Seoul National University short course: An introduction to the structure of crossed product C*-algebras.

Lecture 1: What is a crossed product?

N. Christopher Phillips

University of Oregon

12 December 2009

Comments

There is a related set of notes posted on the web. See the link at:

http://www.uoregon.edu/∼ncp/Courses/LisbonCrossedProducts/
LisbonCrossedProducts.html

This is accessible from my home page:

http://www.uoregon.edu/∼ncp

The notes contain most of what I will say during these lectures, and much more besides.

Comments

There is a related set of notes posted on the web. See the link at:

http://www.uoregon.edu/∼ncp/Courses/LisbonCrossedProducts/
LisbonCrossedProducts.html

This is accessible from my home page:

http://www.uoregon.edu/∼ncp

The notes contain most of what I will say during these lectures, and much more besides. Also see the slides from the Lisbon course, four links on the same website. There is a great deal of overlap.

Comments

There is a related set of notes posted on the web. See the link at:

http://www.uoregon.edu/∼ncp/Courses/LisbonCrossedProducts/
LisbonCrossedProducts.html

This is accessible from my home page:

http://www.uoregon.edu/∼ncp

The notes contain most of what I will say during these lectures, and much more besides. Also see the slides from the Lisbon course, four links on the same website. There is a great deal of overlap.

Please let me know of any misprints, mistakes, etc. found in the notes, slides, etc. I will also post the slides from these lectures on my website, at:

http://www.uoregon.edu/∼ncp/Courses/SeoulCrossedProducts/
SeoulCrossedProducts.html

(also available from a link on my home page).

Outline

There are many directions in the theory of crossed products. These lectures are biased towards the general problem of classifying crossed products,

Outline

There are many directions in the theory of crossed products. These lectures are biased towards the general problem of classifying crossed products, in cases in which they are expected to be simple.

Outline

There are many directions in the theory of crossed products. These lectures are biased towards the general problem of classifying crossed products, in cases in which they are expected to be simple. (However, we will not get very far in that direction.)

Outline

There are many directions in the theory of crossed products. These lectures are biased towards the general problem of classifying crossed products, in cases in which they are expected to be simple. (However, we will not get very far in that direction.) See the end of Section 1 of the notes for other directions.

Outline

There are many directions in the theory of crossed products. These lectures are biased towards the general problem of classifying crossed products, in cases in which they are expected to be simple. (However, we will not get very far in that direction.) See the end of Section 1 of the notes for other directions.

A brief outline of the lectures:

Outline

There are many directions in the theory of crossed products. These lectures are biased towards the general problem of classifying crossed products, in cases in which they are expected to be simple. (However, we will not get very far in that direction.) See the end of Section 1 of the notes for other directions.

A brief outline of the lectures:

- Introductory material, basic definitions, and examples of group actions.

Outline

There are many directions in the theory of crossed products. These lectures are biased towards the general problem of classifying crossed products, in cases in which they are expected to be simple. (However, we will not get very far in that direction.) See the end of Section 1 of the notes for other directions.

A brief outline of the lectures:

- Introductory material, basic definitions, and examples of group actions.
- Construction of the crossed product of an action by a discrete group.

Outline

There are many directions in the theory of crossed products. These lectures are biased towards the general problem of classifying crossed products, in cases in which they are expected to be simple. (However, we will not get very far in that direction.) See the end of Section 1 of the notes for other directions.

A brief outline of the lectures:

- Introductory material, basic definitions, and examples of group actions.
- Construction of the crossed product of an action by a discrete group.
- Examples of some elementary computations of crossed products.

Outline

There are many directions in the theory of crossed products. These lectures are biased towards the general problem of classifying crossed products, in cases in which they are expected to be simple. (However, we will not get very far in that direction.) See the end of Section 1 of the notes for other directions.

A brief outline of the lectures:

- Introductory material, basic definitions, and examples of group actions.
- Construction of the crossed product of an action by a discrete group.
- Examples of some elementary computations of crossed products.
- Simplicity of crossed products by minimal homeomorphisms.

Outline

There are many directions in the theory of crossed products. These lectures are biased towards the general problem of classifying crossed products, in cases in which they are expected to be simple. (However, we will not get very far in that direction.) See the end of Section 1 of the notes for other directions.

A brief outline of the lectures:

- Introductory material, basic definitions, and examples of group actions.
- Construction of the crossed product of an action by a discrete group.
- Examples of some elementary computations of crossed products.
- Simplicity of crossed products by minimal homeomorphisms.
- Toward the classification of crossed products by minimal homeomorphisms.

Outline

There are many directions in the theory of crossed products. These lectures are biased towards the general problem of classifying crossed products, in cases in which they are expected to be simple. (However, we will not get very far in that direction.) See the end of Section 1 of the notes for other directions.

A brief outline of the lectures:

- Introductory material, basic definitions, and examples of group actions.
- Construction of the crossed product of an action by a discrete group.
- Examples of some elementary computations of crossed products.
- Simplicity of crossed products by minimal homeomorphisms.
- Toward the classification of crossed products by minimal homeomorphisms.
- Actions of $\mathbb{Z}^d$: an outline of the subgroupoid subalgebra method.

Actions of groups on C*-algebras

Definition

Let G be a locally compact group, and let A be a C*-algebra.

Actions of groups on C*-algebras

Definition

Let G be a locally compact group, and let A be a C*-algebra. An *action of G on A*

Actions of groups on C*-algebras

Definition

Let G be a locally compact group, and let A be a C*-algebra. An *action of G on A* is a homomorphism $\alpha \colon G \to \mathrm{Aut}(A)$,

Actions of groups on C*-algebras

Definition

Let G be a locally compact group, and let A be a C*-algebra. An *action of G on A* is a homomorphism $\alpha \colon G \to \mathrm{Aut}(A)$, usually written $g \mapsto \alpha_g$,

Actions of groups on C*-algebras

Definition

Let G be a locally compact group, and let A be a C*-algebra. An *action of G on A* is a homomorphism $\alpha \colon G \to \mathrm{Aut}(A)$, usually written $g \mapsto \alpha_g$, such that, for every $a \in A$, the function $g \mapsto \alpha_g(a)$, from G to A, is norm continuous.

Actions of groups on C*-algebras

Definition

Let G be a locally compact group, and let A be a C*-algebra. An *action of G on A* is a homomorphism $\alpha \colon G \to \mathrm{Aut}(A)$, usually written $g \mapsto \alpha_g$, such that, for every $a \in A$, the function $g \mapsto \alpha_g(a)$, from G to A, is norm continuous.

On a von Neumann algebra, substitute the σ-weak operator topology for the norm topology.

Actions of groups on C*-algebras

Definition

Let G be a locally compact group, and let A be a C*-algebra. An *action of G on A* is a homomorphism $\alpha \colon G \to \mathrm{Aut}(A)$, usually written $g \mapsto \alpha_g$, such that, for every $a \in A$, the function $g \mapsto \alpha_g(a)$, from G to A, is norm continuous.

On a von Neumann algebra, substitute the σ-weak operator topology for the norm topology.

The continuity condition is the analog of requiring that a unitary representation of G on a Hilbert space be continuous in the strong operator topology. It is usually much too strong a condition to require that $g \mapsto \alpha_g$ be a norm continuous map from G to the bounded operators on A.

Actions of groups on C*-algebras

Definition

Let G be a locally compact group, and let A be a C*-algebra. An *action of G on A* is a homomorphism $\alpha \colon G \to \mathrm{Aut}(A)$, usually written $g \mapsto \alpha_g$, such that, for every $a \in A$, the function $g \mapsto \alpha_g(a)$, from G to A, is norm continuous.

On a von Neumann algebra, substitute the σ-weak operator topology for the norm topology.

The continuity condition is the analog of requiring that a unitary representation of G on a Hilbert space be continuous in the strong operator topology. It is usually much too strong a condition to require that $g \mapsto \alpha_g$ be a norm continuous map from G to the bounded operators on A.

Of course, if G is discrete, it doesn't matter. In this course, we will concentrate on discrete G.

We will construct crossed products

Given $\alpha \colon G \to \mathrm{Aut}(A)$, we will construct a crossed product C*-algebra $C^*(G, A, \alpha)$

We will construct crossed products

Given $\alpha \colon G \to \mathrm{Aut}(A)$, we will construct a crossed product C*-algebra $C^*(G, A, \alpha)$ and a reduced crossed product C*-algebra $C^*_{\mathrm{r}}(G, A, \alpha)$.

We will construct crossed products

Given $\alpha \colon G \to \mathrm{Aut}(A)$, we will construct a crossed product C*-algebra $C^*(G, A, \alpha)$ and a reduced crossed product C*-algebra $C^*_{\mathrm{r}}(G, A, \alpha)$. (There are many other commonly used notations. See Remark 3.16 in the notes.)

We will construct crossed products

Given $\alpha\colon G \to \mathrm{Aut}(A)$, we will construct a crossed product C*-algebra $C^*(G, A, \alpha)$ and a reduced crossed product C*-algebra $C^*_{\mathrm{r}}(G, A, \alpha)$. (There are many other commonly used notations. See Remark 3.16 in the notes.)

If A is unital and G is discrete, it is a suitable completion of the algebraic skew group ring $A[G]$, with multiplication determined by $gag^{-1} = \alpha_g(a)$ for $g \in G$ and $a \in A$.

Motivation for group actions on C*-algebras and their crossed products

Let G be a locally compact group obtained as a semidirect product $G = N \rtimes H$. The action of H on N gives actions of H on the full and reduced group C*-algebras $C^*(N)$ and $C^*_{\mathrm{r}}(N)$, and one has $C^*(G) \cong C^*(H, C^*(N))$ and $C^*_{\mathrm{r}}(G) \cong C^*_{\mathrm{r}}(H, C^*_{\mathrm{r}}(N))$.

Motivation for group actions on C*-algebras and their crossed products

Let G be a locally compact group obtained as a semidirect product $G = N \rtimes H$. The action of H on N gives actions of H on the full and reduced group C*-algebras $C^*(N)$ and $C^*_{\mathrm{r}}(N)$, and one has $C^*(G) \cong C^*(H, C^*(N))$ and $C^*_{\mathrm{r}}(G) \cong C^*_{\mathrm{r}}(H, C^*_{\mathrm{r}}(N))$.

Probably the most important group action is time evolution: if a C*-algebra A is supposed to represent the possible states of a physical system in some manner, then there should be an action $\alpha\colon \mathbb{R} \to \mathrm{Aut}(A)$ which describes the time evolution of the system. Actions of $\mathbb{Z}$, which are easier to study, can be thought of as "discrete time evolution".

Motivation for group actions on C*-algebras and their crossed products

Let G be a locally compact group obtained as a semidirect product $G = N \rtimes H$. The action of H on N gives actions of H on the full and reduced group C*-algebras $C^*(N)$ and $C^*_{\mathrm{r}}(N)$, and one has $C^*(G) \cong C^*(H, \, C^*(N))$ and $C^*_{\mathrm{r}}(G) \cong C^*_{\mathrm{r}}(H, \, C^*_{\mathrm{r}}(N))$.

Probably the most important group action is time evolution: if a C*-algebra A is supposed to represent the possible states of a physical system in some manner, then there should be an action $\alpha \colon \mathbb{R} \to \mathrm{Aut}(A)$ which describes the time evolution of the system. Actions of $\mathbb{Z}$, which are easier to study, can be thought of as "discrete time evolution".

Crossed products are a common way of constructing simple C*-algebras. We will see some examples later.

Motivation for group actions on C*-algebras and their crossed products (continued)

If one has a homeomorphism h of a locally compact Hausdorff space X, the crossed product $C^*(\mathbb{Z}, X, h)$ sometimes carries considerable information about the dynamics of h. The best known example is the result of Giordano, Putnam, and Skau on minimal homeomorphisms of the Cantor set: isomorphism of the transformation group C*-algebras is equivalent to strong orbit equivalence of the homeomorphisms.

Motivation for group actions on C*-algebras and their crossed products (continued)

If one has a homeomorphism h of a locally compact Hausdorff space X, the crossed product $C^*(\mathbb{Z}, X, h)$ sometimes carries considerable information about the dynamics of h. The best known example is the result of Giordano, Putnam, and Skau on minimal homeomorphisms of the Cantor set: isomorphism of the transformation group C*-algebras is equivalent to strong orbit equivalence of the homeomorphisms.

For compact groups, equivariant indices take values on the equivariant K-theory of a suitable C*-algebra with an action of the group. When the group is not compact, one usually needs instead the K-theory of the crossed product C*-algebra, or of the reduced crossed product C*-algebra. (When the group is compact, this is the same thing.)

Motivation for group actions on C*-algebras and their crossed products (continued)

If one has a homeomorphism h of a locally compact Hausdorff space X, the crossed product $C^*(\mathbb{Z}, X, h)$ sometimes carries considerable information about the dynamics of h. The best known example is the result of Giordano, Putnam, and Skau on minimal homeomorphisms of the Cantor set: isomorphism of the transformation group C*-algebras is equivalent to strong orbit equivalence of the homeomorphisms.

For compact groups, equivariant indices take values on the equivariant K-theory of a suitable C*-algebra with an action of the group. When the group is not compact, one usually needs instead the K-theory of the crossed product C*-algebra, or of the reduced crossed product C*-algebra. (When the group is compact, this is the same thing.)

In other situations as well, the K-theory of the full or reduced crossed product is the appropriate substitute for equivariant K-theory.

The commutative case

Definition

A continuous action of a topological group G on a topological space X is a continuous function $G \times X \to X$, usually written $(g, x) \mapsto g \cdot x$ or $(g, x) \mapsto gx$, such that $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$ and $1 \cdot x = x$ for all $x \in X$.

The commutative case

Definition

A continuous action of a topological group G on a topological space X is a continuous function $G \times X \to X$, usually written $(g, x) \mapsto g \cdot x$ or $(g, x) \mapsto gx$, such that $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$ and $1 \cdot x = x$ for all $x \in X$.

For a continuous action of a locally compact group G on a locally compact Hausdorff space X, there is a corresponding action $\alpha \colon G \to \mathrm{Aut}(C_0(X))$, given by $\alpha_g(f)(x) = f(g^{-1}x)$.

The commutative case

Definition

A continuous action of a topological group G on a topological space X is a continuous function $G \times X \to X$, usually written $(g, x) \mapsto g \cdot x$ or $(g, x) \mapsto gx$, such that $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$ and $1 \cdot x = x$ for all $x \in X$.

For a continuous action of a locally compact group G on a locally compact Hausdorff space X, there is a corresponding action $\alpha \colon G \to \mathrm{Aut}(C_0(X))$, given by $\alpha_g(f)(x) = f(g^{-1}x)$.

(If G is not abelian, the inverse is necessary to get $\alpha_g \circ \alpha_h = \alpha_{gh}$ rather than α_{hg}.)

The commutative case

Definition

A continuous action of a topological group G on a topological space X is a continuous function $G \times X \to X$, usually written $(g, x) \mapsto g \cdot x$ or $(g, x) \mapsto gx$, such that $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$ and $1 \cdot x = x$ for all $x \in X$.

For a continuous action of a locally compact group G on a locally compact Hausdorff space X, there is a corresponding action $\alpha \colon G \to \mathrm{Aut}(C_0(X))$, given by $\alpha_g(f)(x) = f(g^{-1}x)$.

(If G is not abelian, the inverse is necessary to get $\alpha_g \circ \alpha_h = \alpha_{gh}$ rather than α_{hg}.)

One should check that these formulas determine a one to one correspondence between continuous actions of G on X and continuous actions of G on $C_0(X)$.

The commutative case

Definition

A continuous action of a topological group G on a topological space X is a continuous function $G \times X \to X$, usually written $(g, x) \mapsto g \cdot x$ or $(g, x) \mapsto gx$, such that $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$ and $1 \cdot x = x$ for all $x \in X$.

For a continuous action of a locally compact group G on a locally compact Hausdorff space X, there is a corresponding action $\alpha \colon G \to \mathrm{Aut}(C_0(X))$, given by $\alpha_g(f)(x) = f(g^{-1}x)$.

(If G is not abelian, the inverse is necessary to get $\alpha_g \circ \alpha_h = \alpha_{gh}$ rather than α_{hg}.)

One should check that these formulas determine a one to one correspondence between continuous actions of G on X and continuous actions of G on $C_0(X)$. (The main point is to check that the continuity conditions match.)

Examples of group actions on C*-algebras

We will give some examples of actions of a group G on C*-algebras. (Not all of them give interesting crossed products.)

Examples of group actions on C*-algebras

We will give some examples of actions of a group G on C*-algebras. (Not all of them give interesting crossed products.)

We start with examples of group actions on locally compact spaces X, which give rise to examples of group actions on commutative C*-algebras.

Examples of group actions on C*-algebras

We will give some examples of actions of a group G on C*-algebras. (Not all of them give interesting crossed products.)

We start with examples of group actions on locally compact spaces X, which give rise to examples of group actions on commutative C*-algebras.

We will discuss some of their crossed products later, but in some of the examples we state the results immediately.

Examples of group actions on C*-algebras

We will give some examples of actions of a group G on C*-algebras. (Not all of them give interesting crossed products.)

We start with examples of group actions on locally compact spaces X, which give rise to examples of group actions on commutative C*-algebras.

We will discuss some of their crossed products later, but in some of the examples we state the results immediately. As one goes through the commutative examples, note that a closed orbit of the form $Gx \cong G/H$ gives rise to a quotient of the crossed product isomorphic to $K(L^2(G/H)) \otimes C^*(H)$.

Examples of group actions on C*-algebras

We will give some examples of actions of a group G on C*-algebras. (Not all of them give interesting crossed products.)

We start with examples of group actions on locally compact spaces X, which give rise to examples of group actions on commutative C*-algebras.

We will discuss some of their crossed products later, but in some of the examples we state the results immediately. As one goes through the commutative examples, note that a closed orbit of the form $Gx \cong G/H$ gives rise to a quotient of the crossed product isomorphic to $K(L^2(G/H)) \otimes C^*(H)$.

There are more examples in the notes, and there is more detail on these in the Lisbon slides.

Examples of actions on compact spaces

- G is arbitrary, X is a point, and the action is trivial.

Examples of actions on compact spaces

- G is arbitrary, X is a point, and the action is trivial. The full and reduced crossed products are the usual full and reduced group C*-algebras $C^*(G)$ and $C^*_{\mathrm{r}}(G)$.

Examples of actions on compact spaces

- G is arbitrary, X is a point, and the action is trivial. The full and reduced crossed products are the usual full and reduced group C*-algebras $C^*(G)$ and $C^*_{\mathrm{r}}(G)$.
- $X = G$, and the action is given by (left) translation: $g \cdot x = gx$.

Examples of actions on compact spaces

- G is arbitrary, X is a point, and the action is trivial. The full and reduced crossed products are the usual full and reduced group C*-algebras $C^*(G)$ and $C^*_{\mathrm{r}}(G)$.
- $X = G$, and the action is given by (left) translation: $g \cdot x = gx$. The full and reduced crossed products are both isomorphic to $K(L^2(G))$.

