra, and use  $U(M(A))$  with the strict topology in place of  $U(A)$  with the norm topology.

We will see below that the crossed product by an inner action is isomorphic to the crossed product by the trivial action. See Example 4.10 below for the computation when  $G$  is discrete.

An action via inner automorphisms is not necessarily an inner action in the sense of Example 2.9. There are no counterexamples with  $G = \mathbb{Z}$  (trivial) or  $G$  finite cyclic and  $A$  is simple (easy; see Exercise 2.11 below). Here is the smallest counterexample.

**Example 2.10.** Let  $A = M_2$ , let  $G = (\mathbb{Z}/2\mathbb{Z})^2$  with generators  $g_1$  and  $g_2$ , and set

$$\alpha_1 = \text{id}_A, \quad \alpha_{g_1} = \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_{g_2} = \text{Ad} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \alpha_{g_1 g_2} = \text{Ad} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These define an action  $\alpha: G \rightarrow \text{Aut}(A)$  such that  $\alpha_g$  is inner for all  $g \in G$ , but such that there is no homomorphism  $g \mapsto z_g \in U(A)$  such that  $\alpha_g = \text{Ad}(z_g)$  for all  $g \in G$ . The point is that the implementing unitaries for  $\alpha_{g_1}$  and  $\alpha_{g_2}$  commute up to a scalar, but can't be appropriately modified to commute exactly.

See Exercise 4.19 below for the computation of the crossed product.

**Exercise 2.11.** Let  $A$  be a simple unital C\*-algebra, and let  $\alpha: \mathbb{Z}/n\mathbb{Z} \rightarrow \text{Aut}(A)$  be an action such that each automorphism  $\alpha_g$ , for  $g \in \mathbb{Z}/n\mathbb{Z}$ , is an inner automorphism. Prove that  $\alpha$  is an inner action in the sense of Example 2.9.

**Problem 2.12.** Find a counterexample to Exercise 2.11 when  $A$  is not assumed simple. (I presume that a counterexample exists, but I do not know of one.)

**Exercise 2.13.** Prove the statements made in Example 2.10.

**Remark 2.14.** There is a generalization of inner actions that should be mentioned. Actions  $\alpha$  and  $\beta$  of a locally compact group  $G$  on a unital C\*-algebra  $A$  are called *exterior equivalent* if there is a continuous map  $g \mapsto z_g$  from  $G$  to the unitary group of  $A$  such that  $z_{gh} = z_g \alpha_g(z_h)$  and  $\beta_g = \text{Ad}(z_g) \circ \alpha_g$  for  $g, h \in G$ . If  $A$  is not unital, use a strictly continuous map to the unitary group of the multiplier algebra. (See 8.11.3 of [25].) An action is inner if and only if it is exterior equivalent to the trivial action, and it turns out that exterior equivalent actions give isomorphic crossed products. See Exercise 4.11 below for the computation when  $G$  is discrete.

**Example 2.15.** For  $\theta \in \mathbb{R}$ , let  $A_\theta$  be the rotation algebra, which is the universal C\*-algebra generated by two unitaries  $u$  and  $v$  satisfying the commutation relation  $vu = \exp(2\pi i\theta)uv$ . (The convention  $e^{2\pi i\theta}$  instead of  $e^{i\theta}$  has become so standard that it can't be changed.) See Example 4.25 below. (If  $\theta \notin \mathbb{Q}$ , then  $A_\theta$  is known to be simple. This follows from Example 4.25 and Theorem 5.12 below. Thus, one may take *any* C\*-algebra generated by two unitaries satisfying the appropriate commutation relation.) The group  $\text{SL}_2(\mathbb{Z})$  acts on  $A_\theta$  by sending the matrix

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$$

to the automorphism determined by

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}} \quad \text{and} \quad \alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$

The algebra  $A_\theta$  is often considered to be a noncommutative analog of the torus  $S^1 \times S^1$  (more accurately, of  $A_0 \cong C(S^1 \times S^1)$ ), and this action is the analog of the action of  $\text{SL}_2(\mathbb{Z})$  on  $S^1 \times S^1$  in Example 2.8.

The group  $\text{SL}_2(\mathbb{Z})$  has finite subgroups of orders 2, 3, 4, and 6. Restriction of the action gives actions of these groups on rotation algebras. The crossed products by these actions have been intensively studied. Recently, it has been proved [6] that for  $\theta \notin \mathbb{Q}$  they are all AF.

There is a “higher dimensional” analog of  $A_\theta$ , given by the universal C\*-algebra generated by  $d$  unitaries  $u_1, u_2, \dots, u_d$  which commute up to fixed scalars. The resulting C\*-algebra is called a “higher dimensional noncommutative torus”. Unfortunately, there is in general no action of  $\text{SL}_d(\mathbb{Z})$  on this algebra analogous to the action of  $\text{SL}_2(\mathbb{Z})$  on  $A_\theta$ .

**Example 2.16.** Let  $A_\theta$  be generated by unitaries  $u$  and  $v$  as in Example 2.15. For  $\zeta_1, \zeta_2 \in S^1$ , the unitaries  $\zeta_1 u$  and  $\zeta_2 v$  satisfy the same commutation relation. Therefore there is an action  $\alpha: S^1 \times S^1 \rightarrow \text{Aut}(A_\theta)$  determined by  $\alpha_{(\zeta_1, \zeta_2)}(u) = \zeta_1 u$  and  $\alpha_{(\zeta_1, \zeta_2)}(v) = \zeta_2 v$ . (Checking continuity of the action requires a  $3\varepsilon$  argument.)

If we fix  $\zeta_1, \zeta_2 \in S^1$ , then  $\alpha_{(\zeta_1, \zeta_2)}$  generates an action of  $\mathbb{Z}$ . If both have finite order, we get an action of a finite cyclic group. For example, there is an action of  $\mathbb{Z}/n\mathbb{Z}$  generated by the automorphism which sends  $u$  to  $\exp(2\pi i/n)u$  and  $v$  to  $v$ .

This action does generalize to an arbitrary higher dimensional noncommutative torus.

**Problem 2.17.** Find examples of actions of finite groups on higher dimensional noncommutative tori with interesting crossed products. For this purpose, the actions one gets from the higher dimensional version of Example 2.16 are not very interesting, because the crossed product is closely related to another higher dimensional noncommutative torus. The only known general example that is interesting in this sense is the “flip” action of  $\mathbb{Z}/2\mathbb{Z}$ , generated by  $u_k \mapsto u_k^*$  for  $1 \leq k \leq d$ . Whenever the higher dimensional noncommutative torus is simple, the crossed product by this action is known to be AF [6].

**Example 2.18.** Recall that the Cuntz algebra  $\mathcal{O}_n$  is the universal unital  $C^*$ -algebra on generators  $s_1, s_2, \dots, s_n$ , subject to the relations  $s_j^* s_j = 1$  for  $1 \leq j \leq n$  and  $\sum_{j=1}^n s_j s_j^* = 1$ . (It is in fact simple, so any  $C^*$ -algebra generated by elements satisfying these relations is isomorphic to  $\mathcal{O}_n$ .)

There is an action of  $(S^1)^n$  on  $\mathcal{O}_n$  such that  $\alpha_{(\zeta_1, \zeta_2, \dots, \zeta_n)}(s_j) = \zeta_j s_j$  for  $1 \leq j \leq n$ . (Check that the elements  $\zeta_j s_j$  satisfy the required relations.)

In fact, regarding  $(S^1)^n$  as the diagonal unitary matrices, this action extends to an action of the unitary group  $U(M_n)$  on  $\mathcal{O}_n$ , defined as follows. If  $u = (u_{j,k})_{j,k=1}^n \in M_n$  is unitary, then define an automorphism  $\alpha_u$  of  $\mathcal{O}_n$  by the following action on the generating isometries  $s_1, s_2, \dots, s_n$ :

$$\alpha_u(s_j) = \sum_{k=1}^n u_{k,j} s_k.$$

(This action is described, in a different form, in Section 2 of [8].) This determines a continuous action of the compact group  $U(M_n)$  on  $\mathcal{O}_n$ .

Any individual automorphism from this action gives an action of  $\mathbb{Z}$  on  $\mathcal{O}_n$ .

**Exercise 2.19.** Verify that the formula given in Example 2.18 does in fact define a continuous action of  $U(M_n)$  on  $\mathcal{O}_n$ .

**Example 2.20.** Let  $k_1, k_2, \dots$  be integers with all  $k_n \geq 2$ . Consider the UHF algebra  $A$  of type  $\prod_{n=1}^{\infty} k_n$ . We construct it as  $\bigotimes_{n=1}^{\infty} M_{k_n}$ , or, in more detail, as  $\varinjlim A_n$  with  $A_n = M_{k_1} \otimes M_{k_2} \otimes \dots \otimes M_{k_n}$ . Note that  $A_n = A_{n-1} \otimes M_{k_n}$ , and the map  $\varphi_n: A_{n-1} \rightarrow A_n$  is given by  $a \mapsto a \otimes 1_{M_{k_n}}$ .

Let  $G$  be a locally compact group, and let  $\beta^{(n)}: G \rightarrow \text{Aut}(M_{k_n})$  be an action of  $G$  on  $M_{k_n}$ . (The easiest way to get such an action is to choose a unitary representation  $g \mapsto u_n(g)$  on  $\mathbb{C}^{k_n}$ , and set  $\beta_g^{(n)}(a) = u_n(g) a u_n(g)^*$  for  $g \in G$  and  $a \in M_{k_n}$ .) Then define an action  $\alpha^{(n)}: G \rightarrow \text{Aut}(A_n)$  by

$$\alpha_g^{(n)}(a_1 \otimes a_2 \otimes \dots \otimes a_n) = \beta_g^{(1)}(a_1) \otimes \beta_g^{(2)}(a_2) \otimes \dots \otimes \beta_g^{(n)}(a_n).$$

One checks immediately that  $\varphi_n \circ \alpha_g^{(n-1)} = \alpha_g^{(n)} \circ \varphi_n$  for all  $n \in \mathbb{N}$  and  $g \in G$ , so there is a direct limit action  $g \mapsto \alpha_g$  of  $G$  on  $A = \varinjlim A_n$ . (One needs a  $3\varepsilon$  argument to prove continuity. See Exercise 2.21.)

If each  $\beta^{(n)}$  is inner, the resulting action is called a *product type action*. The general case of such actions was first seriously investigated in [11] and [12].

As a specific example, take  $G = \mathbb{Z}/2\mathbb{Z}$ , and for every  $n$  take  $k_n = 2$  and take  $\beta^{(n)}$  to be generated by  $\text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

**Exercise 2.21.** Prove that the actions of Example 2.20 really are continuous.

**Example 2.22.** Let  $A$  be a C\*-algebra. The *tensor flip* is the automorphism  $\varphi \in \text{Aut}(A \otimes_{\max} A)$  of order 2 determined by the formula  $\varphi(a \otimes b) = b \otimes a$  for  $a, b \in A$ . (Use the universal property of  $A \otimes_{\max} A$  in the unital case. Reduce the nonunital case to the unital case.) This gives an action of  $\mathbb{Z}/2\mathbb{Z}$  on  $A \otimes_{\max} A$ .

Similarly, the same formula defines a tensor flip action of  $\mathbb{Z}/2\mathbb{Z}$  on  $A \otimes_{\min} A$ . (Choose an injective representation  $\pi: A \rightarrow L(H)$ , and consider  $\pi \otimes \pi$  as a representation of  $A \otimes_{\min} A$  on  $H \otimes H$ .)

In a similar manner, the symmetric group  $S_n$  acts on the  $n$ -fold maximal and minimal tensor products of  $A$  with itself.

**Example 2.23.** Let  $A$  be a unital C\*-algebra. Let  $B = \bigotimes_{n \in \mathbb{Z}} A$  be the infinite minimal tensor product of copies of  $A$ . We define the *minimal shift* on  $B$  as follows. Set  $B_n = A^{\otimes(2n)}$ , the (minimal) tensor product of  $2n$  copies of  $A$ . (Take  $B_0 = \mathbb{C}$ .) Define  $\varphi_n: B_n \rightarrow B_{n+1}$  by  $\varphi_n(a) = 1_A \otimes a \otimes 1_A$  for  $a \in B_n$ . Identify  $B$  with  $\varinjlim B_n$ , using the maps  $\varphi_n$  in the direct system. Then take  $\sigma: B \rightarrow B$  to be the direct limit of the maps  $\sigma_n: B_n \rightarrow B_{n+1}$  defined by  $\sigma_n(a) = 1_A \otimes 1_A \otimes a$  for  $a \in B_n$ .

We define the *maximal shift* on the infinite maximal tensor product in the same manner.

Example 2.23 is the noncommutative analog of Example 2.4. Indeed, using the notation there, if  $A = \mathbb{C}^2$ , then  $\bigotimes_{n \in \mathbb{Z}} A \cong C(X)$ , and the tensor shift is the automorphism induced by the shift on  $X$ .

### 3. CROSSED PRODUCTS

In this section, we give the definition of both full and reduced crossed products, and prove a few results closely related to the construction.

**Definition 3.1.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a locally compact group  $G$  on a C\*-algebra  $A$ . A *covariant representation* of  $(G, A, \alpha)$  on a Hilbert space  $H$  is a pair  $(v, \pi)$  consisting of a unitary representation  $v: G \rightarrow U(H)$  (the unitary group of  $H$ ) and a representation  $\pi: A \rightarrow L(H)$  (the algebra of all bounded operators on  $H$ ), satisfying the *covariance condition*

$$v(g)\pi(a)v(g)^* = \pi(\alpha_g(a))$$

for all  $g \in G$  and  $a \in A$ . It is called *nondegenerate* if  $\pi$  is nondegenerate.

By convention, unitary representations are strong operator continuous. (This is the usual continuity condition.) Representations of C\*-algebras, and of other \*-algebras (such as the algebras  $L^1(G, A, \alpha)$  and  $C_c(G, A, \alpha)$  introduced below) are \*-representations (and, similarly, homomorphisms are \*-homomorphisms).

The crossed product C\*-algebra  $C^*(G, A, \alpha)$  is the universal C\*-algebra for covariant representations of  $(G, A, \alpha)$ , in essentially the same way that the (full) group

C\*-algebra  $C^*(G)$  is the universal C\*-algebra for unitary representations of  $G$ . We construct it in a similar way to the group C\*-algebra. We start with the analog of  $L^1(G)$ .

For a general locally compact group, one needs an appropriate notion of integration of Banach space valued functions. One must prove that twisted convolution of Definition 3.2 below is well defined, associative, distributive, and satisfies  $(ab)^* = b^*a^*$ . Once one has the appropriate notion of integration, the proofs are similar to the proofs of the corresponding facts about convolution in  $L^1(G)$ . Integration of continuous functions with compact support is much easier than integration of  $L^1$  functions, but the “right” way to do this is to define measurable Banach space valued functions and their integrals. This has been done; one reference is Appendix B of [38]. (Note the systematic misprint there: “separately-valued” should be “separably-valued”.) Things simplify considerably if  $G$  is second countable and  $A$  is separable, but neither of these conditions is necessary for the construction of  $L^1(G, A, \alpha)$  or the crossed product.

**Definition 3.2.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a locally compact group  $G$  on a C\*-algebra  $A$ . We let  $C_c(G, A, \alpha)$  be the \*-algebra of continuous functions  $a: G \rightarrow A$ , with pointwise addition and scalar multiplication. Using Haar measure in the integral, we define multiplication by the following “twisted convolution”:

$$(ab)(g) = \int_G a(h)\alpha_h(b(h^{-1}g)) dh.$$

Let  $\Delta$  be the modular function of  $G$ . We define the adjoint by

$$a^*(g) = \Delta(g)^{-1}\alpha_g(a(g^{-1})^*).$$

We define a norm  $\|\cdot\|_1$  on  $C_c(G, A, \alpha)$  by  $\|a\|_1 = \int_G \|a(g)\| dg$ . One checks that  $\|ab\|_1 \leq \|a\|_1\|b\|_1$  and  $\|a^*\|_1 = \|a\|_1$ . Then  $L^1(G, A, \alpha)$  is the Banach \*-algebra obtained by completing  $C_c(G, A, \alpha)$  in  $\|\cdot\|_1$ .

**Exercise 3.3.** In the situation of Definition 3.2, and assuming suitable versions of Fubini’s Theorem for Banach space valued integrals, check that that multiplication in  $C_c(G, A, \alpha)$  is associative. Further check for  $a, b \in C_c(G, A, \alpha)$  that  $\|ab\|_1 \leq \|a\|_1\|b\|_1$ , that  $(ab)^* = b^*a^*$ , and that  $\|a^*\|_1 = \|a\|_1$ .

**Remark 3.4.** Suppose  $A = C_0(X)$ , and  $\alpha$  comes from an action of  $G$  on  $X$ . Since we complete in a suitable norm later on, it suffices to use only the dense subalgebra  $C_c(X)$  in place of  $C_0(X)$ . There is an obvious identification of  $C_c(G, C_c(X))$  with  $C_c(G \times X)$ . On  $C_c(G \times X)$ , the formulas for multiplication and adjoint become

$$(f_1 f_2)(g, x) = \int_G f_1(h, x) f_2(h^{-1}g, h^{-1}x) dh$$

and

$$f^*(g, x) = \Delta(g)^{-1} \overline{f(g^{-1}, g^{-1}x)}.$$

**Exercise 3.5.** Check the formulas in Remark 3.4.

**Remark 3.6.** If  $G$  is discrete, we choose Haar measure to be counting measure. In this case,  $C_c(G, A, \alpha)$  is, as a vector space, the group ring  $A[G]$ , consisting of all finite formal linear combinations of elements in  $G$  with coefficients in  $A$ . The multiplication and adjoint are given by

$$(a \cdot g)(b \cdot h) = (a[gbg^{-1}]) \cdot (gh) = (a\alpha_g(b)) \cdot (gh) \quad \text{and} \quad (a \cdot g)^* = \alpha_g^{-1}(a^*) \cdot g^{-1}$$

for  $a, b \in A$  and  $g, h \in G$ , extended linearly. This definition makes sense in the purely algebraic situation, where it is called the *skew group ring*.

When  $G$  is discrete, we also often write  $l^1(G, A, \alpha)$  instead of  $L^1(G, A, \alpha)$ .

**Notation 3.7.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a discrete group  $G$  on a C\*-algebra  $A$ . In these notes, we will adopt the following fairly commonly used notation. First, suppose  $A$  is unital. For  $g \in G$ , we let  $u_g$  be the element of  $C_c(G, A, \alpha)$  which takes the value  $1_A$  at  $g$  and 0 at the other elements of  $G$ . We use the same notation for its image in  $l^1(G, A, \alpha)$  (Definition 3.2 above) and in  $C^*(G, A, \alpha)$  and  $C_r^*(G, A, \alpha)$  (Definitions 3.14 and 3.22 below). It is unitary, and we call it the canonical unitary associated with  $g$ .

If  $A$  is not unital, extend the action to an action  $\alpha^+: G \rightarrow \text{Aut}(A^+)$  on the unitization  $A^+$  of  $A$  by  $\alpha_g^+(a + \lambda \cdot 1) = \alpha_g(a) + \lambda \cdot 1$ . Then write  $u_g$  as above. Products  $au_g$ , with  $a \in A$ , are still in  $C_c(G, A, \alpha)$ ,  $l^1(G, A, \alpha)$ ,  $C^*(G, A, \alpha)$ , or  $C_r^*(G, A, \alpha)$ , as appropriate.

