Seoul National University short course: An introduction to the structure of crossed product C*-algebras.

Lecture 2: Explicit computations

N. Christopher Phillips

University of Oregon

14 December 2009

Recall (from the formula for the regular representation when G is discrete):

Corollary

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let $\sigma \colon C^*_{\mathrm{r}}(G, A, \alpha) \to L(H) = L(L^2(G, H_0))$ be the associated regular representation. Let $a = \sum_{g \in G} a_g u_g \in C^*_{\mathrm{r}}(G, A, \alpha)$, with $a_g = 0$ for all but finitely many g. For $g \in G$, let $s_g \in L(H_0, H)$ be the isometry which sends $\eta \in H_0$ to the function $\xi \in L^2(G, H_0)$ given by

$$\xi(h) = \begin{cases} \eta & h = g \\ 0 & h \neq g. \end{cases}$$

Then

$$s_h^* \sigma(a) s_k = \pi_0\big(\alpha_h^{-1}(a_{hk^{-1}})\big)$$

for all $h, k \in G$.

Coefficients in reduced crossed products

Proposition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Then for each $g \in G$, there is a linear map $E_g \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ with $\|E_g\| \leq 1$ such that if $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, then $E_g(a) = a_g$.

Moreover, with s_g as above, we have $s_h^* \sigma(a) s_k = \pi_0\big(\alpha_h^{-1}(E_{hk^{-1}}(a))\big)$ for all $h, k \in G$.

Proof.

The first part is immediate from the inequality $\|a\|_\infty \leq \|a\|_{\mathrm{r}}$ from the last lecture.

The last statement follows by continuity from "picking off coordinates" in the regular representation. □

Coefficients in reduced crossed products: Discussion

Thus, for any $a \in C^*_{\mathrm{r}}(G, A, \alpha)$, and therefore also for $a \in C^*(G, A, \alpha)$, it makes sense to talk about its coefficients a_g. The first point is that if $C^*(G, A, \alpha) \neq C^*_{\mathrm{r}}(G, A, \alpha)$ (which can happen if G is not amenable, but not if G is amenable), and $a \in C^*(G, A, \alpha)$, the coefficients $(a_g)_{g \in G}$ do not even uniquely determine the element a. This is why we are only considering reduced crossed products here. (I do not know of any explicit examples.)

Coefficients in reduced crossed products: Properties

Here are the good things about coefficients.

Proposition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let the maps $E_g \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as in the previous proposition. Then:

1. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $E_g(a) = 0$ for all $g \in G$, then $a = 0$.
2. If $\pi_0 \colon A \to L(H_0)$ is a nondegenerate representation such that $\bigoplus_{g \in G} \pi_0 \circ \alpha_g$ is injective, then the regular representation σ of $C^*_{\mathrm{r}}(G, A, \alpha)$ associated to π_0 is injective.
3. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $E_1(a^* a) = 0$, then $a = 0$.

Proof of the properties of coefficients

(1): Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let the notation be as above. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$.

(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$. So $E_l(a) = 0$. This is true for all $l \in G$, so $a = 0$.

(3): As before, let $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$. Then $a^* a = \sum_{g, h \in G} u_g^* a_g^* a_h u_h$, so

$$E_1(a^* a) = \sum_{g \in G} u_g^* a_g^* a_g u_g = \sum_{g \in G} \alpha_g^{-1}\big(E_g(a)^* E_g(a)\big).$$

In particular, for each fixed g, we have $E_1(a^* a) \geq \alpha_g^{-1}\big(E_g(a)^* E_g(a)\big)$. By continuity, this inequality holds for all $a \in C^*_{\mathrm{r}}(G, A, \alpha)$. Thus, if $E_1(a^* a) = 0$, then $E_g(a)^* E_g(a) = 0$ for all g, so $a = 0$ by Part (1). This completes the proof.

Injective representations of A always give injective regular representations of the reduced crossed product

It is true for general locally compact groups, not just discrete groups, that the regular representation of $C^*_{\mathrm{r}}(G, A, \alpha)$ associated to an injective representation of A is injective. See Theorem 7.7.5 of Pedersen's book.

The conditional expectation

The map E_1 used in Part (3) of the previous proposition is an example of what is called a *conditional expectation* (from $C^*_{\mathrm{r}}(G, A, \alpha)$ to A) that is, it has the properties given in the following exercise. Part (3) of the previous proposition asserts that this conditional expectation is faithful.

Exercise

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $E = E_1 \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as above. Prove that E has the following properties:

1. $E(E(b)) = E(b)$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
2. If $b \geq 0$ then $E(b) \geq 0$.
3. $\|E(b)\| \leq \|b\|$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
4. If $a \in A$ and $b \in C^*_{\mathrm{r}}(G, A, \alpha)$, then $E(ab) = aE(b)$ and $E(ba) = E(b)a$.