Examples of actions on compact spaces

- G is arbitrary, X is a point, and the action is trivial. The full and reduced crossed products are the usual full and reduced group C*-algebras $C^*(G)$ and $C^*_{\mathrm{r}}(G)$.
- $X = G$, and the action is given by (left) translation: $g \cdot x = gx$. The full and reduced crossed products are both isomorphic to $K(L^2(G))$.
- If $H \subset G$ is a closed subgroup, then G acts continuously on G/H by translation.

Examples of actions on compact spaces

- G is arbitrary, X is a point, and the action is trivial. The full and reduced crossed products are the usual full and reduced group C*-algebras $C^*(G)$ and $C^*_{\mathrm{r}}(G)$.
- $X = G$, and the action is given by (left) translation: $g \cdot x = gx$. The full and reduced crossed products are both isomorphic to $K(L^2(G))$.
- If $H \subset G$ is a closed subgroup, then G acts continuously on G/H by translation. It turns out that $C^*(G, G/H) \cong K(L^2(G/H)) \otimes C^*(H)$. Note that there is no "twisting".

Examples of actions on compact spaces

- G is arbitrary, X is a point, and the action is trivial. The full and reduced crossed products are the usual full and reduced group C*-algebras $C^*(G)$ and $C^*_{\mathrm{r}}(G)$.
- $X = G$, and the action is given by (left) translation: $g \cdot x = gx$. The full and reduced crossed products are both isomorphic to $K(L^2(G))$.
- If $H \subset G$ is a closed subgroup, then G acts continuously on G/H by translation. It turns out that $C^*(G, G/H) \cong K(L^2(G/H)) \otimes C^*(H)$. Note that there is no "twisting".
- If $H \subset G$ is a closed subgroup, then H acts continuously on G by translation.

Examples of actions on compact spaces

- G is arbitrary, X is a point, and the action is trivial. The full and reduced crossed products are the usual full and reduced group C*-algebras $C^*(G)$ and $C^*_{\mathrm{r}}(G)$.
- $X = G$, and the action is given by (left) translation: $g \cdot x = gx$. The full and reduced crossed products are both isomorphic to $K(L^2(G))$.
- If $H \subset G$ is a closed subgroup, then G acts continuously on G/H by translation. It turns out that $C^*(G, G/H) \cong K(L^2(G/H)) \otimes C^*(H)$. Note that there is no "twisting".
- If $H \subset G$ is a closed subgroup, then H acts continuously on G by translation. It turns out that $C^*(H, G)$ is stably isomorphic to $K(L^2(H)) \otimes C_0(G/H)$. Stably, there is no "twisting".

Examples of actions on compact spaces

- G is arbitrary, X is a point, and the action is trivial. The full and reduced crossed products are the usual full and reduced group C*-algebras $C^*(G)$ and $C^*_{\mathrm{r}}(G)$.
- $X = G$, and the action is given by (left) translation: $g \cdot x = gx$. The full and reduced crossed products are both isomorphic to $K(L^2(G))$.
- If $H \subset G$ is a closed subgroup, then G acts continuously on G/H by translation. It turns out that $C^*(G, \, G/H) \cong K(L^2(G/H)) \otimes C^*(H)$. Note that there is no "twisting".
- If $H \subset G$ is a closed subgroup, then H acts continuously on G by translation. It turns out that $C^*(H, G)$ is stably isomorphic to $K(L^2(H)) \otimes C_0(G/H)$. Stably, there is no "twisting".
- $X = S^1 = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\}$, $G = \mathbb{Z}$, and the action is rotation by multiples of a fixed angle $2\pi\theta$.

Examples of actions on compact spaces

- G is arbitrary, X is a point, and the action is trivial. The full and reduced crossed products are the usual full and reduced group C*-algebras $C^*(G)$ and $C^*_{\mathrm{r}}(G)$.
- $X = G$, and the action is given by (left) translation: $g \cdot x = gx$. The full and reduced crossed products are both isomorphic to $K(L^2(G))$.
- If $H \subset G$ is a closed subgroup, then G acts continuously on G/H by translation. It turns out that $C^*(G, G/H) \cong K(L^2(G/H)) \otimes C^*(H)$. Note that there is no "twisting".
- If $H \subset G$ is a closed subgroup, then H acts continuously on G by translation. It turns out that $C^*(H, G)$ is stably isomorphic to $K(L^2(H)) \otimes C_0(G/H)$. Stably, there is no "twisting".
- $X = S^1 = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\}$, $G = \mathbb{Z}$, and the action is rotation by multiples of a fixed angle $2\pi\theta$. These are *rational rotations* (for $\theta \in \mathbb{Q}$) or *irrational rotations* (for $\theta \notin \mathbb{Q}$),

Examples of actions on compact spaces

- G is arbitrary, X is a point, and the action is trivial. The full and reduced crossed products are the usual full and reduced group C*-algebras $C^*(G)$ and $C^*_{\mathrm{r}}(G)$.
- $X = G$, and the action is given by (left) translation: $g \cdot x = gx$. The full and reduced crossed products are both isomorphic to $K(L^2(G))$.
- If $H \subset G$ is a closed subgroup, then G acts continuously on G/H by translation. It turns out that $C^*(G, G/H) \cong K(L^2(G/H)) \otimes C^*(H)$. Note that there is no "twisting".
- If $H \subset G$ is a closed subgroup, then H acts continuously on G by translation. It turns out that $C^*(H, G)$ is stably isomorphic to $K(L^2(H)) \otimes C_0(G/H)$. Stably, there is no "twisting".
- $X = S^1 = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\}$, $G = \mathbb{Z}$, and the action is rotation by multiples of a fixed angle $2\pi\theta$. These are *rational rotations* (for $\theta \in \mathbb{Q}$) or *irrational rotations* (for $\theta \notin \mathbb{Q}$), and the crossed products are the well known (rational or irrational) rotation algebras.

Examples of actions on compact spaces (continued)

- Take $X = \{0, 1\}^{\mathbb{Z}}$, with elements being described as $x = (x_n)_{n \in \mathbb{Z}}$. Take $G = \mathbb{Z}$, with action generated by the *shift* homeomorphism $h(x)_n = x_{n-1}$ for $x \in X$ and $n \in \mathbb{Z}$.

Examples of actions on compact spaces (continued)

- Take $X = \{0, 1\}^{\mathbb{Z}}$, with elements being described as $x = (x_n)_{n \in \mathbb{Z}}$. Take $G = \mathbb{Z}$, with action generated by the *shift* homeomorphism $h(x)_n = x_{n-1}$ for $x \in X$ and $n \in \mathbb{Z}$.
- Subshifts: In the previous example, replace X by an invariant subset.

Examples of actions on compact spaces (continued)

- Take $X = \{0, 1\}^{\mathbb{Z}}$, with elements being described as $x = (x_n)_{n \in \mathbb{Z}}$. Take $G = \mathbb{Z}$, with action generated by the *shift* homeomorphism $h(x)_n = x_{n-1}$ for $x \in X$ and $n \in \mathbb{Z}$.
- Subshifts: In the previous example, replace X by an invariant subset.
- More general shifts and subshifts: replace $\{0, 1\}$ by some other compact metric space S.

Examples of actions on compact spaces (continued)

- Take $X = \{0, 1\}^{\mathbb{Z}}$, with elements being described as $x = (x_n)_{n \in \mathbb{Z}}$. Take $G = \mathbb{Z}$, with action generated by the *shift* homeomorphism $h(x)_n = x_{n-1}$ for $x \in X$ and $n \in \mathbb{Z}$.
- Subshifts: In the previous example, replace X by an invariant subset.
- More general shifts and subshifts: replace $\{0, 1\}$ by some other compact metric space S.
- Let $X = \mathbb{Z}_p$, the group of p-adic integers.

Examples of actions on compact spaces (continued)

- Take $X = \{0, 1\}^{\mathbb{Z}}$, with elements being described as $x = (x_n)_{n \in \mathbb{Z}}$. Take $G = \mathbb{Z}$, with action generated by the *shift* homeomorphism $h(x)_n = x_{n-1}$ for $x \in X$ and $n \in \mathbb{Z}$.
- Subshifts: In the previous example, replace X by an invariant subset.
- More general shifts and subshifts: replace $\{0, 1\}$ by some other compact metric space S.
- Let $X = \mathbb{Z}_p$, the group of p-adic integers. It is a compact topological group, and as a metric space it is homeomorphic to the Cantor set.

Examples of actions on compact spaces (continued)

- Take $X = \{0, 1\}^{\mathbb{Z}}$, with elements being described as $x = (x_n)_{n \in \mathbb{Z}}$. Take $G = \mathbb{Z}$, with action generated by the *shift* homeomorphism $h(x)_n = x_{n-1}$ for $x \in X$ and $n \in \mathbb{Z}$.
- Subshifts: In the previous example, replace X by an invariant subset.
- More general shifts and subshifts: replace $\{0, 1\}$ by some other compact metric space S.
- Let $X = \mathbb{Z}_p$, the group of p-adic integers. It is a compact topological group, and as a metric space it is homeomorphic to the Cantor set. Let $h \colon X \to X$ be the homeomorphism defined on the dense subset $\mathbb{Z}$ by $h(n) = n + 1$, and take the action of $\mathbb{Z}$ it generates.

Examples of actions on compact spaces (continued)

- Take $X = \{0, 1\}^{\mathbb{Z}}$, with elements being described as $x = (x_n)_{n \in \mathbb{Z}}$. Take $G = \mathbb{Z}$, with action generated by the *shift* homeomorphism $h(x)_n = x_{n-1}$ for $x \in X$ and $n \in \mathbb{Z}$.
- Subshifts: In the previous example, replace X by an invariant subset.
- More general shifts and subshifts: replace $\{0, 1\}$ by some other compact metric space S.
- Let $X = \mathbb{Z}_p$, the group of p-adic integers. It is a compact topological group, and as a metric space it is homeomorphic to the Cantor set. Let $h \colon X \to X$ be the homeomorphism defined on the dense subset $\mathbb{Z}$ by $h(n) = n + 1$, and take the action of $\mathbb{Z}$ it generates. Many variations are possible.

Examples of actions on compact spaces (continued)

- Take $X = S^n = \{x \in \mathbb{R}^{n+1} \colon \|x\|_2 = 1\}$. Multiplication by -1 generates an action of $\mathbb{Z}/2\mathbb{Z}$.

Examples of actions on compact spaces (continued)

- Take $X = S^n = \{x \in \mathbb{R}^{n+1} \colon \|x\|_2 = 1\}$. Multiplication by -1 generates an action of $\mathbb{Z}/2\mathbb{Z}$. The crossed product turns out to be isomorphic to the section algebra of a locally trivial but nontrivial bundle over the real projective space $\mathbb{R}P^n = S^n/(\mathbb{Z}/2\mathbb{Z})$ with fiber M_2.

Examples of actions on compact spaces (continued)

- Take $X = S^n = \{x \in \mathbb{R}^{n+1} \colon \|x\|_2 = 1\}$. Multiplication by -1 generates an action of $\mathbb{Z}/2\mathbb{Z}$. The crossed product turns out to be isomorphic to the section algebra of a locally trivial but nontrivial bundle over the real projective space $\mathbb{R}P^n = S^n/(\mathbb{Z}/2\mathbb{Z})$ with fiber M_2.
- Complex conjugation generates an action of $\mathbb{Z}/2\mathbb{Z}$ on $S^1 \subset \mathbb{C}$.

Examples of actions on compact spaces (continued)

- Take $X = S^n = \{x \in \mathbb{R}^{n+1} \colon \|x\|_2 = 1\}$. Multiplication by -1 generates an action of $\mathbb{Z}/2\mathbb{Z}$. The crossed product turns out to be isomorphic to the section algebra of a locally trivial but nontrivial bundle over the real projective space $\mathbb{R}P^n = S^n/(\mathbb{Z}/2\mathbb{Z})$ with fiber M_2.
- Complex conjugation generates an action of $\mathbb{Z}/2\mathbb{Z}$ on $S^1 \subset \mathbb{C}$.
- Take $G = \mathrm{SL}_2(\mathbb{Z})$. It acts linearly on $\mathbb{R}^2$ (as a subgroup of $\mathrm{GL}_2(\mathbb{R})$), fixing $\mathbb{Z}^2$,

Examples of actions on compact spaces (continued)

- Take $X = S^n = \{x \in \mathbb{R}^{n+1} \colon \|x\|_2 = 1\}$. Multiplication by -1 generates an action of $\mathbb{Z}/2\mathbb{Z}$. The crossed product turns out to be isomorphic to the section algebra of a locally trivial but nontrivial bundle over the real projective space $\mathbb{R}P^n = S^n/(\mathbb{Z}/2\mathbb{Z})$ with fiber M_2.
- Complex conjugation generates an action of $\mathbb{Z}/2\mathbb{Z}$ on $S^1 \subset \mathbb{C}$.
- Take $G = \mathrm{SL}_2(\mathbb{Z})$. It acts linearly on $\mathbb{R}^2$ (as a subgroup of $\mathrm{GL}_2(\mathbb{R})$), fixing $\mathbb{Z}^2$, so the action is well defined on $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$.

Examples of actions on noncommutative C*-algebras

- There is a trivial action of G on any C*-algebra A.

Examples of actions on noncommutative C*-algebras

- There is a trivial action of G on any C*-algebra A. The full crossed product turns out to be $C^*(G) \otimes_{\mathrm{max}} A$, and the reduced crossed product turns out to be $C^*_{\mathrm{r}}(G) \otimes_{\mathrm{min}} A$.

Examples of actions on noncommutative C*-algebras

- There is a trivial action of G on any C*-algebra A. The full crossed product turns out to be $C^*(G) \otimes_{\max} A$, and the reduced crossed product turns out to be $C^*_{\mathrm{r}}(G) \otimes_{\min} A$.
- If A is unital and $u \in A$ is unitary, let $\mathrm{Ad}(u)$ be the automorphism $a \mapsto uau^*$.

Examples of actions on noncommutative C*-algebras

- There is a trivial action of G on any C*-algebra A. The full crossed product turns out to be $C^*(G) \otimes_{\max} A$, and the reduced crossed product turns out to be $C^*_{\mathrm{r}}(G) \otimes_{\min} A$.
- If A is unital and $u \in A$ is unitary, let $\mathrm{Ad}(u)$ be the automorphism $a \mapsto uau^*$. Now let G be locally compact, let A be unital, and let $g \mapsto z_g$ be a norm continuous group homomorphism from G to the unitary group $U(A)$ of A.

Examples of actions on noncommutative C*-algebras

- There is a trivial action of G on any C*-algebra A. The full crossed product turns out to be $C^*(G) \otimes_{\mathrm{max}} A$, and the reduced crossed product turns out to be $C^*_{\mathrm{r}}(G) \otimes_{\mathrm{min}} A$.
- If A is unital and $u \in A$ is unitary, let $\mathrm{Ad}(u)$ be the automorphism $a \mapsto uau^*$. Now let G be locally compact, let A be unital, and let $g \mapsto z_g$ be a norm continuous group homomorphism from G to the unitary group $U(A)$ of A. Then $g \mapsto \mathrm{Ad}(z_g)$ defines an action of G on A.

Examples of actions on noncommutative C*-algebras

- There is a trivial action of G on any C*-algebra A. The full crossed product turns out to be $C^*(G) \otimes_{\mathrm{max}} A$, and the reduced crossed product turns out to be $C^*_{\mathrm{r}}(G) \otimes_{\mathrm{min}} A$.
- If A is unital and $u \in A$ is unitary, let $\mathrm{Ad}(u)$ be the automorphism $a \mapsto uau^*$. Now let G be locally compact, let A be unital, and let $g \mapsto z_g$ be a norm continuous group homomorphism from G to the unitary group $U(A)$ of A. Then $g \mapsto \mathrm{Ad}(z_g)$ defines an action of G on A. These actions are called *inner*.

Examples of actions on noncommutative C*-algebras

- There is a trivial action of G on any C*-algebra A. The full crossed product turns out to be $C^*(G) \otimes_{\max} A$, and the reduced crossed product turns out to be $C^*_{\mathrm{r}}(G) \otimes_{\min} A$.
- If A is unital and $u \in A$ is unitary, let $\mathrm{Ad}(u)$ be the automorphism $a \mapsto uau^*$. Now let G be locally compact, let A be unital, and let $g \mapsto z_g$ be a norm continuous group homomorphism from G to the unitary group $U(A)$ of A. Then $g \mapsto \mathrm{Ad}(z_g)$ defines an action of G on A. These actions are called *inner*. The crossed products turn out to be the same as for the trivial action.

Examples of actions on noncommutative C*-algebras

- There is a trivial action of G on any C*-algebra A. The full crossed product turns out to be $C^*(G) \otimes_{\max} A$, and the reduced crossed product turns out to be $C^*_{\mathrm{r}}(G) \otimes_{\min} A$.
- If A is unital and $u \in A$ is unitary, let $\mathrm{Ad}(u)$ be the automorphism $a \mapsto uau^*$. Now let G be locally compact, let A be unital, and let $g \mapsto z_g$ be a norm continuous group homomorphism from G to the unitary group $U(A)$ of A. Then $g \mapsto \mathrm{Ad}(z_g)$ defines an action of G on A. These actions are called *inner*. The crossed products turn out to be the same as for the trivial action.
- An action via inner automorphisms is not necessarily an inner action.