In particular,  $l^1(G, A, \alpha)$  is the set of all sums  $\sum_{g \in G} a_g u_g$  with  $a_g \in A$  and  $\sum_{g \in G} \|a_g\| < \infty$ . These sums converge in  $l^1(G, A, \alpha)$ , and hence also in  $C^*(G, A, \alpha)$  and  $C_r^*(G, A, \alpha)$ . A general element of  $C_r^*(G, A, \alpha)$  has such an expansion, but unfortunately the series one writes down generally does not converge. See the discussion at the end of this section.

**Definition 3.8.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a locally compact group  $G$  on a C\*-algebra  $A$ , and let  $(v, \pi)$  be a covariant representation of  $(G, A, \alpha)$  on a Hilbert space  $H$ . Then the *integrated form* of  $(v, \pi)$  is the representation  $\sigma: C_c(G, A, \alpha) \rightarrow L(H)$  given by

$$\sigma(a)\xi = \int_G \pi(a(g))v(g)\xi dg.$$

(This representation is sometimes called  $v \times \pi$  or  $\pi \times v$ .)

One needs to be more careful with the integral here, because  $v$  is generally only strong operator continuous, not norm continuous. Nevertheless, one gets  $\|\sigma(a)\| \leq \|a\|_1$ , so  $\sigma$  extends to a representation of  $L^1(G, A, \alpha)$ . We use the same notation  $\sigma$  for this extension.

Of course, one also needs to check that  $\sigma$  is a representation. When  $G$  is discrete, and using Notation 3.7, the formula for  $\sigma$  comes down to  $\sigma(au_g) = \pi(a)v(g)$  for  $a \in A$  and  $g \in G$ . Then

$$\begin{aligned} \sigma(au_g)\sigma(bu_h) &= \pi(a)v(g)\pi(b)v(g)^*v(g)v(h) = \pi(a)\pi(\alpha_g(b))v(g)v(h) \\ &= \pi(a\alpha_g(b))v(gh) = \sigma([a\alpha_g(b)]u_{gh}) = \sigma((au_g)(bu_h)). \end{aligned}$$

**Exercise 3.9.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a discrete group  $G$  on a C\*-algebra  $A$ , and let  $(v, \pi)$  be a nondegenerate covariant representation of  $(G, A, \alpha)$  on a Hilbert space  $H$ . Starting from the computation above, fill in the details of the proof that the integrated form representation  $\sigma$  of Definition 3.8 really is a nondegenerate representation of  $C_c(G, A, \alpha)$ .

**Theorem 3.10** (Proposition 7.6.4 of [25]). Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a locally compact group  $G$  on a C\*-algebra  $A$ . Then the integrated form construction defines a bijection from the set of covariant representations of  $(G, A, \alpha)$  on a Hilbert space  $H$  to the set of nondegenerate continuous representations of  $L^1(G, A, \alpha)$  on the same Hilbert space.

**Remark 3.11.** In particular, since integrated form representations of  $L^1(G, A, \alpha)$  are necessarily contractive, *all* continuous representations of  $L^1(G, A, \alpha)$  are necessarily contractive.

If  $G$  is discrete and  $A$  is unital, then there are homomorphic images of both  $G$  and  $A$  inside  $C_c(G, A, \alpha)$ , given by  $g \mapsto u_g$  and  $a \mapsto au_1$ , so it is clear how to get a covariant representation of  $(G, A, \alpha)$  from a nondegenerate representation of  $C_c(G, A, \alpha)$ . In general, one must use the multiplier algebra of  $L^1(G, A, \alpha)$ , which contains copies of  $M(A)$  and  $M(L^1(G))$ . The point is that  $M(L^1(G))$  is the measure algebra of  $G$ , and therefore contains the group elements as point masses.

**Exercise 3.12.** Prove Theorem 3.10 when  $G$  is discrete and  $A$  is unital.

For a small taste of the general case, use approximate identities in  $A$  to do the following problem:

**Exercise 3.13.** Prove Theorem 3.10 when  $G$  is discrete but  $A$  is not necessarily unital.

**Definition 3.14.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a locally compact group  $G$  on a  $C^*$ -algebra  $A$ . We define the *universal representation*  $\sigma$  of  $L^1(G, A, \alpha)$  to be the direct sum of all nondegenerate representations of  $L^1(G, A, \alpha)$  on Hilbert spaces. Then we define the *crossed product*  $C^*(G, A, \alpha)$  to be the norm closure of  $\sigma(L^1(G, A, \alpha))$ .

One could of course equally well use the norm closure of  $\sigma(C_c(G, A, \alpha))$ .

There is a minor set theoretic detail: the collection of all nondegenerate representations of  $L^1(G, A, \alpha)$  is not a set. There are several standard ways to deal with this problem, but in these notes we will ignore the issue.

**Exercise 3.15.** Give a set theoretically correct definition of the crossed product. The important point is to preserve the universal property in the next paragraph.

It follows that every covariant representation of  $(G, A, \alpha)$  gives a representation of  $C^*(G, A, \alpha)$ . (Take the integrated form, and restrict elements of  $C^*(G, A, \alpha)$  to the appropriate summand in the direct sum in Definition 3.14.) The crossed product is, essentially by construction, the universal  $C^*$ -algebra for covariant representations of  $(G, A, \alpha)$ , in the same sense that if  $G$  is a locally compact group, then  $C^*(G)$  is the universal  $C^*$ -algebra for unitary representations of  $G$ .

**Remark 3.16.** There are many notations in use for crossed products, including:

- $C^*(G, A, \alpha)$  and  $C_r^*(G, A, \alpha)$ .
- $C^*(A, G, \alpha)$  and  $C_r^*(A, G, \alpha)$ .
- $A \rtimes_\alpha G$  and  $A \rtimes_{\alpha, r} G$  (used in the book [38]).
- $A \times_\alpha G$  and  $A \times_{\alpha, r} G$  (used in the book [4]).
- $G \rtimes_\alpha A$  and  $G \rtimes_{\alpha, r} A$  (used in the book [25]).

In the notation for the reduced crossed products (especially the first two versions), the letter “r” (“reduced”) is sometimes replaced by “ $\lambda$ ” (the conventional name for the left regular representation of a group). The symbol in the third comes from the relation  $C^*(N \rtimes H) \cong C^*(H, C^*(N))$ , and is meant to suggest a generalized semidirect product. The first two make it easy to distinguish  $C^*$ -algebra crossed products from other sorts, such as von Neumann algebra crossed products, smooth crossed products,  $L^1$  crossed products, and purely algebraic crossed products (all of

which will receive short shrift in these notes, but are important in their own right, sometimes in the same paper). I use the order  $C^*(G, A, \alpha)$  because it matches the natural order in  $C_c(G, A, \alpha)$  and  $L^1(G, A, \alpha)$ .

**Theorem 3.17.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a discrete group  $G$  on a unital C\*-algebra  $A$ . Then  $C^*(G, A, \alpha)$  is the universal C\*-algebra generated by a unital copy of  $A$  (that is, the identity of  $A$  is supposed to be the identity of the generated C\*-algebra) and unitaries  $u_g$ , for  $g \in G$ , subject to the relations  $u_g u_h = u_{gh}$  for  $g, h \in G$  and  $u_g a u_g^* = \alpha_g(a)$  for  $a \in A$  and  $g \in G$ .

**Corollary 3.18.** Let  $A$  be a unital C\*-algebra, and let  $\alpha \in \text{Aut}(A)$ . Then the crossed product  $C^*(\mathbb{Z}, A, \alpha)$  is the universal C\*-algebra generated by a copy of  $A$  and a unitary  $u$ , subject to the relations  $u a u^* = \alpha(a)$  for  $a \in A$ .

**Exercise 3.19.** Based on the discussion above, write down a careful proof of Theorem 3.17.

So far, it is not clear that there are any covariant representations.

**Definition 3.20** (7.7.1 of [25]). Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a locally compact group  $G$  on a C\*-algebra  $A$ . Let  $\pi_0: A \rightarrow L(H_0)$  be a representation. We define the *regular covariant representation*  $(v, \pi)$  of  $(G, A, \alpha)$  on the Hilbert space  $H = L^2(G, H_0)$  of  $L^2$  functions from  $G$  to  $H$  as follows. For  $g, h \in G$ , set

$$(v(g)\xi)(h) = \xi(g^{-1}h).$$

For  $a \in A$  and  $g \in G$ , set

$$(\pi(a)\xi)(h) = \pi_0(\alpha_{h^{-1}}(a))(\xi(h)).$$

The integrated form of  $\sigma$ , as in Definition 3.8, will be called a regular representation of any of  $C_c(G, A, \alpha)$ ,  $L^1(G, A, \alpha)$ ,  $C^*(G, A, \alpha)$ , and (when we have defined it; see Definition 3.22)  $C_r^*(G, A, \alpha)$ .

The easy way to construct  $L^2(G, H)$  is to take it to be the completion of  $C_c(G, H)$  in the norm coming from the scalar product

$$\langle \xi, \eta \rangle = \int_G \langle \xi(g), \eta(g) \rangle dg.$$

**Exercise 3.21.** In Definition 3.20, suppose that  $G$  is discrete. Prove that  $(v, \pi)$  really is a covariant representation.

Note that, if  $A = \mathbb{C}$ ,  $H_0 = \mathbb{C}$ , and  $\pi_0$  is the obvious representation of  $A$  on  $H_0$ , then the representation of Definition 3.20 is the usual left regular representation of  $G$ .

**Definition 3.22.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a locally compact group  $G$  on a C\*-algebra  $A$ . Let  $\lambda: L^1(G, A, \alpha) \rightarrow L(H)$  be the direct sum of all regular representations of  $L^1(G, A, \alpha)$ . We define the *reduced crossed product*  $C_r^*(G, A, \alpha)$  to be the norm closure of  $\lambda(L^1(G, A, \alpha))$ .

As with crossed products, in these notes we ignore the set theoretic difficulty.

Implicit in the definition of  $C_r^*(G, A, \alpha)$  is a representation of  $L^1(G, A, \alpha)$ , hence of  $C^*(G, A, \alpha)$ . Thus, there is a homomorphism  $C^*(G, A, \alpha) \rightarrow C_r^*(G, A, \alpha)$ . By construction, it has dense range, and is therefore surjective. Moreover, by construction, any regular representation of  $L^1(G, A, \alpha)$  extends to a representation of  $C_r^*(G, A, \alpha)$ .

**Theorem 3.23** (Theorem 7.7.7 of [25]). Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a locally compact group  $G$  on a  $C^*$ -algebra  $A$ . If  $G$  is amenable, then  $C^*(G, A, \alpha) \rightarrow C_r^*(G, A, \alpha)$  is an isomorphism.

The converse is true for  $A = \mathbb{C}$ : if  $C^*(G) \rightarrow C_r^*(G)$  is an isomorphism, then  $G$  is amenable. But it is not true in general. For example, if  $G$  acts on itself by translation, then  $C^*(G, C_0(G)) \rightarrow C_r^*(G, C_0(G))$  is an isomorphism for every  $G$ . See Example 4.13 for the case of a discrete group.

**Theorem 3.24.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a locally compact group  $G$  on a  $C^*$ -algebra  $A$ . Then  $C_c(G, A, \alpha) \rightarrow C_r^*(G, A, \alpha)$  is injective.

We will prove this below in the case of a discrete group. The proof of the general case can be found in Lemma 2.26 of [38]. It is, I believe, true that  $L^1(G, A, \alpha) \rightarrow C_r^*(G, A, \alpha)$  is injective, and this can probably be proved by working a little harder in the proof of Lemma 2.26 of [38], but I have not carried out the details and I do not know a reference.

We now further analyze the reduced crossed product  $C_r^*(G, A, \alpha)$  when  $G$  is discrete. The main tool is the structure of regular representations of  $C_r^*(G, A, \alpha)$ . When  $G$  is discrete, we can write  $L^2(G, H_0)$  as a Hilbert space direct sum  $\bigoplus_{g \in G} H_0$ , and elements of it can be thought of as families  $(\xi_g)_{g \in G}$ .

**Lemma 3.25.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a discrete group  $G$  on a  $C^*$ -algebra  $A$ . Let  $\pi_0: A \rightarrow L(H_0)$  be a representation, and let  $\sigma: C_r^*(G, A, \alpha) \rightarrow H = L^2(G, H_0)$  be the associated regular representation. Let  $a = \sum_{g \in G} a_g u_g \in C_r^*(G, A, \alpha)$ , with  $a_g = 0$  for all but finitely many  $g$ . For  $\xi \in H$ , we then have

$$(\sigma(a)\xi)(h) = \sum_{g \in G} \pi_0(\alpha_h^{-1}(a_g))(\xi(g^{-1}h))$$

for all  $h \in G$ .

*Proof.* This is a calculation.  $\square$

**Corollary 3.26.** Let the hypotheses be as in Lemma 3.25, and let  $a = \sum_{g \in G} a_g u_g \in C_r^*(G, A, \alpha)$  as there. For  $g \in G$ , let  $s_g \in L(H_0, H)$  be the isometry which sends  $\eta \in H_0$  to the function  $\xi \in L^2(G, H_0)$  which takes the value  $\eta$  at  $g$  and is zero elsewhere. Then  $s_h^* \sigma(a) s_k = \pi_0(\alpha_h^{-1}(a_{hk^{-1}}))$  for all  $h, k \in G$ .

*Proof.* This is an easy calculation from Lemma 3.25.  $\square$

**Lemma 3.27.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a discrete group  $G$  on a  $C^*$ -algebra  $A$ . Let  $\|\cdot\|$  be the  $C^*$ -algebra norm on  $C^*(G, A, \alpha)$  restricted to  $C_c(G, A, \alpha)$ , let  $\|\cdot\|_r$  be the  $C^*$ -algebra norm on  $C_r^*(G, A, \alpha)$  restricted to  $C_c(G, A, \alpha)$ , and let  $\|\cdot\|_\infty$  be the supremum norm. Then for every  $a \in C_c(G, A, \alpha)$ , we have  $\|a\|_\infty \leq \|a\|_r \leq \|a\| \leq \|a\|_1$ .

*Proof.* The middle of this inequality follows from the definitions.

The last part follows from the observation in Remark 3.11 that all continuous representations of  $L^1(G, A, \alpha)$  are norm reducing. Here is a direct proof: for  $a = \sum_{g \in G} a_g u_g \in C_c(G, A, \alpha)$ , with all but finitely many of the  $a_g$  equal to zero, we have

$$\left\| \sum_{g \in G} a_g u_g \right\| \leq \sum_{g \in G} \|a_g\| \cdot \|u_g\| = \sum_{g \in G} \|a_g\| = \left\| \sum_{g \in G} a_g u_g \right\|_1.$$

We prove the first part of this inequality. Let  $a = \sum_{g \in G} a_g u_g$ , with all but finitely many of the  $a_g$  equal to zero, and let  $g \in G$ . Let  $\pi_0: A \rightarrow L(H_0)$  be an injective nondegenerate representation. With the notation of Corollary 3.26, we have

$$\|a_g\| = \|\pi_0(a_g)\| = \|s_g^* \sigma(a) s_1\| \leq \|\sigma(a)\| \leq \|a\|_r.$$

This completes the proof.  $\square$

**Remark 3.28.** Lemma 3.27 implies that the map  $a \mapsto au_1$ , from  $A$  to  $C_r^*(G, A, \alpha)$ , is injective. We routinely identify  $A$  with its image in  $C_r^*(G, A, \alpha)$  under this map, thus treating it as a subalgebra of  $C_r^*(G, A, \alpha)$ .

Of course, we can do the same with the full crossed product  $C^*(G, A, \alpha)$ .

**Corollary 3.29.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a finite group  $G$  on a C\*-algebra  $A$ . Then the maps  $C_c(G, A, \alpha) \rightarrow C^*(G, A, \alpha) \rightarrow C_r^*(G, A, \alpha)$  are bijective.

*Proof.* When  $G$  is finite,  $\|\cdot\|_1$  is equivalent to  $\|\cdot\|_\infty$  as defined in Lemma 3.27, and is complete in both. Lemma 3.27 implies that both C\* norms are equivalent to these norms, so  $C_c(G, A, \alpha)$  is complete in both C\* norms.  $\square$

When  $G$  is discrete but not finite, things are much more complicated. We can get started:

**Proposition 3.30.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a discrete group  $G$  on a C\*-algebra  $A$ . Then for each  $g \in G$ , there is a linear map  $E_g: C_r^*(G, A, \alpha) \rightarrow A$  with  $\|E_g\| \leq 1$  such that if  $a = \sum_{g \in G} a_g u_g \in C_c(G, A, \alpha)$ , then  $E_g(a) = a_g$ . Moreover, with  $s_g$  as in Corollary 3.26, we have  $s_h^* \sigma(a) s_k = \pi_0(\alpha_h^{-1}(E_{hk^{-1}}(a)))$  for all  $h, k \in G$ .

*Proof.* This is immediate from the first inequality in Lemma 3.27. The last statement follows from Corollary 3.26 by continuity.  $\square$

Thus, for any  $a \in C_r^*(G, A, \alpha)$ , and therefore also for  $a \in C^*(G, A, \alpha)$ , it makes sense to talk about its coefficients  $a_g$ . The first point is that if  $C^*(G, A, \alpha) \neq C_r^*(G, A, \alpha)$  (which can happen if  $G$  is not amenable, but not if  $G$  is amenable), the coefficients  $(a_g)_{g \in G}$  do not even uniquely determine the element  $a$ . This is why we are only considering reduced crossed products here.

**Proposition 3.31.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a discrete group  $G$  on a C\*-algebra  $A$ . Let the maps  $E_g: C_r^*(G, A, \alpha) \rightarrow A$  be as in Proposition 3.30. Then:

- (1) If  $a \in C_r^*(G, A, \alpha)$  and  $E_g(a) = 0$  for all  $g \in G$ , then  $a = 0$ .
- (2) If  $\pi_0: A \rightarrow L(H_0)$  is a nondegenerate representation such that  $\bigoplus_{g \in G} \pi_0 \circ \alpha_g$  is injective, then the regular representation  $\sigma$  of  $C_r^*(G, A, \alpha)$  associated to  $\pi_0$  is injective.
- (3) If  $a \in C_r^*(G, A, \alpha)$  and  $E_1(a^*a) = 0$ , then  $a = 0$ .

*Proof.* We prove (1). Let  $\pi_0: A \rightarrow L(H_0)$  be a representation, and let the notation be as in Corollary 3.26. If  $a \in C_r^*(G, A, \alpha)$  satisfies  $E_g(a) = 0$  for all  $g \in G$ , then  $s_h^* \sigma(a) s_k = 0$  for all  $h, k \in G$ , whence  $\sigma(a) = 0$ . Since  $\pi_0$  is arbitrary, it follows that  $a = 0$ . This proves (1).

For (2), suppose  $a \in C_r^*(G, A, \alpha)$  and  $\sigma(a) = 0$ . Fix  $l \in G$ . Taking  $h = g^{-1}$  and  $k = l^{-1}g^{-1}$  in Proposition 3.30, we get  $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$  for all  $g \in G$ . So  $E_l(a) = 0$ . This is true for all  $l \in G$ , so  $a = 0$ .

We now prove (3). As before, let  $a = \sum_{g \in G} a_g u_g \in C_c(G, A, \alpha)$ . Then  $a^*a = \sum_{g, h \in G} u_g^* a_g^* a_h u_h$ , so

$$E_1(a^*a) = \sum_{g \in G} u_g^* a_g^* a_g u_g = \sum_{g \in G} \alpha_g^{-1}(E_g(a)^* E_g(a)).$$

In particular, for each fixed  $g$ , we have  $E_1(a^*a) \geq \alpha_g^{-1}(E_g(a)^* E_g(a))$ . By continuity, this inequality holds for all  $a \in C_r^*(G, A, \alpha)$ . Thus, if  $E_1(a^*a) = 0$ , then  $E_g(a)^* E_g(a) = 0$  for all  $g$ , so  $a = 0$  by Part (1).  $\square$

It is true for general locally compact groups, not just discrete groups, that the regular representation of  $C_r^*(G, A, \alpha)$  associated to an injective representation of  $A$  is injective. See Theorem 7.7.5 of [25].