The limits of coefficients

Unfortunately, in general $\sum_{g \in G} a_g u_g$ does not converge in $C^*_{\mathrm{r}}(G, A, \alpha)$, and it is very difficult to tell exactly which families of coefficients correspond to elements of $C^*_{\mathrm{r}}(G, A, \alpha)$. In fact, the situation is intractable even for the case of the trivial action of $\mathbb{Z}$ on $\mathbb{C}$. In this case, $l^1(\mathbb{Z}, A, \alpha) = l^1(\mathbb{Z})$. The crossed product is the group C*-algebra $C^*(\mathbb{Z})$, which can be identified with $C(S^1)$. The map $l^1(\mathbb{Z}) \to C(S^1)$ is given by Fourier series: the sequence $a = (a_n)_{n \in \mathbb{Z}_{>0}}$ goes to the function $\zeta \mapsto \sum_{n \in \mathbb{Z}} a_n \zeta^n$. (This looks more familiar when expressed in terms of 2π-periodic functions on $\mathbb{R}$: it is $t \mapsto \sum_{n \in \mathbb{Z}} a_n e^{int}$.) Failure of convergence of $\sum_{n \in \mathbb{Z}} a_n u_n$ corresponds to the fact that the Fourier series of a continuous function need not converge uniformly. Identifying the coefficient sequences which correspond to elements of the crossed product corresponds to giving a criterion for exactly when a sequence $(a_n)_{n \in \mathbb{Z}_{>0}}$ of complex numbers is the sequence of Fourier coefficients of some continuous function on S^1, a problem for which I know of no satisfactory solution.

The limits of coefficients (continued)

Let's pursue this a little farther. The regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$ gives an injective map $\lambda \colon C^*(\mathbb{Z}) \to L(l^2(\mathbb{Z}))$. Let $\delta_n \in l^2(\mathbb{Z})$ be the function

$$\delta_n(k) = \begin{cases} 1 & k = n \\ 0 & k \neq n. \end{cases}$$

Then the Fourier coefficient a_n is recovered as $a_n = \langle \lambda(a)\delta_0, \delta_n \rangle$. That is, $\lambda(a)\delta_0 \in l^2(\mathbb{Z})$ is given by $\lambda(a)\delta_0 = \sum_{n \in \mathbb{Z}} a_n \delta_n$. Thus, the sequence of Fourier coefficients of a continuous function is always in $l^2(\mathbb{Z})$. (Of course, we already know this, but the calculation here can be applied to more general crossed products.) Unfortunately, this fact is essentially useless for the study of the group C*-algebra. Not only is the Fourier series of a continuous function always in $l^2(\mathbb{Z})$, but the Fourier series of a function in $L^\infty(S^1)$, which is the group von Neumann algebra of $\mathbb{Z}$, is also always in $l^2(\mathbb{Z})$. One will get essentially no useful information from a criterion which can't even exclude any elements of $L^\infty(S^1)$.

The limits of coefficients (continued)

Even if one understands completely what all the elements of $C^*_{\mathrm{r}}(G)$ are, and even if the action is trivial, understanding the elements of the reduced crossed product requires that one understand all the elements of the completed tensor product $C^*_{\mathrm{r}}(G) \otimes_{\min} A$. As far as I know, this problem is also in general intractable.

There is just one bright spot: although we will not prove it here, there is an analog for general crossed products by $\mathbb{Z}$ of the fact that the Cesaro means of the Fourier series of a continuous function always converge uniformly to the function. See Theorem 8.2.2 of Davidson's book.

The discussion above is meant to point out the difficulties in dealing with crossed products by infinite groups. Despite all this, for some problems, finite groups are harder. Computing the K-theory of a crossed product by $\mathbb{Z}/2\mathbb{Z}$ is harder than computing the K-theory of a crossed product by any of $\mathbb{Z}$, $\mathbb{R}$, or even a (nonabelian) free group!

Preliminaries for computing crossed products

We will shortly do some explicit computations of examples. First, though, we give some useful preliminaries:

- Equivariant maps and functoriality.
- Crossed products of exact sequences.
- Crossed products and direct limits.
- Notation for matrix units.

In many of the cases we consider, the ideal structure of the crossed product can be derived from the Gootman-Rosenberg theorem. (See below for a little more about this theorem.) In some cases, one can then use this information to determine the entire structure of the crossed product.

Equivariant homomorphisms

Let G be a locally compact group. A C*-algebra A equipped with an action $G \to \mathrm{Aut}(A)$ will be called a *G-algebra*. We sometimes refer to (G, A, α) as a G-algebra.

Definition

If (G, A, α) and (G, B, β) are G-algebras, then a homomorphism $\varphi\colon A \to B$ is said to be *equivariant* (or G-equivariant if the group must be specified) if for every $g \in G$, we have $\varphi \circ \alpha_g = \beta_g \circ \varphi$.

For a fixed locally compact group G, the G-algebras and equivariant homomorphisms form a category.

The crossed product construction is functorial for equivariant homomorphisms

Theorem

Let G be a locally compact group. If (G, A, α) and (G, B, β) are G-algebras and $\varphi\colon A \to B$ is an equivariant homomorphism, then there is a homomorphism $\psi\colon C_{\mathrm{c}}(G, A, \alpha) \to C_{\mathrm{c}}(G, B, \beta)$ given by the formula $\psi(a)(g) = \varphi(a(g))$ for $a \in C_{\mathrm{c}}(G, A, \alpha)$ and $g \in G$, and this homomorphism extends by continuity to a homomorphism $L^1(G, A, \alpha) \to L^1(G, B, \beta)$, and then to homomorphisms

$$C^*(G, A, \alpha) \to C^*(G, B, \beta) \quad \text{and} \quad C^*_{\mathrm{r}}(G, A, \alpha) \to C^*_{\mathrm{r}}(G, B, \beta).$$

This construction makes the crossed product and reduced crossed product constructions functors from the category of G-algebras to the category of C*-algebras.

This is straightforward. See the notes for details.