Examples of actions on noncommutative C*-algebras

- There is a trivial action of G on any C*-algebra A. The full crossed product turns out to be $C^*(G) \otimes_{\max} A$, and the reduced crossed product turns out to be $C^*_{\mathrm{r}}(G) \otimes_{\min} A$.
- If A is unital and $u \in A$ is unitary, let $\mathrm{Ad}(u)$ be the automorphism $a \mapsto uau^*$. Now let G be locally compact, let A be unital, and let $g \mapsto z_g$ be a norm continuous group homomorphism from G to the unitary group $U(A)$ of A. Then $g \mapsto \mathrm{Ad}(z_g)$ defines an action of G on A. These actions are called *inner*. The crossed products turn out to be the same as for the trivial action.
- An action via inner automorphisms is not necessarily an inner action. Let $A = M_2$, let $G = (\mathbb{Z}/2\mathbb{Z})^2$ with generators g_1 and g_2, and set

Examples of actions on noncommutative C*-algebras

- There is a trivial action of G on any C*-algebra A. The full crossed product turns out to be $C^*(G) \otimes_{\max} A$, and the reduced crossed product turns out to be $C^*_{\mathrm{r}}(G) \otimes_{\min} A$.
- If A is unital and $u \in A$ is unitary, let $\mathrm{Ad}(u)$ be the automorphism $a \mapsto uau^*$. Now let G be locally compact, let A be unital, and let $g \mapsto z_g$ be a norm continuous group homomorphism from G to the unitary group $U(A)$ of A. Then $g \mapsto \mathrm{Ad}(z_g)$ defines an action of G on A. These actions are called *inner*. The crossed products turn out to be the same as for the trivial action.
- An action via inner automorphisms is not necessarily an inner action. Let $A = M_2$, let $G = (\mathbb{Z}/2\mathbb{Z})^2$ with generators g_1 and g_2, and set

$$\alpha_1 = \mathrm{id}_A, \quad \alpha_{g_1} = \mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right), \quad \alpha_{g_2} = \mathrm{Ad}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right), \quad \alpha_{g_1 g_2} = \mathrm{Ad}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right).$$

Examples of actions on noncommutative C*-algebras

- There is a trivial action of G on any C*-algebra A. The full crossed product turns out to be $C^*(G) \otimes_{\max} A$, and the reduced crossed product turns out to be $C^*_{\mathrm{r}}(G) \otimes_{\min} A$.
- If A is unital and $u \in A$ is unitary, let $\mathrm{Ad}(u)$ be the automorphism $a \mapsto uau^*$. Now let G be locally compact, let A be unital, and let $g \mapsto z_g$ be a norm continuous group homomorphism from G to the unitary group $U(A)$ of A. Then $g \mapsto \mathrm{Ad}(z_g)$ defines an action of G on A. These actions are called *inner*. The crossed products turn out to be the same as for the trivial action.
- An action via inner automorphisms is not necessarily an inner action. Let $A = M_2$, let $G = (\mathbb{Z}/2\mathbb{Z})^2$ with generators g_1 and g_2, and set

$$\alpha_1 = \mathrm{id}_A, \quad \alpha_{g_1} = \mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad \alpha_{g_2} = \mathrm{Ad}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \quad \alpha_{g_1 g_2} = \mathrm{Ad}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right).$$

  The point is that the implementing unitaries for α_{g_1} and α_{g_2} commute up to a scalar,

Examples of actions on noncommutative C*-algebras

- There is a trivial action of G on any C*-algebra A. The full crossed product turns out to be $C^*(G) \otimes_{\max} A$, and the reduced crossed product turns out to be $C^*_{\mathrm{r}}(G) \otimes_{\min} A$.
- If A is unital and $u \in A$ is unitary, let $\mathrm{Ad}(u)$ be the automorphism $a \mapsto uau^*$. Now let G be locally compact, let A be unital, and let $g \mapsto z_g$ be a norm continuous group homomorphism from G to the unitary group $U(A)$ of A. Then $g \mapsto \mathrm{Ad}(z_g)$ defines an action of G on A. These actions are called *inner*. The crossed products turn out to be the same as for the trivial action.
- An action via inner automorphisms is not necessarily an inner action. Let $A = M_2$, let $G = (\mathbb{Z}/2\mathbb{Z})^2$ with generators g_1 and g_2, and set

$$\alpha_1 = \mathrm{id}_A, \quad \alpha_{g_1} = \mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad \alpha_{g_2} = \mathrm{Ad}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \quad \alpha_{g_1 g_2} = \mathrm{Ad}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right).$$

  The point is that the implementing unitaries for α_{g_1} and α_{g_2} commute up to a scalar, but can't be appropriately modified to commute exactly.

Examples of actions on noncommutative C*-algebras

- There is a trivial action of G on any C*-algebra A. The full crossed product turns out to be $C^*(G) \otimes_{\max} A$, and the reduced crossed product turns out to be $C^*_{\mathrm{r}}(G) \otimes_{\min} A$.
- If A is unital and $u \in A$ is unitary, let $\mathrm{Ad}(u)$ be the automorphism $a \mapsto uau^*$. Now let G be locally compact, let A be unital, and let $g \mapsto z_g$ be a norm continuous group homomorphism from G to the unitary group $U(A)$ of A. Then $g \mapsto \mathrm{Ad}(z_g)$ defines an action of G on A. These actions are called *inner*. The crossed products turn out to be the same as for the trivial action.
- An action via inner automorphisms is not necessarily an inner action. Let $A = M_2$, let $G = (\mathbb{Z}/2\mathbb{Z})^2$ with generators g_1 and g_2, and set

$$\alpha_1 = \mathrm{id}_A, \quad \alpha_{g_1} = \mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad \alpha_{g_2} = \mathrm{Ad}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \quad \alpha_{g_1 g_2} = \mathrm{Ad}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right).$$

  The point is that the implementing unitaries for α_{g_1} and α_{g_2} commute up to a scalar, but can't be appropriately modified to commute exactly. The crossed product turns out to be isomorphic to M_4.

Examples of actions on C*-algebras (continued)

- For $\theta \in \mathbb{R}$, let A_θ be the rotation algebra,

Examples of actions on C*-algebras (continued)

- For $\theta \in \mathbb{R}$, let A_θ be the rotation algebra, the universal C*-algebra generated by unitaries u and v satisfying $vu = \exp(2\pi i\theta)uv$.

Examples of actions on C*-algebras (continued)

- For $\theta \in \mathbb{R}$, let A_θ be the rotation algebra, the universal C*-algebra generated by unitaries u and v satisfying $vu = \exp(2\pi i\theta)uv$. The group $G = \mathrm{SL}_2(\mathbb{Z})$ acts on A_θ by sending the matrix $n = \left(\begin{smallmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{smallmatrix}\right)$ to the automorphism

Examples of actions on C*-algebras (continued)

- For $\theta \in \mathbb{R}$, let A_θ be the rotation algebra, the universal C*-algebra generated by unitaries u and v satisfying $vu = \exp(2\pi i\theta)uv$. The group $G = \mathrm{SL}_2(\mathbb{Z})$ acts on A_θ by sending the matrix $n = \left(\begin{smallmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{smallmatrix} \right)$ to the automorphism

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}}, \quad \alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$

Examples of actions on C*-algebras (continued)

- For $\theta \in \mathbb{R}$, let A_θ be the rotation algebra, the universal C*-algebra generated by unitaries u and v satisfying $vu = \exp(2\pi i\theta)uv$. The group $G = \mathrm{SL}_2(\mathbb{Z})$ acts on A_θ by sending the matrix $n = \left(\begin{smallmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{smallmatrix}\right)$ to the automorphism

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1}\theta) u^{n_{1,1}} v^{n_{2,1}}, \quad \alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2}\theta) u^{n_{1,2}} v^{n_{2,2}}.$$

This is the noncommutative version of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $S^1 \times S^1$ above.

Examples of actions on C*-algebras (continued)

- For $\theta \in \mathbb{R}$, let A_θ be the rotation algebra, the universal C*-algebra generated by unitaries u and v satisfying $vu = \exp(2\pi i\theta)uv$. The group $G = \mathrm{SL}_2(\mathbb{Z})$ acts on A_θ by sending the matrix $n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$ to the automorphism

  $$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1}\theta) u^{n_{1,1}} v^{n_{2,1}}, \quad \alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2}\theta) u^{n_{1,2}} v^{n_{2,2}}.$$

  This is the noncommutative version of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $S^1 \times S^1$ above.
- Restrict the action of the previous example to finite subgroups.

Examples of actions on C*-algebras (continued)

- For $\theta \in \mathbb{R}$, let A_θ be the rotation algebra, the universal C*-algebra generated by unitaries u and v satisfying $vu = \exp(2\pi i\theta)uv$. The group $G = \mathrm{SL}_2(\mathbb{Z})$ acts on A_θ by sending the matrix $n = \left(\begin{smallmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{smallmatrix}\right)$ to the automorphism

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1}\theta) u^{n_{1,1}} v^{n_{2,1}}, \quad \alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2}\theta) u^{n_{1,2}} v^{n_{2,2}}.$$

  This is the noncommutative version of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $S^1 \times S^1$ above.
- Restrict the action of the previous example to finite subgroups. We now know that for $\theta \notin \mathbb{Q}$ the crossed products are all AF.

Examples of actions on C*-algebras (continued)

- For $\theta \in \mathbb{R}$, let A_θ be the rotation algebra, the universal C*-algebra generated by unitaries u and v satisfying $vu = \exp(2\pi i\theta)uv$. The group $G = \mathrm{SL}_2(\mathbb{Z})$ acts on A_θ by sending the matrix $n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$ to the automorphism

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1}\theta) u^{n_{1,1}} v^{n_{2,1}}, \quad \alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2}\theta) u^{n_{1,2}} v^{n_{2,2}}.$$

  This is the noncommutative version of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $S^1 \times S^1$ above.

- Restrict the action of the previous example to finite subgroups. We now know that for $\theta \notin \mathbb{Q}$ the crossed products are all AF.
- There is an action $\alpha \colon S^1 \times S^1 \to \mathrm{Aut}(A_\theta)$ determined by

$$\alpha_{(\zeta_1,\zeta_2)}(u) = \zeta_1 u \quad \text{and} \quad \alpha_{(\zeta_1,\zeta_2)}(v) = \zeta_2 v.$$

Examples of actions on C*-algebras (continued)

- For $\theta \in \mathbb{R}$, let A_θ be the rotation algebra, the universal C*-algebra generated by unitaries u and v satisfying $vu = \exp(2\pi i \theta) uv$. The group $G = \mathrm{SL}_2(\mathbb{Z})$ acts on A_θ by sending the matrix $n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$ to the automorphism

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}}, \quad \alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$

  This is the noncommutative version of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $S^1 \times S^1$ above.
- Restrict the action of the previous example to finite subgroups. We now know that for $\theta \notin \mathbb{Q}$ the crossed products are all AF.
- There is an action $\alpha \colon S^1 \times S^1 \to \mathrm{Aut}(A_\theta)$ determined by

$$\alpha_{(\zeta_1, \zeta_2)}(u) = \zeta_1 u \quad \text{and} \quad \alpha_{(\zeta_1, \zeta_2)}(v) = \zeta_2 v.$$

- Restrict the previous action to subgroups of $S^1 \times S^1$.

Examples of actions on C*-algebras (continued)

- For $\theta \in \mathbb{R}$, let A_θ be the rotation algebra, the universal C*-algebra generated by unitaries u and v satisfying $vu = \exp(2\pi i\theta)uv$. The group $G = \mathrm{SL}_2(\mathbb{Z})$ acts on A_θ by sending the matrix $n = \left(\begin{smallmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{smallmatrix}\right)$ to the automorphism

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1}\theta) u^{n_{1,1}} v^{n_{2,1}}, \quad \alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2}\theta) u^{n_{1,2}} v^{n_{2,2}}.$$

  This is the noncommutative version of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $S^1 \times S^1$ above.
- Restrict the action of the previous example to finite subgroups. We now know that for $\theta \notin \mathbb{Q}$ the crossed products are all AF.
- There is an action $\alpha \colon S^1 \times S^1 \to \mathrm{Aut}(A_\theta)$ determined by

$$\alpha_{(\zeta_1,\zeta_2)}(u) = \zeta_1 u \quad \text{and} \quad \alpha_{(\zeta_1,\zeta_2)}(v) = \zeta_2 v.$$

- Restrict the previous action to subgroups of $S^1 \times S^1$. For example, a single such automorphism generates an action of $\mathbb{Z}$.

Examples of actions on C*-algebras (continued)

- Let $s_1, s_2, \ldots, s_n$ be the standard generators of the Cuntz algebra $\mathcal{O}_n$, satisfying $s_j^* s_j = 1$ for $1 \leq j \leq n$ and $\sum_{j=1}^{n} s_j s_j^* = 1$.

Examples of actions on C*-algebras (continued)

- Let $s_1, s_2, \ldots, s_n$ be the standard generators of the Cuntz algebra $\mathcal{O}_n$, satisfying $s_j^* s_j = 1$ for $1 \leq j \leq n$ and $\sum_{j=1}^{n} s_j s_j^* = 1$. There is an action of $(S^1)^n$ on $\mathcal{O}_n$ such that $\alpha_{(\zeta_1, \zeta_2, \ldots, \zeta_n)}(s_j) = \zeta_j s_j$ for $1 \leq j \leq n$.

Examples of actions on C*-algebras (continued)

- Let $s_1, s_2, \ldots, s_n$ be the standard generators of the Cuntz algebra $\mathcal{O}_n$, satisfying $s_j^* s_j = 1$ for $1 \leq j \leq n$ and $\sum_{j=1}^{n} s_j s_j^* = 1$. There is an action of $(S^1)^n$ on $\mathcal{O}_n$ such that $\alpha_{(\zeta_1, \zeta_2, \ldots, \zeta_n)}(s_j) = \zeta_j s_j$ for $1 \leq j \leq n$.
- Regarding $(S^1)^n$ as the diagonal unitary matrices, this action extends to an action of the unitary group $U(M_n)$ on $\mathcal{O}_n$.

Examples of actions on C*-algebras (continued)

- Let $s_1, s_2, \ldots, s_n$ be the standard generators of the Cuntz algebra $\mathcal{O}_n$, satisfying $s_j^* s_j = 1$ for $1 \leq j \leq n$ and $\sum_{j=1}^{n} s_j s_j^* = 1$. There is an action of $(S^1)^n$ on $\mathcal{O}_n$ such that $\alpha_{(\zeta_1, \zeta_2, \ldots, \zeta_n)}(s_j) = \zeta_j s_j$ for $1 \leq j \leq n$.
- Regarding $(S^1)^n$ as the diagonal unitary matrices, this action extends to an action of the unitary group $U(M_n)$ on $\mathcal{O}_n$. If $u = (u_{j,k})_{j,k=1}^{n} \in M_n$ is unitary, then $\alpha_u \in \mathrm{Aut}(\mathcal{O}_n)$ is determined by

Examples of actions on C*-algebras (continued)

- Let $s_1, s_2, \ldots, s_n$ be the standard generators of the Cuntz algebra $\mathcal{O}_n$, satisfying $s_j^* s_j = 1$ for $1 \leq j \leq n$ and $\sum_{j=1}^n s_j s_j^* = 1$. There is an action of $(S^1)^n$ on $\mathcal{O}_n$ such that $\alpha_{(\zeta_1, \zeta_2, \ldots, \zeta_n)}(s_j) = \zeta_j s_j$ for $1 \leq j \leq n$.
- Regarding $(S^1)^n$ as the diagonal unitary matrices, this action extends to an action of the unitary group $U(M_n)$ on $\mathcal{O}_n$. If $u = (u_{j,k})_{j,k=1}^n \in M_n$ is unitary, then $\alpha_u \in \mathrm{Aut}(\mathcal{O}_n)$ is determined by

$$\alpha_u(s_j) = \sum_{k=1}^n u_{k,j} s_k.$$

Examples of actions on C*-algebras (continued)

- Let $s_1, s_2, \ldots, s_n$ be the standard generators of the Cuntz algebra $\mathcal{O}_n$, satisfying $s_j^* s_j = 1$ for $1 \leq j \leq n$ and $\sum_{j=1}^{n} s_j s_j^* = 1$. There is an action of $(S^1)^n$ on $\mathcal{O}_n$ such that $\alpha_{(\zeta_1, \zeta_2, \ldots, \zeta_n)}(s_j) = \zeta_j s_j$ for $1 \leq j \leq n$.
- Regarding $(S^1)^n$ as the diagonal unitary matrices, this action extends to an action of the unitary group $U(M_n)$ on $\mathcal{O}_n$. If $u = (u_{j,k})_{j,k=1}^{n} \in M_n$ is unitary, then $\alpha_u \in \mathrm{Aut}(\mathcal{O}_n)$ is determined by

$$\alpha_u(s_j) = \sum_{k=1}^{n} u_{k,j} s_k.$$

- Any individual automorphism from this action gives an action of $\mathbb{Z}$ on $\mathcal{O}_n$.

Examples of actions on C*-algebras (continued)

- Let $s_1, s_2, \ldots, s_n$ be the standard generators of the Cuntz algebra $\mathcal{O}_n$, satisfying $s_j^* s_j = 1$ for $1 \leq j \leq n$ and $\sum_{j=1}^n s_j s_j^* = 1$. There is an action of $(S^1)^n$ on $\mathcal{O}_n$ such that $\alpha_{(\zeta_1, \zeta_2, \ldots, \zeta_n)}(s_j) = \zeta_j s_j$ for $1 \leq j \leq n$.
- Regarding $(S^1)^n$ as the diagonal unitary matrices, this action extends to an action of the unitary group $U(M_n)$ on $\mathcal{O}_n$. If $u = (u_{j,k})_{j,k=1}^n \in M_n$ is unitary, then $\alpha_u \in \mathrm{Aut}(\mathcal{O}_n)$ is determined by

$$\alpha_u(s_j) = \sum_{k=1}^n u_{k,j} s_k.$$

- Any individual automorphism from this action gives an action of $\mathbb{Z}$ on $\mathcal{O}_n$.
- The first example on this slide generalizes to give gauge actions on graph C*-algebras.

Examples of actions on C*-algebras (continued)

- Let A be the UHF algebra $\bigotimes_{n=1}^{\infty} M_{k_n}$, let G be a locally compact group, and let $\beta^{(n)} \colon G \to \mathrm{Aut}(M_{k_n})$ be an action of G on M_{k_n}.

Examples of actions on C*-algebras (continued)

- Let A be the UHF algebra $\bigotimes_{n=1}^{\infty} M_{k_n}$, let G be a locally compact group, and let $\beta^{(n)} \colon G \to \mathrm{Aut}(M_{k_n})$ be an action of G on M_{k_n}. Define an action $\alpha \colon G \to \mathrm{Aut}(A)$ by

$$\alpha_g(a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \cdots) = \beta_g^{(1)}(a_1) \otimes \cdots \otimes \beta_g^{(n)}(a_n) \otimes 1 \otimes \cdots .$$

Examples of actions on C*-algebras (continued)

- Let A be the UHF algebra $\bigotimes_{n=1}^{\infty} M_{k_n}$, let G be a locally compact group, and let $\beta^{(n)} \colon G \to \mathrm{Aut}(M_{k_n})$ be an action of G on M_{k_n}. Define an action $\alpha \colon G \to \mathrm{Aut}(A)$ by

$$\alpha_g(a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \cdots) = \beta_g^{(1)}(a_1) \otimes \cdots \otimes \beta_g^{(n)}(a_n) \otimes 1 \otimes \cdots .$$

- If each $\beta^{(n)}$ above is the inner action coming from a unitary representation of G on $\mathbb{C}^{k_n}$, then α is called a *product type action*.

Examples of actions on C*-algebras (continued)

- Let A be the UHF algebra $\bigotimes_{n=1}^{\infty} M_{k_n}$, let G be a locally compact group, and let $\beta^{(n)} \colon G \to \mathrm{Aut}(M_{k_n})$ be an action of G on M_{k_n}. Define an action $\alpha \colon G \to \mathrm{Aut}(A)$ by

$$\alpha_g(a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \cdots) = \beta_g^{(1)}(a_1) \otimes \cdots \otimes \beta_g^{(n)}(a_n) \otimes 1 \otimes \cdots .$$

- If each $\beta^{(n)}$ above is the inner action coming from a unitary representation of G on $\mathbb{C}^{k_n}$, then α is called a *product type action*.
- As a specific example, take $G = \mathbb{Z}/2\mathbb{Z}$, and for every n take $k_n = 2$ and take $\beta^{(n)}$ to be generated by $\mathrm{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Examples of actions on C*-algebras (continued)

- Let A be the UHF algebra $\bigotimes_{n=1}^{\infty} M_{k_n}$, let G be a locally compact group, and let $\beta^{(n)} \colon G \to \mathrm{Aut}(M_{k_n})$ be an action of G on M_{k_n}. Define an action $\alpha \colon G \to \mathrm{Aut}(A)$ by

$$\alpha_g(a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \cdots) = \beta_g^{(1)}(a_1) \otimes \cdots \otimes \beta_g^{(n)}(a_n) \otimes 1 \otimes \cdots .$$

- If each $\beta^{(n)}$ above is the inner action coming from a unitary representation of G on $\mathbb{C}^{k_n}$, then α is called a *product type action*.
- As a specific example, take $G = \mathbb{Z}/2\mathbb{Z}$, and for every n take $k_n = 2$ and take $\beta^{(n)}$ to be generated by $\mathrm{Ad} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$.
- Let A be a unital C*-algebra. The *tensor flip* is the action of $\mathbb{Z}/2\mathbb{Z}$ on $A \otimes_{\mathrm{max}} A$ generated by $a \otimes b \mapsto b \otimes a$.