The map  $E_1$  used in Proposition 3.31(3) is an example of what is called a *conditional expectation* (from  $C_r^*(G, A, \alpha)$  to  $A$ ) that is, it has the properties given in the following exercise. Proposition 3.31(3) asserts that this conditional expectation is faithful.

**Exercise 3.32.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a discrete group  $G$  on a  $C^*$ -algebra  $A$ . Let  $E = E_1: C_r^*(G, A, \alpha) \rightarrow A$  be as in Proposition 3.30. Prove that  $E$  has the following properties:

- (1)  $E(E(b)) = E(b)$  for all  $b \in C_r^*(G, A, \alpha)$ .
- (2) If  $b \geq 0$  then  $E(b) \geq 0$ .
- (3)  $\|E(b)\| \leq \|b\|$  for all  $b \in C_r^*(G, A, \alpha)$ .
- (4) If  $a \in A$  and  $b \in C_r^*(G, A, \alpha)$ , then  $E(ab) = aE(b)$  and  $E(ba) = E(b)a$ .

Unfortunately, in general  $\sum_{g \in G} a_g u_g$  does not converge in  $C_r^*(G, A, \alpha)$ , and it is very difficult to tell exactly which families of coefficients correspond to elements of  $C_r^*(G, A, \alpha)$ . In fact, the situation is intractable even for the case of the trivial action of  $\mathbb{Z}$  on  $\mathbb{C}$ . In this case,  $l^1(\mathbb{Z}, A, \alpha)$  is the usual convolution algebra  $l^1(\mathbb{Z})$ . The crossed product is the group  $C^*$ -algebra  $C^*(\mathbb{Z})$ , which can be identified with  $C(S^1)$ . (We write  $C^*(\mathbb{Z})$  because,  $\mathbb{Z}$  being amenable, we have  $C_r^*(\mathbb{Z}) = C^*(\mathbb{Z})$ .) The map from  $l^1(\mathbb{Z})$  to  $C(S^1)$  is given by Fourier series: the sequence  $a = (a_n)_{n \in \mathbb{N}}$  goes to the function  $\zeta \mapsto \sum_{n \in \mathbb{Z}} a_n \zeta^n$ . (This looks more familiar when expressed in terms of  $2\pi$ -periodic functions on  $\mathbb{R}$ : it is  $t \mapsto \sum_{n \in \mathbb{Z}} a_n e^{int}$ .) Failure of convergence of  $\sum_{n \in \mathbb{Z}} a_n u_n$  corresponds to the fact that the Fourier series of a continuous function need not converge uniformly. Identifying the coefficient sequences which correspond to elements of the crossed product corresponds to giving a criterion for exactly when a sequence  $(a_n)_{n \in \mathbb{N}}$  of complex numbers is the sequence of Fourier coefficients of some continuous function on  $S^1$ , a problem for which I know of no satisfactory solution.

Let's pursue this a little farther. The regular representation of  $\mathbb{Z}$  on  $l^2(\mathbb{Z})$  gives an injective map  $\lambda: C^*(\mathbb{Z}) \rightarrow L(l^2(\mathbb{Z}))$ . Let  $\delta_n \in l^2(\mathbb{Z})$  be the function  $\delta_n(n) = 1$  and  $\delta_n(k) = 0$  for  $k \neq n$ . Then the Fourier coefficient  $a_n$  is recovered as  $a_n = \langle \lambda(a)\delta_0, \delta_n \rangle$ . That is,  $\lambda(a)\delta_0 \in l^2(\mathbb{Z})$  is given by  $\lambda(a)\delta_0 = \sum_{n \in \mathbb{Z}} a_n \delta_n$ . Thus, the sequence of Fourier coefficients of a continuous function is always in  $l^2(\mathbb{Z})$ . (Of course, we already know this, but the calculation here can be applied to more general crossed products.) Unfortunately, this fact is essentially useless for the study of the group  $C^*$ -algebra. Not only is the Fourier series of a continuous function always in  $l^2(\mathbb{Z})$ , but the Fourier series of a function in  $L^\infty(S^1)$ , which is the group von

Neumann algebra of  $\mathbb{Z}$ , is also always in  $l^2(\mathbb{Z})$ . One will get essentially no useful information from a criterion which can't even exclude any elements of  $L^\infty(S^1)$ .

Even if one understands completely what all the elements of  $C_r^*(G)$  are, and even if the action is trivial, understanding the elements of the reduced crossed product requires that one understand all the elements of the completed tensor product  $C_r^*(G) \otimes_{\min} A$ . As far as I know, this problem is also in general intractable.

There is just one bright spot: although we will not prove it here, there is an analog for general crossed products by  $\mathbb{Z}$  of the fact that the Cesaro means of the Fourier series of a continuous function always converge uniformly to the function. See Theorem 8.2.2 of [4].

The discussion above is meant to point out the difficulties in dealing with crossed products by infinite groups. Despite all this, for some problems, finite groups are harder. Computing the K-theory of a crossed product by  $\mathbb{Z}/2\mathbb{Z}$  is harder than computing the K-theory of a crossed product by any of  $\mathbb{Z}$ ,  $\mathbb{R}$ , or even a (nonabelian) free group!

#### 4. COMPUTATION OF SOME EXAMPLES

We give some explicit elementary computations of crossed products, mostly involving finite groups. We begin with several useful general results.

**Definition 4.1.** Let  $G$  be a locally compact group. A C\*-algebra  $A$  equipped with an action  $G \rightarrow \text{Aut}(A)$  will be called a  $G$ -algebra, or a  $G$ -C\*-algebra. We sometimes refer to  $(G, A, \alpha)$  as a  $G$ -algebra or  $G$ -C\*-algebra.

If  $(G, A, \alpha)$  and  $(G, B, \beta)$  are  $G$ -algebras, then a homomorphism  $\varphi: A \rightarrow B$  is said to be *equivariant* (or  $G$ -equivariant if the group must be specified) if for every  $g \in G$ , we have  $\varphi \circ \alpha_g = \beta_g \circ \varphi$ .

**Remark 4.2.** For a fixed locally compact group  $G$ , the  $G$ -algebras and equivariant homomorphisms form a category.

**Theorem 4.3.** Let  $G$  be a locally compact group. If  $(G, A, \alpha)$  and  $(G, B, \beta)$  are  $G$ -algebras and  $\varphi: A \rightarrow B$  is an equivariant homomorphism, then there is a homomorphism  $\psi: C_c(G, A, \alpha) \rightarrow C_c(G, B, \beta)$  given by the formula  $\psi(a)(g) = \varphi(a(g))$  for  $a \in C_c(G, A, \alpha)$  and  $g \in G$ , and this homomorphism extends by continuity to a homomorphism  $L^1(G, A, \alpha) \rightarrow L^1(G, B, \beta)$ , and then to homomorphisms

$$C^*(G, A, \alpha) \rightarrow C^*(G, B, \beta) \quad \text{and} \quad C_r^*(G, A, \alpha) \rightarrow C_r^*(G, B, \beta).$$

This construction makes the crossed product and reduced crossed product constructions functors from the category of  $G$ -algebras to the category of C\*-algebras.

*Proof.* One checks directly that  $\psi$  preserves multiplication and adjoint, and that  $\|\psi(a)\|_1 \leq \|a\|_1$  for all  $a \in C_c(G, A, \alpha)$ . The extension to the  $L^1$ -algebras and full crossed products is now immediate. To extend to the reduced crossed products, one observes that if  $\pi_0: B \rightarrow L(H_0)$  is a representation, and if  $\sigma: C_c(G, B, \beta) \rightarrow L(L^2(G, H_0))$  is the associated regular representation, then  $\sigma \circ \psi$  is the regular representation associated with the representation  $\pi_0 \circ \varphi: A \rightarrow L(H_0)$ .

The properties of a functor are easy to check. □

**Theorem 4.4** (Lemma 2.8.2 of [26]; Theorem 2.6 of [36]; Proposition 3.9 of [38]). Let  $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$  be an exact sequence of  $G$ -algebras, with actions  $\gamma$  on  $J$ ,

$\alpha$  on  $A$ , and  $\beta$  on  $B$ . Then the sequence

$$0 \longrightarrow C^*(G, J, \gamma) \longrightarrow C^*(G, A, \alpha) \longrightarrow C^*(G, B, \beta) \longrightarrow 0$$

is exact.

The analog for reduced crossed products is in general false.

**Theorem 4.5.** Let  $((G, A_i, \alpha^{(i)})_{i \in I}, (\varphi_{j,i})_{i \leq j})$  be a direct system of  $G$ -algebras. Let  $A = \varinjlim A_i$ , with action  $\alpha: G \rightarrow \text{Aut}(A)$  given by  $\alpha_g = \varinjlim \alpha_g^{(i)}$ . Let

$$\psi_{j,i}: C^*(G, A_i, \alpha^{(i)}) \rightarrow C^*(G, A_j, \alpha^{(j)})$$

be the map obtained from  $\varphi_{j,i}$ . Using these maps in the direct system of crossed products, there is a natural isomorphism  $C^*(G, A, \alpha) \cong \varinjlim C^*(G, A_i, \alpha^{(i)})$ .

*Proof.* We show that  $C^*(G, A, \alpha)$  satisfies the universal property which defines  $\varinjlim C^*(G, A_i, \alpha^{(i)})$ . First, for  $i \in I$  let  $\varphi_i: A_i \rightarrow A$  be the canonical map for the direct limit of the system  $((A_i)_{i \in I}, (\varphi_{j,i})_{i \leq j})$ . We have maps  $\psi_i: C^*(G, A_i, \alpha^{(i)}) \rightarrow C^*(G, A, \alpha)$  obtained from the  $\varphi_i$  by forming crossed products. Clearly  $\psi_j \circ \psi_{j,i} = \psi_i$  whenever  $i \leq j$ .

Now suppose we have a  $C^*$ -algebra  $B$  and homomorphisms  $\mu_i: C^*(G, A_i, \alpha^{(i)}) \rightarrow B$  such that  $\mu_j \circ \psi_{j,i} = \mu_i$  whenever  $i \leq j$ . We need to prove that there is a unique  $\mu: C^*(G, A, \alpha) \rightarrow B$  such that  $\mu \circ \psi_i = \mu_i$  for all  $i \in I$ . Without loss of generality  $B$  is a nondegenerate subalgebra of  $L(H)$  for some Hilbert space  $H$ . Then, for each  $i \in I$ , the map  $\mu_i$  is the integrated form of a covariant representation  $(v_i, \pi_i)$  of  $(G, A_i, \alpha^{(i)})$ . For  $i \leq j$ , using  $\mu_i = \mu_j \circ \psi_{j,i}$ , one checks that  $\mu_i$  is also equal to the integrated form of  $(v_j, \pi_j \circ \varphi_{j,i})$ . Uniqueness in the construction of the integrated form implies that  $v_j = v_i$  and  $\pi_j \circ \varphi_{j,i} = \pi_i$ . In particular, all the  $v_i$  are the same representation of  $G$ , so say  $v_i = v$  for all  $i \in I$ . By the universal property of  $\varinjlim A_i$ , there is  $\pi: A \rightarrow L(H)$  such that  $\pi \circ \varphi_i = \pi_i$  for all  $i \in I$ , and moreover  $(v, \pi)$  is a covariant representation. Let  $\mu: C^*(G, A, \alpha) \rightarrow L(H)$  be the integrated form of  $(v, \pi)$ . Then one gets  $\mu \circ \psi_i = \mu_i$  for all  $i \in I$ . Since  $A$  is generated by the images of the  $A_i$ , it follows that  $\mu(C^*(G, A, \alpha)) \subset B$ . Uniqueness of  $\mu$  follows from uniqueness of the integrated form of a covariant representation.  $\square$

Our final preliminary is to establish notation for matrix units.

**Notation 4.6.** For any index set  $S$ , let  $\delta_s \in l^2(S)$  be the standard basis vector, determined by  $\delta_s(s) = 1$  and  $\delta_s(t) = 0$  for  $t \neq s$ . For  $j, k \in S$ , we let the ‘‘matrix unit’’  $e_{j,k}$  be the rank one operator on  $l^2(S)$  given by  $e_{j,k}\xi = \langle \xi, \delta_k \rangle \delta_j$ . This gives the product formula  $e_{j,k}e_{l,m} = \delta_{k,l}e_{j,m}$ . Conventional matrix units for  $M_n$  are obtained by taking  $S = \{1, 2, \dots, n\}$ , but we will sometimes want to take  $S$  to be a discrete (even finite) group. For  $S = \{1, 2\}$ , with the obvious choice of matrix representation, we get

$$e_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad e_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Example 4.7.** If  $G$  acts trivially on the  $C^*$ -algebra  $A$ , then

$$C^*(G, A) \cong C^*(G) \otimes_{\max} A \quad \text{and} \quad C_r^*(G, A) \cong C_r^*(G) \otimes_{\min} A.$$

For the full crossed product, first assume  $G$  is discrete and  $A$  is unital. Then Theorem 3.17 implies that  $C^*(G, A)$  is the universal unital  $C^*$ -algebra generated

by a unital copy of  $A$  and a commuting unitary representation of  $G$  in the algebra. Since  $C^*(G)$  is the universal unital C\*-algebra generated by a unitary representation of  $G$  in the algebra, this is exactly the universal property of the maximal tensor product.

The proof for the general case is essentially the same. Omitting the technicalities, a covariant representation consists of commuting representations of  $A$  and  $G$ , and hence of  $A$  and  $C^*(G)$ .

For the reduced crossed product, the point is that a regular covariant representation of  $(G, A)$  has the form  $(\lambda \otimes 1_{H_0}, 1_{L^2(G)} \otimes \pi_0)$  for  $\pi_0: A \rightarrow L(H_0)$  an arbitrary nondegenerate representation and with  $\lambda: G \rightarrow U(L^2(G))$  being the left regular representation. By Proposition 3.31(2), it suffices to take  $\pi_0$  to be a single injective representation. Now we are looking at  $C_r^*(G)$  on one Hilbert space and  $A$  on another, and taking the tensor product of the Hilbert spaces. This is exactly how one gets the minimal tensor product of two C\*-algebras.

Note how full and reduced crossed products parallel maximal and minimal tensor products.

**Remark 4.8.** More generally, if  $A$  and  $B$  are C\*-algebras,  $\alpha: G \rightarrow \text{Aut}(A)$  is any action, and  $\beta: G \rightarrow \text{Aut}(B)$  is the trivial action, then one gets actions  $\alpha \otimes_{\max} \beta$  of  $G$  on  $A \otimes_{\max} B$  and  $\alpha \otimes_{\min} \beta$  of  $G$  on  $A \otimes_{\min} B$  (so far, it does not matter what  $\beta$  is), and one has

$$C^*(G, A \otimes_{\max} B, \alpha \otimes_{\max} \beta) \cong C^*(G, A, \alpha) \otimes_{\max} B$$

and

$$C_r^*(G, A \otimes_{\min} B, \alpha \otimes_{\min} \beta) \cong C_r^*(G, A, \alpha) \otimes_{\min} B.$$

**Exercise 4.9.** Prove Remark 4.8 when  $G$  is discrete and  $A$  and  $B$  are both unital.

**Example 4.10.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an inner action of a discrete group  $G$  on a unital C\*-algebra  $A$ . Thus, there is a homomorphism  $g \mapsto z_g$  from  $G$  to  $U(A)$  such that  $\alpha_g(a) = z_g a z_g^*$  for all  $g \in G$  and  $a \in A$ . (See Example 2.9.) We claim that  $C^*(G, A, \alpha) \cong C^*(G) \otimes_{\max} A$ . (This is true even if  $G$  is not discrete. See Exercise 4.11.)

Let  $\iota: G \rightarrow \text{Aut}(A)$  be the trivial action of  $G$  on  $A$ . As usual, for  $g \in G$  let  $u_g \in C_c(G, A, \alpha)$  be the standard unitary, but let  $v_g \in C_c(G, A, \iota)$  be the standard unitary in the crossed product by the trivial action. Define  $\varphi_0: C_c(G, A, \alpha) \rightarrow C_c(G, A, \iota)$  by  $\varphi_0(au_g) = az_g v_g$  for  $a \in A$  and  $g \in G$ , and extend linearly. This map is obviously bijective (the inverse sends  $av_g$  to  $az_g^* u_g$ ) and isometric for  $\|\cdot\|_1$ . For multiplicativity, it suffices to check the following, for  $a, b \in A$  and  $g, h \in H$ , using the fact that  $v_g$  commutes with all elements of  $A$ :

$$\begin{aligned} \varphi_0(au_g)\varphi_0(bu_h) &= az_g v_g b z_h v_h = az_g b z_g^* z_{gh} v_g v_h \\ &= a\alpha_g(b) z_{gh} v_{gh} = \varphi_0(a\alpha_g(b)u_{gh}) = \varphi_0((au_g)(bu_h)). \end{aligned}$$

Also,

$$\begin{aligned} \varphi_0(au_g)^* &= (az_g v_g)^* = v_g^* z_g^* a^* = (z_g^* a^* z_g) z_g^* v_g^* \\ &= \alpha_{g^{-1}}(a^*) z_{g^{-1}} v_{g^{-1}} = \varphi_0(\alpha_{g^{-1}}(a^*)u_{g^{-1}}) = \varphi_0((au_g)^*). \end{aligned}$$

So  $\varphi_0$  is an isometric isomorphism of \*-algebras, and therefore extends to an isomorphism of the universal C\*-algebras as in Theorem 3.17. Now use Example 4.7.

For use in Example 4.22, we write out explicitly what happens when  $G = \mathbb{Z}/2\mathbb{Z}$ .

Let  $v_0 \in C^*(\mathbb{Z}/2\mathbb{Z})$  be the image of the nontrivial element of the group. Then  $\lambda + \mu v_0 \mapsto (\lambda + \mu, \lambda - \mu)$  is an isomorphism from  $C^*(\mathbb{Z}/2\mathbb{Z})$  to  $\mathbb{C} \oplus \mathbb{C}$ . (The algebra  $C^*(\mathbb{Z}/2\mathbb{Z})$  is the universal C\*-algebra generated by a unitary with square 1, and the corresponding unitary in  $\mathbb{C} \oplus \mathbb{C}$  is  $(1, -1)$ . But one can check directly that the map above is an isomorphism.)

For the crossed product of a unital C\*-algebra  $A$  by the trivial action  $\iota$  of  $\mathbb{Z}/2\mathbb{Z}$ , let  $v \in C^*(\mathbb{Z}/2\mathbb{Z}, A, \iota)$  be the standard unitary associated to the nontrivial element of the group. Then  $a + bv \mapsto (a + b, a - b)$  is an isomorphism from  $C^*(\mathbb{Z}/2\mathbb{Z}, A, \iota)$  to  $A \oplus A$ . This map is a homomorphism because the copy  $\{(a, a) : a \in A\} \subset A \oplus A$  of  $A$  and the unitary  $(1, -1) \in A \oplus A$  satisfy the appropriate commutation relations. One proves that this map is an isomorphism from  $l^1(\mathbb{Z}/2\mathbb{Z}, A, \iota)$  to  $A \oplus A$  by explicitly writing down an inverse. Corollary 3.29 now shows it is an isomorphism from  $l^1(\mathbb{Z}/2\mathbb{Z}, A, \iota)$  to  $A \oplus A$ . (For a faster proof, just tensor the isomorphism of the previous paragraph with  $\text{id}_A$ .)