Full crossed products preserve exact sequences

Theorem

Let $0 \to J \to A \to B \to 0$ be an exact sequence of G-algebras, with actions γ on J, α on A, and β on B. Then the sequence

$$0 \longrightarrow C^*(G, J, \gamma) \longrightarrow C^*(G, A, \alpha) \longrightarrow C^*(G, B, \beta) \longrightarrow 0$$

is exact.

Proofs can be found in the three places listed in the notes.

The analog for reduced crossed products is in general false.

Full crossed products preserve direct limits

Theorem

Let $\left(\left(G, A_i, \alpha^{(i)}\right)_{i \in I}, (\varphi_{j,i})_{i \leq j} \right)$ be a direct system of G-algebras. Let $A = \varinjlim A_i$, with action $\alpha\colon G \to \mathrm{Aut}(A)$ given by $\alpha_g = \varinjlim \alpha_g^{(i)}$. Let

$$\psi_{j,i}\colon C^*\left(G, A_i, \alpha^{(i)}\right) \to C^*\left(G, A_j, \alpha^{(j)}\right)$$

be the map obtained from $\varphi_{j,i}$. Using these maps in the direct system of crossed products, there is a natural isomorphism $C^*(G, A, \alpha) \cong \varinjlim C^*\left(G, A_i, \alpha^{(i)}\right)$.

The proof is done by combining the universal properties of direct limits and crossed products. See the notes.

Notation for matrix units

For any index set S, let $\delta_s \in l^2(S)$ be the standard basis vector, determined by
$$\delta_s(t) = \begin{cases} 1 & t = s \\ 0 & t \neq s. \end{cases}$$
For $j, k \in S$, we let the "matrix unit" $e_{j,k}$ be the rank one operator on $l^2(S)$ given by $e_{j,k}\xi = \langle \xi, \delta_k \rangle \delta_j$. This gives the product formula $e_{j,k}e_{l,m} = \delta_{k,l}e_{j,m}$. Conventional matrix units for M_n are obtained by taking $S = \{1, 2, \ldots, n\}$, but we will sometimes want to take S to be a discrete (even finite) group. For $S = \{1, 2\}$, with the obvious choice of matrix representation, we get

$$e_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad e_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Example: The trivial action

Example

If G acts trivially on the C*-algebra A, then
$$C^*(G, A) \cong C^*(G) \otimes_{\max} A \quad \text{and} \quad C^*_{\mathrm{r}}(G, A) \cong C^*_{\mathrm{r}}(G) \otimes_{\min} A.$$

We describe how to see this when G is discrete and A is unital. Then $C^*(G, A)$ is the universal unital C*-algebra generated by a unital copy of A and a commuting unitary representation of G in the algebra. Since $C^*(G)$ is the universal unital C*-algebra generated by a unitary representation of G in the algebra, this is exactly the universal property of the maximal tensor product.

Example: The trivial action (continued)

For the reduced crossed product, the point is that a regular covariant representation of (G, A) has the form $(\lambda \otimes 1_{H_0}, 1_{L^2(G)} \otimes \pi_0)$ for $\pi_0 \colon A \to L(H_0)$ an arbitrary nondegenerate representation and with $\lambda \colon G \to U(L^2(G))$ being the left regular representation. As we saw above, it suffices to take π_0 to be a single injective representation. Now we are looking at $C^*_{\mathrm{r}}(G)$ on one Hilbert space and A on another, and taking the tensor product of the Hilbert spaces. This is exactly how one gets the minimal tensor product of two C*-algebras.

Note how full and reduced crossed products parallel maximal and minimal tensor products.

Example: Inner actions

Example

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an inner action of a discrete group G on a unital C*-algebra A. Thus, there is a homomorphism $g \mapsto z_g$ from G to $U(A)$ such that $\alpha_g(a) = z_g a z_g^*$ for all $g \in G$ and $a \in A$. Then $C^*(G, A, \alpha) \cong C^*(G) \otimes_{\max} A$. (This is true even if G is not discrete.)

One shows that the crossed product is the same as for the trivial action. Let $\iota \colon G \to \mathrm{Aut}(A)$ be the trivial action of G on A. As usual, for $g \in G$ let $u_g \in C_{\mathrm{c}}(G, A, \alpha)$ be the standard unitary, but let $v_g \in C_{\mathrm{c}}(G, A, \iota)$ be the standard unitary in the crossed product by the trivial action. Define $\varphi_0 \colon C_{\mathrm{c}}(G, A, \alpha) \to C_{\mathrm{c}}(G, A, \iota)$ by $\varphi_0(a u_g) = a z_g v_g$ for $a \in A$ and $g \in G$, and extend linearly. This map is obviously bijective (the inverse sends $a v_g$ to $a z_g^* u_g$) and isometric for $\| \cdot \|_1$.

Example: Inner actions (continued)

For multiplicativity, it suffices to check the following, for $a, b \in A$ and $g, h \in H$, using the fact that v_g commutes with all elements of A:

$$\varphi_0(au_g)\varphi_0(bu_h) = az_g v_g b z_h v_h = az_g b z_g^* z_{gh} v_g v_h$$
$$= a\alpha_g(b) z_{gh} v_{gh} = \varphi_0\big(a\alpha_g(b)u_{gh}\big) = \varphi_0\big((au_g)(bu_h)\big).$$

For preservation of adjoints:

$$\varphi_0(au_g)^* = (az_g v_g)^* = v_g^* z_g^* a^* = (z_g^* a^* z_g) z_g^* v_g^*$$
$$= \alpha_{g^{-1}}(a^*) z_{g^{-1}} v_{g^{-1}} = \varphi_0\big(\alpha_{g^{-1}}(a^*) u_{g^{-1}}\big) = \varphi_0\big((au_g)^*\big).$$

So φ_0 is an isometric isomorphism of *-algebras, and therefore extends to an isomorphism of the universal C*-algebras.