Examples of actions on C*-algebras (continued)

- Let A be the UHF algebra $\bigotimes_{n=1}^{\infty} M_{k_n}$, let G be a locally compact group, and let $\beta^{(n)} \colon G \to \mathrm{Aut}(M_{k_n})$ be an action of G on M_{k_n}. Define an action $\alpha \colon G \to \mathrm{Aut}(A)$ by

$$\alpha_g(a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \cdots) = \beta_g^{(1)}(a_1) \otimes \cdots \otimes \beta_g^{(n)}(a_n) \otimes 1 \otimes \cdots .$$

- If each $\beta^{(n)}$ above is the inner action coming from a unitary representation of G on $\mathbb{C}^{k_n}$, then α is called a *product type action*.
- As a specific example, take $G = \mathbb{Z}/2\mathbb{Z}$, and for every n take $k_n = 2$ and take $\beta^{(n)}$ to be generated by $\mathrm{Ad} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$.
- Let A be a unital C*-algebra. The *tensor flip* is the action of $\mathbb{Z}/2\mathbb{Z}$ on $A \otimes_{\max} A$ generated by $a \otimes b \mapsto b \otimes a$.
- There is also a tensor flip on $A \otimes_{\min} A$.

Examples of actions on C*-algebras (continued)

- Let A be the UHF algebra $\bigotimes_{n=1}^{\infty} M_{k_n}$, let G be a locally compact group, and let $\beta^{(n)} \colon G \to \mathrm{Aut}(M_{k_n})$ be an action of G on M_{k_n}. Define an action $\alpha \colon G \to \mathrm{Aut}(A)$ by

$$\alpha_g(a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \cdots) = \beta_g^{(1)}(a_1) \otimes \cdots \otimes \beta_g^{(n)}(a_n) \otimes 1 \otimes \cdots .$$

- If each $\beta^{(n)}$ above is the inner action coming from a unitary representation of G on $\mathbb{C}^{k_n}$, then α is called a *product type action*.
- As a specific example, take $G = \mathbb{Z}/2\mathbb{Z}$, and for every n take $k_n = 2$ and take $\beta^{(n)}$ to be generated by $\mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.
- Let A be a unital C*-algebra. The *tensor flip* is the action of $\mathbb{Z}/2\mathbb{Z}$ on $A \otimes_{\max} A$ generated by $a \otimes b \mapsto b \otimes a$.
- There is also a tensor flip on $A \otimes_{\min} A$.
- The symmetric group S_n acts on the n-fold maximal and minimal tensor products of A with itself.

Examples of actions on C*-algebras (continued)

- Let A be the UHF algebra $\bigotimes_{n=1}^{\infty} M_{k_n}$, let G be a locally compact group, and let $\beta^{(n)} \colon G \to \mathrm{Aut}(M_{k_n})$ be an action of G on M_{k_n}. Define an action $\alpha \colon G \to \mathrm{Aut}(A)$ by

$$\alpha_g(a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \cdots) = \beta_g^{(1)}(a_1) \otimes \cdots \otimes \beta_g^{(n)}(a_n) \otimes 1 \otimes \cdots .$$

- If each $\beta^{(n)}$ above is the inner action coming from a unitary representation of G on $\mathbb{C}^{k_n}$, then α is called a *product type action*.
- As a specific example, take $G = \mathbb{Z}/2\mathbb{Z}$, and for every n take $k_n = 2$ and take $\beta^{(n)}$ to be generated by $\mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.
- Let A be a unital C*-algebra. The *tensor flip* is the action of $\mathbb{Z}/2\mathbb{Z}$ on $A \otimes_{\max} A$ generated by $a \otimes b \mapsto b \otimes a$.
- There is also a tensor flip on $A \otimes_{\min} A$.
- The symmetric group S_n acts on the n-fold maximal and minimal tensor products of A with itself.
- There is also a "tensor shift", a noncommutative analog, defined on $\bigotimes_{n \in \mathbb{Z}} A$, of the shift on $S^{\mathbb{Z}}$.

Covariant representations

To define the crossed product, we need:

Covariant representations

To define the crossed product, we need:

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A.

Covariant representations

To define the crossed product, we need:

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. A *covariant representation* of (G, A, α) on a Hilbert space H is

Covariant representations

To define the crossed product, we need:

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. A *covariant representation* of (G, A, α) on a Hilbert space H is a pair (v, π) consisting of a unitary representation $v\colon G \to U(H)$ (the unitary group of H)

Covariant representations

To define the crossed product, we need:

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. A *covariant representation* of (G, A, α) on a Hilbert space H is a pair (v, π) consisting of a unitary representation $v\colon G \to U(H)$ (the unitary group of H) and a representation $\pi\colon A \to L(H)$ (the algebra of all bounded operators on H),

Covariant representations

To define the crossed product, we need:

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. A *covariant representation* of (G, A, α) on a Hilbert space H is a pair (v, π) consisting of a unitary representation $v \colon G \to U(H)$ (the unitary group of H) and a representation $\pi \colon A \to L(H)$ (the algebra of all bounded operators on H), satisfying the *covariance condition*

$$v(g)\pi(a)v(g)^* = \pi(\alpha_g(a))$$

for all $g \in G$ and $a \in A$.

Covariant representations

To define the crossed product, we need:

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. A *covariant representation* of (G, A, α) on a Hilbert space H is a pair (v, π) consisting of a unitary representation $v \colon G \to U(H)$ (the unitary group of H) and a representation $\pi \colon A \to L(H)$ (the algebra of all bounded operators on H), satisfying the *covariance condition*

$$v(g)\pi(a)v(g)^* = \pi(\alpha_g(a))$$

for all $g \in G$ and $a \in A$. It is called *nondegenerate* if π is nondegenerate.

Covariant representations

To define the crossed product, we need:

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. A *covariant representation* of (G, A, α) on a Hilbert space H is a pair (v, π) consisting of a unitary representation $v \colon G \to U(H)$ (the unitary group of H) and a representation $\pi \colon A \to L(H)$ (the algebra of all bounded operators on H), satisfying the *covariance condition*

$$v(g)\pi(a)v(g)^* = \pi(\alpha_g(a))$$

for all $g \in G$ and $a \in A$. It is called *nondegenerate* if π is nondegenerate.

By convention, unitary representations are strong operator continuous.

Covariant representations

To define the crossed product, we need:

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. A *covariant representation* of (G, A, α) on a Hilbert space H is a pair (v, π) consisting of a unitary representation $v \colon G \to U(H)$ (the unitary group of H) and a representation $\pi \colon A \to L(H)$ (the algebra of all bounded operators on H), satisfying the *covariance condition*

$$v(g)\pi(a)v(g)^* = \pi(\alpha_g(a))$$

for all $g \in G$ and $a \in A$. It is called *nondegenerate* if π is nondegenerate.

By convention, unitary representations are strong operator continuous. Representations of C*-algebras, and of other *-algebras are *-representations (and, similarly, homomorphisms are *-homomorphisms).

Remarks on Banach space valued integration

The crossed product C*-algebra $C^*(G, A, \alpha)$ is the universal C*-algebra for covariant representations of (G, A, α),

Remarks on Banach space valued integration

The crossed product C*-algebra $C^*(G, A, \alpha)$ is the universal C*-algebra for covariant representations of (G, A, α), in essentially the same way that the (full) group C*-algebra $C^*(G)$ is the universal C*-algebra for unitary representations of G.

Remarks on Banach space valued integration

The crossed product C*-algebra $C^*(G, A, \alpha)$ is the universal C*-algebra for covariant representations of (G, A, α), in essentially the same way that the (full) group C*-algebra $C^*(G)$ is the universal C*-algebra for unitary representations of G. We construct it in a similar way to the group C*-algebra. We start with the analog of $L^1(G)$.

Remarks on Banach space valued integration

The crossed product C*-algebra $C^*(G, A, \alpha)$ is the universal C*-algebra for covariant representations of (G, A, α), in essentially the same way that the (full) group C*-algebra $C^*(G)$ is the universal C*-algebra for unitary representations of G. We construct it in a similar way to the group C*-algebra. We start with the analog of $L^1(G)$.

For a general locally compact group, one needs an appropriate notion of integration of Banach space valued functions.

Remarks on Banach space valued integration

The crossed product C*-algebra $C^*(G, A, \alpha)$ is the universal C*-algebra for covariant representations of (G, A, α), in essentially the same way that the (full) group C*-algebra $C^*(G)$ is the universal C*-algebra for unitary representations of G. We construct it in a similar way to the group C*-algebra. We start with the analog of $L^1(G)$.

For a general locally compact group, one needs an appropriate notion of integration of Banach space valued functions. One must prove that twisted convolution below is well defined, associative, distributive, and satisfies $(ab)^* = b^* a^*$.

Remarks on Banach space valued integration

The crossed product C*-algebra $C^*(G, A, \alpha)$ is the universal C*-algebra for covariant representations of (G, A, α), in essentially the same way that the (full) group C*-algebra $C^*(G)$ is the universal C*-algebra for unitary representations of G. We construct it in a similar way to the group C*-algebra. We start with the analog of $L^1(G)$.

For a general locally compact group, one needs an appropriate notion of integration of Banach space valued functions. One must prove that twisted convolution below is well defined, associative, distributive, and satisfies $(ab)^* = b^* a^*$. Once one has the appropriate notion of integration, the proofs are similar to the proofs of the corresponding facts about convolution in $L^1(G)$.

Remarks on Banach space valued integration

The crossed product C*-algebra $C^*(G, A, \alpha)$ is the universal C*-algebra for covariant representations of (G, A, α), in essentially the same way that the (full) group C*-algebra $C^*(G)$ is the universal C*-algebra for unitary representations of G. We construct it in a similar way to the group C*-algebra. We start with the analog of $L^1(G)$.

For a general locally compact group, one needs an appropriate notion of integration of Banach space valued functions. One must prove that twisted convolution below is well defined, associative, distributive, and satisfies $(ab)^* = b^* a^*$. Once one has the appropriate notion of integration, the proofs are similar to the proofs of the corresponding facts about convolution in $L^1(G)$. Integration of continuous functions with compact support is much easier than integration of L^1 functions,

Remarks on Banach space valued integration

The crossed product C*-algebra $C^*(G, A, \alpha)$ is the universal C*-algebra for covariant representations of (G, A, α), in essentially the same way that the (full) group C*-algebra $C^*(G)$ is the universal C*-algebra for unitary representations of G. We construct it in a similar way to the group C*-algebra. We start with the analog of $L^1(G)$.

For a general locally compact group, one needs an appropriate notion of integration of Banach space valued functions. One must prove that twisted convolution below is well defined, associative, distributive, and satisfies $(ab)^* = b^* a^*$. Once one has the appropriate notion of integration, the proofs are similar to the proofs of the corresponding facts about convolution in $L^1(G)$. Integration of continuous functions with compact support is much easier than integration of L^1 functions, but the "right" way to do this is to define measurable Banach space valued functions and their integrals.

Remarks on Banach space valued integration

The crossed product C*-algebra $C^*(G, A, \alpha)$ is the universal C*-algebra for covariant representations of (G, A, α), in essentially the same way that the (full) group C*-algebra $C^*(G)$ is the universal C*-algebra for unitary representations of G. We construct it in a similar way to the group C*-algebra. We start with the analog of $L^1(G)$.

For a general locally compact group, one needs an appropriate notion of integration of Banach space valued functions. One must prove that twisted convolution below is well defined, associative, distributive, and satisfies $(ab)^* = b^* a^*$. Once one has the appropriate notion of integration, the proofs are similar to the proofs of the corresponding facts about convolution in $L^1(G)$. Integration of continuous functions with compact support is much easier than integration of L^1 functions, but the "right" way to do this is to define measurable Banach space valued functions and their integrals. This has been done; one reference is Appendix B of the book of Williams. Things simplify considerably if G is second countable and A is separable, but neither of these conditions is necessary.

Twisted convolution

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A.

Twisted convolution

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We let $C_{\mathrm{c}}(G, A, \alpha)$ be the *-algebra of continuous functions $a \colon G \to A$ with compact support, with pointwise addition and scalar multiplication.

Twisted convolution

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We let $C_{\mathrm{c}}(G, A, \alpha)$ be the *-algebra of continuous functions $a\colon G \to A$ with compact support, with pointwise addition and scalar multiplication. Using Haar measure in the integral, we define multiplication by the following "twisted convolution":

$$(ab)(g) = \int_G a(h)\alpha_h(b(h^{-1}g))\, dh.$$

Twisted convolution

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We let $C_{\mathrm{c}}(G, A, \alpha)$ be the *-algebra of continuous functions $a\colon G \to A$ with compact support, with pointwise addition and scalar multiplication. Using Haar measure in the integral, we define multiplication by the following "twisted convolution":

$$(ab)(g) = \int_G a(h)\alpha_h(b(h^{-1}g))\, dh.$$

Let Δ be the modular function of G. We define the adjoint by

$$a^*(g) = \Delta(g)^{-1}\alpha_g(a(g^{-1})^*).$$

Twisted convolution

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We let $C_{\mathrm{c}}(G, A, \alpha)$ be the *-algebra of continuous functions $a\colon G \to A$ with compact support, with pointwise addition and scalar multiplication. Using Haar measure in the integral, we define multiplication by the following "twisted convolution":

$$(ab)(g) = \int_G a(h)\alpha_h(b(h^{-1}g))\, dh.$$

Let Δ be the modular function of G. We define the adjoint by

$$a^*(g) = \Delta(g)^{-1}\alpha_g(a(g^{-1})^*).$$

We define a norm $\| \cdot \|_1$ on $C_{\mathrm{c}}(G, A, \alpha)$ by $\|a\|_1 = \int_G \|a(g)\|\, dg$.

Twisted convolution

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We let $C_{\mathrm{c}}(G, A, \alpha)$ be the *-algebra of continuous functions $a \colon G \to A$ with compact support, with pointwise addition and scalar multiplication. Using Haar measure in the integral, we define multiplication by the following "twisted convolution":

$$(ab)(g) = \int_G a(h)\alpha_h(b(h^{-1}g))\, dh.$$

Let Δ be the modular function of G. We define the adjoint by

$$a^*(g) = \Delta(g)^{-1}\alpha_g(a(g^{-1})^*).$$

We define a norm $\| \cdot \|_1$ on $C_{\mathrm{c}}(G, A, \alpha)$ by $\|a\|_1 = \int_G \|a(g)\|\, dg$. One checks that $\|ab\|_1 \leq \|a\|_1 \|b\|_1$ and $\|a^*\|_1 = \|a\|_1$.

Twisted convolution

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We let $C_{\mathrm{c}}(G, A, \alpha)$ be the *-algebra of continuous functions $a\colon G \to A$ with compact support, with pointwise addition and scalar multiplication. Using Haar measure in the integral, we define multiplication by the following "twisted convolution":

$$(ab)(g) = \int_G a(h)\alpha_h(b(h^{-1}g))\, dh.$$

Let Δ be the modular function of G. We define the adjoint by

$$a^*(g) = \Delta(g)^{-1}\alpha_g(a(g^{-1})^*).$$

We define a norm $\|\cdot\|_1$ on $C_{\mathrm{c}}(G, A, \alpha)$ by $\|a\|_1 = \int_G \|a(g)\|\, dg$. One checks that $\|ab\|_1 \leq \|a\|_1\|b\|_1$ and $\|a^*\|_1 = \|a\|_1$. Then $L^1(G, A, \alpha)$ is the Banach *-algebra obtained by completing $C_{\mathrm{c}}(G, A, \alpha)$ in $\|\cdot\|_1$.

Twisted convolution (continued)

Exercise

Assuming suitable versions of Fubini's Theorem for Banach space valued integrals,

Twisted convolution (continued)

Exercise

Assuming suitable versions of Fubini's Theorem for Banach space valued integrals, check that that multiplication in $C_{\mathrm{c}}(G, A, \alpha)$ is associative.

Twisted convolution (continued)

Exercise

Assuming suitable versions of Fubini's Theorem for Banach space valued integrals, check that that multiplication in $C_{\mathrm{c}}(G, A, \alpha)$ is associative. Further check for $a, b \in C_{\mathrm{c}}(G, A, \alpha)$ that $\|ab\|_1 \leq \|a\|_1 \|b\|_1$,

Twisted convolution (continued)

Exercise

Assuming suitable versions of Fubini's Theorem for Banach space valued integrals, check that that multiplication in $C_{\mathrm{c}}(G, A, \alpha)$ is associative. Further check for $a, b \in C_{\mathrm{c}}(G, A, \alpha)$ that $\|ab\|_1 \leq \|a\|_1 \|b\|_1$, that $(ab)^* = b^* a^*$,

Twisted convolution (continued)

Exercise

Assuming suitable versions of Fubini's Theorem for Banach space valued integrals, check that that multiplication in $C_{\mathrm{c}}(G, A, \alpha)$ is associative. Further check for $a, b \in C_{\mathrm{c}}(G, A, \alpha)$ that $\|ab\|_1 \leq \|a\|_1 \|b\|_1$, that $(ab)^* = b^* a^*$, and that $\|a^*\|_1 = \|a\|_1$.

When G is discrete

If G is discrete, we choose Haar measure to be counting measure.

When G is discrete

If G is discrete, we choose Haar measure to be counting measure. In this case, $C_{\mathrm{c}}(G, A, \alpha)$ is, as a vector space, the group ring $A[G]$, consisting of all finite formal linear combinations of elements in G with coefficients in A.

When G is discrete

If G is discrete, we choose Haar measure to be counting measure. In this case, $C_{\mathrm{c}}(G, A, \alpha)$ is, as a vector space, the group ring $A[G]$, consisting of all finite formal linear combinations of elements in G with coefficients in A. The multiplication and adjoint are given by

$$(a \cdot g)(b \cdot h) = (a[gbg^{-1}]) \cdot (gh) = (a \alpha_g(b)) \cdot (gh) \quad \text{and} \quad (a \cdot g)^* = \alpha_g^{-1}(a^*) \cdot g^{-1}$$

for $a, b \in A$ and $g, h \in G$,

When G is discrete

If G is discrete, we choose Haar measure to be counting measure. In this case, $C_{\mathrm{c}}(G, A, \alpha)$ is, as a vector space, the group ring $A[G]$, consisting of all finite formal linear combinations of elements in G with coefficients in A. The multiplication and adjoint are given by

$$(a \cdot g)(b \cdot h) = (a[gbg^{-1}]) \cdot (gh) = (a\alpha_g(b)) \cdot (gh) \quad \text{and} \quad (a \cdot g)^* = \alpha_g^{-1}(a^*) \cdot g^{-1}$$

for $a, b \in A$ and $g, h \in G$, extended linearly.