Now suppose that  $z \in A$  is a unitary of order 2. Let  $g_0 \in \mathbb{Z}/2\mathbb{Z}$  be the nontrivial group element, and let  $\alpha : \mathbb{Z}/2\mathbb{Z} \rightarrow \text{Aut}(A)$  be the action such that  $\alpha_{g_0} = \text{Ad}(z)$ . Let  $u = u_{g_0} \in C^*(\mathbb{Z}/2\mathbb{Z}, A, \alpha)$ . Then  $a + bu \mapsto (a + bz, a - bz)$  is an isomorphism from  $l^1(\mathbb{Z}/2\mathbb{Z}, A, \alpha)$  to  $A \oplus A$ . (Of course, once one has the formula, one can prove this directly.)

**Exercise 4.11.** Prove the following generalization of Example 4.10. Let  $\alpha, \beta : G \rightarrow \text{Aut}(A)$  be two actions of a locally compact group  $G$  on a C\*-algebra  $A$  which are exterior equivalent in the sense of Remark 2.14. Prove that

$$C^*(G, A, \alpha) \cong C^*(G, A, \beta) \quad \text{and} \quad C_r^*(G, A, \alpha) \cong C_r^*(G, A, \beta).$$

**Exercise 4.12.** Let  $\alpha : (\mathbb{Z}/2\mathbb{Z})^2 \rightarrow \text{Aut}(M_2)$  be as in Example 2.10. Prove that the crossed product  $C^*((\mathbb{Z}/2\mathbb{Z})^2, M_2, \alpha)$  is isomorphic to  $M_4$ .

**Example 4.13.** We prove that if  $G$  is discrete and acts on itself by translation, then the crossed product is  $K(l^2(G))$ . More generally (compare with Remark 4.8, but we will not give a proof), if  $G$  acts on  $G \times X$  by translation on the first factor and trivially on the second factor, then

$$C^*(G, G \times X) \cong K(l^2(G)) \otimes C_0(X) \cong C_0(X, K(l^2(G))).$$

In fact, the action on  $X$  need not be trivial. The map  $(h, x) \mapsto (h, h^{-1}x)$  is an isomorphism from  $G \times X$  with a general action of  $G$  on  $X$  to  $G \times X$  with the trivial action of  $G$  on  $X$ .

Let  $\alpha : G \rightarrow \text{Aut}(C_0(G))$  denote the action. For  $g \in G$ , we let  $u_g$  be the standard unitary as in Notation 3.7, and we let  $\delta_g \in C_0(G)$  be the function  $\chi_{\{g\}}$ . Note that  $\alpha_g(\delta_h) = \delta_{gh}$  for  $g, h \in G$ . Also,  $\text{span}(\{\delta_g : g \in G\})$  is dense in  $C_0(G)$ . For  $g, h \in G$ , the element  $v_{g,h} = \delta_g u_{gh^{-1}}$  is in  $C^*(G, C_0(G), \alpha)$ . Moreover,

$$\begin{aligned} v_{g_1, h_1} v_{g_2, h_2} &= \delta_{g_1} u_{g_1 h_1^{-1}} \delta_{g_2} u_{g_2 h_2^{-1}} \\ &= \delta_{g_1} \alpha_{g_1 h_1^{-1}}(\delta_{g_2}) u_{g_1 h_1^{-1} g_2 h_2^{-1}} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}. \end{aligned}$$

Thus, if  $g_2 \neq h_1$ , the answer is zero, while if  $g_2 = h_1$ , the answer is  $v_{g_1, h_2}$ . Similarly,  $v_{g,h}^* = v_{h,g}$ . That is, the elements  $v_{g,h}$  satisfy the relations for a system of matrix units indexed by  $G$ . Also,  $\text{span}(\{v_{g,h} : g, h \in G\})$  is dense in  $l^1(G, C_0(G), \alpha)$ , and hence in  $C^*(G, C_0(G), \alpha)$ .

For any finite set  $F \subset G$ , we thus get a homomorphism

$$\psi_F: L(l^2(F)) \rightarrow C_c(G, C_0(G), \alpha)$$

sending the matrix unit  $e_{g,h} \in L(l^2(F))$  to  $v_{g,h}$ . Let

$$\varphi_F: L(l^2(F)) \rightarrow C^*(G, C_0(G), \alpha)$$

be the result of composing with the map from  $C_c(G, C_0(G), \alpha)$  to  $C^*(G, C_0(G), \alpha)$ . Set

$$K_0 = \bigcup_{F \subset G \text{ finite}} L(l^2(F)).$$

Putting our homomorphisms together gives a homomorphism

$$\varphi^{(0)}: K_0 \rightarrow C^*(G, C_0(G), \alpha).$$

Since the restriction to each  $L(l^2(F))$  is a homomorphism of C\*-algebras, it follows that  $\|\varphi^{(0)}(x)\| \leq \|x\|$  for all  $x \in K_0$ . Therefore  $\varphi^{(0)}$  extends by continuity to a homomorphism

$$\varphi: K(l^2(G)) \rightarrow C^*(G, C_0(G), \alpha).$$

The homomorphism  $\varphi$  is surjective because it has dense range, and it is injective because  $K(l^2(G))$  is simple.

**Example 4.14.** Fix  $n \in \mathbb{N}$ , and consider the action of  $G = \mathbb{Z}/n\mathbb{Z}$  on  $S^1$  generated by rotation by  $2\pi/n$ , that is, the homeomorphism  $h(\zeta) = e^{2\pi i/n}\zeta$  for  $\zeta \in S^1$ . (This action is from Example 2.3.)

We describe what to expect. Every point in  $S^1$  has a closed invariant neighborhood which is equivariantly homeomorphic to  $G \times I$  for some closed interval  $I \subset \mathbb{R}$ , with the translation action on  $G$  and the trivial action on  $I$ . This leads to quotients of  $C^*(G, S^1, h)$  isomorphic to  $M_n \otimes C(I)$ . (See Theorem 4.4 and the general version of Example 4.13.) Since  $S^1$  itself is not such a product, one does not immediately get an isomorphism  $C^*(G, S^1, h) \cong M_n \otimes C(Y)$  for any  $Y$ . Instead, one gets the section algebra of a locally trivial bundle over  $Y$  with fiber  $M_n$ . However, the appropriate  $Y$  is the orbit space  $S^1/G \cong S^1$ , and all locally trivial bundles over  $S^1$  with fiber  $M_n$  are in fact trivial. Thus, one gets  $C^*(G, S^1, h) \cong C(S^1, M_n)$  after all.

We carry out the details. Let  $\alpha \in \text{Aut}(C(S^1))$  be the order  $n$  automorphism  $\alpha(f) = f \circ h^{-1}$ . Thus,  $\alpha(f)(\zeta) = f(e^{-2\pi i/n}\zeta)$  for  $\zeta \in S^1$ . Let  $s \in M_n$  be the shift unitary

$$s = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

The key computation, which we leave to the reader, is

$$(1) \quad s \text{diag}(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n) s^* = \text{diag}(\lambda_n, \lambda_1, \lambda_2, \dots, \lambda_{n-1}).$$

Set

$$B = \{f \in C([0, 1], M_n): f(0) = sf(1)s^*\}.$$

Define  $\varphi_0: C(S^1) \rightarrow B$  by sending  $f \in C(S^1)$  to the continuously varying diagonal matrix

$$\varphi_0(f)(t) = \text{diag}(f(e^{2\pi it/n}), f(e^{2\pi i(t+1)/n}), \dots, f(e^{2\pi i(t+n-1)/n})).$$

The diagonal entries of  $f(0)$  are gotten from those of  $f(1)$  by a forwards cyclic shift, so  $\varphi_0(f)$  really is in  $B$ . For the same reason, we get

$$\begin{aligned} \varphi_0(\alpha(f))(t) &= \text{diag}(f(e^{2\pi i(t-1)/n}), f(e^{2\pi it/n}), \dots, f(e^{2\pi i(t+n-2)/n})) \\ &= s\varphi_0(f)(t)s^*. \end{aligned}$$

Now let  $v \in C([0, 1], M_n)$  be the constant function with value  $s$ . Note that  $v \in B$ . The calculation just done implies that  $\varphi(\alpha^k(f)) = v^k \varphi(f) v^{-k}$  for  $0 \leq k \leq n-1$ . Also clearly  $v^n = 1$ . We write the group elements as  $0, 1, \dots, n-1$ , by abuse of notation treating them as integers when convenient. The universal property of the crossed product therefore implies that there is a homomorphism  $\varphi: C^*(G, S^1, h) \rightarrow B$  such that  $\varphi|_{C(S^1)} = \varphi_0$  and  $\varphi(u_k) = v^k$  for  $0 \leq k \leq n-1$ .

We prove directly that  $\varphi$  is bijective. By Corollary 3.29, we can rewrite  $\varphi$  as the map  $C(\mathbb{Z}/n\mathbb{Z} \times S^1) \rightarrow B$  given by

$$\varphi(f) = \sum_{k=0}^{n-1} \varphi_0(f(k, -))v^k.$$

Injectivity now reduces to the easily verified fact that if  $a_0, a_1, \dots, a_{n-1} \in M_n$  are diagonal matrices, and  $\sum_{k=0}^{n-1} a_k v^k = 0$ , then  $a_0 = a_1 = \dots = a_{n-1} = 0$ .

For surjectivity, let  $a \in B$ , and write

$$a(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) & \cdots & a_{1,n}(t) \\ a_{2,1}(t) & a_{2,2}(t) & \cdots & a_{2,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(t) & a_{n,2}(t) & \cdots & a_{n,n}(t) \end{pmatrix}$$

with  $a_{j,k} \in C([0, 1])$  for  $1 \leq j, k \leq n$ . The condition  $a \in B$  implies that, taking the indices mod  $n$  in  $\{1, 2, \dots, n\}$ , we have  $a_{j,k}(1) = a_{j+1, k+1}(0)$  for all  $j$  and  $k$ . This implies that the formula

$$f(l, e^{2\pi i(t+j)/n}) = a_{l+j, j}(t)$$

for  $t \in [0, 1]$ ,  $1 \leq j \leq n$ , and  $0 \leq l \leq n-1$ , with  $l+j$  taken mod  $n$  in  $\{1, 2, \dots, n\}$ , gives a well defined element of  $C(\mathbb{Z}/n\mathbb{Z} \times S^1)$ . One checks that  $\varphi(f) = a$ .

It remains to prove that  $B \cong C(S^1, M_n)$ . Since  $U(M_n)$  is connected, there is a unitary path  $t \mapsto s_t$ , for  $t \in [0, 1, ]$  such that  $s_0 = 1$  and  $s_1 = s$ . Define  $\psi: C(S^1, M_n) \rightarrow B$  by  $\psi(f)(t) = s_t^* f(e^{2\pi it}) s_t$ . For  $f \in C(S^1, M_n)$ , we have  $\psi(f)(1) = s^* f(1) s = s^* f(0) s = s^* \psi(f)(0) s$ , so  $\psi(f)$  really is in  $B$ . It is easily checked that  $\psi$  is bijective.

**Example 4.15.** Let  $X = S^n = \{x \in \mathbb{R}^{n+1}: \|x\|_2 = 1\}$ , and let  $\mathbb{Z}/2\mathbb{Z}$  act by sending the nontrivial group element to the order 2 homeomorphism  $x \mapsto -x$ . (This is Example 2.6.) The ‘‘local structure’’ of the crossed product  $C^*(\mathbb{Z}/2\mathbb{Z}, X)$  is the same as in Example 4.14. However, for  $n \geq 2$  the resulting bundle is no longer trivial. The crossed product is isomorphic to the section algebra of a locally trivial but nontrivial bundle over the real projective space  $\mathbb{R}P^n = S^n/(\mathbb{Z}/2\mathbb{Z})$  with fiber  $M_2$ . See Proposition 4.15 of [38].

The bundles one gets from free proper actions are, however, often stably trivial. Theorem 14 of [10] implies that the bundle always comes from a bundle of Hilbert spaces, and, if the algebra is separable, Theorem 10.7.15 of [5] implies that the Dixmier-Douady invariant is zero. If the fibers are infinite dimensional, and if the quotient space has finite covering dimension or if the map  $X \rightarrow X/G$  is locally trivial, then the crossed product is  $K \otimes C_0(X/G)$ . See Theorems 10.8.4 and 10.8.8 of [5], and Corollary 15 of [10].

**Example 4.16.** Let  $X = \mathbb{Z}/n\mathbb{Z}$ , and let  $\mathbb{Z}$  act on  $X$  by translation. We show that  $C^*(\mathbb{Z}, X) \cong M_n \otimes C(S^1)$ . This is a special case of Example 2.2. In the general case, it turns out that  $C^*(G, G/H) \cong K(L^2(G/H)) \otimes C^*(H)$ . Note that there is no twisting.

Identify  $\mathbb{Z}/n\mathbb{Z}$  with  $\{1, 2, \dots, n\}$ . (We start at 1 instead of 0 to be consistent with common matrix unit notation.) Let  $\alpha \in \text{Aut}(C(\mathbb{Z}/n\mathbb{Z}))$  be  $\alpha(f)(k) = f(k-1)$ , with indices taken mod  $n$  in  $\{1, 2, \dots, n\}$ . (Equivalently,  $\alpha(\chi_{\{k\}}) = \chi_{\{k+1\}}$ , with  $k+1$  taken to be 1 when  $k = n$ .) In  $C(S^1)$  let  $z$  be the function  $z(\zeta) = \zeta$  for all  $\zeta$ . In  $M_n(C(S^1)) \cong M_n \otimes C(S^1)$ , abbreviate  $e_{j,k} \otimes 1$  to  $e_{j,k}$ , and let  $v$  be the unitary

$$v = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & z \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

(This unitary differs from the unitary  $s$  in Example 4.14 only in that here the upper right corner entry is  $z$  instead of 1.)

Define  $\varphi_0: C(\mathbb{Z}/n\mathbb{Z}) \rightarrow M_n \otimes C(S^1)$  by  $\varphi_0(\chi_{\{k\}}) = e_{k,k}$ . Then one checks that  $v\varphi_0(f)v^* = \varphi_0(\alpha(f))$  for all  $f \in C(\mathbb{Z}/n\mathbb{Z})$ . Therefore there is a homomorphism  $\varphi: C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \rightarrow M_n \otimes C(S^1)$  such that  $\varphi|_{C(\mathbb{Z}/n\mathbb{Z})} = \varphi_0$  and  $\varphi(v) = v$ . We claim that  $\varphi$  is an isomorphism.

The following description of  $M_n \otimes C(S^1)$  will be useful: it is the universal unital C\*-algebra generated by a system  $(e_{j,k})_{1 \leq j,k \leq n}$  of matrix units such that  $\sum_{j=1}^n e_{j,j} = 1$  and a central unitary  $y$ . The  $e_{j,k}$  are the matrix units we have already used, and the central unitary is  $1 \otimes z$ . (Proof: Exercise 4.17 below.)

To prove that  $\varphi$  is surjective, it suffices to prove that its image contains  $1 \otimes z$  and contains  $e_{j,k}$  for  $1 \leq j, k \leq n$ . The image contains  $1 \otimes z$  because  $v^n = 1 \otimes z$ . For  $1 \leq j \leq n$ , the image contains  $e_{j,j} = \varphi_0(\chi_{\{j\}})$ . It follows that the image also contains  $e_{j+1,j} = e_{j+1,j+1}v e_{j,j}$  for  $1 \leq j \leq n-1$ . It now easily follows that the image contains  $e_{j,k}$  for all  $j$  and  $k$ .

To prove injectivity, it suffices to prove that whenever  $A$  is a unital C\*-algebra,  $\psi_0: C(\mathbb{Z}/n\mathbb{Z}) \rightarrow A$  is a unital homomorphism, and  $w \in A$  is a unitary such that  $w\psi_0(f)w^* = \psi_0(\alpha(f))$  for all  $f \in C(\mathbb{Z}/n\mathbb{Z})$ , then there is a homomorphism  $\gamma: M_n \otimes C(S^1) \rightarrow A$  such that  $\gamma \circ \varphi_0 = \psi_0$  and  $\gamma(v) = w$ . (If  $\varphi$  were not injective, taking  $\psi_0$  and  $w$  to come from  $\text{id}_{C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})}$  would yield a contradiction.) It suffices to define  $\gamma$  on the generators. For  $1 \leq j \leq n$ , we define  $f_{j,j} = \psi_0(\chi_{\{j\}})$ . For  $1 \leq k < j \leq n$ , we define  $f_{j,k} = f_{j,j}w^{j-k}f_{k,k}$  and  $f_{k,j} = f_{j,k}^*$ . One easily checks that  $(f_{j,k})_{1 \leq j,k \leq n}$  is a system of matrix units such that  $\sum_{j=1}^n f_{j,j} = 1$ , and that  $w^n$  is a unitary which

commutes with  $f_{j,k}$  for  $1 \leq j, k \leq n$ . Accordingly, we may define  $\gamma$  by  $\gamma(1 \otimes z) = w^n$  and  $\gamma(e_{j,k}) = f_{j,k}$  for  $1 \leq j, k \leq n$ . It is obvious that  $\gamma \circ \varphi_0 = \psi_0$ . To compute  $(\gamma \circ \varphi)(u)$ , we observe that

$$\varphi(u) = v = (1 \otimes z)e_{1,n} + \sum_{j=1}^{n-1} e_{j+1,j}.$$

Therefore, using the definitions of the  $f_{j,k}$  for  $j \neq k$  at the third step and the relations  $f_{j+1,j+1} = wf_{j,j}w^*$  for  $1 \leq j \leq n-1$  and  $f_{1,1} = w^{-(n-1)}f_{n,n}w^{n-1}$  at the fourth step, we get

$$\begin{aligned} (\gamma \circ \varphi)(u) &= \gamma(v) = w^n f_{1,n} + \sum_{j=1}^{n-1} f_{j+1,j} \\ &= w^n f_{1,1} w^{-(n-1)} f_{n,n} + \sum_{j=1}^{n-1} f_{j+1,j+1} w f_{j,j} = w f_{n,n} + \sum_{j=1}^{n-1} w f_{j,j} = w. \end{aligned}$$

This completes the proof.

**Exercise 4.17.** Prove the description of  $M_n \otimes C(S^1)$  in terms of generators and relations used in Example 4.16.

**Remark 4.18.** In Examples 4.14 and 4.16, we have seen two sources of ideals in a reduced crossed product  $C_r^*(G, A, \alpha)$ : invariant ideals in  $A$ , and group elements which act trivially on  $A$ . There is a theorem due to Gootman and Rosenberg which gives a description of the primitive ideals of any crossed product  $C^*(G, A)$  with  $G$  amenable, and which, very roughly, says that they all come from some combination of these two sources. (One does not even need to restrict to discrete groups.) To be a little more precise, every primitive ideal in  $C^*(G, A)$  is “induced” from an ideal  $J$  in a crossed product by the stabilizer subgroup of some primitive ideal  $P$  of  $A$ , with  $J$  closely related to  $P$ . The theorem is Theorem 8.21 of [38]; see Definition 8.18 of [38] for the terminology. The proof of the Gootman-Rosenberg Theorem is quite long. (Starting from about the same assumed background as these notes, it occupies a large part of the book [38].)