Example: G acting on itself by translation

Example

If G is discrete and acts on itself by translation, then the crossed product is $K(l^2(G))$. (This is actually true for general G.)

Let $\alpha \colon G \to \mathrm{Aut}(C_0(G))$ denote the action. For $g \in G$, let u_g be the standard unitary, and let $\delta_g \in C_0(G)$ be the function $\chi_{\{g\}}$. Thus $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$. Also, $\mathrm{span}\big(\{\delta_g \colon g \in G\}\big)$ is dense in $C_0(G)$. For $g, h \in G$, we have $v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, \, C_0(G), \, \alpha)$. Moreover,

$$v_{g_1,h_1} v_{g_2,h_2} = \delta_{g_1} u_{g_1 h_1^{-1}} \delta_{g_2} u_{g_2 h_2^{-1}}$$
$$= \delta_{g_1} \alpha_{g_1 h_1^{-1}}(\delta_{g_2}) u_{g_1 h_1^{-1}} u_{g_2 h_2^{-1}} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}.$$

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is v_{g_1,h_2}. Similarly, $v_{g,h}^* = v_{h,g}$. That is, the elements $v_{g,h}$ satisfy the relations for a system of matrix units indexed by G. Also, $\mathrm{span}\big(\{v_{g,h} \colon g, h \in G\}\big)$ is dense in $l^1(G, \, C_0(G), \, \alpha)$, and hence in $C^*(G, \, C_0(G), \, \alpha)$.

For any finite set $F \subset G$, we thus get a homomorphism

$$\psi_F \colon L(l^2(F)) \to C_{\mathrm{c}}(G, \, C_0(G), \, \alpha)$$

sending the matrix unit $e_{g,h} \in L(l^2(F))$ to $v_{g,h}$. If G is finite, we have a surjective homomorphism $L(l^2(G)) \to C^*(G, \, C_0(G), \, \alpha)$, necessarily injective since $L(l^2(G))$ is simple.

In general, one assembles the maps ψ_F to get an isomorphism $K(l^2(G)) \to C^*(G, \, C_0(G), \, \alpha)$. For details, see the notes.

Since the full crossed product is simple, the map to the reduced crossed product is an isomorphism.

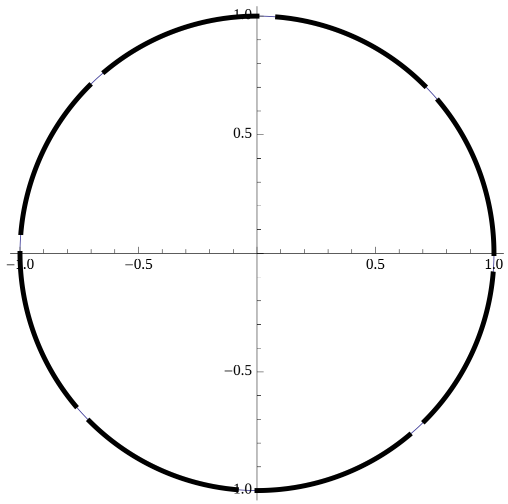
If G acts on $G \times X$ by translation in the first factor and trivially in the second factor, we get the crossed product $C(X, \, K(l^2(G)))$. (In fact, this is true for an arbitrary action of G on X. See the notes.)

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$

Fix $n \in \mathbb{Z}_{>0}$, and let $G = \mathbb{Z}/n\mathbb{Z}$ act on S^1 via the rotation by $2\pi/n$, that is, with generator the homeomorphism $h(\zeta) = e^{2\pi i/n}\zeta$ for $\zeta \in S^1$.

We describe what to expect. Every point in S^1 has a closed invariant neighborhood which is equivariantly homeomorphic to $G \times I$ for some closed interval $I \subset \mathbb{R}$, with the translation action on G and the trivial action on I. This leads to quotients of $C^*(G, S^1, h)$ isomorphic to $M_n \otimes C(I)$. Since S^1 itself is not such a product, one does not immediately get an isomorphism $C^*(G, S^1, h) \cong M_n \otimes C(Y)$ for any Y. Instead, one gets the section algebra of a locally trivial bundle over Y with fiber M_n. However, the appropriate Y is the orbit space $S^1/G \cong S^1$, and all locally trivial bundles over S^1 with fiber M_n are in fact trivial. Thus, one gets $C^*(G, S^1, h) \cong C(S^1, M_n)$ after all.

Example: $n = 8$



The heavy sections are the 8 translates of the closed arc Y from 1 to $\exp(2\pi i/9)$. They give a quotient of the crossed product isomorphic to $C(Y, M_8)$.