When G is discrete

If G is discrete, we choose Haar measure to be counting measure. In this case, $C_{\mathrm{c}}(G, A, \alpha)$ is, as a vector space, the group ring $A[G]$, consisting of all finite formal linear combinations of elements in G with coefficients in A. The multiplication and adjoint are given by

$$(a \cdot g)(b \cdot h) = (a[gbg^{-1}]) \cdot (gh) = (a\alpha_g(b)) \cdot (gh) \quad \text{and} \quad (a \cdot g)^* = \alpha_g^{-1}(a^*) \cdot g^{-1}$$

for $a, b \in A$ and $g, h \in G$, extended linearly. This definition makes sense in the purely algebraic situation, where it is called the *skew group ring*.

When G is discrete

If G is discrete, we choose Haar measure to be counting measure. In this case, $C_{\mathrm{c}}(G, A, \alpha)$ is, as a vector space, the group ring $A[G]$, consisting of all finite formal linear combinations of elements in G with coefficients in A. The multiplication and adjoint are given by

$$(a \cdot g)(b \cdot h) = (a[gbg^{-1}]) \cdot (gh) = (a\alpha_g(b)) \cdot (gh) \quad \text{and} \quad (a \cdot g)^* = \alpha_g^{-1}(a^*) \cdot g^{-1}$$

for $a, b \in A$ and $g, h \in G$, extended linearly. This definition makes sense in the purely algebraic situation, where it is called the *skew group ring*.

We also often write $l^1(G, A, \alpha)$ instead of $L^1(G, A, \alpha)$.

When G is discrete (continued)

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A.

When G is discrete (continued)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. In these notes, we will adopt the following fairly commonly used notation. For $g \in G$, we let u_g be the element of $C_{\mathrm{c}}(G, A, \alpha)$ which takes the value 1_A at g and 0 at the other elements of G.

When G is discrete (continued)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. In these notes, we will adopt the following fairly commonly used notation. For $g \in G$, we let u_g be the element of $C_{\mathrm{c}}(G, A, \alpha)$ which takes the value 1_A at g and 0 at the other elements of G. We use the same notation for its image in $l^1(G, A, \alpha)$ (above)

When G is discrete (continued)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. In these notes, we will adopt the following fairly commonly used notation. For $g \in G$, we let u_g be the element of $C_{\mathrm{c}}(G, A, \alpha)$ which takes the value 1_A at g and 0 at the other elements of G. We use the same notation for its image in $l^1(G, A, \alpha)$ (above) and in $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$ (defined below). It is unitary, and we call it the canonical unitary associated with g.

When G is discrete (continued)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. In these notes, we will adopt the following fairly commonly used notation. For $g \in G$, we let u_g be the element of $C_{\mathrm{c}}(G, A, \alpha)$ which takes the value 1_A at g and 0 at the other elements of G. We use the same notation for its image in $l^1(G, A, \alpha)$ (above) and in $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$ (defined below). It is unitary, and we call it the canonical unitary associated with g.

In particular, $l^1(G, A, \alpha)$ is the set of all sums $\sum_{g \in G} a_g u_g$ with $a_g \in A$ and $\sum_{g \in G} \|a_g\| < \infty$.

When G is discrete (continued)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. In these notes, we will adopt the following fairly commonly used notation. For $g \in G$, we let u_g be the element of $C_{\mathrm{c}}(G, A, \alpha)$ which takes the value 1_A at g and 0 at the other elements of G. We use the same notation for its image in $l^1(G, A, \alpha)$ (above) and in $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$ (defined below). It is unitary, and we call it the canonical unitary associated with g.

In particular, $l^1(G, A, \alpha)$ is the set of all sums $\sum_{g \in G} a_g u_g$ with $a_g \in A$ and $\sum_{g \in G} \|a_g\| < \infty$. These sums converge in $l^1(G, A, \alpha)$, and hence also in $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$.

When G is discrete (continued)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. In these notes, we will adopt the following fairly commonly used notation. For $g \in G$, we let u_g be the element of $C_{\mathrm{c}}(G, A, \alpha)$ which takes the value 1_A at g and 0 at the other elements of G. We use the same notation for its image in $l^1(G, A, \alpha)$ (above) and in $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$ (defined below). It is unitary, and we call it the canonical unitary associated with g.

In particular, $l^1(G, A, \alpha)$ is the set of all sums $\sum_{g \in G} a_g u_g$ with $a_g \in A$ and $\sum_{g \in G} \|a_g\| < \infty$. These sums converge in $l^1(G, A, \alpha)$, and hence also in $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$. A general element of $C^*_{\mathrm{r}}(G, A, \alpha)$ has such an expansion,

When G is discrete (continued)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. In these notes, we will adopt the following fairly commonly used notation. For $g \in G$, we let u_g be the element of $C_{\mathrm{c}}(G, A, \alpha)$ which takes the value 1_A at g and 0 at the other elements of G. We use the same notation for its image in $l^1(G, A, \alpha)$ (above) and in $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$ (defined below). It is unitary, and we call it the canonical unitary associated with g.

In particular, $l^1(G, A, \alpha)$ is the set of all sums $\sum_{g \in G} a_g u_g$ with $a_g \in A$ and $\sum_{g \in G} \|a_g\| < \infty$. These sums converge in $l^1(G, A, \alpha)$, and hence also in $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$. A general element of $C^*_{\mathrm{r}}(G, A, \alpha)$ has such an expansion, but unfortunately the series one writes down generally does not converge. See the discussion later.

The integrated form of a covariant representation

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A, and let (v, π) be a covariant representation of (G, A, α) on a Hilbert space H.

The integrated form of a covariant representation

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A, and let (v, π) be a covariant representation of (G, A, α) on a Hilbert space H. Then the *integrated form* of (v, π) is the representation $\sigma \colon C_{\mathrm{c}}(G, A, \alpha) \to L(H)$ given by

The integrated form of a covariant representation

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A, and let (v, π) be a covariant representation of (G, A, α) on a Hilbert space H. Then the *integrated form* of (v, π) is the representation $\sigma \colon C_{\mathrm{c}}(G, A, \alpha) \to L(H)$ given by

$$\sigma(a)\xi = \int_G \pi(a(g))v(g)\xi \, dg.$$

The integrated form of a covariant representation

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A, and let (v, π) be a covariant representation of (G, A, α) on a Hilbert space H. Then the *integrated form* of (v, π) is the representation $\sigma \colon C_{\mathrm{c}}(G, A, \alpha) \to L(H)$ given by

$$\sigma(a)\xi = \int_G \pi(a(g))v(g)\xi \, dg.$$

(This representation is sometimes called $v \times \pi$ or $\pi \times v$.)

The integrated form of a covariant representation

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A, and let (v, π) be a covariant representation of (G, A, α) on a Hilbert space H. Then the *integrated form* of (v, π) is the representation $\sigma \colon C_{\mathrm{c}}(G, A, \alpha) \to L(H)$ given by

$$\sigma(a)\xi = \int_G \pi(a(g))v(g)\xi \, dg.$$

(This representation is sometimes called $v \times \pi$ or $\pi \times v$.)

One needs to be more careful with the integral here, because v is generally only strong operator continuous, not norm continuous.

The integrated form of a covariant representation

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A, and let (v, π) be a covariant representation of (G, A, α) on a Hilbert space H. Then the *integrated form* of (v, π) is the representation $\sigma \colon C_{\mathrm{c}}(G, A, \alpha) \to L(H)$ given by

$$\sigma(a)\xi = \int_G \pi(a(g))v(g)\xi \, dg.$$

(This representation is sometimes called $v \times \pi$ or $\pi \times v$.)

One needs to be more careful with the integral here, because v is generally only strong operator continuous, not norm continuous. Nevertheless, one gets $\|\sigma(a)\| \leq \|a\|_1$, so σ extends to a representation of $L^1(G, A, \alpha)$. We use the same notation σ for this extension.

The integrated form of a covariant representation (continued)

One needs to check that σ is a representation.

The integrated form of a covariant representation (continued)

One needs to check that σ is a representation. When G is discrete and A is unital, the formula for σ comes down to $\sigma(au_g) = \pi(a)v(g)$ for $a \in A$ and $g \in G$.

The integrated form of a covariant representation (continued)

One needs to check that σ is a representation. When G is discrete and A is unital, the formula for σ comes down to $\sigma(au_g) = \pi(a)v(g)$ for $a \in A$ and $g \in G$. Then

$$\begin{aligned}\sigma(au_g)\sigma(bu_h) &= \pi(a)v(g)\pi(b)v(g)^* v(g)v(h) = \pi(a)\pi(\alpha_g(b))v(g)v(h) \\ &= \pi(a\alpha_g(b))v(gh) = \sigma\big([a\alpha_g(b)]u_{gh}\big) = \sigma\big((au_g)(bu_h)\big).\end{aligned}$$

The integrated form of a covariant representation (continued)

One needs to check that σ is a representation. When G is discrete and A is unital, the formula for σ comes down to $\sigma(au_g) = \pi(a)v(g)$ for $a \in A$ and $g \in G$. Then

$$\begin{aligned}\sigma(au_g)\sigma(bu_h) &= \pi(a)v(g)\pi(b)v(g)^* v(g)v(h) = \pi(a)\pi(\alpha_g(b))v(g)v(h) \\ &= \pi(a\alpha_g(b))v(gh) = \sigma\big([a\alpha_g(b)]u_{gh}\big) = \sigma\big((au_g)(bu_h)\big).\end{aligned}$$

Exercise

Starting from this computation, fill in the details of the proof that the integrated form representation σ really is a nondegenerate representation of $C_{\mathrm{c}}(G, A, \alpha)$.

The integrated form of a covariant representation (continued)

Theorem (Proposition 7.6.4 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A.

The integrated form of a covariant representation (continued)

Theorem (Proposition 7.6.4 of Pedersen's book)

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Then the integrated form construction defines a bijection from the set of covariant representations of (G, A, α) on a Hilbert space H to the set of nondegenerate continuous representations of $L^1(G, A, \alpha)$ on the same Hilbert space.

The integrated form of a covariant representation (continued)

Theorem (Proposition 7.6.4 of Pedersen's book)

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Then the integrated form construction defines a bijection from the set of covariant representations of (G, A, α) on a Hilbert space H to the set of nondegenerate continuous representations of $L^1(G, A, \alpha)$ on the same Hilbert space.

In particular, since integrated form representations of $L^1(G, A, \alpha)$ are necessarily contractive,

The integrated form of a covariant representation (continued)

Theorem (Proposition 7.6.4 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Then the integrated form construction defines a bijection from the set of covariant representations of (G, A, α) on a Hilbert space H to the set of nondegenerate continuous representations of $L^1(G, A, \alpha)$ on the same Hilbert space.

In particular, since integrated form representations of $L^1(G, A, \alpha)$ are necessarily contractive, *all* continuous representations of $L^1(G, A, \alpha)$ are necessarily contractive.

The integrated form when G is discrete

If G is discrete and A is unital, then there are homomorphic images of both G and A inside $C_{\mathrm{c}}(G, A, \alpha)$,

The integrated form when G is discrete

If G is discrete and A is unital, then there are homomorphic images of both G and A inside $C_{\mathrm{c}}(G, A, \alpha)$, given by $g \mapsto u_g$ and $a \mapsto a u_1$,

The integrated form when G is discrete

If G is discrete and A is unital, then there are homomorphic images of both G and A inside $C_{\mathrm{c}}(G, A, \alpha)$, given by $g \mapsto u_g$ and $a \mapsto a u_1$, so it is clear how to get a covariant representation of (G, A, α) from a nondegenerate representation of $C_{\mathrm{c}}(G, A, \alpha)$.

The integrated form when G is discrete

If G is discrete and A is unital, then there are homomorphic images of both G and A inside $C_{\mathrm{c}}(G, A, \alpha)$, given by $g \mapsto u_g$ and $a \mapsto a u_1$, so it is clear how to get a covariant representation of (G, A, α) from a nondegenerate representation of $C_{\mathrm{c}}(G, A, \alpha)$. In general, one must use the multiplier algebra of $L^1(G, A, \alpha)$, which contains copies of $M(A)$ and $M(L^1(G))$.

The integrated form when G is discrete

If G is discrete and A is unital, then there are homomorphic images of both G and A inside $C_{\mathrm{c}}(G, A, \alpha)$, given by $g \mapsto u_g$ and $a \mapsto a u_1$, so it is clear how to get a covariant representation of (G, A, α) from a nondegenerate representation of $C_{\mathrm{c}}(G, A, \alpha)$. In general, one must use the multiplier algebra of $L^1(G, A, \alpha)$, which contains copies of $M(A)$ and $M(L^1(G))$. The point is that $M(L^1(G))$ is the measure algebra of G, and therefore contains the group elements as point masses.

The integrated form when G is discrete

If G is discrete and A is unital, then there are homomorphic images of both G and A inside $C_{\mathrm{c}}(G, A, \alpha)$, given by $g \mapsto u_g$ and $a \mapsto a u_1$, so it is clear how to get a covariant representation of (G, A, α) from a nondegenerate representation of $C_{\mathrm{c}}(G, A, \alpha)$. In general, one must use the multiplier algebra of $L^1(G, A, \alpha)$, which contains copies of $M(A)$ and $M(L^1(G))$. The point is that $M(L^1(G))$ is the measure algebra of G, and therefore contains the group elements as point masses.

Exercise

Prove the theorem on the previous slide when G is discrete and A is unital.

The integrated form when G is discrete

If G is discrete and A is unital, then there are homomorphic images of both G and A inside $C_{\mathrm{c}}(G, A, \alpha)$, given by $g \mapsto u_g$ and $a \mapsto a u_1$, so it is clear how to get a covariant representation of (G, A, α) from a nondegenerate representation of $C_{\mathrm{c}}(G, A, \alpha)$. In general, one must use the multiplier algebra of $L^1(G, A, \alpha)$, which contains copies of $M(A)$ and $M(L^1(G))$. The point is that $M(L^1(G))$ is the measure algebra of G, and therefore contains the group elements as point masses.

Exercise

Prove the theorem on the previous slide when G is discrete and A is unital.

For a small taste of the general case, use approximate identities in A to generalize to the case in which A is not necessarily unital.

The universal representation and the crossed product

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We define the *universal representation* σ of $L^1(G, A, \alpha)$

The universal representation and the crossed product

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We define the *universal representation* σ of $L^1(G, A, \alpha)$ to be the direct sum of all nondegenerate representations of $L^1(G, A, \alpha)$ on Hilbert spaces.

The universal representation and the crossed product

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We define the *universal representation* σ of $L^1(G, A, \alpha)$ to be the direct sum of all nondegenerate representations of $L^1(G, A, \alpha)$ on Hilbert spaces. Then we define the *crossed product* $C^*(G, A, \alpha)$ to be the norm closure of $\sigma(L^1(G, A, \alpha))$.

The universal representation and the crossed product

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We define the *universal representation* σ of $L^1(G, A, \alpha)$ to be the direct sum of all nondegenerate representations of $L^1(G, A, \alpha)$ on Hilbert spaces. Then we define the *crossed product* $C^*(G, A, \alpha)$ to be the norm closure of $\sigma(L^1(G, A, \alpha))$.

One could of course equally well use the norm closure of $\sigma(C_{\mathrm{c}}(G, A, \alpha))$.

The universal representation and the crossed product

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We define the *universal representation* σ of $L^1(G, A, \alpha)$ to be the direct sum of all nondegenerate representations of $L^1(G, A, \alpha)$ on Hilbert spaces. Then we define the *crossed product* $C^*(G, A, \alpha)$ to be the norm closure of $\sigma(L^1(G, A, \alpha))$.

One could of course equally well use the norm closure of $\sigma(C_{\mathrm{c}}(G, A, \alpha))$.

There is a minor set theoretic detail: the collection of all nondegenerate representations of $L^1(G, A, \alpha)$ is not a set.

The universal representation and the crossed product

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We define the *universal representation* σ of $L^1(G, A, \alpha)$ to be the direct sum of all nondegenerate representations of $L^1(G, A, \alpha)$ on Hilbert spaces. Then we define the *crossed product* $C^*(G, A, \alpha)$ to be the norm closure of $\sigma(L^1(G, A, \alpha))$.

One could of course equally well use the norm closure of $\sigma(C_{\mathrm{c}}(G, A, \alpha))$.

There is a minor set theoretic detail: the collection of all nondegenerate representations of $L^1(G, A, \alpha)$ is not a set. There are several standard ways to deal with this problem, but in these notes we will ignore the issue.

The universal representation and the crossed product

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We define the *universal representation* σ of $L^1(G, A, \alpha)$ to be the direct sum of all nondegenerate representations of $L^1(G, A, \alpha)$ on Hilbert spaces. Then we define the *crossed product* $C^*(G, A, \alpha)$ to be the norm closure of $\sigma(L^1(G, A, \alpha))$.

One could of course equally well use the norm closure of $\sigma(C_{\mathrm{c}}(G, A, \alpha))$.

There is a minor set theoretic detail: the collection of all nondegenerate representations of $L^1(G, A, \alpha)$ is not a set. There are several standard ways to deal with this problem, but in these notes we will ignore the issue.

Exercise

Give a set theoretically correct definition of the crossed product.

The universal representation and the crossed product

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We define the *universal representation* σ of $L^1(G, A, \alpha)$ to be the direct sum of all nondegenerate representations of $L^1(G, A, \alpha)$ on Hilbert spaces. Then we define the *crossed product* $C^*(G, A, \alpha)$ to be the norm closure of $\sigma(L^1(G, A, \alpha))$.

One could of course equally well use the norm closure of $\sigma(C_{\mathrm{c}}(G, A, \alpha))$.

There is a minor set theoretic detail: the collection of all nondegenerate representations of $L^1(G, A, \alpha)$ is not a set. There are several standard ways to deal with this problem, but in these notes we will ignore the issue.

Exercise

Give a set theoretically correct definition of the crossed product.

The important point is to preserve the universal property below.

The universal representation and the crossed product (continued)

It follows that every covariant representation of (G, A, α) gives a representation of $C^*(G, A, \alpha)$.

The universal representation and the crossed product (continued)

It follows that every covariant representation of (G, A, α) gives a representation of $C^*(G, A, \alpha)$. (Take the integrated form, and restrict elements of $C^*(G, A, \alpha)$ to the appropriate summand in the direct sum in the definition above.)

The universal representation and the crossed product (continued)

It follows that every covariant representation of (G, A, α) gives a representation of $C^*(G, A, \alpha)$. (Take the integrated form, and restrict elements of $C^*(G, A, \alpha)$ to the appropriate summand in the direct sum in the definition above.) The crossed product is, essentially by construction, the universal C*-algebra for covariant representations of (G, A, α),

The universal representation and the crossed product (continued)

It follows that every covariant representation of (G, A, α) gives a representation of $C^*(G, A, \alpha)$. (Take the integrated form, and restrict elements of $C^*(G, A, \alpha)$ to the appropriate summand in the direct sum in the definition above.) The crossed product is, essentially by construction, the universal C*-algebra for covariant representations of (G, A, α), in the same sense that if G is a locally compact group, then $C^*(G)$ is the universal C*-algebra for unitary representations of G.

The universal representation and the crossed product (continued)

It follows that every covariant representation of (G, A, α) gives a representation of $C^*(G, A, \alpha)$. (Take the integrated form, and restrict elements of $C^*(G, A, \alpha)$ to the appropriate summand in the direct sum in the definition above.) The crossed product is, essentially by construction, the universal C*-algebra for covariant representations of (G, A, α), in the same sense that if G is a locally compact group, then $C^*(G)$ is the universal C*-algebra for unitary representations of G.