**Example 4.19.** Take  $X = S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ , and let  $\mathbb{Z}/2\mathbb{Z}$  act by sending the nontrivial group element to the order 2 homeomorphism  $\zeta \mapsto \bar{\zeta}$ . (This is Example 2.7.) Let  $\alpha \in \text{Aut}(C(S^1))$  be the corresponding automorphism. We compute the crossed product, but we first describe what to expect. By considering Theorem 4.4 and Examples 4.7 and 4.14, we should expect that the points 1 and  $-1$  contribute quotients isomorphic to  $\mathbb{C} \oplus \mathbb{C}$ , and that for  $\zeta \neq \pm 1$ , the pair of points  $(\zeta, \bar{\zeta})$  contributes a quotient isomorphic to  $M_2$ . We will in fact show that  $C^*(\mathbb{Z}/2\mathbb{Z}, X)$  is isomorphic to the  $C^*$ -algebra

$$B = \{f \in C([-1, 1], M_2) : f(1) \text{ and } f(-1) \text{ are diagonal matrices}\}.$$

First, let  $C_0 \subset M_2$  be the subalgebra consisting of all matrices of the form  $\begin{pmatrix} \lambda & \mu \\ \mu & \lambda \end{pmatrix}$  with  $\lambda, \mu \in \mathbb{C}$ . Then define

$$C = \{f : [-1, 1] \rightarrow M_2 : f \text{ is continuous and } f(1), f(-1) \in C_0\}.$$

Let  $v \in C$  be the constant function  $v(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  for all  $t \in [-1, 1]$ . Define  $\varphi_0: C(S^1) \rightarrow C$  by

$$\varphi_0(f)(t) = \begin{pmatrix} f(t + i\sqrt{1-t^2}) & 0 \\ 0 & f(t - i\sqrt{1-t^2}) \end{pmatrix}$$

for  $f \in C(S^1)$  and  $t \in [-1, 1]$ . One checks that the conditions at  $\pm 1$  for membership in  $C$  are satisfied. Moreover,  $v^2 = 1$  and  $v\varphi_0(f)v^* = \varphi_0(\alpha(f))$  for  $f \in C(S^1)$ . Therefore there is a homomorphism  $\varphi: C^*(\mathbb{Z}/2\mathbb{Z}, X) \rightarrow C$  such that  $\varphi|_{C(S^1)} = \varphi_0$  and  $\varphi$  sends the standard unitary  $u$  in  $C^*(\mathbb{Z}/2\mathbb{Z}, X)$  to  $v$ . It is given by the formula

$$\varphi(f_0 + f_1u)(t) = \begin{pmatrix} f_0(t + i\sqrt{1-t^2}) & f_1(t + i\sqrt{1-t^2}) \\ f_1(t - i\sqrt{1-t^2}) & f_0(t - i\sqrt{1-t^2}) \end{pmatrix}$$

for  $f_1, f_2 \in C(S^1)$  and  $t \in [-1, 1]$ .

We claim that  $\varphi$  is an isomorphism. Since

$$C^*(\mathbb{Z}/2\mathbb{Z}, X) = \{f_0 + f_1u: f_1, f_2 \in C(S^1)\}$$

by Corollary 3.29, it is easy to check injectivity. For surjectivity, let

$$a(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) \\ a_{2,1}(t) & a_{2,2}(t) \end{pmatrix}$$

define an element  $a \in C$ . Then

$$(2) \quad a_{1,1}(-1) = a_{2,2}(-1) \quad \text{and} \quad a_{2,1}(-1) = a_{1,2}(-1),$$

and

$$(3) \quad a_{1,1}(1) = a_{2,2}(1) \quad \text{and} \quad a_{2,1}(1) = a_{1,2}(1).$$

Now set

$$f_0(\zeta) = \begin{cases} a_{1,1}(\operatorname{Re}(\zeta)) & \operatorname{Im}(\zeta) \geq 0 \\ a_{2,2}(\operatorname{Re}(\zeta)) & \operatorname{Im}(\zeta) \leq 0 \end{cases}$$

and

$$f_1(\zeta) = \begin{cases} a_{1,2}(\operatorname{Re}(\zeta)) & \operatorname{Im}(\zeta) \geq 0 \\ a_{2,1}(\operatorname{Re}(\zeta)) & \operatorname{Im}(\zeta) \leq 0 \end{cases}$$

for  $\zeta \in S^1$ . The relations (2) and (3) ensure that  $f_0$  and  $f_1$  are well defined at  $\pm 1$ , and are continuous. One easily checks that  $\varphi(f_0 + f_1u) = a$ . This proves surjectivity.

The algebra  $C$  is not quite what was promised. Set

$$w = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

which is a unitary in  $M_2$ . Then the isomorphism  $\psi: C^*(\mathbb{Z}/2\mathbb{Z}, X) \rightarrow B$  is given by  $\psi(a)(t) = w\varphi(a)(t)w^*$ . (Check this!)

In this example, one choice of matrix units in  $M_2$  was convenient for the free orbits, while another choice was convenient for the fixed points. It seemed better to compute everything in terms of the choice convenient for the free orbits, and convert afterwards.

**Exercise 4.20.** Let  $\mathbb{Z}/2\mathbb{Z}$  act on  $[-1, 1]$  via  $x \mapsto -x$ . Compute the crossed product.

**Exercise 4.21.** Let  $\mathbb{Z}/2\mathbb{Z}$  act on

$$S^n = \{(x_1, x_2, \dots, x_{n+1}) : x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1\}$$

via  $(x_1, x_2, \dots, x_n, x_{n+1}) \mapsto (x_1, x_2, \dots, x_n, -x_{n+1})$ . Compute the crossed product.

**Example 4.22.** We consider the specific case at the end of Example 2.20, which is the action of  $\mathbb{Z}/2\mathbb{Z}$  on the  $2^\infty$  UHF algebra  $A$  generated by  $\bigotimes_{n=1}^\infty \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . We simply write  $\alpha$  for the automorphism given by the nontrivial group element.

Write  $A = \varinjlim M_{2^n}$ , with maps  $\varphi_n : M_{2^n} \rightarrow M_{2^{n+1}}$  given by  $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ . Define unitaries  $z_n \in M_{2^n}$  inductively by  $z_0 = 1$  and  $z_{n+1} = \begin{pmatrix} z_n & 0 \\ 0 & -z_n \end{pmatrix}$ . (In tensor product notation, and with an appropriate choice of isomorphism  $M_{2^n} \otimes M_2 \rightarrow M_{2^{n+1}}$ , these are  $\varphi_n(a) = a \otimes 1_{M_2}$  and  $z_{n+1} = z_n \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .)

From the discussion at the end of Example 4.10, we get the isomorphisms

$$\sigma_n : C^*(\mathbb{Z}/2\mathbb{Z}, M_{2^n}, \text{Ad}(z_n)) \rightarrow M_{2^n} \oplus M_{2^n}$$

given by  $a + bu_n \mapsto (a + bz_n, a - bz_n)$ . We now need a map

$$\psi_n : M_{2^n} \oplus M_{2^n} \rightarrow M_{2^{n+1}} \oplus M_{2^{n+1}}$$

which sends

$$\sigma_n(a + bu_n) = (a + bz_n, a - bz_n)$$

to

$$\begin{aligned} & \sigma_{n+1}(\varphi_n(a) + \varphi_n(b)u_{n+1}) \\ &= \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} z_n & 0 \\ 0 & -z_n \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} - \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} z_n & 0 \\ 0 & -z_n \end{pmatrix} \right) \\ &= \left( \begin{pmatrix} a + bz_n & 0 \\ 0 & a - bz_n \end{pmatrix}, \begin{pmatrix} a - bz_n & 0 \\ 0 & a + bz_n \end{pmatrix} \right). \end{aligned}$$

So we take

$$\psi_n(b, c) = \left( \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix} \right).$$

Those familiar with Bratteli diagrams will now be able to write down the Bratteli diagram for the crossed product. Here, we give a direct identification. Inductively define unitaries  $x_n, y_n \in M_{2^n}$  by  $x_0 = y_0 = 1$  and

$$x_{n+1} = \begin{pmatrix} x_n & 0 \\ 0 & y_n \end{pmatrix} \quad \text{and} \quad y_{n+1} = \begin{pmatrix} 0 & y_n \\ x_n & 0 \end{pmatrix}$$

for  $n \geq 0$ . Then define  $\lambda_n : M_{2^n} \rightarrow M_{2^n} \oplus M_{2^n}$  by  $\lambda_n(a) = (x_n a x_n^*, y_n a y_n^*)$  for  $a \in M_{2^n}$ , and define  $\mu_n : M_{2^n} \oplus M_{2^n} \rightarrow M_{2^{n+1}}$  by

$$\mu_n(b, c) = \begin{pmatrix} x_n^* b x_n & 0 \\ 0 & y_n^* c y_n \end{pmatrix}$$

for  $b, c \in M_{2^n}$ . Then one checks that  $\mu_n \circ \lambda_n = \varphi_n$  and  $\lambda_{n+1} \circ \mu_n = \psi_n$  for all  $n$ . It follows that the direct limit of the system

$$\mathbb{C} \oplus \mathbb{C} \xrightarrow{\psi_0} M_2 \oplus M_2 \xrightarrow{\psi_1} M_4 \oplus M_4 \xrightarrow{\psi_2} M_8 \oplus M_8 \xrightarrow{\psi_3} \dots,$$

which is the crossed product  $C^*(\mathbb{Z}/2\mathbb{Z}, A, \alpha)$ , is isomorphic to the direct limit of the system

$$\mathbb{C} \xrightarrow{\varphi_0} M_2 \xrightarrow{\varphi_1} M_4 \xrightarrow{\varphi_2} M_8 \xrightarrow{\varphi_3} \dots,$$

which is the original algebra  $A$ .

The fact that we got the same algebra back in Example 4.22 is somewhat special, but the general principle of the computation is much more generally applicable.

Since the crossed product is simple, the action is not inner. (Compare with Example 4.10.)

The theorem of Gootman and Rosenberg described in Remark 4.18 gives no information here.

The following two exercises are much harder than most of the exercises in these notes. The first combines the methods of Example 4.19 (see Exercise 4.21) and Example 4.22, and the second uses Example 4.15 in place of Exercise 4.21. The computations asked for in the exercises are an important part of Propositions 4.6 and 4.2 of [30], which describe the properties of two significant examples of crossed products.

**Exercise 4.23.** Let  $m \in \mathbb{N}$ . Define  $h: S^{2m} \rightarrow S^{2m}$  by

$$h(x_0, x_1, \dots, x_{2m}) = (-x_0, x_1, \dots, x_{2m})$$

for  $x = (x_0, x_1, \dots, x_{2m}) \in S^{2m}$ , and let  $\beta \in \text{Aut}(C(S^{2m}))$  be the corresponding automorphism of order 2. For  $r \in \mathbb{N}$  and  $b \in S^{2m}$ , define  $\psi_{r,b}: C(S^{2m}) \rightarrow M_{2r+1} \otimes C(S^{2m})$  by

$$\psi_{r,b}(f)(x) = \text{diag}(f(x), f(b), f(h(b)), f(b), f(h(b)), \dots, f(b), f(h(b)))$$

for  $x \in S^{2m}$ , where  $f(b)$  and  $f(h(b))$  each occur  $r$  times. Choose a dense sequence  $(x(n))_{n \in \mathbb{N}}$  in  $S^{2m}$ , such that no point  $x_n$  is a fixed point of  $h$ , and choose a sequence  $(r(n))_{n \in \mathbb{N}}$  of strictly positive integers. Set

$$s(n) = [2r(1) + 1][2r(2) + 1] \cdots [2r(n) + 1],$$

and set  $A_n = M_{s(n)} \otimes C(S^{2m})$ , which, when appropriate, we think of as

$$M_{2r(1)+1} \otimes M_{2r(2)+1} \otimes \cdots \otimes M_{2r(n)+1} \otimes C(S^{2m}).$$

Define  $\varphi_n: A_{n-1} \rightarrow A_n$  by  $\varphi_n = \text{id}_{M_{s(n-1)}} \otimes \psi_{x(n), r(n)}$ . Then set  $A = \varinjlim A_n$ .

For  $r \in \mathbb{N}$  define a unitary  $w_r \in M_{2r+1}$  by

$$w_r = \text{diag} \left( 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

Then define an automorphism  $\alpha_n \in \text{Aut}(A_n)$  of order 2 by

$$\alpha_n = \text{Ad}(w_{r(1)} \otimes w_{r(2)} \otimes \cdots \otimes w_{r(n)}) \otimes \beta.$$

One checks that  $\varphi_n \circ \alpha_{n-1} = \alpha_n \circ \varphi_n$ , so that the automorphisms  $\alpha_n$  define an automorphism  $\alpha \in \text{Aut}(A)$  of order 2.

Compute the crossed product  $C^*(\mathbb{Z}/2\mathbb{Z}, A, \alpha)$ , at least sufficiently well to determine its K-theory.

**Exercise 4.24.** Repeat Exercise 4.23, with just one change: the formula for  $h$  is now  $h(x) = -x$  for all  $x \in S^{2m}$ . The algebras in the direct system for the crossed product are harder to describe, since they are section algebras of nontrivial bundles, but the full description is not needed in order to compute the K-theory of the resulting direct limit.

**Example 4.25.** Let  $\theta \in \mathbb{R}$ . Recall from Example 2.15 that the rotation algebra  $A_\theta$  is the universal C\*-algebra generated by unitaries  $u$  and  $v$  satisfying  $vu = e^{2\pi i \theta} uv$ .

Let  $h_\theta: S^1 \rightarrow S^1$  be the homeomorphism  $h_\theta(\zeta) = e^{2\pi i \theta} \zeta$ . (Recall Example 2.3.) We claim that there is an isomorphism  $\varphi: A_\theta \rightarrow C^*(\mathbb{Z}, S^1, h_\theta)$  which sends  $u$  to the

standard unitary  $u_1$  in the crossed product, and sends  $v$  to the function  $z \in C(S^1)$  defined by  $z(\zeta) = \zeta$  for all  $\zeta \in S^1$ . (In Corollary 3.18, the unitary  $u_1$  was called  $u$ , so we are essentially sending  $u$  to  $u$ .)

The proof of the claim is by comparison of universal properties. First, one checks that  $zu_1 = e^{2\pi i\theta}u_1z$ , so at least there is a homomorphism  $\varphi$  with the properties claimed. Next, define a homomorphism  $\psi_0: C(S^1) \rightarrow A_\theta$  by  $\psi_0(f) = f(v)$  (continuous functional calculus) for  $f \in C(S^1)$ . For  $n \in \mathbb{Z}$ , we have

$$u\psi_0(z^n)u^* = (uvu^*)^n = e^{2\pi in\theta}v^n = \psi_0(e^{2\pi in\theta}z^n) = \psi_0(z^n \circ h_\theta^{-1}).$$

Since the functions  $z^n$  span a dense subspace of  $C(S^1)$ , it follows that  $u\psi_0(f)u^* = \psi_0(f \circ h_\theta^{-1})$  for all  $f \in C(S^1)$ . By Corollary 3.18, there is a homomorphism  $\psi: C^*(\mathbb{Z}, S^1, h_\theta) \rightarrow A_\theta$  such that  $\psi|_{C(S^1)} = \psi_0$  and  $\psi(u_1) = u$ .

We have  $(\psi \circ \varphi)(u) = u$  and  $(\psi \circ \varphi)(v) = v$ . Since  $u$  and  $v$  generate  $A_\theta$ , we conclude that  $\psi \circ \varphi = \text{id}_{A_\theta}$ . Similarly,  $(\varphi \circ \psi)(z) = z$  and  $(\varphi \circ \psi)(u_1) = u_1$ , the elements  $z$  and  $u_1$  generate  $C^*(\mathbb{Z}, S^1, h_\theta)$  (since  $z$  generates  $C(S^1)$ ), and therefore  $\varphi \circ \psi = \text{id}_{C^*(\mathbb{Z}, S^1, h_\theta)}$ .

We will see below that for  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ , the algebra  $C^*(\mathbb{Z}, S^1, h_\theta)$  is simple. (See Theorems 5.12 and 5.13 below, and also Proposition 2.56 of [38].) On the other hand, if  $\theta = p/q$  in lowest terms, with  $q > 0$ , then  $A_\theta$  turns out to be the section algebra of a locally trivial bundle over  $S^1 \times S^1$  with fiber  $M_q$ . (See Example 8.46 of [38].) The bundles have trivial Dixmier-Douady class, so are stably trivial, but they are not trivial. They are analyzed in [14].

## 5. MINIMAL HOMEOMORPHISMS

In this section, we give a sufficient criterion for simplicity of reduced transformation group  $C^*$ -algebras.

**Definition 5.1.** Let a locally compact group  $G$  act continuously on a locally compact space  $X$ . The action is called *minimal* if whenever  $T \subset X$  is a closed subset such that  $gT \subset T$  for all  $g \in G$ , then  $T = \emptyset$  or  $T = X$ .

In short, there are no nontrivial invariant closed subsets. For those knowing some ergodic theory, this is the topological analog of an ergodic action on a measure space.

If the action of  $G$  on  $X$  is not minimal, then there is a nontrivial invariant closed subset  $T \subset X$ , and  $C^*(G, X \setminus T)$  turns out to be a nontrivial ideal in  $C^*(G, X)$ . Thus  $C^*(G, X)$  is not simple.

For the case  $G = \mathbb{Z}$ , the conventional terminology is a bit different.

**Definition 5.2.** Let  $X$  be a locally compact Hausdorff space, and let  $h: X \rightarrow X$  be a homeomorphism. Then  $h$  is called *minimal* if whenever  $T \subset X$  is a closed subset such that  $h(T) = T$ , then  $T = \emptyset$  or  $T = X$ .

Almost all work on minimal homeomorphisms has been on compact spaces. For these, we have the following equivalent conditions.

**Lemma 5.3.** Let  $X$  be a compact Hausdorff space, and let  $h: X \rightarrow X$  be a homeomorphism. Then the following are equivalent:

- (1)  $h$  is minimal.
- (2) Whenever  $T \subset X$  is a closed subset such that  $h(T) \subset T$ , then  $T = \emptyset$  or  $T = X$ .

- (3) Whenever  $U \subset X$  is an open subset such that  $h(U) = U$ , then  $U = \emptyset$  or  $U = X$ .
- (4) Whenever  $U \subset X$  is an open subset such that  $h(U) \subset U$ , then  $U = \emptyset$  or  $U = X$ .
- (5) For every  $x \in X$ , the orbit  $\{h^n(x) : n \in \mathbb{Z}\}$  is dense in  $X$ .
- (6) For every  $x \in X$ , the forward orbit  $\{h^n(x) : n \geq 0\}$  is dense in  $X$ .

Conditions (1), (3), and (5) are equivalent even when  $X$  is only locally compact, and in fact there is an analog for actions of arbitrary groups. Minimality does not imply the other three conditions without compactness, as can be seen by considering the homeomorphism  $n \mapsto n+1$  of  $\mathbb{Z}$ . Also, even for compact  $X$ , it isn't good enough to merely have the existence of some dense orbit, as can be seen by considering the homeomorphism  $n \mapsto n+1$  on the two point compactification  $\mathbb{Z} \cup \{\pm\infty\}$  of  $\mathbb{Z}$ .

**Exercise 5.4.** Prove Lemma 5.3.

We give a few examples.

**Example 5.5.** Let  $G$  be a locally compact group, let  $H \subset G$  be a closed subgroup, and let  $G$  act on  $G/H$  by translation, as in Example 2.2. This action is minimal: there are no nontrivial invariant subsets, closed or not.

Example 5.5 is a “trivial” example of a minimal action. Here are several more interesting ones.

**Example 5.6.** The irrational rotations in Example 2.3 are minimal homeomorphisms.

The proof is contained in the following lemma.

**Lemma 5.7.** Let  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ . Then  $\{e^{2\pi i n \theta} : n \in \mathbb{Z}\}$  is dense in  $S^1$ .