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

Here are the details. Let $\alpha \in \mathrm{Aut}(C(S^1))$ be the order n automorphism $\alpha(f) = f \circ h^{-1}$. Thus, $\alpha(f)(\zeta) = f(e^{-2\pi i/n}\zeta)$ for $\zeta \in S^1$. Let $s \in M_n$ be the shift unitary

$$s = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

The key computation is

$$s \, \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \, s^* = \mathrm{diag}(\lambda_n, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}).$$

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

Set

$$B = \{ f \in C([0,1], M_n) \colon f(0) = s f(1) s^* \}.$$

Define $\varphi_0 \colon C(S^1) \to B$ by sending $f \in C(S^1)$ to the continuously varying diagonal matrix

$$\varphi_0(f)(t) = \mathrm{diag}\big(f\big(e^{2\pi i t/n}\big), \, f\big(e^{2\pi i (t+1)/n}\big), \, \ldots, \, f\big(e^{2\pi i (t+n-1)/n}\big) \big).$$

(For fixed t, the diagonal entries are obtained by evaluating f at the points in the orbit of $e^{2\pi i t/n}$.) The diagonal entries of $f(0)$ are gotten from those of $f(1)$ by a forwards cyclic shift, so $\varphi_0(f)$ really is in B. For the same reason, we get

$$\begin{aligned} \varphi_0(\alpha(f))(t) &= \mathrm{diag}\big(f\big(e^{2\pi i (t-1)/n}\big), \, f\big(e^{2\pi i t/n}\big), \, \ldots, \, f\big(e^{2\pi i (t+n-2)/n}\big) \big) \\ &= s \varphi_0(f)(t) s^*. \end{aligned}$$

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

Now let $v \in C([0,1], M_n)$ be the constant function with value s. Note that $v \in B$. The calculation just done implies that

$$\varphi_0(\alpha^k(f)) = v^k \varphi_0(f) v^{-k}$$

for $0 \leq k \leq n-1$. Also clearly $v^n = 1$. We write the group elements as $0, \, 1, \, \ldots, \, n-1$, by abuse of notation treating them as integers when convenient. The universal property of the crossed product therefore implies that there is a homomorphism $\varphi \colon C^*(G, S^1, h) \to B$ such that $\varphi|_{C(S^1)} = \varphi_0$ and $\varphi(u_k) = v^k$ for $0 \leq k \leq n-1$.

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

We prove directly that φ is bijective. We can rewrite φ as the map $C(\mathbb{Z}/n\mathbb{Z} \times S^1) \to B$ given by

$$\varphi(f) = \sum_{k=0}^{n-1} \varphi_0(f(k, -))v^k.$$

Injectivity now reduces to the easily verified fact that if $a_0, a_1, \ldots, a_{n-1} \in M_n$ are diagonal matrices, and $\sum_{k=0}^{n-1} a_k v^k = 0$, then $a_0 = a_1 = \cdots = a_{n-1} = 0$.

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

For surjectivity, let $a \in B$, and write

$$a(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) & \cdots & a_{1,n}(t) \\ a_{2,1}(t) & a_{2,2}(t) & \cdots & a_{2,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(t) & a_{n,2}(t) & \cdots & a_{n,n}(t) \end{pmatrix}$$

with $a_{j,k} \in C([0,1])$ for $1 \leq j, k \leq n$. The condition $a \in B$ implies that, taking the indices mod n in $\{1, 2, \ldots, n\}$, we have $a_{j,k}(1) = a_{j+1,\,k+1}(0)$ for all j and k. So we get a well defined element of $C(\mathbb{Z}/n\mathbb{Z} \times S^1)$ via

$$f\left(l,\ e^{2\pi i(t+j)/n}\right) = a_{l+j,\,j}(t)$$

for $t \in [0,1]$, $1 \leq j \leq n$, and $0 \leq l \leq n-1$, with $l + j$ taken mod n in $\{1, 2, \ldots, n\}$. One checks that $\varphi(f) = a$.

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

It remains to prove that $B \cong C(S^1, M_n)$. Since $U(M_n)$ is connected, there is a unitary path $t \mapsto s_t$, for $t \in [0,1]$, such that $s_0 = 1$ and $s_1 = s$. Define $\psi \colon C(S^1, M_n) \to B$ by $\psi(f)(t) = s_t^* f(e^{2\pi i t}) s_t$. For $f \in C(S^1, M_n)$, we have

$$\psi(f)(1) = s^* f(1) s = s^* f(0) s = s^* \psi(f)(0) s,$$

so $\psi(f)$ really is in B. It is easily checked that ψ is bijective.

Example: $x \mapsto -x$ on S^n

Example

Let $X = S^n = \{x \in \mathbb{R}^{n+1} \colon \|x\|_2 = 1\}$, and let $\mathbb{Z}/2\mathbb{Z}$ act by sending the nontrivial group element to the order 2 homeomorphism $x \mapsto -x$.

The "local structure" of the crossed product $C^*(\mathbb{Z}/2\mathbb{Z}, X)$ is the same as in the previous example. However, for $n \geq 2$ the resulting bundle is no longer trivial. The crossed product is isomorphic to the section algebra of a locally trivial but nontrivial bundle over the real projective space $\mathbb{R}P^n = S^n/(\mathbb{Z}/2\mathbb{Z})$ with fiber M_2. See Proposition 4.15 of Williams' book.

Example: Complex conjugation on S^1

Example

Take $X = S^1 = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\}$, and let $\mathbb{Z}/2\mathbb{Z}$ act by sending the nontrivial group element to the order 2 homeomorphism $\zeta \mapsto \overline{\zeta}$. Let $\alpha \in \mathrm{Aut}(C(S^1))$ be the corresponding automorphism.