There are many notations in use for crossed products, including:

- $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$.
- $C^*(A, G, \alpha)$ and $C^*_{\mathrm{r}}(A, G, \alpha)$.
- $A \rtimes_{\alpha} G$ and $A \rtimes_{\alpha,\mathrm{r}} G$ (used in Williams' book).
- $A \times_{\alpha} G$ and $A \times_{\alpha,\mathrm{r}} G$ (used in Davidson's book).
- $G \times_{\alpha} A$ and $G \times_{\alpha,\mathrm{r}} A$ (used in Pedersen's book).

The universal representation and the crossed product when G is discrete

Theorem

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A.

The universal representation and the crossed product when G is discrete

Theorem

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. Then $C^*(G, A, \alpha)$ is the universal C*-algebra generated by a unital copy of A (that is, the identity of A is supposed to be the identity of the generated C*-algebra) and

The universal representation and the crossed product when G is discrete

Theorem

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. Then $C^*(G, A, \alpha)$ is the universal C*-algebra generated by a unital copy of A (that is, the identity of A is supposed to be the identity of the generated C*-algebra) and unitaries u_g, for $g \in G$,

The universal representation and the crossed product when G is discrete

Theorem

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. Then $C^*(G, A, \alpha)$ is the universal C*-algebra generated by a unital copy of A (that is, the identity of A is supposed to be the identity of the generated C*-algebra) and unitaries u_g, for $g \in G$, subject to the relations $u_g u_h = u_{gh}$ for $g, h \in G$ and $u_g a u_g^* = \alpha_g(a)$ for $a \in A$ and $g \in G$.

The universal representation and the crossed product when G is discrete

Theorem

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. Then $C^*(G, A, \alpha)$ is the universal C*-algebra generated by a unital copy of A (that is, the identity of A is supposed to be the identity of the generated C*-algebra) and unitaries u_g, for $g \in G$, subject to the relations $u_g u_h = u_{gh}$ for $g, h \in G$ and $u_g a u_g^* = \alpha_g(a)$ for $a \in A$ and $g \in G$.

Corollary

Let A be a unital C*-algebra, and let $\alpha \in \mathrm{Aut}(A)$. Then the crossed product $C^*(\mathbb{Z}, A, \alpha)$ is the universal C*-algebra generated by a copy of A and a unitary u, subject to the relations $uau^* = \alpha(a)$ for $a \in A$.

The universal representation and the crossed product when G is discrete (continued)

Exercise

Based on the discussion above, write down a careful proof of the theorem.

Regular covariant representations

So far, it is not clear that there are any covariant representations.

Regular covariant representations

So far, it is not clear that there are any covariant representations.

Definition (7.7.1 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\pi_0 \colon A \to L(H_0)$ be a representation.

Regular covariant representations

So far, it is not clear that there are any covariant representations.

Definition (7.7.1 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\pi_0 \colon A \to L(H_0)$ be a representation. We define the *regular covariant representation* (v, π) of (G, A, α) on the Hilbert space $H = L^2(G, H_0)$ of L^2 functions from G to H_0 as follows.

Regular covariant representations

So far, it is not clear that there are any covariant representations.

Definition (7.7.1 of Pedersen's book)

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\pi_0\colon A \to L(H_0)$ be a representation. We define the *regular covariant representation* (v, π) of (G, A, α) on the Hilbert space $H = L^2(G, H_0)$ of L^2 functions from G to H_0 as follows. For $g, h \in G$, set

$$(v(g)\xi)(h) = \xi(g^{-1}h).$$

Regular covariant representations

So far, it is not clear that there are any covariant representations.

Definition (7.7.1 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\pi_0 \colon A \to L(H_0)$ be a representation. We define the *regular covariant representation* (v, π) of (G, A, α) on the Hilbert space $H = L^2(G, H_0)$ of L^2 functions from G to H_0 as follows. For $g, h \in G$, set

$$(v(g)\xi)(h) = \xi(g^{-1}h).$$

For $a \in A$ and $g \in G$, set

$$(\pi(a)\xi)(h) = \pi_0(\alpha_{h^{-1}}(a))(\xi(h)).$$

Regular covariant representations

So far, it is not clear that there are any covariant representations.

Definition (7.7.1 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\pi_0 \colon A \to L(H_0)$ be a representation. We define the *regular covariant representation* (v, π) of (G, A, α) on the Hilbert space $H = L^2(G, H_0)$ of L^2 functions from G to H_0 as follows. For $g, h \in G$, set

$$(v(g)\xi)(h) = \xi(g^{-1}h).$$

For $a \in A$ and $g \in G$, set

$$(\pi(a)\xi)(h) = \pi_0(\alpha_{h^{-1}}(a))(\xi(h)).$$

The integrated form of σ will be called a regular representation

Regular covariant representations

So far, it is not clear that there are any covariant representations.

Definition (7.7.1 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\pi_0 \colon A \to L(H_0)$ be a representation. We define the *regular covariant representation* (v, π) of (G, A, α) on the Hilbert space $H = L^2(G, H_0)$ of L^2 functions from G to H_0 as follows. For $g, h \in G$, set

$$(v(g)\xi)(h) = \xi(g^{-1}h).$$

For $a \in A$ and $g \in G$, set

$$(\pi(a)\xi)(h) = \pi_0(\alpha_{h^{-1}}(a))(\xi(h)).$$

The integrated form of σ will be called a regular representation of any of $C_{\mathrm{c}}(G, A, \alpha)$, $L^1(G, A, \alpha)$, $C^*(G, A, \alpha)$, and (when defined) $C^*_{\mathrm{r}}(G, A, \alpha)$.

The Hilbert space of the regular covariant representation

The easy way to construct $L^2(G, H_0)$ is to take it to be the completion of $C_{\mathrm{c}}(G, H_0)$ in the norm coming from the scalar product

$$\langle \xi, \eta \rangle = \int_G \langle \xi(g), \eta(g) \rangle \, dg.$$

Reduced crossed products

Exercise

Suppose that G is discrete. Prove that a regular representation really is a covariant representation.

Reduced crossed products

Exercise

Suppose that G is discrete. Prove that a regular representation really is a covariant representation.

If $A = \mathbb{C}$, $H_0 = \mathbb{C}$, and π_0 is the obvious representation of A on H_0, then the regular representation is the usual left regular representation of G.

Reduced crossed products

Exercise

Suppose that G is discrete. Prove that a regular representation really is a covariant representation.

If $A = \mathbb{C}$, $H_0 = \mathbb{C}$, and π_0 is the obvious representation of A on H_0, then the regular representation is the usual left regular representation of G.

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A.

Reduced crossed products

Exercise

Suppose that G is discrete. Prove that a regular representation really is a covariant representation.

If $A = \mathbb{C}$, $H_0 = \mathbb{C}$, and π_0 is the obvious representation of A on H_0, then the regular representation is the usual left regular representation of G.

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\lambda\colon L^1(G, A, \alpha) \to L(H)$ be the direct sum of all regular representations of $L^1(G, A, \alpha)$.

Reduced crossed products

Exercise

Suppose that G is discrete. Prove that a regular representation really is a covariant representation.

If $A = \mathbb{C}$, $H_0 = \mathbb{C}$, and π_0 is the obvious representation of A on H_0, then the regular representation is the usual left regular representation of G.

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\lambda\colon L^1(G, A, \alpha) \to L(H)$ be the direct sum of all regular representations of $L^1(G, A, \alpha)$. We define the *reduced crossed product* $C^*_{\mathrm{r}}(G, A, \alpha)$ to be the norm closure of $\lambda(L^1(G, A, \alpha))$.

Reduced crossed products

Exercise

Suppose that G is discrete. Prove that a regular representation really is a covariant representation.

If $A = \mathbb{C}$, $H_0 = \mathbb{C}$, and π_0 is the obvious representation of A on H_0, then the regular representation is the usual left regular representation of G.

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\lambda \colon L^1(G, A, \alpha) \to L(H)$ be the direct sum of all regular representations of $L^1(G, A, \alpha)$. We define the *reduced crossed product* $C^*_{\mathrm{r}}(G, A, \alpha)$ to be the norm closure of $\lambda(L^1(G, A, \alpha))$.

As with crossed products, in these notes we ignore the set theoretic difficulty.

The relationship between reduced and full crossed products

Implicit in the definition of $C^*_{\mathrm{r}}(G, A, \alpha)$ is a representation of $L^1(G, A, \alpha)$,

The relationship between reduced and full crossed products

Implicit in the definition of $C^*_{\mathrm{r}}(G, A, \alpha)$ is a representation of $L^1(G, A, \alpha)$, hence of $C^*(G, A, \alpha)$.

The relationship between reduced and full crossed products

Implicit in the definition of $C^*_{\mathrm{r}}(G, A, \alpha)$ is a representation of $L^1(G, A, \alpha)$, hence of $C^*(G, A, \alpha)$. Thus, there is a homomorphism $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$. By construction, it has dense range, and is therefore surjective.

The relationship between reduced and full crossed products

Implicit in the definition of $C^*_{\mathrm{r}}(G, A, \alpha)$ is a representation of $L^1(G, A, \alpha)$, hence of $C^*(G, A, \alpha)$. Thus, there is a homomorphism $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$. By construction, it has dense range, and is therefore surjective. Moreover, by construction, any regular representation of $L^1(G, A, \alpha)$ extends to a representation of $C^*_{\mathrm{r}}(G, A, \alpha)$.

The relationship between reduced and full crossed products

Implicit in the definition of $C^*_{\mathrm{r}}(G, A, \alpha)$ is a representation of $L^1(G, A, \alpha)$, hence of $C^*(G, A, \alpha)$. Thus, there is a homomorphism $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$. By construction, it has dense range, and is therefore surjective. Moreover, by construction, any regular representation of $L^1(G, A, \alpha)$ extends to a representation of $C^*_{\mathrm{r}}(G, A, \alpha)$.

Theorem (Theorem 7.7.7 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A.

The relationship between reduced and full crossed products

Implicit in the definition of $C^*_{\mathrm{r}}(G, A, \alpha)$ is a representation of $L^1(G, A, \alpha)$, hence of $C^*(G, A, \alpha)$. Thus, there is a homomorphism $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$. By construction, it has dense range, and is therefore surjective. Moreover, by construction, any regular representation of $L^1(G, A, \alpha)$ extends to a representation of $C^*_{\mathrm{r}}(G, A, \alpha)$.

Theorem (Theorem 7.7.7 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. If G is amenable, then $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is an isomorphism.

The relationship between reduced and full crossed products

Implicit in the definition of $C^*_{\mathrm{r}}(G, A, \alpha)$ is a representation of $L^1(G, A, \alpha)$, hence of $C^*(G, A, \alpha)$. Thus, there is a homomorphism $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$. By construction, it has dense range, and is therefore surjective. Moreover, by construction, any regular representation of $L^1(G, A, \alpha)$ extends to a representation of $C^*_{\mathrm{r}}(G, A, \alpha)$.

Theorem (Theorem 7.7.7 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. If G is amenable, then $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is an isomorphism.

The converse is true for $A = \mathbb{C}$: if $C^*(G) \to C^*_{\mathrm{r}}(G)$ is an isomorphism, then G is amenable.

The relationship between reduced and full crossed products

Implicit in the definition of $C^*_{\mathrm{r}}(G, A, \alpha)$ is a representation of $L^1(G, A, \alpha)$, hence of $C^*(G, A, \alpha)$. Thus, there is a homomorphism $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$. By construction, it has dense range, and is therefore surjective. Moreover, by construction, any regular representation of $L^1(G, A, \alpha)$ extends to a representation of $C^*_{\mathrm{r}}(G, A, \alpha)$.

Theorem (Theorem 7.7.7 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. If G is amenable, then $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is an isomorphism.

The converse is true for $A = \mathbb{C}$: if $C^*(G) \to C^*_{\mathrm{r}}(G)$ is an isomorphism, then G is amenable. But it is not true in general. For example, if G acts on itself by translation,

The relationship between reduced and full crossed products

Implicit in the definition of $C^*_{\mathrm{r}}(G, A, \alpha)$ is a representation of $L^1(G, A, \alpha)$, hence of $C^*(G, A, \alpha)$. Thus, there is a homomorphism $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$. By construction, it has dense range, and is therefore surjective. Moreover, by construction, any regular representation of $L^1(G, A, \alpha)$ extends to a representation of $C^*_{\mathrm{r}}(G, A, \alpha)$.

Theorem (Theorem 7.7.7 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. If G is amenable, then $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is an isomorphism.

The converse is true for $A = \mathbb{C}$: if $C^*(G) \to C^*_{\mathrm{r}}(G)$ is an isomorphism, then G is amenable. But it is not true in general. For example, if G acts on itself by translation, then $C^*(G, \, C_0(G)) \to C^*_{\mathrm{r}}(G, \, C_0(G))$ is an isomorphism for every G.

The relationship between reduced and full crossed products

Implicit in the definition of $C^*_{\mathrm{r}}(G, A, \alpha)$ is a representation of $L^1(G, A, \alpha)$, hence of $C^*(G, A, \alpha)$. Thus, there is a homomorphism $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$. By construction, it has dense range, and is therefore surjective. Moreover, by construction, any regular representation of $L^1(G, A, \alpha)$ extends to a representation of $C^*_{\mathrm{r}}(G, A, \alpha)$.

Theorem (Theorem 7.7.7 of Pedersen's book)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. If G is amenable, then $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is an isomorphism.

The converse is true for $A = \mathbb{C}$: if $C^*(G) \to C^*_{\mathrm{r}}(G)$ is an isomorphism, then G is amenable. But it is not true in general. For example, if G acts on itself by translation, then $C^*(G, \, C_0(G)) \to C^*_{\mathrm{r}}(G, \, C_0(G))$ is an isomorphism for every G. (We will do this below for a discrete group.)

The crossed product is not too small

Theorem

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A.

The crossed product is not too small

Theorem

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Then $C_{\mathrm{c}}(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is injective.

The crossed product is not too small

Theorem

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Then $C_{\mathrm{c}}(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is injective.

We will prove this below in the case of a discrete group. The proof of the general case can be found in Lemma 2.26 of the book of Williams.

The crossed product is not too small

Theorem

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Then $C_{\mathrm{c}}(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is injective.

We will prove this below in the case of a discrete group. The proof of the general case can be found in Lemma 2.26 of the book of Williams. It is, I believe, true that $L^1(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is injective, and this can probably be proved by working a little harder in the proof of Lemma 2.26 of the book of Williams, but I have not carried out the details and I do not know a reference.

When G is discrete: integrated form of a regular representation

We specialize to the case of discrete G.

When G is discrete: integrated form of a regular representation

We specialize to the case of discrete G. The main tool is the structure of regular representations. When G is discrete, we can write $L^2(G, H_0)$ as a Hilbert space direct sum $\bigoplus_{g \in G} H_0$,

When G is discrete: integrated form of a regular representation

We specialize to the case of discrete G. The main tool is the structure of regular representations. When G is discrete, we can write $L^2(G, H_0)$ as a Hilbert space direct sum $\bigoplus_{g \in G} H_0$, and elements of it can be thought of as families $(\xi_g)_{g \in G}$.

When G is discrete: integrated form of a regular representation

We specialize to the case of discrete G. The main tool is the structure of regular representations. When G is discrete, we can write $L^2(G, H_0)$ as a Hilbert space direct sum $\bigoplus_{g \in G} H_0$, and elements of it can be thought of as families $(\xi_g)_{g \in G}$. The following formula for the integrated form of a regular representation is just a calculation.

When G is discrete: integrated form of a regular representation

We specialize to the case of discrete G. The main tool is the structure of regular representations. When G is discrete, we can write $L^2(G, H_0)$ as a Hilbert space direct sum $\bigoplus_{g \in G} H_0$, and elements of it can be thought of as families $(\xi_g)_{g \in G}$. The following formula for the integrated form of a regular representation is just a calculation.

Lemma

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A.

When G is discrete: integrated form of a regular representation

We specialize to the case of discrete G. The main tool is the structure of regular representations. When G is discrete, we can write $L^2(G, H_0)$ as a Hilbert space direct sum $\bigoplus_{g \in G} H_0$, and elements of it can be thought of as families $(\xi_g)_{g \in G}$. The following formula for the integrated form of a regular representation is just a calculation.

Lemma

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let $\sigma \colon C^*_{\mathrm{r}}(G, A, \alpha) \to L(H) = L(L^2(G, H_0))$ be the associated regular representation.

When G is discrete: integrated form of a regular representation

We specialize to the case of discrete G. The main tool is the structure of regular representations. When G is discrete, we can write $L^2(G, H_0)$ as a Hilbert space direct sum $\bigoplus_{g \in G} H_0$, and elements of it can be thought of as families $(\xi_g)_{g \in G}$. The following formula for the integrated form of a regular representation is just a calculation.

Lemma

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let $\sigma \colon C^*_{\mathrm{r}}(G, A, \alpha) \to L(H) = L(L^2(G, H_0))$ be the associated regular representation. Let $a = \sum_{g \in G} a_g u_g \in C^*_{\mathrm{r}}(G, A, \alpha)$, with $a_g = 0$ for all but finitely many g.

When G is discrete: integrated form of a regular representation

We specialize to the case of discrete G. The main tool is the structure of regular representations. When G is discrete, we can write $L^2(G, H_0)$ as a Hilbert space direct sum $\bigoplus_{g \in G} H_0$, and elements of it can be thought of as families $(\xi_g)_{g \in G}$. The following formula for the integrated form of a regular representation is just a calculation.

Lemma

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let $\sigma \colon C^*_{\mathrm{r}}(G, A, \alpha) \to L(H) = L(L^2(G, H_0))$ be the associated regular representation. Let $a = \sum_{g \in G} a_g u_g \in C^*_{\mathrm{r}}(G, A, \alpha)$, with $a_g = 0$ for all but finitely many g. For $\xi \in H$ and $h \in G$, we then have

$$(\sigma(a)\xi)(h) = \sum_{g \in G} \pi_0(\alpha_h^{-1}(a_g))\big(\xi(g^{-1}h)\big).$$

When G is discrete: integrated form of a regular representation (continued)

In particular, picking off coordinates in $L^2(G, H_0)$ gives:

When G is discrete: integrated form of a regular representation (continued)

In particular, picking off coordinates in $L^2(G, H_0)$ gives:

Corollary

Let the hypotheses be as in the Lemma, and let
$a = \sum_{g \in G} a_g u_g \in C^*_{\mathrm{r}}(G, A, \alpha)$.

When G is discrete: integrated form of a regular representation (continued)

In particular, picking off coordinates in $L^2(G, H_0)$ gives:

Corollary

Let the hypotheses be as in the Lemma, and let $a = \sum_{g \in G} a_g u_g \in C^*_{\mathrm{r}}(G, A, \alpha)$. For $g \in G$, let $s_g \in L(H_0, H)$ be the isometry which sends $\eta \in H_0$ to the function $\xi \in L^2(G, H_0)$ given by

$$\xi(h) = \begin{cases} \eta & h = g \\ 0 & h \neq g. \end{cases}$$

When G is discrete: integrated form of a regular representation (continued)

In particular, picking off coordinates in $L^2(G, H_0)$ gives:

Corollary

Let the hypotheses be as in the Lemma, and let $a = \sum_{g \in G} a_g u_g \in C^*_{\mathrm{r}}(G, A, \alpha)$. For $g \in G$, let $s_g \in L(H_0, H)$ be the isometry which sends $\eta \in H_0$ to the function $\xi \in L^2(G, H_0)$ given by

$$\xi(h) = \begin{cases} \eta & h = g \\ 0 & h \neq g. \end{cases}$$

Then

$$s_h^* \sigma(a) s_k = \pi_0\big(\alpha_h^{-1}(a_{hk^{-1}})\big)$$

for all $h, k \in G$.