There are a number of ways to prove this. In [38] (see the proof of Lemma 3.29), the basic idea is that all proper closed subgroups of  $S^1$  are finite. One can also get the result from number theory: there is a constant  $c$  such that there are pairs  $(p, q)$  of integers, with  $q$  arbitrarily large, such that  $|\theta - p/q| < cq^{-2}$ . We give here a proof close to that of [38].

*Proof of Lemma 5.7.* It suffices to prove that  $\mathbb{Z} + \theta\mathbb{Z}$  is dense in  $\mathbb{R}$ . Suppose not. Because we are dealing with groups, there is an open set  $U \subset \mathbb{R}$  such that  $U \cap (\mathbb{Z} + \theta\mathbb{Z}) = \{0\}$ . Let  $t = \inf(\{x \in \overline{\mathbb{Z} + \theta\mathbb{Z}} : x > 0\})$ . Then  $t > 0$ . Since  $\overline{\mathbb{Z} + \theta\mathbb{Z}}$  is a group, one checks that  $\overline{\mathbb{Z} + \theta\mathbb{Z}} = \mathbb{Z}t$ . Then we must have  $\mathbb{Z} + \theta\mathbb{Z} = \mathbb{Z}t$ . It follows that both 1 and  $\theta$  are integer multiples of  $t$ , so that  $\theta \in \mathbb{Q}$ .  $\square$

**Example 5.8.** The homeomorphism  $x \mapsto x+1$  on the  $p$ -adic integers (Example 2.5) is minimal. The orbit of 0 is  $\mathbb{Z}$ , which is dense, essentially by definition. Every other orbit is a translate of this one, so also dense.

**Example 5.9.** The shift homeomorphism (Example 2.4) and the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $S^1 \times S^1$  (Example 2.8) are not minimal. In fact, they have fixed points.

Minimal actions are plentiful: a Zorn's Lemma argument shows that every nonempty compact  $G$ -space  $X$  contains a nonempty invariant closed subset on which the restricted action is minimal.

The C\*-algebras associated with a number of other minimal homeomorphisms have been studied, although we do not have time to go into the definitions of these

homeomorphisms: Furstenberg transformations ([24]; [16]; [18]; Example 4.9 of [28]; Sections 2 and 3 of [29]; [35]), restricted Denjoy homeomorphisms ([34]), certain irrational time maps of suspension flows ([15]), and many others.

The transformation group C\*-algebra of a minimal action need not be simple. Consider, for example, the trivial action of a group  $G$  (particularly an abelian group) on a one point space, for which the transformation group C\*-algebra is  $C^*(G)$ .

**Definition 5.10.** Let a locally compact group  $G$  act continuously on a locally compact space  $X$ . The action is called *free* if whenever  $g \in G \setminus \{1\}$  and  $x \in X$ , then  $gx \neq x$ . The action is called *essentially free* if whenever  $g \in G \setminus \{1\}$ , the set  $\{x \in X : gx = x\}$  has empty interior.

**Remark 5.11.** Let  $X$  be an infinite compact Hausdorff space, and let  $h: X \rightarrow X$  be a minimal homeomorphism. Then the corresponding action of  $\mathbb{Z}$  on  $X$  is free. Indeed, if for some  $n \neq 0$  and  $x \in X$ , we have  $h^n(x) = x$ , then the orbit of  $x$  is finite, hence closed, and is clearly invariant. Now minimality contradicts infiniteness of  $X$ .

Of course, nothing like Remark 5.11 is true for general groups. For example, let  $G$  act freely and minimally on  $X$ , let  $H$  be some other group, and let  $G \times H$  act on  $X$  via  $(g, h)x = gx$ .

There is one other thing that is true: an essentially free minimal action of an abelian group is free. This is because the fixed point set of any group element is invariant under the action.

The following result is essentially a special case of the corollary at the end of [1]; see the discussion before the corollary and the Remark before Lemma 1 of [1]. We state a much more general result from [1] below (Theorem 5.16).

**Theorem 5.12.** Let a discrete group  $G$  act minimally and essentially freely on a locally compact space  $X$ . Then  $C_r^*(G, X)$  is simple.

Our proof will follow [1], but we first discuss some special cases and different proofs.

First, we point out that, when  $G$  is amenable and the action is free, and probably even when the action is only essentially free, Theorem 5.12 can be derived from the theorem of Gootman and Rosenberg described in Remark 4.18. See Corollary 8.22 of [38] for the free case.

Next, we give a simple proof for the special case of an irrational rotation on the circle. It introduces some important ideas which we, regrettably, will not develop further. (Also see Proposition 2.56 of [38].)

**Theorem 5.13.** Let  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ . Let  $h_\theta: S^1 \rightarrow S^1$  be the homeomorphism  $h_\theta(\zeta) = e^{2\pi i\theta}\zeta$ . Then  $C^*(\mathbb{Z}, S^1, h_\theta)$  is simple.

*Proof.* Following Example 4.25, we identify  $C^*(\mathbb{Z}, S^1, h_\theta)$  with the universal C\*-algebra  $A_\theta$  generated by unitaries  $u$  and  $v$  satisfying  $vu = e^{2\pi i\theta}uv$ , by identifying  $v$  with the function  $\zeta \mapsto \zeta$  on  $S^1$  and identifying  $u$  with the standard unitary of the crossed product.

Following Example 2.16, let  $\beta: S^1 \rightarrow \text{Aut}(A_\theta)$  be the action such that  $\beta_\zeta(u) = \zeta u$  and  $\beta_\zeta(v) = v$  for  $\zeta \in S^1$ . Using normalized Haar measure in the integral, we define a linear map  $E: A_\theta \rightarrow A_\theta$  by  $E(a) = \int_{S^1} \beta_\zeta(a) d\zeta$ . (The special case of Banach space valued integration theory needed here, essentially for continuous functions on

a compact interval with respect to Lebesgue measure, is easily treated by elementary methods.) One checks that  $E(v^n u^m) = v^n$  for  $m, n \in \mathbb{Z}$ . Since the elements  $v^n u^m$  span a dense subset of  $A_\theta$ , it follows that  $E$  is equal to the conditional expectation  $E_1$  of Proposition 3.31(3).

Now let  $I \subset A_\theta$  be a nonzero closed ideal. We claim that  $E(I) \subset I$ . First, check that, for  $\zeta = e^{2\pi i k \theta}$  with  $k \in \mathbb{Z}$ , and for  $m, n \in \mathbb{Z}$ , we have  $\beta_\zeta(v^n u^m) = e^{2\pi i k m \theta} v^n u^m = v^k (v^n u^m) v^{-k}$ . Therefore  $\beta_\zeta(a) = v^k a v^{-k}$  for all  $a \in A_\theta$ . In particular,  $\beta_\zeta(I) \subset I$ . Since  $\{e^{2\pi i k \theta} : k \in \mathbb{Z}\}$  is dense in  $S^1$  (by Lemma 5.7), it follows from continuity of the action that  $\beta_\zeta(I) \subset I$  for all  $\zeta \in S^1$ . The claim now follows by integration.

We finish the proof by showing that  $I = A_\theta$ . Choose a nonzero positive element  $a \in I$ . Let  $f = E(a)$ , which is a nonzero nonnegative function in  $I \cap C(S^1)$ . Then  $u^k f u^{-k}$ , which is the function in  $C(S^1)$  given by  $\zeta \mapsto f(e^{-2\pi i k \theta} \zeta)$ , is also in  $I \cap C(S^1)$ . Let  $U = \{\zeta \in S^1 : f(\zeta) \neq 0\}$ . Then  $u^k f u^{-k}$  is strictly positive on  $e^{2\pi i k \theta} U$ . The set  $\bigcup_{k \in \mathbb{Z}} e^{2\pi i k \theta} U$  is a nonempty invariant open subset of  $S^1$ , and it therefore equal to  $S^1$ . By compactness, there is a finite set  $S \subset \mathbb{Z}$  such that  $\bigcup_{k \in S} e^{2\pi i k \theta} U = S^1$ . Then  $\sum_{k \in S} u^k f u^{-k}$  is a strictly positive function on  $S^1$ , and is hence invertible. Since it is in  $I$ , we conclude that  $I = A_\theta$ .  $\square$

The action  $\beta: S^1 \rightarrow \text{Aut}(A_\theta)$  used in the proof of Theorem 5.13 is a special case of the dual action on a crossed product by an abelian group. We describe dual actions without proof; see [37] for details. Let  $A$  be any C\*-algebra, let  $G$  be a locally compact abelian group, and let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action. Let  $\widehat{G}$  be the Pontryagin dual of  $G$ . For  $\sigma \in \widehat{G}$ , there is an automorphism  $\widehat{\alpha}_\sigma$  of  $C^*(G, A, \alpha)$  given on  $L^1(G, A, \alpha)$  by  $\widehat{\alpha}_\sigma(f)(g) = \sigma(g)f(g)$ . Moreover,  $\widehat{\alpha}: \widehat{G} \rightarrow \text{Aut}(C^*(G, A, \alpha))$  is a continuous action of  $\widehat{G}$  on  $C^*(G, A, \alpha)$ , called the dual action. If  $G$  is discrete then  $\widehat{G}$  is compact, and the conditional expectation  $E: C^*(G, A, \alpha) \rightarrow A$  is given by  $E(a) = \int_{\widehat{G}} \widehat{\alpha}_\sigma(a) d\sigma$ , using normalized Haar measure in the integral. Whether or not  $G$  is discrete, the crossed product by the dual action is  $K(L^2(G)) \otimes A$ . This result is Takai duality. It is a generalization of the abelian case of Example 4.13: if  $A = \mathbb{C}$ , then  $C^*(G, A, \alpha) = C^*(G) \cong C_0(\widehat{G})$ , and the dual action is just translation on  $\widehat{G}$ .

The usual proof of Theorem 5.12 for  $G = \mathbb{Z}$  (see, for example, Theorem VIII.3.9 of [4]) is similar to the proof given for Theorem 5.13 above. However, it is much harder to prove that  $E(I) \subset I$ , since there is no analog of the automorphism  $\text{Ad}(v)$ . The usual method depends on Rokhlin type arguments. (See the proof of Lemma VIII.3.7 of [4].) We have avoided such arguments in this section. However, to obtain more information about simple transformation group C\*-algebras, such arguments are necessary, at least with the current state of knowledge. Examples show that, in the absence of some form of the Rokhlin property, stronger structural properties of crossed products of noncommutative C\*-algebras need not hold, even when they are simple.

We now turn to the proof of Theorem 5.12. We need several lemmas, which are special cases of the corresponding lemmas in [1].

**Lemma 5.14.** Let  $A$  be a C\*-algebra, let  $B \subset A$  be a subalgebra, and let  $\omega$  be a state on  $A$  such that  $\omega|_B$  is multiplicative. Then for all  $a \in A$  and  $b \in B$ , we have  $\omega(ab) = \omega(a)\omega(b)$  and  $\omega(ba) = \omega(b)\omega(a)$ .

This is a special case of Theorem 3.1 of [3]. (The corresponding lemma in [1] also follows from Theorem 3.1 of [3].)

*Proof of Lemma 5.14.* We prove  $\omega(ab) = \overline{\omega(a)}\omega(b)$ . The other equation will follow by using adjoints and the relation  $\omega(c^*) = \overline{\omega(c)}$ .

If  $A$  is not unital, then  $\omega$  extends to a state on the unitization  $A^+$ . Thus, we may assume that  $A$  is unital. Also, if  $\omega$  is multiplicative on  $B$ , one easily checks that  $\omega$  is multiplicative on  $B + \mathbb{C} \cdot 1$ . Thus, we may assume that  $1 \in B$ .

We recall from the Cauchy-Schwarz inequality that  $|\omega(x^*y)|^2 \leq \omega(y^*y)\omega(x^*x)$ . Replacing  $x$  by  $x^*$ , we get  $|\omega(xy)|^2 \leq \omega(y^*y)\omega(xx^*)$ . Now let  $a \in A$  and  $b \in B$ . Then

$$|\omega(ab) - \omega(a)\omega(b)|^2 = |\omega(a(b - \omega(b) \cdot 1))|^2 \leq \omega((b - \omega(b) \cdot 1)^*(b - \omega(b) \cdot 1))\omega(aa^*).$$

Since  $\omega$  is multiplicative on  $B$ , we have

$$\omega((b - \omega(b) \cdot 1)^*(b - \omega(b) \cdot 1)) = \omega((b - \omega(b) \cdot 1)^*)\omega(b - \omega(b) \cdot 1) = 0.$$

So  $|\omega(ab) - \omega(a)\omega(b)|^2 = 0$ .  $\square$

**Lemma 5.15.** Let a discrete group  $G$  act on a locally compact space  $X$ . Let  $x \in X$ , let  $g \in G$ , and assume that  $gx \neq x$ . Let  $\text{ev}_x: C_0(X) \rightarrow \mathbb{C}$  be the evaluation map  $\text{ev}_x(f) = f(x)$  for all  $f \in C_0(X)$ , and let  $\omega$  be a state on  $C_r^*(G, X)$  which extends  $\text{ev}_x$ . Then  $\omega(fu_g) = 0$  for all  $f \in C_0(X)$ .

*Proof.* Let  $\alpha: G \rightarrow \text{Aut}(C_0(X))$  be  $\alpha_g(f)(x) = f(g^{-1}x)$  for  $f \in C_0(X)$ ,  $g \in G$ , and  $x \in X$ . Choose  $f_0 \in C_0(X)$  such that  $f_0(x) = 1$  and  $f_0(gx) = 0$ . Applying Lemma 5.14 to  $\omega$  at the second and fourth steps, and using  $\omega(f_0) = 1$  at the first step, we have

$$\begin{aligned} \omega(fu_g) &= \omega(f_0)\omega(fu_g) = \omega(f_0fu_g) = \omega(fu_g\alpha_g^{-1}(f_0)) \\ &= \omega(fu_g)\omega(\alpha_g^{-1}(f_0)) = \omega(fu_g)f_0(gx) = 0. \end{aligned}$$

This completes the proof.  $\square$

*Proof of Theorem 5.12.* Let  $I \subset C_r^*(G, X)$  be a nonzero closed ideal.

First suppose  $I \cap C_0(X) = 0$ . Choose  $a \in I$  with  $a \neq 0$ . Let

$$E: C_r^*(G, C_0(X), \alpha) \rightarrow C_0(X)$$

be the map  $E_1$  of Proposition 3.30. Then  $E(a^*a) \neq 0$  by Proposition 3.31(3). Choose  $b \in C_c(G, C_0(X), \alpha)$  such that  $\|b - a^*a\| < \frac{1}{4}\|E(a^*a)\|$ . We can write  $b = \sum_{g \in S} b_g u_g$  for some finite set  $S \subset G$  and with  $b_g \in C_0(X)$  for  $g \in S$ . Without loss of generality  $1 \in S$ . Since  $E(a^*a)$  is a positive element of  $C_0(X)$ , there is  $x_0 \in X$  such that  $E(a^*a)(x_0) = \|E(a^*a)\|$ . Essential freeness implies that

$$\{x \in X: gx \neq x \text{ for all } g \in S \setminus \{1\}\}$$

is dense in  $X$ . In particular, there is  $x \in X$  so close to  $x_0$  that  $E(a^*a)(x) > \frac{3}{4}\|E(a^*a)\|$ , and also satisfying  $gx \neq x$  for all  $g \in S$ .

The set  $C_0(X) + I$  is a  $C^*$ -subalgebra of  $C_r^*(G, X)$ . Let  $\omega_0: C_0(X) + I \rightarrow \mathbb{C}$  be the following composition:

$$C_0(X) + I \longrightarrow (C_0(X) + I)/I \xrightarrow{\cong} C_0(X)/(C_0(X) \cap I) = C_0(X) \xrightarrow{\text{ev}_x} \mathbb{C}.$$

Then  $\omega_0$  is a homomorphism. Use the Hahn-Banach Theorem in the usual way to get a state  $\omega: C_r^*(G, X) \rightarrow \mathbb{C}$  which extends  $\omega_0$ . Note that  $\omega(a^*a) = 0$ .

We now have, using Lemma 5.15 at the fifth step,

$$\begin{aligned} \frac{1}{4}\|E(a^*a)\| &> \|b - a^*a\| \geq |\omega(b - a^*a)| = |\omega(b)| = \left| \sum_{g \in S} \omega(b_g u_g) \right| \\ &= |\omega(b_1)| = |\omega_0(b_1)| = |b_1(x)| \geq E(a^*a)(x) - \|E(a^*a) - b_1\| \\ &\geq E(a^*a)(x) - \|a^*a - b\| > \frac{3}{4}\|E(a^*a)\| - \frac{1}{4}\|E(a^*a)\| = \frac{1}{2}\|E(a^*a)\|. \end{aligned}$$

This contradiction shows that  $I \cap C_0(X) \neq 0$ .

Since  $I \cap C_0(X)$  is an ideal in  $C_0(X)$ , it has the form  $C_0(U)$  for some nonempty open set  $U \subset X$ . We claim that  $U$  is  $G$ -invariant. Let  $g \in G$  and let  $f \in C_0(U)$ . Let  $(e_\lambda)_{\lambda \in \Lambda}$  be an approximate identity for  $C_0(X)$ . Then the elements  $e_\lambda u_g$  are in  $C_r^*(G, X)$ , and we have  $(e_\lambda u_g)f(e_\lambda u_g)^* = e_\lambda \alpha_g(f) e_\lambda$ , which converges to  $\alpha_g(f)$ . We also have  $(e_\lambda u_g)f(e_\lambda u_g)^* \in I \cap C_0(X)$ , since  $I$  is an ideal. So  $\alpha_g(C_0(U)) \subset C_0(U)$  for all  $g \in G$ , and the claim follows.

Since  $U$  is open, invariant, and nonempty, we have  $U = X$ . One easily checks that an approximate identity for  $C_0(X)$  is also an approximate identity for  $C_r^*(G, X)$ , so  $I = C_r^*(G, X)$ , as desired.  $\square$

Theorem 5.12 generalizes, with essentially the same proof, to crossed products of actions of discrete groups on noncommutative C\*-algebras  $A$  satisfying a kind of essential freeness condition for the action on the space of unitary equivalence classes of irreducible representations of  $A$ . Here is the general statement; it is the corollary after Theorem 1 in [1].

**Theorem 5.16.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a discrete group  $G$  on a C\*-algebra  $A$ . Suppose that  $\alpha$  is minimal, that is,  $A$  has no nontrivial  $\alpha$ -invariant ideals. Suppose further that  $\alpha$  is topologically free, that is, with  $\widehat{A}$  being the space of unitary equivalence classes of irreducible representations of  $A$  with the hull-kernel topology, the following property holds: for every finite set  $F \subset G \setminus \{1\}$ , the set

$$\{x \in \widehat{A}: gx \neq x \text{ for all } g \in F\}$$

is dense in  $\widehat{A}$ . Then  $C_r^*(G, A, \alpha)$  is simple.

The changes to the proof include using irreducible representations in place of the maps  $\text{ev}_x$ , and completely positive maps to  $L(H)$  in place of states.

As discussed in the remark in [1] after corollary after Theorem 1, this result implies Theorem 3.1 of [17]. We state the following important special case:

**Theorem 5.17.** Let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a discrete group  $G$  on a simple C\*-algebra  $A$ . Suppose that  $\alpha_g$  is outer for every  $g \in G \setminus \{1\}$ . Then  $C_r^*(G, A, \alpha)$  is simple.

## 6. CLASSIFICATION

In this section, we give a proof that the transformation group C\*-algebra of a minimal homeomorphism of the Cantor set has tracial rank zero in the sense of Definition 3.1 of [20], equivalently (by Theorem 7.1(a) of [20]), is tracially AF in the sense of Definition 2.1 of [19]. It is implicit in Section 8 of [13], with the main step having been done in [33], that these transformation group C\*-algebras are AT algebras (direct limits of circle algebras) with real rank zero. The result we prove is weaker, but general theory (Lin's classification theorem for C\*-algebras with

tracial rank zero, Theorem 5.2 of [21], a K-theory calculation using the Pimsner-Voiculescu exact sequence [31], and results on the range of the Elliott invariant) show that the AT algebra result follows from the theorem we prove here.