We compute the crossed product, but we first describe what to expect. We should expect that the points 1 and -1 contribute quotients isomorphic to $\mathbb{C} \oplus \mathbb{C}$, and that for $\zeta \neq \pm 1$, the pair of points $(\zeta, \overline{\zeta})$ contributes a quotient isomorphic to M_2. We will in fact show that $C^*(\mathbb{Z}/2\mathbb{Z}, X)$ is isomorphic to the C*-algebra

$$B = \{f \in C([-1, 1], M_2) \colon f(1) \text{ and } f(-1) \text{ are diagonal matrices}\}.$$

Example: Complex conjugation on S^1 (continued)

Here are the details. First, let $C_0 \subset M_2$ be the subalgebra consisting of all matrices of the form $\left(\begin{smallmatrix} \lambda & \mu \\ \mu & \lambda \end{smallmatrix}\right)$ with $\lambda, \mu \in \mathbb{C}$. Then define

$$C = \{f \colon [-1, 1] \to M_2 \colon f \text{ is continuous and } f(1), f(-1) \in C_0\}.$$

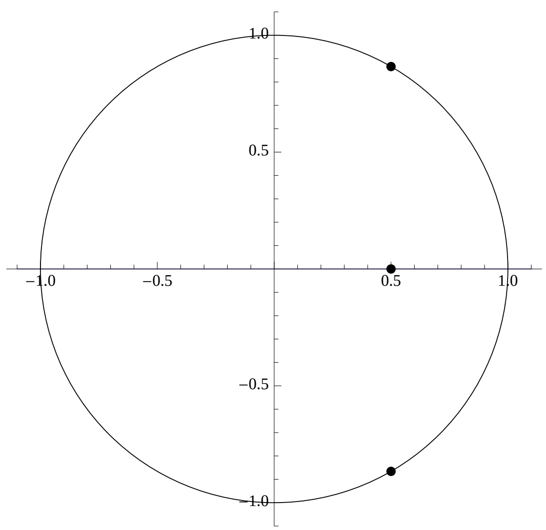
Let $v \in C$ be the constant function $v(t) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for all $t \in [-1, 1]$. Define $\varphi_0 \colon C(S^1) \to C$ by

$$\varphi_0(f)(t) = \begin{pmatrix} f\left(t + i\sqrt{1 - t^2}\right) & 0 \\ 0 & f\left(t - i\sqrt{1 - t^2}\right) \end{pmatrix}$$

for $f \in C(S^1)$ and $t \in [-1, 1]$. One checks that the conditions at ± 1 for membership in C are satisfied. Moreover, $v^2 = 1$ and $v\varphi_0(f)v^* = \varphi_0(\alpha(f))$ for $f \in C(S^1)$.

Picture for the definition of $\varphi_0(f)$)

$$\varphi_0(f)(t) = \begin{pmatrix} f\left(t + i\sqrt{1 - t^2}\right) & 0 \\ 0 & f\left(t - i\sqrt{1 - t^2}\right) \end{pmatrix}.$$



Example: Complex conjugation on S^1 (continued)

Therefore there is a homomorphism $\varphi \colon C^*(\mathbb{Z}/2\mathbb{Z}, X) \to C$ such that $\varphi|_{C(S^1)} = \varphi_0$ and φ sends the standard unitary u in $C^*(\mathbb{Z}/2\mathbb{Z}, X)$ to v. It is given by the formula

$$\varphi(f_0 + f_1 u)(t) = \begin{pmatrix} f_0\left(t + i\sqrt{1 - t^2}\right) & f_1\left(t + i\sqrt{1 - t^2}\right) \\ f_1\left(t - i\sqrt{1 - t^2}\right) & f_0\left(t - i\sqrt{1 - t^2}\right) \end{pmatrix}$$

for $f_1, f_2 \in C(S^1)$ and $t \in [-1, 1]$.

We claim that φ is an isomorphism. Since

$$C^*(\mathbb{Z}/2\mathbb{Z}, X) = \{f_0 + f_1 u \colon f_1, f_2 \in C(S^1)\},$$

it is easy to check injectivity.

Example: Complex conjugation on S^1 (continued)

For surjectivity, let

$$a(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) \\ a_{2,1}(t) & a_{2,2}(t) \end{pmatrix}$$

define an element $a \in C$. Then

$$a_{1,1}(-1) = a_{2,2}(-1) \quad \text{and} \quad a_{2,1}(-1) = a_{1,2}(-1), \tag{1}$$

and

$$a_{1,1}(1) = a_{2,2}(1) \quad \text{and} \quad a_{2,1}(1) = a_{1,2}(1). \tag{2}$$

Now set

$$f_0(\zeta) = \begin{cases} a_{1,1}(\mathrm{Re}(\zeta)) & \mathrm{Im}(\zeta) \geq 0 \\ a_{2,2}(\mathrm{Re}(\zeta)) & \mathrm{Im}(\zeta) \leq 0 \end{cases}$$

and

$$f_1(\zeta) = \begin{cases} a_{1,2}(\mathrm{Re}(\zeta)) & \mathrm{Im}(\zeta) \geq 0 \\ a_{2,1}(\mathrm{Re}(\zeta)) & \mathrm{Im}(\zeta) \leq 0 \end{cases}$$

for $\zeta \in S^1$. The relations (1) and (2) ensure that f_0 and f_1 are well defined at ± 1, and are continuous. One easily checks that $\varphi(f_0 + f_1 u) = a$. This proves surjectivity.

Example: Complex conjugation on S^1 (continued)

The algebra C is not quite what was promised. Set

$$w = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

which is a unitary in M_2. Then the isomorphism $\psi \colon C^*(\mathbb{Z}/2\mathbb{Z}, X) \to B$ is given by $\psi(a)(t) = w\varphi(a)(t)w^*$. (Check this!)

In this example, one choice of matrix units in M_2 was convenient for the free orbits, while another choice was convenient for the fixed points. It seemed better to compute everything in terms of the choice convenient for the free orbits, and convert afterwards.