Comparing norms

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Define norms on $C_{\mathrm{c}}(G, A, \alpha)$ as follows:

Comparing norms

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Define norms on $C_{\mathrm{c}}(G, A, \alpha)$ as follows:

- $\| \cdot \|_{\infty}$ is the supremum norm.

Comparing norms

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Define norms on $C_{\mathrm{c}}(G, A, \alpha)$ as follows:

- $\| \cdot \|_{\infty}$ is the supremum norm.
- $\| \cdot \|_1$ is the l^1 norm.

Comparing norms

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Define norms on $C_{\mathrm{c}}(G, A, \alpha)$ as follows:

- $\| \cdot \|_{\infty}$ is the supremum norm.
- $\| \cdot \|_{1}$ is the l^1 norm.
- $\| \cdot \|$ is the restriction of the C*-algebra norm on $C^*(G, A, \alpha)$.

Comparing norms

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Define norms on $C_{\mathrm{c}}(G, A, \alpha)$ as follows:

- $\| \cdot \|_{\infty}$ is the supremum norm.
- $\| \cdot \|_{1}$ is the l^1 norm.
- $\| \cdot \|$ is the restriction of the C*-algebra norm on $C^*(G, A, \alpha)$.
- $\| \cdot \|_{\mathrm{r}}$ is the restriction of the C*-algebra norm on $C^*_{\mathrm{r}}(G, A, \alpha)$.

Comparing norms

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Define norms on $C_{\mathrm{c}}(G, A, \alpha)$ as follows:

- $\| \cdot \|_{\infty}$ is the supremum norm.
- $\| \cdot \|_{1}$ is the l^1 norm.
- $\| \cdot \|$ is the restriction of the C*-algebra norm on $C^*(G, A, \alpha)$.
- $\| \cdot \|_{\mathrm{r}}$ is the restriction of the C*-algebra norm on $C^*_{\mathrm{r}}(G, A, \alpha)$.

Lemma

For every $a \in C_{\mathrm{c}}(G, A, \alpha)$, we have $\|a\|_{\infty} \leq \|a\|_{\mathrm{r}} \leq \|a\| \leq \|a\|_{1}$.

Comparing norms: the proof

The middle of this inequality follows from the definitions.

Comparing norms: the proof

The middle of this inequality follows from the definitions.

The last part follows from the observation above that all continuous representations of $L^1(G, A, \alpha)$ are norm reducing.

Comparing norms: the proof

The middle of this inequality follows from the definitions.
The last part follows from the observation above that all continuous representations of $L^1(G, A, \alpha)$ are norm reducing. Here is a direct proof: for $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, with all but finitely many of the a_g equal to zero, we have

Comparing norms: the proof

The middle of this inequality follows from the definitions.
The last part follows from the observation above that all continuous representations of $L^1(G, A, \alpha)$ are norm reducing. Here is a direct proof: for $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, with all but finitely many of the a_g equal to zero, we have

$$\Big\| \sum_{g \in G} a_g u_g \Big\| \leq \sum_{g \in G} \|a_g\| \cdot \|u_g\| = \sum_{g \in G} \|a_g\| = \Big\| \sum_{g \in G} a_g u_g \Big\|_1 .$$

Comparing norms: the proof

The middle of this inequality follows from the definitions.
The last part follows from the observation above that all continuous representations of $L^1(G, A, \alpha)$ are norm reducing. Here is a direct proof: for $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, with all but finitely many of the a_g equal to zero, we have

$$\left\| \sum_{g \in G} a_g u_g \right\| \leq \sum_{g \in G} \|a_g\| \cdot \|u_g\| = \sum_{g \in G} \|a_g\| = \left\| \sum_{g \in G} a_g u_g \right\|_1 .$$

We prove the first part of this inequality. Let $a = \sum_{g \in G} a_g u_g$, with all but finitely many of the a_g equal to zero, and let $g \in G$.

Comparing norms: the proof

The middle of this inequality follows from the definitions.
The last part follows from the observation above that all continuous representations of $L^1(G, A, \alpha)$ are norm reducing. Here is a direct proof: for $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, with all but finitely many of the a_g equal to zero, we have

$$\left\| \sum\nolimits_{g \in G} a_g u_g \right\| \leq \sum\nolimits_{g \in G} \| a_g \| \cdot \| u_g \| = \sum\nolimits_{g \in G} \| a_g \| = \left\| \sum\nolimits_{g \in G} a_g u_g \right\|_1 .$$

We prove the first part of this inequality. Let $a = \sum_{g \in G} a_g u_g$, with all but finitely many of the a_g equal to zero, and let $g \in G$. Let $\pi_0 \colon A \to L(H_0)$ be an injective nondegenerate representation.

Comparing norms: the proof

The middle of this inequality follows from the definitions.
The last part follows from the observation above that all continuous representations of $L^1(G, A, \alpha)$ are norm reducing. Here is a direct proof: for $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, with all but finitely many of the a_g equal to zero, we have

$$\left\| \sum_{g \in G} a_g u_g \right\| \leq \sum_{g \in G} \|a_g\| \cdot \|u_g\| = \sum_{g \in G} \|a_g\| = \left\| \sum_{g \in G} a_g u_g \right\|_1 .$$

We prove the first part of this inequality. Let $a = \sum_{g \in G} a_g u_g$, with all but finitely many of the a_g equal to zero, and let $g \in G$. Let $\pi_0 \colon A \to L(H_0)$ be an injective nondegenerate representation. With the notation of the previous corollary, we have

Comparing norms: the proof

The middle of this inequality follows from the definitions.
The last part follows from the observation above that all continuous representations of $L^1(G, A, \alpha)$ are norm reducing. Here is a direct proof: for $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, with all but finitely many of the a_g equal to zero, we have

$$\left\| \sum_{g \in G} a_g u_g \right\| \leq \sum_{g \in G} \|a_g\| \cdot \|u_g\| = \sum_{g \in G} \|a_g\| = \left\| \sum_{g \in G} a_g u_g \right\|_1 .$$

We prove the first part of this inequality. Let $a = \sum_{g \in G} a_g u_g$, with all but finitely many of the a_g equal to zero, and let $g \in G$. Let $\pi_0 \colon A \to L(H_0)$ be an injective nondegenerate representation. With the notation of the previous corollary, we have

$$\|a_g\| = \|\pi_0(a_g)\| = \|s_g^* \sigma(a) s_1\| \leq \|\sigma(a)\| \leq \|a\|_{\mathrm{r}}.$$

This completes the proof.

A is a subalgebra of the reduced crossed product

The lemma implies that the map $a \mapsto a u_1$, from A to $C^*_{\mathrm{r}}(G, A, \alpha)$, is injective.

A is a subalgebra of the reduced crossed product

The lemma implies that the map $a \mapsto a u_1$, from A to $C^*_{\mathrm{r}}(G, A, \alpha)$, is injective. We routinely identify A with its image in $C^*_{\mathrm{r}}(G, A, \alpha)$ under this map, thus treating it as a subalgebra of $C^*_{\mathrm{r}}(G, A, \alpha)$.

A is a subalgebra of the reduced crossed product

The lemma implies that the map $a \mapsto au_1$, from A to $C^*_{\mathrm{r}}(G, A, \alpha)$, is injective. We routinely identify A with its image in $C^*_{\mathrm{r}}(G, A, \alpha)$ under this map, thus treating it as a subalgebra of $C^*_{\mathrm{r}}(G, A, \alpha)$.

Of course, we can do the same with the full crossed product $C^*(G, A, \alpha)$.

For finite groups, no completion is needed

Corollary

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A.

For finite groups, no completion is needed

Corollary

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. Then the maps $C_{\mathrm{c}}(G, A, \alpha) \to C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ are bijective.

For finite groups, no completion is needed

Corollary

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. Then the maps $C_{\mathrm{c}}(G, A, \alpha) \to C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ are bijective.

Proof.

When G is finite, $\| \cdot \|_1$ (the l^1 norm) is equivalent to $\| \cdot \|_\infty$ (the supremum norm), and is complete in both.

For finite groups, no completion is needed

Corollary

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. Then the maps $C_{\mathrm{c}}(G, A, \alpha) \to C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ are bijective.

Proof.

When G is finite, $\| \cdot \|_1$ (the l^1 norm) is equivalent to $\| \cdot \|_\infty$ (the supremum norm), and is complete in both. The lemma now implies that both C* norms are equivalent to these norms, so $C_{\mathrm{c}}(G, A, \alpha)$ is complete in both C* norms. □

Coefficients in reduced crossed products

When G is discrete but not finite, things are much more complicated. We can get started:

Coefficients in reduced crossed products

When G is discrete but not finite, things are much more complicated. We can get started:

Proposition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A.

Coefficients in reduced crossed products

When G is discrete but not finite, things are much more complicated. We can get started:

Proposition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Then for each $g \in G$, there is a linear map $E_g\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ with $\|E_g\| \leq 1$

Coefficients in reduced crossed products

When G is discrete but not finite, things are much more complicated. We can get started:

Proposition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Then for each $g \in G$, there is a linear map $E_g\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ with $\|E_g\| \leq 1$ such that if $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, then $E_g(a) = a_g$.

Coefficients in reduced crossed products

When G is discrete but not finite, things are much more complicated. We can get started:

Proposition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Then for each $g \in G$, there is a linear map $E_g\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ with $\|E_g\| \leq 1$ such that if $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, then $E_g(a) = a_g$.

Moreover, with s_g as above, we have $s_h^* \sigma(a) s_k = \pi_0\big(\alpha_h^{-1}(E_{hk^{-1}}(a))\big)$ for all $h, k \in G$.

Coefficients in reduced crossed products

When G is discrete but not finite, things are much more complicated. We can get started:

Proposition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Then for each $g \in G$, there is a linear map $E_g \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ with $\|E_g\| \leq 1$ such that if $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, then $E_g(a) = a_g$.

Moreover, with s_g as above, we have $s_h^* \sigma(a) s_k = \pi_0\big(\alpha_h^{-1}(E_{hk^{-1}}(a))\big)$ for all $h, k \in G$.

Proof.

The first part is immediate from the inequality $\|a\|_\infty \leq \|a\|_{\mathrm{r}}$ above.

Coefficients in reduced crossed products

When G is discrete but not finite, things are much more complicated. We can get started:

Proposition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Then for each $g \in G$, there is a linear map $E_g \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ with $\|E_g\| \leq 1$ such that if $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, then $E_g(a) = a_g$.

Moreover, with s_g as above, we have $s_h^* \sigma(a) s_k = \pi_0\big(\alpha_h^{-1}(E_{hk^{-1}}(a))\big)$ for all $h, k \in G$.

Proof.

The first part is immediate from the inequality $\|a\|_\infty \leq \|a\|_{\mathrm{r}}$ above.

The last statement follows by continuity from "picking off coordinates" in the regular representation. □

Coefficients in reduced crossed products: Discussion

Thus, for any $a \in C^*_{\mathrm{r}}(G, A, \alpha)$,

Coefficients in reduced crossed products: Discussion

Thus, for any $a \in C^*_{\mathrm{r}}(G, A, \alpha)$, and therefore also for $a \in C^*(G, A, \alpha)$,

Coefficients in reduced crossed products: Discussion

Thus, for any $a \in C^*_{\mathrm{r}}(G, A, \alpha)$, and therefore also for $a \in C^*(G, A, \alpha)$, it makes sense to talk about its coefficients a_g.

Coefficients in reduced crossed products: Discussion

Thus, for any $a \in C^*_{\mathrm{r}}(G, A, \alpha)$, and therefore also for $a \in C^*(G, A, \alpha)$, it makes sense to talk about its coefficients a_g. The first point is that if $C^*(G, A, \alpha) \neq C^*_{\mathrm{r}}(G, A, \alpha)$

Coefficients in reduced crossed products: Discussion

Thus, for any $a \in C^*_{\mathrm{r}}(G, A, \alpha)$, and therefore also for $a \in C^*(G, A, \alpha)$, it makes sense to talk about its coefficients a_g. The first point is that if $C^*(G, A, \alpha) \neq C^*_{\mathrm{r}}(G, A, \alpha)$ (which can happen if G is not amenable, but not if G is amenable),

Coefficients in reduced crossed products: Discussion

Thus, for any $a \in C^*_{\mathrm{r}}(G, A, \alpha)$, and therefore also for $a \in C^*(G, A, \alpha)$, it makes sense to talk about its coefficients a_g. The first point is that if $C^*(G, A, \alpha) \neq C^*_{\mathrm{r}}(G, A, \alpha)$ (which can happen if G is not amenable, but not if G is amenable), the coefficients $(a_g)_{g \in G}$ do not even uniquely determine the element a.

Coefficients in reduced crossed products: Discussion

Thus, for any $a \in C^*_{\mathrm{r}}(G, A, \alpha)$, and therefore also for $a \in C^*(G, A, \alpha)$, it makes sense to talk about its coefficients a_g. The first point is that if $C^*(G, A, \alpha) \neq C^*_{\mathrm{r}}(G, A, \alpha)$ (which can happen if G is not amenable, but not if G is amenable), the coefficients $(a_g)_{g \in G}$ do not even uniquely determine the element a. This is why we are only considering reduced crossed products here.

Coefficients in reduced crossed products: Properties

Here are the good things about coefficients.

Coefficients in reduced crossed products: Properties

Here are the good things about coefficients.

Proposition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A.

Coefficients in reduced crossed products: Properties

Here are the good things about coefficients.

Proposition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let the maps $E_g \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as in the previous proposition. Then:

Coefficients in reduced crossed products: Properties

Here are the good things about coefficients.

Proposition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let the maps $E_g \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as in the previous proposition. Then:

1. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $E_g(a) = 0$ for all $g \in G$, then $a = 0$.

Coefficients in reduced crossed products: Properties

Here are the good things about coefficients.

Proposition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let the maps $E_g \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as in the previous proposition. Then:

1. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $E_g(a) = 0$ for all $g \in G$, then $a = 0$.
2. If $\pi_0 \colon A \to L(H_0)$ is a nondegenerate representation such that $\bigoplus_{g \in G} \pi_0 \circ \alpha_g$ is injective, then the regular representation σ of $C^*_{\mathrm{r}}(G, A, \alpha)$ associated to π_0 is injective.

Coefficients in reduced crossed products: Properties

Here are the good things about coefficients.

Proposition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let the maps $E_g \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as in the previous proposition. Then:

1. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $E_g(a) = 0$ for all $g \in G$, then $a = 0$.
2. If $\pi_0 \colon A \to L(H_0)$ is a nondegenerate representation such that $\bigoplus_{g \in G} \pi_0 \circ \alpha_g$ is injective, then the regular representation σ of $C^*_{\mathrm{r}}(G, A, \alpha)$ associated to π_0 is injective.
3. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $E_1(a^* a) = 0$, then $a = 0$.

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above.

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$,

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$.

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$.

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$.

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in the previous proposition,

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$.

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$. So $E_l(a) = 0$.

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$. So $E_l(a) = 0$. This is true for all $l \in G$, so $a = 0$.

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$. So $E_l(a) = 0$. This is true for all $l \in G$, so $a = 0$.

(3): As before, let $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$.

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$. So $E_l(a) = 0$. This is true for all $l \in G$, so $a = 0$.

(3): As before, let $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$. Then $a^* a = \sum_{g, h \in G} u_g^* a_g^* a_h u_h$, so

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$. So $E_l(a) = 0$. This is true for all $l \in G$, so $a = 0$.

(3): As before, let $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$. Then $a^* a = \sum_{g, h \in G} u_g^* a_g^* a_h u_h$, so

$$E_1(a^* a) = \sum_{g \in G} u_g^* a_g^* a_g u_g = \sum_{g \in G} \alpha_g^{-1} \big(E_g(a)^* E_g(a) \big).$$

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$. So $E_l(a) = 0$. This is true for all $l \in G$, so $a = 0$.

(3): As before, let $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$. Then $a^* a = \sum_{g, h \in G} u_g^* a_g^* a_h u_h$, so

$$E_1(a^* a) = \sum_{g \in G} u_g^* a_g^* a_g u_g = \sum_{g \in G} \alpha_g^{-1} \big(E_g(a)^* E_g(a) \big).$$

In particular, for each fixed g, we have $E_1(a^* a) \geq \alpha_g^{-1} \big(E_g(a)^* E_g(a) \big)$.

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$. So $E_l(a) = 0$. This is true for all $l \in G$, so $a = 0$.

(3): As before, let $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$. Then $a^* a = \sum_{g, h \in G} u_g^* a_g^* a_h u_h$, so

$$E_1(a^* a) = \sum_{g \in G} u_g^* a_g^* a_g u_g = \sum_{g \in G} \alpha_g^{-1}\big(E_g(a)^* E_g(a)\big).$$

In particular, for each fixed g, we have $E_1(a^* a) \geq \alpha_g^{-1}\big(E_g(a)^* E_g(a)\big)$. By continuity, this inequality holds for all $a \in C^*_{\mathrm{r}}(G, A, \alpha)$.

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$. So $E_l(a) = 0$. This is true for all $l \in G$, so $a = 0$.

(3): As before, let $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$. Then $a^* a = \sum_{g, h \in G} u_g^* a_g^* a_h u_h$, so

$$E_1(a^* a) = \sum_{g \in G} u_g^* a_g^* a_g u_g = \sum_{g \in G} \alpha_g^{-1}\big(E_g(a)^* E_g(a)\big).$$

In particular, for each fixed g, we have $E_1(a^* a) \geq \alpha_g^{-1}\big(E_g(a)^* E_g(a)\big)$. By continuity, this inequality holds for all $a \in C^*_{\mathrm{r}}(G, A, \alpha)$. Thus, if $E_1(a^* a) = 0$, then $E_g(a)^* E_g(a) = 0$ for all g,

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$. So $E_l(a) = 0$. This is true for all $l \in G$, so $a = 0$.

(3): As before, let $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$. Then $a^* a = \sum_{g, h \in G} u_g^* a_g^* a_h u_h$, so

$$E_1(a^* a) = \sum_{g \in G} u_g^* a_g^* a_g u_g = \sum_{g \in G} \alpha_g^{-1}\big(E_g(a)^* E_g(a)\big).$$

In particular, for each fixed g, we have $E_1(a^* a) \geq \alpha_g^{-1}\big(E_g(a)^* E_g(a)\big)$. By continuity, this inequality holds for all $a \in C^*_{\mathrm{r}}(G, A, \alpha)$. Thus, if $E_1(a^* a) = 0$, then $E_g(a)^* E_g(a) = 0$ for all g, so $a = 0$ by Part (1).

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$. So $E_l(a) = 0$. This is true for all $l \in G$, so $a = 0$.