We follow the proof of [22], where the following more general result is proved. For any unital C\*-algebra  $A$ , we let  $T(A)$  be the tracial state space of  $A$ . For any compact convex set  $\Delta$ , we let  $\text{Aff}(\Delta)$  be the space of real valued continuous affine functions on  $\Delta$ , with the supremum norm. Further let  $\rho = \rho_A: K_0(A) \rightarrow \text{Aff}(T(A))$  be the homomorphism determined by  $\rho([p])(\tau) = \tau(p)$  for  $\tau \in T(A)$  and  $p$  a projection in some matrix algebra over  $A$ .

**Theorem 6.1** (Theorem 4.6 of [22]). Let  $X$  be an infinite compact metric space with finite covering dimension, and let  $h: X \rightarrow X$  be a minimal homeomorphism. Suppose that  $\rho(K_0(C^*(\mathbb{Z}, X, h)))$  is dense in  $\text{Aff}(T(C^*(\mathbb{Z}, X, h)))$ . Then  $C^*(\mathbb{Z}, X, h)$  is a simple unital C\*-algebra with tracial rank zero which satisfies the Universal Coefficient Theorem.

There is machinery available to compute the range of  $\rho$  in the above theorem without computing  $C^*(\mathbb{Z}, X, h)$ . See, for example, [7].

**Corollary 6.2** (Corollary 4.7 of [22]). Let  $X$  be an infinite compact metric space with finite covering dimension, and let  $h: X \rightarrow X$  be a minimal homeomorphism. Suppose that  $\rho(K_0(C^*(\mathbb{Z}, X, h)))$  is dense in  $\text{Aff}(T(C^*(\mathbb{Z}, X, h)))$ . Then  $C^*(\mathbb{Z}, X, h)$  is a simple AH algebra with no dimension growth and with real rank zero.

The restriction to the Cantor set simplifies the argument greatly. In particular, one need not deal with recursive subhomogeneous C\*-algebras.

We use the following definition of tracial rank zero. We use the notation  $[a, b]$  for the commutator  $ab - ba$ .

**Definition 6.3.** Let  $A$  be a simple unital C\*-algebra. Then  $A$  has tracial rank zero if for every finite subset  $F \subset A$ , every  $\varepsilon > 0$ , and every nonzero positive element  $c \in A$ , there exists a unital finite dimensional subalgebra  $D \subset pAp$  and a projection  $p \in B$  such that:

- (1)  $\|[a, p]\| < \varepsilon$  for all  $a \in F$ .
- (2)  $\text{dist}(pap, D) < \varepsilon$  for all  $a \in F$ .
- (3)  $1 - p$  is Murray-von Neumann equivalent to a projection in  $\overline{cAc}$ .

That this definition is equivalent to the original one follows from Proposition 3.8 of [19].

**Lemma 6.4.** Let  $A$  be a simple unital C\*-algebra. Suppose that for every finite subset  $F \subset A$ , every  $\varepsilon > 0$ , and every nonzero positive element  $c \in A$ , there exists a projection  $p \in A$  and a unital AF subalgebra  $B \subset A$  with  $p \in B$  such that:

- (1)  $\|[a, p]\| < \varepsilon$  for all  $a \in F$ .
- (2)  $\text{dist}(pap, pBp) < \varepsilon$  for all  $a \in F$ .
- (3)  $1 - p$  is Murray-von Neumann equivalent to a projection in  $\overline{cAc}$ .

Then  $A$  has tracial rank zero.

*Proof.* Let  $F \subset A$  be a finite subset, let  $\varepsilon > 0$ , and let  $c \in A$  be a nonzero positive element. Choose  $p$  and  $B$  as in the hypotheses, with  $F$  and  $c$  as given and with  $\frac{1}{2}\varepsilon$  in place of  $\varepsilon$ . Let  $F_0 \subset pBp$  be a finite set such that  $\text{dist}(pap, F_0) < \frac{1}{2}\varepsilon$  for all  $a \in F$ . Since  $p \in B$ , the algebra  $pBp$  is also AF. Choose a unital finite dimensional

subalgebra  $D \subset pBp$  such that  $\text{dist}(b, D) < \frac{1}{2}\varepsilon$  for all  $b \in F_0$ . Then  $\text{dist}(pap, D) < \varepsilon$  for all  $a \in F$ .  $\square$

**Exercise 6.5.** Let  $A$  be a unital C\*-algebra, and let  $S \subset A$  be a subset which generates  $A$  as a C\*-algebra. Assume that the condition of Lemma 6.4 holds for all finite subsets  $F \subset S$ . Prove that  $A$  has tracial rank zero.

**Notation 6.6.** Let  $X$  be a compact metric space, and let  $h: X \rightarrow X$  be a homeomorphism. In the transformation group C\*-algebra  $C^*(\mathbb{Z}, X, h)$ , we normally write  $u$  for the standard unitary representing the generator of  $\mathbb{Z}$ . For a closed subset  $Y \subset X$ , we define the C\*-subalgebra  $C^*(\mathbb{Z}, X, h)_Y$  to be

$$C^*(\mathbb{Z}, X, h)_Y = C^*(C(X), uC_0(X \setminus Y)) \subset C^*(\mathbb{Z}, X, h).$$

**Remark 6.7.** Although we will not use formally groupoids in these notes, it should be pointed out that  $C^*(\mathbb{Z}, X, h)_Y$  is the C\*-algebra of a subgroupoid of the transformation group groupoid  $\mathbb{Z} \times X$  made from the action of  $\mathbb{Z}$  on  $X$  generated by  $h$ . Informally, we “break” every orbit each time it goes through  $Y$ . More formally, for  $n < 0$  the pair  $(n, x)$  is in the subgroupoid only if all of  $h^n(x), h^{n+1}(x), \dots, h^{-1}(x)$  are in  $X \setminus Y$ , and for  $n > 0$  the pair  $(n, x)$  is in the subgroupoid only if all of  $x, h(x), \dots, h^{n-1}(x)$  are in  $X \setminus Y$ .

For actions of  $\mathbb{Z}^d$ , it appears to be necessary to use subalgebras of the crossed product for which the only nice description is in terms of subgroupoids of the transformation group groupoid. See [27].

The following lemma is Theorem 3.3 of [32].

**Lemma 6.8.** Let  $X$  be the Cantor set, and let  $h: X \rightarrow X$  be a minimal homeomorphism. Let  $Y \subset X$  be a nonempty compact open subset. Then  $C^*(\mathbb{Z}, X, h)_Y$  is an AF algebra.

*Proof.* The proof depends on the construction of Rokhlin towers, which is a crucial element of many structure results for crossed products.

We first claim that there is  $N \in \mathbb{N}$  such that  $\bigcup_{n=1}^N h^{-n}(Y) = X$ . Set  $Z = X \setminus \bigcup_{n=1}^\infty h^{-n}(Y)$ . Then  $Z$  is closed, and  $h(Z) \subset Z$ . By (1) implies (2) of Lemma 5.3, we have  $Z = X$ . The claim now follows from compactness of  $X$ .

It follows that for each fixed  $y \in Y$ , the sequence of iterates  $h(y), h^2(y), \dots$  of  $y$  under  $h$  must return to  $Y$  in at most  $N$  steps. Define the *first return time*  $r(y)$  to be

$$r(y) = \min\{n \geq 1: h^n(y) \in Y\} \leq N.$$

Let  $n(0) < n(1) < \dots < n(l) \leq N$  be the values of  $r$ . Set

$$Y_k = \{y \in Y: r(y) = n(k)\}.$$

Then the sets  $Y_k$  are compact, open, and partition  $Y$ , and the sets  $h^j(Y_k)$ , for  $1 \leq j \leq n(k)$ , partition  $X$ :

$$Y = \bigsqcup_{k=0}^l Y_k \quad \text{and} \quad X = \bigsqcup_{k=0}^l \bigsqcup_{j=1}^{n(k)} h^j(Y_k).$$

Each finite sequence  $h(Y_k), h^2(Y_k), \dots, h^{n(k)}(Y_k)$  is a *Rokhlin tower* with base  $h(Y_k)$  and height  $n(k)$ . (It is more common to let the power of  $h$  run from 0 to  $n(k)-1$ . The choice made here, effectively taking the base of the collection of Rokhlin towers to be

$h(Y)$  rather than  $Y$ , is more convenient for use with our definition of  $C^*(\mathbb{Z}, X, h)_Y$ .) Further set  $X_k = \bigcup_{j=1}^{n(k)} h^j(Y_k)$ . The sets  $X_k$  then also partition  $X$ .

Define  $p_k \in C(X) \subset C^*(\mathbb{Z}, X, h)_Y$  by  $p_k = \chi_{X_k}$ . Then  $p_k$  clearly commutes with every element of  $C(X)$ . Moreover, if  $f \in C(X)$  vanishes on  $Y$ , we use  $Y_k, h^{n(k)}(Y_k) \subset Y$  at the second step to get

$$p_k u f = u(p_k + \chi_{Y_k} - \chi_{h^{n(k)}(Y_k)}) f = u p_k f.$$

It follows that  $p_k$  commutes with all elements of  $C^*(\mathbb{Z}, X, h)_Y$ . It therefore suffices to prove that  $p_k C^*(\mathbb{Z}, X, h)_Y p_k$  is AF for each  $k$ .

Now  $p_k C^*(\mathbb{Z}, X, h)_Y p_k$  is the C\*-algebra generated by  $C(X_k)$  and

$$u \chi_{X \setminus Y} p_k = u \chi_{X_k \setminus h^{n(k)}(Y_k)} = \sum_{j=1}^{n(k)-1} (\chi_{h^{j+1}(Y_k)}) u (\chi_{h^j(Y_k)}).$$

One can now check, although it is a bit tedious to write out the details, that there is an isomorphism  $\psi_k: p_k C^*(\mathbb{Z}, X, h)_Y p_k \rightarrow M_{n(k)} \otimes C(Y_k)$  such that for  $f \in C(X_k)$  we have

$$\psi_k(f) = \text{diag}(f \circ h|_{Y_k}, f \circ h^2|_{Y_k}, \dots, f \circ h^{n(k)}|_{Y_k})$$

and for  $1 \leq j \leq n(k) - 1$  we have

$$\psi_k(\chi_{h^{j+1}(Y_k)} u \chi_{h^j(Y_k)}) = e_{j+1, j} \otimes 1.$$

The algebra  $M_{n(k)} \otimes C(Y_k)$  is AF because  $Y_k$  is totally disconnected.  $\square$

**Notation 6.9.** Let  $A$  be a C\*-algebra, and let  $p, q \in A$  be projections. We write  $p \sim q$  to mean that  $p$  and  $q$  are Murray-von Neumann equivalent in  $A$ , that is, there exists  $v \in A$  such that  $v^*v = p$  and  $vv^* = q$ . We write  $p \lesssim q$  if  $p$  is Murray-von Neumann equivalent to a subprojection of  $q$ .

There are two other commonly used equivalence relations on projections, namely homotopy and unitary equivalence, so one needs to be careful with the meaning of  $p \sim q$  when reading papers. However, its most common meaning is Murray-von Neumann equivalence.

**Lemma 6.10.** Let  $X$  be the Cantor set, and let  $h: X \rightarrow X$  be a minimal homeomorphism. Let  $Y \subset X$  be a nonempty compact open subset. Let  $N \in \mathbb{N}$ , and suppose that  $Y, h(Y), \dots, h^N(Y)$  are disjoint. Then the projections  $\chi_Y$  and  $\chi_{h^N(Y)}$  are Murray-von Neumann equivalent in  $C^*(\mathbb{Z}, X, h)_Y$ .

*Proof.* First, observe that if  $Z \subset X$  is a compact open subset such that  $Y \cap Z = \emptyset$ , then  $v = u \chi_Z \in C^*(\mathbb{Z}, X, h)_Y$  and satisfies  $v^*v = \chi_Z$  and  $vv^* = \chi_{h(Z)}$ . Thus  $\chi_Z \sim \chi_{h(Z)}$ .

An induction argument now shows that  $\chi_{h(Y)} \sim \chi_{h^N(Y)}$ . Also,  $\chi_{X \setminus Y} \sim \chi_{X \setminus h(Y)}$ . Since  $C^*(\mathbb{Z}, X, h)_Y$  is an AF algebra, it follows that  $\chi_Y \sim \chi_{h(Y)}$ . The result follows by transitivity.  $\square$

**Lemma 6.11.** Let  $X$  be the Cantor set, and let  $h: X \rightarrow X$  be a minimal homeomorphism. Let  $c \in C^*(\mathbb{Z}, X, h)$  be a nonzero positive element. Then there exists a nonzero projection  $p \in C(X)$  such that  $p$  is Murray-von Neumann equivalent in  $C^*(\mathbb{Z}, X, h)$  to a projection in  $c C^*(\mathbb{Z}, X, h) c$ .

*Proof.* Let  $E: C^*(\mathbb{Z}, X, h) \rightarrow C(X)$  be the conditional expectation  $E_1$  of Proposition 3.30. It follows from Proposition 3.31(3) and Exercise 3.32(2) that  $E(c)$  is a nonzero positive element of  $C(X)$ . Choose a nonempty compact open subset  $K_0 \subset X$  and  $\delta > 0$  such that the function  $E(c)$  satisfies  $E(c)(x) > 4\delta$  for all  $x \in K_0$ . Choose a finite sum  $b = \sum_{n=-N}^N b_n u^n \in C^*(\mathbb{Z}, X, h)$  such that  $\|b - c\| < \delta$ . Since the action of  $\mathbb{Z}$  induced by  $h$  is free, there is a nonempty compact open subset  $K \subset K_0$  such that the sets  $h^{-N}(K), h^{-N+1}(K), \dots, h^N(K)$  are disjoint. Set  $p = \chi_K \in C(X)$ . For  $n \in \{-N, -N+1, \dots, N\} \setminus \{0\}$ , the disjointness condition implies that  $pu^n p = 0$ . Therefore  $pbp = pb_0 p = pE(b)p$ . Using this equation at the first step and Exercise 3.32(3) at the second step, we get

$$(4) \quad \|pcp - pE(c)p\| \leq \|pcp - pbp\| + \|pE(b)p - pE(c)p\| \leq 2\|c - b\| < 2\delta.$$

Since  $K \subset K_0$ , the function  $pE(c)p$  is invertible in  $pC(X)p$ . In the following, we take inverses in  $pC^*(\mathbb{Z}, X, h)p$ . Then, in fact,  $\|[pE(c)p]^{-1}\| < \frac{1}{4}\delta^{-1}$ . The estimate (4) now implies that  $pcp$  is invertible in  $pC^*(\mathbb{Z}, X, h)p$ . Let  $a = (pcp)^{-1/2}$ , calculated in  $pC^*(\mathbb{Z}, X, h)p$ . Set  $v = apc^{1/2}$ . Then

$$vv^* = apcpa = (pcp)^{-1/2}(pcp)(pcp)^{-1/2} = p$$

and

$$v^*v = c^{1/2}pa^2pc^{1/2} \in \overline{cC^*(\mathbb{Z}, X, h)c}.$$

This completes the proof.  $\square$

Most of the proof of the following lemma is taken from [22].

**Lemma 6.12.** Let  $X$  be the Cantor set, and let  $h: X \rightarrow X$  be a minimal homeomorphism. Let  $y \in X$ . Then for any  $\varepsilon > 0$ , any nonempty open set  $U \subset X$ , and any finite subset  $F \subset C(X)$ , there is a compact open set  $Y \subset X$  containing  $y$  and a projection  $p \in C^*(\mathbb{Z}, X, h)_Y$  such that:

- (1)  $\|pa - ap\| < \varepsilon$  for all  $a \in F \cup \{u\}$ .
- (2)  $pap \in pC^*(\mathbb{Z}, X, h)_Y p$  for all  $a \in F \cup \{u\}$ .
- (3) There is a compact open set  $Z \subset U$  such that  $1 - p \preceq \chi_Z$  in  $C^*(\mathbb{Z}, X, h)$ .

*Proof.* Let  $d$  be the metric on  $X$ . Choose  $N_0 \in \mathbb{N}$  so large that  $4\pi/N_0 < \varepsilon$ . Choose  $\delta_0 > 0$  with  $\delta_0 < \frac{1}{2}\varepsilon$  and so small that  $d(x_1, x_2) < 4\delta_0$  implies  $|f(x_1) - f(x_2)| < \frac{1}{4}\varepsilon$  for all  $f \in F$ . Choose  $\delta > 0$  with  $\delta \leq \delta_0$  and such that whenever  $d(x_1, x_2) < \delta$  and  $0 \leq k \leq N_0$ , then  $d(h^{-k}(x_1), h^{-k}(x_2)) < \delta_0$ .

Since  $h$  is minimal, there is  $N > N_0 + 1$  such that  $d(h^N(y), y) < \delta$ . Choose  $N + N_0 + 1$  disjoint nonempty open subsets  $U_{-N_0}, U_{-N_0+1}, \dots, U_N \subset U$ . Using minimality again, choose  $r_{-N_0}, r_{-N_0+1}, \dots, r_N \in \mathbb{Z}$  such that  $h^{r_l}(y) \in U_l$  for  $-N_0 \leq l \leq N$ . Since  $h$  is free, there is a compact open set  $Y \subset X$  containing  $y$  such that

$$h^{-N_0}(Y), h^{-N_0+1}(Y), \dots, Y, h(Y), \dots, h^N(Y)$$

are disjoint and all have diameter less than  $\delta$ . We may also require that  $h^{r_l}(Y) \subset U_l$  for  $-N_0 \leq l \leq N$ .

Set  $q_0 = \chi_Y$ . For  $-N_0 \leq n \leq N$  set

$$T_n = h^n(Y) \quad \text{and} \quad q_n = u^n q_0 u^{-n} = \chi_{h^n(Y)}.$$

Then the  $q_n$  are mutually orthogonal projections in  $C(X)$ .

We now have a sequence of projections, in principle going to infinity in both directions:

$$\dots, q_{-N_0}, \dots, q_{-1}, q_0, q_1, \dots, q_{N-N_0}, \dots, q_{N-1}, q_N, \dots$$

The ones shown are orthogonal, and conjugation by  $u$  is the shift. The projections  $q_0$  and  $q_N$  are the characteristic functions of compact open sets which are disjoint but close to each other, and similarly for the pairs  $q_{-1}$  and  $q_{N-1}$  down to  $q_{-N_0}$  and  $q_{N-N_0}$ . We are now going to use Berg's technique [2] to splice this sequence along the pairs of indices  $(-N_0, N - N_0)$  through  $(0, N)$ , obtaining a loop of length  $N$  on which conjugation by  $u$  is approximately the cyclic shift. Lemma 6.10 provides a partial isometry  $w \in C^*(\mathbb{Z}, X, h)_Y$  such that  $w^*w = q_0$  and  $ww^* = q_N$ .

For  $t \in [0, 1]$  define  $v(t) = \cos(\pi t/2)(q_0 + q_N) + \sin(\pi t/2)(w - w^*)$ . Then  $v(t)$  is a unitary in the corner  $(q_0 + q_N)C^*(\mathbb{Z}, X, h)_Y(q_0 + q_N)$  whose matrix with respect to the obvious block decomposition is

$$v(t) = \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

For  $0 \leq k \leq N_0$  define  $z_k = u^{-k}v(k/N_0)u^k$ .