Example: $x \mapsto -x$ on $[-1, 1]$

Exercise

Let $\mathbb{Z}/2\mathbb{Z}$ act on $[-1, 1]$ via $x \mapsto -x$. Compute the crossed product.

Example: $(x_1, x_2, \ldots, x_n, x_{n+1}) \mapsto (x_1, x_2, \ldots, x_n, -x_{n+1})$ on S^n

Exercise

Let $\mathbb{Z}/2\mathbb{Z}$ act on

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \colon x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

via $(x_1, x_2, \ldots, x_n, x_{n+1}) \mapsto (x_1, x_2, \ldots, x_n, -x_{n+1})$. Compute the crossed product.

Example: $\mathbb{Z}$ acting on $\mathbb{Z}/n\mathbb{Z}$ by translation

Example

Let $X = \mathbb{Z}/n\mathbb{Z}$, and let $\mathbb{Z}$ act on X by translation. We show that $C^*(\mathbb{Z}, X) \cong M_n \otimes C(S^1)$.

This is a special case of G acting on G/H by translation. In the general case, it turns out that $C^*(G,\ G/H) \cong K(L^2(G/H)) \otimes C^*(H)$. Note that there is no twisting.

We will be sketchy. See the notes for details.

Identify $\mathbb{Z}/n\mathbb{Z}$ with $\{1, 2, \ldots, n\}$. (We start at 1 instead of 0 to be consistent with common matrix unit notation.) Let $\alpha \in \mathrm{Aut}(C(\mathbb{Z}/n\mathbb{Z}))$ be $\alpha(f)(k) = f(k-1)$, with indices taken mod n in $\{1, 2, \ldots, n\}$. (Equivalently, $\alpha(\chi_{\{k\}}) = \chi_{\{k+1\}}$, with $k+1$ taken to be 1 when $k = n$.)

Example: $\mathbb{Z}$ acting on $\mathbb{Z}/n\mathbb{Z}$ by translation (continued)

In $C(S^1)$ let z be the function $z(\zeta) = \zeta$ for all ζ. In $M_n(C(S^1)) \cong M_n \otimes C(S^1)$, abbreviate $e_{j,k} \otimes 1$ to $e_{j,k}$, and let v be the unitary

$$v = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & z \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

(This unitary differs from the unitary s used before only in that here the upper right corner entry is z instead of 1.)

Example: $\mathbb{Z}$ acting on $\mathbb{Z}/n\mathbb{Z}$ by translation (continued)

Define $\varphi_0 \colon C(\mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ by $\varphi_0(\chi_{\{k\}}) = e_{k,k}$. Then one checks that $v\varphi_0(f)v^* = \varphi_0(\alpha(f))$ for all $f \in C(\mathbb{Z}/n\mathbb{Z})$. Therefore there is a homomorphism $\varphi \colon C^*(\mathbb{Z},\ \mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ such that $\varphi|_{C(\mathbb{Z}/n\mathbb{Z})} = \varphi_0$ and $\varphi(u) = v$. We claim that φ is an isomorphism.

We use the following description of $M_n \otimes C(S^1)$: it is the universal unital C*-algebra generated by a system $(e_{j,k})_{1 \le j,k \le n}$ of matrix units such that $\sum_{j=1}^{n} e_{j,j} = 1$ and a central unitary y. The $e_{j,k}$ are the matrix units we have already used, and the central unitary is $1 \otimes z$. (Proof: Exercise.)

Example: $\mathbb{Z}$ acting on $\mathbb{Z}/n\mathbb{Z}$ by translation (continued)

To prove that φ is surjective, it suffices to prove that its image contains the generators above. This is easy; see the notes.

To prove injectivity, it suffices to prove that whenever A is a unital C*-algebra, $\psi_0 \colon C(\mathbb{Z}/n\mathbb{Z}) \to A$ is a unital homomorphism, and $w \in A$ is a unitary such that $w\psi_0(f)w^* = \psi_0(\alpha(f))$ for all $f \in C(\mathbb{Z}/n\mathbb{Z})$, then there is a homomorphism $\gamma \colon M_n \otimes C(S^1) \to A$ such that $\gamma \circ \varphi_0 = \psi_0$ and $\gamma(v) = w$. That is, we are showing that $M_n \otimes C(S^1)$ satisfies the universal property of the crossed product. If φ were not injective, taking ψ_0 and w to come from $\mathrm{id}_{C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})}$ would yield a contradiction. It suffices to define γ on the generators above.

The rest of the details are omitted; see the notes. The main point is to use the description of $M_n \otimes C(S^1)$ as the universal C*-algebra on the generators and relations above.

Where ideals in crossed products come from

We have implicitly seen two sources of ideals in a reduced crossed product $C^*_{\mathrm{r}}(G, A, \alpha)$: invariant ideals in A, and group elements which act trivially on A. There is a theorem due to Gootman and Rosenberg which gives a description of the primitive ideals of any crossed product $C^*(G, A)$ with G amenable, and which, very roughly, says that they all come from some combination of these two sources. (One does not even need to restrict to discrete groups.) There is a bit more discussion, with references, in the notes.

Example: A product type action

Example

We consider the action of $\mathbb{Z}/2\mathbb{Z}$ on the 2^∞ UHF algebra A generated by $\bigotimes_{n=1}^{\infty} \mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Again, we will be sketchy. See the notes for more details.