(3): As before, let $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$. Then $a^* a = \sum_{g, h \in G} u_g^* a_g^* a_h u_h$, so

$$E_1(a^* a) = \sum_{g \in G} u_g^* a_g^* a_g u_g = \sum_{g \in G} \alpha_g^{-1}\big(E_g(a)^* E_g(a)\big).$$

In particular, for each fixed g, we have $E_1(a^* a) \geq \alpha_g^{-1}\big(E_g(a)^* E_g(a)\big)$. By continuity, this inequality holds for all $a \in C^*_{\mathrm{r}}(G, A, \alpha)$. Thus, if $E_1(a^* a) = 0$, then $E_g(a)^* E_g(a) = 0$ for all g, so $a = 0$ by Part (1). This completes the proof.

Injective representations of A always give injective regular representations of the reduced crossed product

It is true for general locally compact groups, not just discrete groups, that the regular representation of $C^*_{\mathrm{r}}(G, A, \alpha)$ associated to an injective representation of A is injective. See Theorem 7.7.5 of Pedersen's book.

The conditional expectation

The map E_1 used in Part (3) of the previous proposition is an example of what is called a *conditional expectation* (from $C^*_{\mathrm{r}}(G, A, \alpha)$ to A)

The conditional expectation

The map E_1 used in Part (3) of the previous proposition is an example of what is called a *conditional expectation* (from $C^*_{\mathrm{r}}(G, A, \alpha)$ to A) that is, it has the properties given in the following exercise.

The conditional expectation

The map E_1 used in Part (3) of the previous proposition is an example of what is called a *conditional expectation* (from $C^*_{\mathrm{r}}(G, A, \alpha)$ to A) that is, it has the properties given in the following exercise. Part (3) of the previous proposition asserts that this conditional expectation is faithful.

The conditional expectation

The map E_1 used in Part (3) of the previous proposition is an example of what is called a *conditional expectation* (from $C^*_{\mathrm{r}}(G, A, \alpha)$ to A) that is, it has the properties given in the following exercise. Part (3) of the previous proposition asserts that this conditional expectation is faithful.

Exercise

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A.

The conditional expectation

The map E_1 used in Part (3) of the previous proposition is an example of what is called a *conditional expectation* (from $C^*_{\mathrm{r}}(G, A, \alpha)$ to A) that is, it has the properties given in the following exercise. Part (3) of the previous proposition asserts that this conditional expectation is faithful.

Exercise

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $E = E_1\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as above. Prove that E has the following properties:

The conditional expectation

The map E_1 used in Part (3) of the previous proposition is an example of what is called a *conditional expectation* (from $C^*_{\mathrm{r}}(G, A, \alpha)$ to A) that is, it has the properties given in the following exercise. Part (3) of the previous proposition asserts that this conditional expectation is faithful.

Exercise

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $E = E_1\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as above. Prove that E has the following properties:

1. $E(E(b)) = E(b)$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.

The conditional expectation

The map E_1 used in Part (3) of the previous proposition is an example of what is called a *conditional expectation* (from $C^*_{\mathrm{r}}(G, A, \alpha)$ to A) that is, it has the properties given in the following exercise. Part (3) of the previous proposition asserts that this conditional expectation is faithful.

Exercise

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $E = E_1 \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as above. Prove that E has the following properties:

1. $E(E(b)) = E(b)$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
2. If $b \geq 0$ then $E(b) \geq 0$.

The conditional expectation

The map E_1 used in Part (3) of the previous proposition is an example of what is called a *conditional expectation* (from $C^*_{\mathrm{r}}(G, A, \alpha)$ to A) that is, it has the properties given in the following exercise. Part (3) of the previous proposition asserts that this conditional expectation is faithful.

Exercise

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $E = E_1 \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as above. Prove that E has the following properties:

1. $E(E(b)) = E(b)$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
2. If $b \geq 0$ then $E(b) \geq 0$.
3. $\|E(b)\| \leq \|b\|$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.

The conditional expectation

The map E_1 used in Part (3) of the previous proposition is an example of what is called a *conditional expectation* (from $C^*_{\mathrm{r}}(G, A, \alpha)$ to A) that is, it has the properties given in the following exercise. Part (3) of the previous proposition asserts that this conditional expectation is faithful.

Exercise

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $E = E_1\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as above. Prove that E has the following properties:

1. $E(E(b)) = E(b)$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
2. If $b \geq 0$ then $E(b) \geq 0$.
3. $\|E(b)\| \leq \|b\|$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
4. If $a \in A$ and $b \in C^*_{\mathrm{r}}(G, A, \alpha)$, then $E(ab) = aE(b)$ and $E(ba) = E(b)a$.

The limits of coefficients

Unfortunately, in general $\sum_{g \in G} a_g u_g$ does not converge in $C^*_{\mathrm{r}}(G, A, \alpha)$, and it is very difficult to tell exactly which families of coefficients correspond to elements of $C^*_{\mathrm{r}}(G, A, \alpha)$.

The limits of coefficients

Unfortunately, in general $\sum_{g \in G} a_g u_g$ does not converge in $C^*_{\mathrm{r}}(G, A, \alpha)$, and it is very difficult to tell exactly which families of coefficients correspond to elements of $C^*_{\mathrm{r}}(G, A, \alpha)$. In fact, the situation is intractable even for the case of the trivial action of $\mathbb{Z}$ on $\mathbb{C}$. In this case, $l^1(\mathbb{Z}, A, \alpha) = l^1(\mathbb{Z})$. The crossed product is the group C*-algebra $C^*(\mathbb{Z})$, which can be identified with $C(S^1)$. The map $l^1(\mathbb{Z}) \to C(S^1)$ is given by Fourier series: the sequence $a = (a_n)_{n \in \mathbb{Z}_{>0}}$ goes to the function $\zeta \mapsto \sum_{n \in \mathbb{Z}} a_n \zeta^n$.

The limits of coefficients

Unfortunately, in general $\sum_{g \in G} a_g u_g$ does not converge in $C^*_{\mathrm{r}}(G, A, \alpha)$, and it is very difficult to tell exactly which families of coefficients correspond to elements of $C^*_{\mathrm{r}}(G, A, \alpha)$. In fact, the situation is intractable even for the case of the trivial action of $\mathbb{Z}$ on $\mathbb{C}$. In this case, $l^1(\mathbb{Z}, A, \alpha) = l^1(\mathbb{Z})$. The crossed product is the group C*-algebra $C^*(\mathbb{Z})$, which can be identified with $C(S^1)$. The map $l^1(\mathbb{Z}) \to C(S^1)$ is given by Fourier series: the sequence $a = (a_n)_{n \in \mathbb{Z}_{>0}}$ goes to the function $\zeta \mapsto \sum_{n \in \mathbb{Z}} a_n \zeta^n$. (This looks more familiar when expressed in terms of 2π-periodic functions on $\mathbb{R}$: it is $t \mapsto \sum_{n \in \mathbb{Z}} a_n e^{int}$.)

The limits of coefficients

Unfortunately, in general $\sum_{g \in G} a_g u_g$ does not converge in $C^*_{\mathrm{r}}(G, A, \alpha)$, and it is very difficult to tell exactly which families of coefficients correspond to elements of $C^*_{\mathrm{r}}(G, A, \alpha)$. In fact, the situation is intractable even for the case of the trivial action of $\mathbb{Z}$ on $\mathbb{C}$. In this case, $l^1(\mathbb{Z}, A, \alpha) = l^1(\mathbb{Z})$. The crossed product is the group C*-algebra $C^*(\mathbb{Z})$, which can be identified with $C(S^1)$. The map $l^1(\mathbb{Z}) \to C(S^1)$ is given by Fourier series: the sequence $a = (a_n)_{n \in \mathbb{Z}_{>0}}$ goes to the function $\zeta \mapsto \sum_{n \in \mathbb{Z}} a_n \zeta^n$. (This looks more familiar when expressed in terms of 2π-periodic functions on $\mathbb{R}$: it is $t \mapsto \sum_{n \in \mathbb{Z}} a_n e^{int}$.) Failure of convergence of $\sum_{n \in \mathbb{Z}} a_n u_n$ corresponds to the fact that the Fourier series of a continuous function need not converge uniformly.

The limits of coefficients

Unfortunately, in general $\sum_{g \in G} a_g u_g$ does not converge in $C^*_{\mathrm{r}}(G, A, \alpha)$, and it is very difficult to tell exactly which families of coefficients correspond to elements of $C^*_{\mathrm{r}}(G, A, \alpha)$. In fact, the situation is intractable even for the case of the trivial action of $\mathbb{Z}$ on $\mathbb{C}$. In this case, $l^1(\mathbb{Z}, A, \alpha) = l^1(\mathbb{Z})$. The crossed product is the group C*-algebra $C^*(\mathbb{Z})$, which can be identified with $C(S^1)$. The map $l^1(\mathbb{Z}) \to C(S^1)$ is given by Fourier series: the sequence $a = (a_n)_{n \in \mathbb{Z}_{>0}}$ goes to the function $\zeta \mapsto \sum_{n \in \mathbb{Z}} a_n \zeta^n$. (This looks more familiar when expressed in terms of 2π-periodic functions on $\mathbb{R}$: it is $t \mapsto \sum_{n \in \mathbb{Z}} a_n e^{int}$.) Failure of convergence of $\sum_{n \in \mathbb{Z}} a_n u_n$ corresponds to the fact that the Fourier series of a continuous function need not converge uniformly. Identifying the coefficient sequences which correspond to elements of the crossed product corresponds to giving a criterion for exactly when a sequence $(a_n)_{n \in \mathbb{Z}_{>0}}$ of complex numbers is the sequence of Fourier coefficients of some continuous function on S^1,

The limits of coefficients

Unfortunately, in general $\sum_{g \in G} a_g u_g$ does not converge in $C^*_{\mathrm{r}}(G, A, \alpha)$, and it is very difficult to tell exactly which families of coefficients correspond to elements of $C^*_{\mathrm{r}}(G, A, \alpha)$. In fact, the situation is intractable even for the case of the trivial action of $\mathbb{Z}$ on $\mathbb{C}$. In this case, $l^1(\mathbb{Z}, A, \alpha) = l^1(\mathbb{Z})$. The crossed product is the group C*-algebra $C^*(\mathbb{Z})$, which can be identified with $C(S^1)$. The map $l^1(\mathbb{Z}) \to C(S^1)$ is given by Fourier series: the sequence $a = (a_n)_{n \in \mathbb{Z}_{>0}}$ goes to the function $\zeta \mapsto \sum_{n \in \mathbb{Z}} a_n \zeta^n$. (This looks more familiar when expressed in terms of 2π-periodic functions on $\mathbb{R}$: it is $t \mapsto \sum_{n \in \mathbb{Z}} a_n e^{int}$.) Failure of convergence of $\sum_{n \in \mathbb{Z}} a_n u_n$ corresponds to the fact that the Fourier series of a continuous function need not converge uniformly. Identifying the coefficient sequences which correspond to elements of the crossed product corresponds to giving a criterion for exactly when a sequence $(a_n)_{n \in \mathbb{Z}_{>0}}$ of complex numbers is the sequence of Fourier coefficients of some continuous function on S^1, a problem for which I know of no satisfactory solution.

The limits of coefficients (continued)

Let's pursue this a little farther. The regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$ gives an injective map $\lambda \colon C^*(\mathbb{Z}) \to L(l^2(\mathbb{Z}))$.

The limits of coefficients (continued)

Let's pursue this a little farther. The regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$ gives an injective map $\lambda \colon C^*(\mathbb{Z}) \to L(l^2(\mathbb{Z}))$. Let $\delta_n \in l^2(\mathbb{Z})$ be the function

$$\delta_n(k) = \begin{cases} 1 & k = n \\ 0 & k \neq n. \end{cases}$$

The limits of coefficients (continued)

Let's pursue this a little farther. The regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$ gives an injective map $\lambda \colon C^*(\mathbb{Z}) \to L(l^2(\mathbb{Z}))$. Let $\delta_n \in l^2(\mathbb{Z})$ be the function

$$\delta_n(k) = \begin{cases} 1 & k = n \\ 0 & k \neq n. \end{cases}$$

Then the Fourier coefficient a_n is recovered as $a_n = \langle \lambda(a)\delta_0, \, \delta_n \rangle$. That is, $\lambda(a)\delta_0 \in l^2(\mathbb{Z})$ is given by $\lambda(a)\delta_0 = \sum_{n \in \mathbb{Z}} a_n \delta_n$.

The limits of coefficients (continued)

Let's pursue this a little farther. The regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$ gives an injective map $\lambda \colon C^*(\mathbb{Z}) \to L(l^2(\mathbb{Z}))$. Let $\delta_n \in l^2(\mathbb{Z})$ be the function

$$\delta_n(k) = \begin{cases} 1 & k = n \\ 0 & k \neq n. \end{cases}$$

Then the Fourier coefficient a_n is recovered as $a_n = \langle \lambda(a)\delta_0, \, \delta_n \rangle$. That is, $\lambda(a)\delta_0 \in l^2(\mathbb{Z})$ is given by $\lambda(a)\delta_0 = \sum_{n \in \mathbb{Z}} a_n \delta_n$. Thus, the sequence of Fourier coefficients of a continuous function is always in $l^2(\mathbb{Z})$. (Of course, we already know this, but the calculation here can be applied to more general crossed products.)

The limits of coefficients (continued)

Let's pursue this a little farther. The regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$ gives an injective map $\lambda \colon C^*(\mathbb{Z}) \to L(l^2(\mathbb{Z}))$. Let $\delta_n \in l^2(\mathbb{Z})$ be the function

$$\delta_n(k) = \begin{cases} 1 & k = n \\ 0 & k \neq n. \end{cases}$$

Then the Fourier coefficient a_n is recovered as $a_n = \langle \lambda(a)\delta_0, \, \delta_n \rangle$. That is, $\lambda(a)\delta_0 \in l^2(\mathbb{Z})$ is given by $\lambda(a)\delta_0 = \sum_{n \in \mathbb{Z}} a_n \delta_n$. Thus, the sequence of Fourier coefficients of a continuous function is always in $l^2(\mathbb{Z})$. (Of course, we already know this, but the calculation here can be applied to more general crossed products.) Unfortunately, this fact is essentially useless for the study of the group C*-algebra. Not only is the Fourier series of a continuous function always in $l^2(\mathbb{Z})$, but the Fourier series of a function in $L^\infty(S^1)$, which is the group von Neumann algebra of $\mathbb{Z}$, is also always in $l^2(\mathbb{Z})$.

The limits of coefficients (continued)

Let's pursue this a little farther. The regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$ gives an injective map $\lambda \colon C^*(\mathbb{Z}) \to L(l^2(\mathbb{Z}))$. Let $\delta_n \in l^2(\mathbb{Z})$ be the function

$$\delta_n(k) = \begin{cases} 1 & k = n \\ 0 & k \neq n. \end{cases}$$

Then the Fourier coefficient a_n is recovered as $a_n = \langle \lambda(a)\delta_0, \, \delta_n \rangle$. That is, $\lambda(a)\delta_0 \in l^2(\mathbb{Z})$ is given by $\lambda(a)\delta_0 = \sum_{n \in \mathbb{Z}} a_n \delta_n$. Thus, the sequence of Fourier coefficients of a continuous function is always in $l^2(\mathbb{Z})$. (Of course, we already know this, but the calculation here can be applied to more general crossed products.) Unfortunately, this fact is essentially useless for the study of the group C*-algebra. Not only is the Fourier series of a continuous function always in $l^2(\mathbb{Z})$, but the Fourier series of a function in $L^\infty(S^1)$, which is the group von Neumann algebra of $\mathbb{Z}$, is also always in $l^2(\mathbb{Z})$. One will get essentially no useful information from a criterion which can't even exclude any elements of $L^\infty(S^1)$.

The limits of coefficients (continued)

Even if one understands completely what all the elements of $C^*_{\mathrm{r}}(G)$ are, and even if the action is trivial, understanding the elements of the reduced crossed product requires that one understand all the elements of the completed tensor product $C^*_{\mathrm{r}}(G) \otimes_{\min} A$.

The limits of coefficients (continued)

Even if one understands completely what all the elements of $C^*_{\mathrm{r}}(G)$ are, and even if the action is trivial, understanding the elements of the reduced crossed product requires that one understand all the elements of the completed tensor product $C^*_{\mathrm{r}}(G) \otimes_{\min} A$. As far as I know, this problem is also in general intractable.

The limits of coefficients (continued)

Even if one understands completely what all the elements of $C^*_{\mathrm{r}}(G)$ are, and even if the action is trivial, understanding the elements of the reduced crossed product requires that one understand all the elements of the completed tensor product $C^*_{\mathrm{r}}(G) \otimes_{\min} A$. As far as I know, this problem is also in general intractable.

There is just one bright spot: although we will not prove it here, there is an analog for general crossed products by $\mathbb{Z}$ of the fact that the Cesaro means of the Fourier series of a continuous function always converge uniformly to the function. See Theorem 8.2.2 of Davidson's book.

The limits of coefficients (continued)

Even if one understands completely what all the elements of $C^*_{\mathrm{r}}(G)$ are, and even if the action is trivial, understanding the elements of the reduced crossed product requires that one understand all the elements of the completed tensor product $C^*_{\mathrm{r}}(G) \otimes_{\min} A$. As far as I know, this problem is also in general intractable.

There is just one bright spot: although we will not prove it here, there is an analog for general crossed products by $\mathbb{Z}$ of the fact that the Cesaro means of the Fourier series of a continuous function always converge uniformly to the function. See Theorem 8.2.2 of Davidson's book.

The discussion above is meant to point out the difficulties in dealing with crossed products by infinite groups.

The limits of coefficients (continued)

Even if one understands completely what all the elements of $C^*_{\mathrm{r}}(G)$ are, and even if the action is trivial, understanding the elements of the reduced crossed product requires that one understand all the elements of the completed tensor product $C^*_{\mathrm{r}}(G) \otimes_{\min} A$. As far as I know, this problem is also in general intractable.

There is just one bright spot: although we will not prove it here, there is an analog for general crossed products by $\mathbb{Z}$ of the fact that the Cesaro means of the Fourier series of a continuous function always converge uniformly to the function. See Theorem 8.2.2 of Davidson's book.

The discussion above is meant to point out the difficulties in dealing with crossed products by infinite groups. Despite all this, for some problems, finite groups are harder.

The limits of coefficients (continued)

Even if one understands completely what all the elements of $C^*_{\mathrm{r}}(G)$ are, and even if the action is trivial, understanding the elements of the reduced crossed product requires that one understand all the elements of the completed tensor product $C^*_{\mathrm{r}}(G) \otimes_{\mathrm{min}} A$. As far as I know, this problem is also in general intractable.

There is just one bright spot: although we will not prove it here, there is an analog for general crossed products by $\mathbb{Z}$ of the fact that the Cesaro means of the Fourier series of a continuous function always converge uniformly to the function. See Theorem 8.2.2 of Davidson's book.

The discussion above is meant to point out the difficulties in dealing with crossed products by infinite groups. Despite all this, for some problems, finite groups are harder. Computing the K-theory of a crossed product by $\mathbb{Z}/2\mathbb{Z}$ is harder than computing the K-theory of a crossed product by any of $\mathbb{Z}$, $\mathbb{R}$, or even a (nonabelian) free group!