We claim that  $z_k \in C^*(\mathbb{Z}, X, h)_Y$  for  $0 \leq k \leq N_0$ . For  $k = 0$  this is true by construction. For  $1 \leq k \leq N_0$ , with

$$a_k = q_0u^k = (uq_{-1})(uq_{-2}) \cdots (uq_{-k}) \in C^*(\mathbb{Z}, X, h)_Y$$

and

$$b_k = q_Nu^k = (uq_{N-1})(uq_{N-2}) \cdots (uq_{N-k}) \in C^*(\mathbb{Z}, X, h)_Y$$

(because  $N_0 < N$ ), and using  $T_{-k} \cap T_{N-k} = T_0 \cap T_N = \emptyset$ , we can write

$$z_k = (a_k + b_k)^*v(k/N_0)(a_k + b_k) \in C^*(\mathbb{Z}, X, h)_Y.$$

Therefore  $z_k$  is a unitary in the corner  $(q_{-k} + q_{N-k})C^*(\mathbb{Z}, X, h)_Y(q_{-k} + q_{N-k})$ .

Moreover, adding estimates on the differences of the matrix entries at the second step,

$$\|uz_{k+1}u^* - z_k\| = \|v(k/N_0) - v((k-1)/N_0)\| \leq 2\pi/N_0 < \frac{1}{2}\varepsilon.$$

Now define  $e_n = q_n$  for  $0 \leq n \leq N - N_0$ , and for  $N - N_0 \leq n \leq N$  write  $k = N - n$  and set  $e_n = z_kq_{-k}z_k^*$ . The two definitions for  $n = N - N_0$  agree because  $z_{N_0}q_{-N_0}z_{N_0}^* = q_{N-N_0}$ , and moreover  $e_N = e_0$ . Therefore  $ue_{n-1}u^* = e_n$  for  $1 \leq n \leq N - N_0$ , and also  $ue_Nu^* = e_1$ , while for  $N - N_0 < n \leq N$  we have

$$\|ue_{n-1}u^* - e_n\| \leq 2\|uz_{N-n+1}u^* - z_{N-n}\| < \varepsilon.$$

Also, clearly  $e_n \in C^*(\mathbb{Z}, X, h)_Y$  for all  $n$ .

Set  $e = \sum_{n=1}^N e_n$  and  $p = 1 - e$ . We verify that  $p$  satisfies (1) through (3).

First,

$$p - upu^* = ueu^* - e = \sum_{n=N_0+1}^N (ue_{n-1}u^* - e_n).$$

The terms in the sum are orthogonal and have norm less than  $\varepsilon$ , so  $\|upu^* - p\| < \varepsilon$ . Furthermore, since  $p \leq 1 - q_0 = 1 - \chi_Y$ , we get  $pup \in C^*(\mathbb{Z}, X, h)_Y$ . This is (1) and (2) for the element  $u \in F$ .

Next, let  $f \in F$ . The sets  $T_0, T_1, \dots, T_N$  all have diameter less than  $\delta$ . We have  $d(h^N(y), y) < \delta$ , so the choice of  $\delta$  implies that  $d(h^n(y), h^{n-N}(y)) < \delta_0$  for  $N - N_0 \leq n \leq N$ . Also,  $T_{n-N} = h^{n-N}(T_0)$  has diameter less than  $\delta$ . Therefore  $T_{n-N} \cup T_n$

has diameter less than  $2\delta + \delta_0 \leq 3\delta_0$ . Since  $f$  varies by at most  $\frac{1}{4}\varepsilon$  on any set with diameter less than  $4\delta_0$ , and since the sets

$$S_1 = T_1, S_2 = T_2, \dots, S_{N-N_0-1} = T_{N-N_0-1},$$

$S_{N-N_0} = T_{N-N_0} \cup T_{-N_0}$ ,  $S_{N-N_0+1} = T_{N-N_0+1} \cup T_{-N_0+1}$ ,  $\dots$ ,  $S_N = T_N \cup T_0$  are disjoint, there is  $g \in C(X)$  which is constant on each of these sets and satisfies  $\|f - g\| < \frac{1}{2}\varepsilon$ .

Let the values of  $g$  on these sets be  $\lambda_1$  on  $S_1$  through  $\lambda_N$  on  $S_N$ . Then  $ge_n = e_n g = \lambda_n e_n$  for  $0 \leq n \leq N - N_0$ . For  $N - N_0 < n \leq N$  we use  $e_n \in (q_{n-N} + q_n)C^*(\mathbb{Z}, X, h)_Y(q_{n-N} + q_n)$  to get, using the same calculations as above at the third and fourth steps,

$$ge_n = g(q_{n-N} + q_n)e_n = \lambda_n(q_{n-N} + q_n)e_n = e_n(q_{n-N} + q_n)g = e_n g.$$

Since  $\|f - g\| < \frac{1}{2}\varepsilon$  and  $ge = eg$ , it follows that

$$\|pf - fp\| = \|fe - ef\| < \varepsilon.$$

This is (1) for  $f$ . That  $pfp \in C^*(\mathbb{Z}, X, h)_Y$  follows from the fact that  $f$  and  $p$  are in this subalgebra. So we also have (2) for  $f$ .

It remains only to verify (3). Using  $h^{r_l}(Y) \subset U_l$  for  $-N_0 \leq l \leq N$  at the third step, and with  $Z = \bigcup_{l=-N_0}^N h^{r_l}(Y) \subset U$ , we get (with Murray-von Neumann equivalence in  $C^*(\mathbb{Z}, X, h)$ )

$$1 - p = e \leq \sum_{l=-N_0}^N q_l \sim \sum_{l=-N_0}^N \chi_{h^{r_l}(Y)} = \chi_Z.$$

This completes the proof.  $\square$

The proof of Theorem 6.1 for the case that  $X$  is the Cantor set now follows by combining Lemmas 6.8 and 6.12 with Lemma 6.4 and Exercise 6.5.

In the general case, one major main complication is that one can't choose the Rokhlin towers to consist of compact open sets. It turns out that one must take  $Y$  to be closed with nonempty interior, and replace the sets  $Y_k$  by their closures. Then they are no longer disjoint. The algebra  $C^*(\mathbb{Z}, X, h)_Y$  is now a very complicated subalgebra of  $\bigoplus_{k=0}^l M_{n(k)} \otimes C(Y_k)$ . It is what is known as a "recursive subhomogeneous algebra". Such algebras may have few or no nontrivial projections, and are generally not AF. The hypothesis on the range of  $\rho$  (which was not used above, although it is automatic when  $X$  is the Cantor set) must be used to produce sufficiently many nonzero projections and approximating finite dimensional subalgebras.

## 7. MISCELLANEOUS ITEMS FOR PLANNED ADDITIONS

**Lemma 7.1.** Let  $X$  be an infinite compact metric space, and let  $h: X \rightarrow X$  be a minimal homeomorphism. Let  $y \in X$ . Then the restriction map  $T(C^*(\mathbb{Z}, X, h)) \rightarrow T(C^*(\mathbb{Z}, X, h)_{\{y\}})$  is a bijection.

The proof follows from the following two results. The first is well known, and holds in much greater generality. The proofs are roughly the same, but the proof of the second is more complicated, since the algebra is smaller.

**Proposition 7.2.** Let  $X$  be an infinite compact metric space, and let  $h: X \rightarrow X$  be a minimal homeomorphism. Then the restriction map  $T(C^*(\mathbb{Z}, X, h)) \rightarrow T(C(X))$  is a bijection from  $T(C^*(\mathbb{Z}, X, h))$  to the set of  $h$ -invariant Borel probability measures on  $X$ . Moreover, if  $E: C^*(\mathbb{Z}, X, h) \rightarrow C(X)$  is the conditional expectation  $E_1$  of Proposition 3.30, and  $\mu$  is an  $h$ -invariant Borel probability measure on  $X$ , then the tracial state  $\tau_\mu$  on  $C^*(\mathbb{Z}, X, h)$  which restricts to  $\mu$  is given by the formula  $\tau_\mu(a) = \int_X E(a) d\mu$  for  $a \in C^*(\mathbb{Z}, X, h)$ .

*Proof.* It is immediate that  $\tau_\mu$  is positive, and that  $\tau_\mu(1) = 1$ . Therefore  $\tau_\mu$  is a state.

We prove  $\tau_\mu(ab) = \tau_\mu(ba)$  for all  $a, b \in C^*(\mathbb{Z}, X, h)$ . Since  $\tau_\mu$  is continuous, and since  $C^*(\mathbb{Z}, X, h)$  is the closed linear span of all elements  $fu^m$  with  $f \in C(X)$  and  $m \in \mathbb{Z}$ , it suffices to prove  $\tau_\mu((fu^m)(gu^n)) = \tau_\mu((gu^n)(fu^m))$  for all  $f, g \in C(X)$  and  $m, n \in \mathbb{Z}$ . For  $n \neq -m$ , we have

$$\tau_\mu((fu^m)(gu^n)) = \tau_\mu((gu^n)(fu^m)) = 0,$$

while for  $n = -m$ , we have, using  $h$ -invariance of  $\mu$  at the second step,

$$\begin{aligned} \tau_\mu((fu^m)(gu^{-m})) &= \tau_\mu(f(u^m gu^{-m})) = \int_X f(g \circ h^{-m}) d\mu \\ &= \int_X (f \circ h^m)g d\mu = \tau_\mu(g(u^{-m} fu^m)) = \tau_\mu((gu^{-m})(fu^m)). \end{aligned}$$

This completes the proof that  $\tau_\mu$  is a tracial state on  $C^*(\mathbb{Z}, X, h)$ .

Now let  $\tau$  be any tracial state on  $C^*(\mathbb{Z}, X, h)$ . Let  $\mu$  be the Borel probability measure on  $X$  determined by  $\tau(f) = \int_X f d\mu$  for  $f \in C(X)$ . We complete the proof by showing that  $\mu$  is  $h$ -invariant, and that  $\tau = \tau_\mu$ .

For the first, for every  $f \in C(X)$ , we use the trace property at the second step to get

$$\int_X (f \circ h^{-1}) d\mu = \tau(ufu^*) = \tau(u^*(uf)) = \tau(f) = \int_X f d\mu.$$

Since  $f \in C(X)$  is arbitrary, it follows that  $\mu$  is  $h$ -invariant.

For the second, and since  $C^*(\mathbb{Z}, X, h)$  is the closed linear span of all elements  $fu^n$  with  $f \in C(X)$  and  $m \in \mathbb{Z}$ , it suffices to prove  $\tau(fu^n) = 0$  for  $f \in C(X)$  and  $n \in \mathbb{Z} \setminus \{0\}$ . Since  $h^n$  has no fixed points, there is an open cover of  $X$  consisting of sets  $U$  such that  $h^n(U) \cap U = \emptyset$ . Choose  $g_1, g_2, \dots, g_m \in C(X)$  which form a partition of unity subordinate to this cover. In particular, the supports of  $g_j$  and  $g_j \circ h^{-n}$  are disjoint for all  $j$ . For  $1 \leq j \leq m$  we have, using the trace property at the second step, the relation  $u^n g_j u^{-n} = g_j \circ h^{-n}$  at the third step,

$$\tau(g_j fu^n) = \tau(g_j^{1/2} fu^n g_j^{1/2}) = \tau(g_j^{1/2} f (g_j^{1/2} \circ h^{-n}) u^n) = \tau(0) = 0.$$

Summing over  $j$  gives  $\tau(fu^n) = 0$ .  $\square$

**Lemma 7.3** (Proposition 16 of [23]). Let  $X$  be an infinite compact metric space, and let  $h: X \rightarrow X$  be a minimal homeomorphism. Let  $y \in X$ . Then the restriction map  $T(C^*(\mathbb{Z}, X, h)_{\{y\}}) \rightarrow T(C(X))$  is a bijection from  $T(C^*(\mathbb{Z}, X, h)_{\{y\}})$  to the set of  $h$ -invariant Borel probability measures on  $X$ .

*Proof.* Applying Proposition 7.2 and restricting from  $C^*(\mathbb{Z}, X, h)$  to  $C^*(\mathbb{Z}, X, h)_{\{y\}}$ , we see that every  $h$ -invariant Borel probability measure on  $X$  gives a tracial state on  $C^*(\mathbb{Z}, X, h)_{\{y\}}$ .

Now let  $\tau$  be any tracial state on  $C^*(\mathbb{Z}, X, h)_{\{y\}}$ . Let  $\mu$  be the Borel probability measure on  $X$  determined by  $\tau(f) = \int_X f d\mu$  for  $f \in C(X)$ . As in the proof of Proposition 7.2, we complete the proof by showing that  $\mu$  is  $h$ -invariant, and that  $\tau = \tau_\mu|_{C^*(\mathbb{Z}, X, h)_{\{y\}}}$ .

For the first, we again show that  $\int_X (f \circ h^{-1}) d\mu = \int_X f d\mu$  for every  $f \in C(X)$ . This is clearly true for constant functions  $f$ . Therefore, it suffices to consider functions  $f$  such that  $f(y) = 0$ . For such a function  $f$ , write  $f = f_1 f_2^*$  with  $f_1, f_2 \in C(X)$  such that  $f_1(y) = f_2(y) = 0$ . (For example, take  $f_1 = f|f|^{-1/2}$  and  $f_2 = |f|^{1/2}$ .) Then  $u f_1, u f_2 \in C^*(\mathbb{Z}, X, h)_{\{y\}}$ . So

$$f \circ h^{-1} = u f u^* = (u f_1)(u f_2)^* \in C^*(\mathbb{Z}, X, h)_{\{y\}}.$$

We now use the trace property at the second step to get

$$\int_X (f \circ h^{-1}) d\mu = \tau((u f_1)(u f_2)^*) = \tau((u f_2)^*(u f_1)) = \tau(f) = \int_X f d\mu.$$

Thus  $\mu$  is  $h$ -invariant.

For the second, we first claim that  $C^*(\mathbb{Z}, X, h)_{\{y\}}$  is the closed linear span of all elements of the form  $f u^m$ , with  $f \in C(X)$  and  $m \in \mathbb{Z}$ , which actually happen to be in  $C^*(\mathbb{Z}, X, h)_{\{y\}}$ . The claim follows from three facts: all the generators listed in the definition of  $C^*(\mathbb{Z}, X, h)_{\{y\}}$  have this form (using  $u f = (f \circ h^{-1})u$ ), the product of two things of this form again has this form (use  $(f u^m)(g u^n) = [f(g \circ h^{-m})]u^{m+n}$ , which is again in  $C^*(\mathbb{Z}, X, h)_{\{y\}}$  because  $C^*(\mathbb{Z}, X, h)_{\{y\}}$  is closed under multiplication), and the adjoint of something of this form again has this form (by a similar argument).

It now suffices to prove that if  $f u^n \in C^*(\mathbb{Z}, X, h)_{\{y\}}$  and  $n \neq 0$ , then  $\tau(f u^n) = 0$ . The argument is the same as in the proof of Proposition 7.2; one merely needs to note that the elements  $g_j$  used there are in  $C(X) \subset C^*(\mathbb{Z}, X, h)_{\{y\}}$ .  $\square$

The following is a generalization of Lemma 6.11. It is both more elementary and more general than the corresponding argument in [22] (the main part of the proof of Theorem 4.5 there).

**Proof of the following needs to be rechecked.**

**Lemma 7.4.** Let  $X$  be a compact metric space, and let  $h: X \rightarrow X$  be a minimal homeomorphism. Let  $B \subset C^*(\mathbb{Z}, X, h)$  be a unital subalgebra which contains  $C(X)$  and has Property (SP). Let  $c \in C^*(\mathbb{Z}, X, h)$  be a nonzero positive element. Then there exists a nonzero projection  $p \in \overline{B}$  such that  $p$  is Murray-von Neumann equivalent in  $C^*(\mathbb{Z}, X, h)$  to a projection in  $c C^*(\mathbb{Z}, X, h) c$ .

*Proof.* Let  $E: C^*(\mathbb{Z}, X, h) \rightarrow C(X)$  be the conditional expectation  $E_1$  of Proposition 3.30. It follows from Proposition 3.31(3) and Exercise 3.32(2) that  $E(c)$  is a nonzero positive element of  $C(X)$ . Set  $\delta = \frac{1}{7} \|E(c)\|$ . Let  $f \in C(X)$  be the pointwise minimum  $f(x) = \min(6\delta, E(c)(x))$ . Then

$$(5) \quad \|f - E(c)\| = \delta.$$

Set

$$U_0 = \{x \in X: E(c)(x) > 6\delta\},$$

which is a nonempty open set.

Choose a finite sum  $a = \sum_{n=-N}^N a_n u^n \in C^*(\mathbb{Z}, X, h)$  such that  $\|a - c\| < \delta$ . Let  $d$  be the metric on  $X$ . Choose  $\varepsilon > 0$  so small that whenever  $x_1, x_2 \in X$  satisfy  $d(x_1, x_2) < \varepsilon$ , then

$$|a_n(x_1) - a_n(x_2)| < \frac{\delta}{2N+1}$$

for  $-N \leq n \leq N$ . Using freeness of the action of  $\mathbb{Z}$  induced by  $h$ , choose a nonempty open subset  $U \subset U_0$  such that the sets  $h^{-N}(U), h^{-N+1}(U), \dots, h^N(U)$  are disjoint, and such that  $\overline{U}$  has diameter less than  $\varepsilon$ . Then there are  $b_{-N}, b_{-N+1}, \dots, b_N \in C(X)$  such that  $b_n$  is constant on  $\overline{U}$  and

$$\|b_n - a_n\| < \frac{\delta}{2N+1}$$

for  $-N \leq n \leq N$ . Set  $b = \sum_{n=-N}^N b_n u^n$ , and note that  $\|b - a\| < \delta$ . Thus,  $\|b - c\| < 2\delta$ .

Choose continuous functions  $g_0, g_1: X \rightarrow [0, 1]$  such that  $\text{supp}(g_0) \subset U_0$ ,  $g_0 g_1 = g_1$ , and  $g_1 \neq 0$ . Use the hypotheses on  $B$  to choose a nonzero projection  $p \in g_1 B g_1$ . We clearly have  $g_0 p = p g_0 = p$ . It follows that  $f p = f g_0 p = 6\delta g_0 p = 6\delta p$ , and similarly  $p f = 6\delta p$ . The same reasoning shows that  $p b_n = b_n p$  for  $-N \leq n \leq N$ . Also, for  $n \in \{-N, -N+1, \dots, N\} \setminus \{0\}$ , the disjointness condition implies that  $g_0 u^n g_0 = 0$ , whence  $p u^n p = p g_0 u^n g_0 p = 0$ . It follows that

$$p b p = \sum_{n=-N}^N p b_n u^n p = \sum_{n=-N}^N b_n p u^n p = b_0 p = p E(b) p.$$

Using this last equation at the second step and Exercise 3.32(3) at the third step, we get

$$\begin{aligned} \|p c p - 6\delta p\| &= \|p c p - p f p\| \\ &\leq \|p c p - p b p\| + \|p E(b) p - p E(c) p\| + \|p E(c) p - p f p\| \\ &\leq 2\|c - b\| + \|E(c) - f\| < 5\delta. \end{aligned}$$

It follows that  $p c p$  is invertible in  $p C^*(\mathbb{Z}, X, h) p$ . Let  $a = (p c p)^{-1/2}$ , calculated in  $p C^*(\mathbb{Z}, X, h) p$ . Set  $v = a p c^{1/2}$ . Then

$$v v^* = a p c p a = (p c p)^{-1/2} (p c p) (p c p)^{-1/2} = p$$

and

$$v^* v = c^{1/2} p a^2 p c^{1/2} \in \overline{c C^*(\mathbb{Z}, X, h) c}.$$

This completes the proof.  $\square$

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