Write $A = \varinjlim M_{2^n}$, with maps $\varphi_n \colon M_{2^n} \to M_{2^{n+1}}$ given by $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$. Define unitaries $z_n \in M_{2^n}$ inductively by $z_0 = 1$ and $z_{n+1} = \left(\begin{smallmatrix} z_n & 0 \\ 0 & -z_n \end{smallmatrix}\right)$. In tensor product notation, and with an appropriate choice of isomorphism $M_{2^n} \otimes M_2 \to M_{2^{n+1}}$, these are

$$\varphi_n(a) = a \otimes 1_{M_2} \quad \text{and} \quad z_{n+1} = z_n \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

Let

$$\overline{\varphi}_n \colon C^*(\mathbb{Z}/2\mathbb{Z}, \, M_{2^n}, \, \mathrm{Ad}(z_n)) \to C^*(\mathbb{Z}/2\mathbb{Z}, \, M_{2^{n+1}}, \, \mathrm{Ad}(z_{n+1}))$$

be the corresponding map on the crossed products.

Example: A product type action (continued)

From what we did with inner actions, we get isomorphisms

$$\sigma_n \colon C^*(\mathbb{Z}/2\mathbb{Z}, \, M_{2^n}, \, \mathrm{Ad}(z_n)) \to M_{2^n} \oplus M_{2^n}$$

given by $a + bu_n \mapsto (a + bz_n, \, a - bz_n)$. We now need a map

$$\psi_n \colon M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$$

which makes the following diagram commute:

$$\begin{array}{ccc} C^*(\mathbb{Z}/2\mathbb{Z}, \, M_{2^n}, \, \mathrm{Ad}(z_n)) & \xrightarrow{\sigma_n} & M_{2^n} \oplus M_{2^n} \\ {\scriptstyle \overline{\varphi}_n}\big\downarrow & & \big\downarrow{\scriptstyle \psi_n} \\ C^*(\mathbb{Z}/2\mathbb{Z}, \, M_{2^{n+1}}, \, \mathrm{Ad}(z_{n+1})) & \xrightarrow{\sigma_{n+1}} & M_{2^{n+1}} \oplus M_{2^{n+1}} \end{array}$$

Example: A product type action (continued)

We need a map

$$\psi_n \colon M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$$

which makes the diagram on the previous slide commute. That is, ψ_n sends

$$\sigma_n(a + bu_n) = (a + bz_n, \, a - bz_n)$$

to

$$\sigma_{n+1}(\varphi_n(a) + \varphi_n(b)u_{n+1}).$$

We have enough information to write an explicit formula for this expression (see the notes), and it turns out that we can take

$$\psi_n(b, c) = \left(\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix} \right).$$

Example: A product type action (continued)

Those familiar with Bratteli diagrams will now be able to write down the Bratteli diagram for the crossed product. There is a direct computation in the notes. It turns out that the crossed product is isomorphic to A.

Remarks on the product type example

The fact that we got the same algebra back is somewhat special, but the general principle of the computation is much more generally applicable.

Since the crossed product is simple, the action is not inner.

The theorem of Gootman and Rosenberg described above gives no information here.

Exercises 4.23 and 4.24 in the notes combine direct limit methods with computations of the sort done above.

Example: The irrational rotation algebras

Let $\theta \in \mathbb{R}$. Recall that the rotation algebra A_θ is the universal C*-algebra generated by unitaries u and v satisfying $vu = e^{2\pi i \theta} uv$.

Let $h_\theta \colon S^1 \to S^1$ be the homeomorphism $h_\theta(\zeta) = e^{2\pi i \theta}\zeta$. We claim that there is an isomorphism $\varphi \colon A_\theta \to C^*(\mathbb{Z}, S^1, h_\theta)$ which sends u to the standard unitary u_1 in the crossed product, and sends v to the function $z \in C(S^1)$ defined by $z(\zeta) = \zeta$ for all $\zeta \in S^1$.

Example: The irrational rotation algebras (continued)

The proof of the claim is by comparison of universal properties. First, one checks that $zu_1 = e^{2\pi i \theta} u_1 z$, so at least there is a homomorphism φ as claimed.

Next, define $\psi_0 \colon C(S^1) \to A_\theta$ by $\psi_0(f) = f(v)$ (continuous functional calculus) for $f \in C(S^1)$. For $n \in \mathbb{Z}$, we have

$$u\psi_0(z^n)u^* = (uvu^*)^n = e^{2\pi i n\theta} v^n = \psi_0\big(e^{2\pi i n\theta} z^n\big) = \psi_0\big(z^n \circ h_\theta^{-1}\big).$$

Since the functions z^n span a dense subspace of $C(S^1)$, it follows that $u\psi_0(f)u^* = \psi_0\big(f \circ h_\theta^{-1}\big)$ for all $f \in C(S^1)$. By the universal property of the crossed product, there is a homomorphism $\psi \colon C^*(\mathbb{Z}, S^1, h_\theta) \to A_\theta$ such that $\psi|_{C(S^1)} = \psi_0$ and $\psi(u_1) = u$.

We have $(\psi \circ \varphi)(u) = u$ and $(\psi \circ \varphi)(v) = v$. Since u and v generate A_θ, we conclude that $\psi \circ \varphi = \mathrm{id}_{A_\theta}$. Similarly, one proves $\varphi \circ \psi = \mathrm{id}_{C^*(\mathbb{Z}, S^1, h_\theta)}$.

We will see below that for $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the algebra $C^*(\mathbb{Z}, S^1, h_\theta)$ is simple.