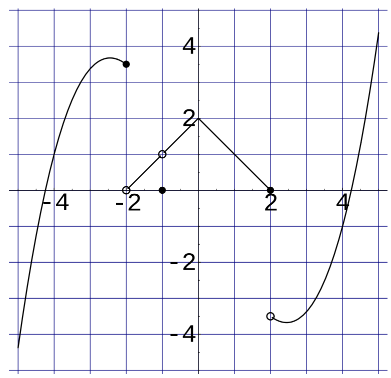
MATH 251 (PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 3.

This sheet is part of the homework for Week 3, and is due in class on Friday 25 January 2008.

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WebAssign), give exact answers (not decimal approximations; again, unlike WebAssign), and use correct notation. Some of the grade will be based on correctness of notation in the work shown.

1. For the function $y = f(x)$ graphed below, answer the following questions:



(a) List all numbers a in the interval $(-5, 5)$ such that $f'(a)$ does not exist. Give brief reasons for your choices.

Solution: The numbers are -2, -1, 0, and 2. The derivative does not exist at -2, -1, and 2 because the function is discontinuous at all those points. (You can see directly that the derivative doesn't exist at those points by drawing secant lines.) The derivative does not exist at 0 because the graph has a corner there.

Note that it is not correct to say that f is discontinuous at $(-1, 0)$ (or at $(-1, 1)$). A function is continuous or discontinuous, as the case may be, at a number in its *domain*. Similarly, it is correct to say that f is not differentiable at 0, but it is not correct to say that f is not differentiable at $(0, 2)$. Moreover, it is the *function* that is continuous, not continuous, differentiable, etc. It is not correct to say that the point -1, or the point $(-1, 0)$ or $(-1, 1)$, is discontinuous.

(b) Which of the following best describes $f'(3)$? Why?
(1) $f'(3)$ does not exist.
(2) $f'(3)$ is close to 0.
(3) $f'(3)$ is positive and not close to 0.
(4) $f'(3)$ is negative and not close to 0.

Solution: $f'(3)$ is the slope of the tangent line to the graph of $y = f(x)$ at $x = 3$. You can tell from inspection that this slope is positive and not close to 0 (choice (3) above). If you actually draw a tangent line on the graph, you should get a slope of somewhere near 1.

2. Let $w(t)$ be the water flow at time t in a river at a particular measuring station. Assume that t is measured in days, and that $w(t)$ is measured in $\mathrm{m}^3/\mathrm{sec}$.

(a) What are the units of $w'(t)$?

Solution: $\mathrm{m}^3/\mathrm{sec}$ per day, or $\mathrm{m}^3/(\mathrm{sec}\cdot\mathrm{day})$.

(b) During the beginning of the rainy season, do you expect $w'(t)$ to be positive or negative? Why?

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Solution: Positive, because the amount of water flowing in the river should be increasing.

(c) Explain the practical significance of the statement $w'(t_0) = -27$ for a particular time t_0.

Solution: The amount of water flowing in the river is decreasing at time t_0. going down by 27 m^3/sec per day. In particular, if it decreased at the same rate for a whole day, the amount of water flowing in the river would be 27 m^3/sec less a day after time t_0.

3. Express the following statement in terms of calculus. Be sure to define everything that appears in your formulas.

"The average income of lawyers is decreasing."

Solution: Let $L(t)$ be the average income of lawyers at time t. Let $t = t_0$ be the time referred to in the statement. Then $L'(t_0) < 0$ since the average income is decreasing.

(Obviously this problem does not reflect reality.)

4. If $f(x) = \sqrt{x+7}$, compute the derivative $f'(9)$ *directly from the definition*. (If you have looked ahead, you can check your answer using the differentiation formula, but no credit will be given for just using the formula.)

Solution: We find the limit of the difference quotient, using the technique of rationalizing the numerator to handle the resulting expression:

$$f'(9) = \lim_{h\to 0} \frac{f(9+h) - f(9)}{h} = \lim_{h\to 0} \frac{\sqrt{9+h+7} - \sqrt{9+7}}{h} = \lim_{h\to 0} \frac{\sqrt{16+h} - 4}{h} = \lim_{h\to 0} \frac{\left(\sqrt{16+h} - 4\right)\left(\sqrt{16+h} + 4\right)}{h\left(\sqrt{16+h} + 4\right)}$$

$$= \lim_{h\to 0} \frac{(16+h) - 16}{h\left(\sqrt{16+h} + 4\right)} = \lim_{h\to 0} \frac{h}{h\left(\sqrt{16+h} + 4\right)} = \lim_{h\to 0} \frac{1}{\sqrt{16+h} + 4} = \frac{1}{\sqrt{16} + 4} = \frac{1}{8}.$$

We can check using the differentiation formulas: $f(x) = (x+7)^{1/2}$, so $f'(x) = \frac{1}{2}(x+7)^{-1/2}$, and $f'(9) = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{7+9}}\right) = \frac{1}{8}$. (However, you get no credit if this is the only thing you do.)