# Seoul National University short course: An introduction to the structure of crossed product C*-algebras.

## Lecture 3: Crossed products by minimal homeomorphisms

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# Minimal actions

We will prove that reduced crossed products by free minimal actions of countable discrete groups on compact metric spaces are simple.

**Definition**

Let a locally compact group $G$ act continuously on a locally compact space $X$. The action is called *minimal* if whenever $T \subset X$ is a closed subset such that $gT \subset T$ for all $g \in G$, then $T = \varnothing$ or $T = X$.

In short, there are no nontrivial invariant closed subsets. For those knowing some ergodic theory, this is the topological analog of an ergodic action on a measure space.

If the action of $G$ on $X$ is not minimal, then there is a nontrivial invariant closed subset $T \subset X$, and $C^*_{\mathrm{r}}(G, X \setminus T)$ turns out to be a nontrivial ideal in $C^*_{\mathrm{r}}(G, X)$. Thus $C^*_{\mathrm{r}}(G, X)$ is not simple.



# Minimal homeomorphisms

For the case $G = \mathbb{Z}$, the conventional terminology is a bit different.

**Definition**

Let $X$ be a locally compact Hausdorff space, and let $h \colon X \to X$ be a homeomorphism. Then $h$ is called *minimal* if whenever $T \subset X$ is a closed subset such that $h(T) = T$, then $T = \varnothing$ or $T = X$.

Almost all work on minimal homeomorphisms has been on compact spaces. For these, we have the following equivalent conditions.



# Minimal homeomorphisms (continued)

**Lemma**

Let $X$ be a compact Hausdorff space, and let $h \colon X \to X$ be a homeomorphism. Then the following are equivalent:

1. $h$ is minimal.
2. Whenever $T \subset X$ is a closed subset such that $h(T) \subset T$, then $T = \varnothing$ or $T = X$.
3. Whenever $U \subset X$ is an open subset such that $h(U) = U$, then $U = \varnothing$ or $U = X$.
4. Whenever $U \subset X$ is an open subset such that $h(U) \subset U$, then $U = \varnothing$ or $U = X$.
5. For every $x \in X$, the orbit $\{h^n(x) \colon n \in \mathbb{Z}\}$ is dense in $X$.
6. For every $x \in X$, the forward orbit $\{h^n(x) \colon n \geq 0\}$ is dense in $X$.

## Minimal homeomorphisms (continued)

Conditions (1), (3), and (5) are equivalent even when $X$ is only locally compact, and in fact there is an analog for actions of arbitrary groups. Minimality does not imply the other three conditions without compactness, as can be seen by considering the homeomorphism $n \mapsto n+1$ of $\mathbb{Z}$. Also, even for compact $X$, it isn't good enough to merely have the existence of some dense orbit, as can be seen by considering the homeomorphism $n \mapsto n+1$ on the two point compactification $\mathbb{Z} \cup \{\pm\infty\}$ of $\mathbb{Z}$.

**Exercise**

Prove the lemma.



## Examples of minimal actions

**Example**

Let $G$ be a locally compact group, let $H \subset G$ be a closed subgroup, and let $G$ act on $G/H$ by translation. This action is minimal: there are no nontrivial invariant subsets, closed or not.

This is a "trivial" example of a minimal action. Here are several more interesting ones.

**Example**

Irrational rotations of the circle are minimal.



## Minimality of irrational rotations

Minimality of irrational rotations follows from the following lemma.

**Lemma**

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then $\left\{ e^{2\pi i n\theta} \colon n \in \mathbb{Z} \right\}$ is dense in $S^1$.

**Proof.**

It suffices to prove that $\mathbb{Z} + \theta\mathbb{Z}$ is dense in $\mathbb{R}$. Suppose not. Because we are dealing with groups, there is an open set $U \subset \mathbb{R}$ such that $U \cap (\mathbb{Z} + \theta\mathbb{Z}) = \{0\}$. Let $t = \inf\left( \left\{ x \in \overline{\mathbb{Z} + \theta\mathbb{Z}} \colon x > 0 \right\} \right)$. Then $t > 0$. Since $\overline{\mathbb{Z} + \theta\mathbb{Z}}$ is a group, one checks that $\overline{\mathbb{Z} + \theta\mathbb{Z}} = \mathbb{Z}t$. Then we must have $\mathbb{Z} + \theta\mathbb{Z} = \mathbb{Z}t$. It follows that both 1 and $\theta$ are integer multiples of $t$, so that $\theta \in \mathbb{Q}$. □

There are other proofs of minimality.



## More examples of minimal homeomorphisms

**Example**

The homeomorphism $x \mapsto x+1$ on the $p$-adic integers is minimal. The orbit of 0 is $\mathbb{Z}$, which is dense, essentially by definition. Every other orbit is a translate of this one, so also dense.

**Example**

The shift homeomorphism on $\{0,1\}^{\mathbb{Z}}$ and the action of $\mathrm{SL}_2(\mathbb{Z})$ on $S^1 \times S^1$ are not minimal. In fact, they have fixed points.

Other examples of minimal homeomorphisms include Furstenberg transformations, restrictions of Denjoy homeomorphisms of the circle to their minimal sets, and certain irrational time maps of suspension flows.

## Minimal actions are plentiful

Minimal actions are plentiful: a Zorn's Lemma argument shows that every nonempty compact $G$-space $X$ contains a nonempty invariant closed subset on which the restricted action is minimal.



## Free actions

The transformation group C*-algebra of a minimal action need not be simple. Consider, for example, the trivial action of a group $G$ (particularly an abelian group) on a one point space, for which the transformation group C*-algebra is $C^*(G)$.

**Definition**

Let a locally compact group $G$ act continuously on a locally compact space $X$. The action is called *free* if whenever $g \in G \setminus \{1\}$ and $x \in X$, then $gx \neq x$. The action is called *essentially free* if whenever $g \in G \setminus \{1\}$, the set $\{x \in X \colon gx = x\}$ has empty interior.



## Remarks on freeness

Let $X$ be an infinite compact Hausdorff space, and let $h \colon X \to X$ be a minimal homeomorphism. Then one easily checks that the corresponding action of $\mathbb{Z}$ on $X$ is free.

An essentially free minimal action of an abelian group is free. This is because the fixed point set of any group element is invariant under the action.



## Simplicity of the transformation group C*-algebra

**Theorem (Archbold-Spielberg)**

Let a discrete group $G$ act minimally and essentially freely on a locally compact space $X$. Then $C^*_{\mathrm{r}}(G, X)$ is simple.

Archbold and Spielberg actually proved something stronger, involving actions on not necessarily commutative C*-algebras. See the notes, and also see the notes for discussions of other proofs and related results.

The usual proof for $G = \mathbb{Z}$ depends on Rokhlin type arguments. See the proof of Lemma VIII.3.7 of Davidson's book. We have avoided such arguments here, but will use them later. To obtain more information about simple transformation group C*-algebras, such arguments are necessary, at least with the current state of knowledge. Examples show that, in the absence of some form of the Rokhlin property, stronger structural properties of crossed products of noncommutative C*-algebras need not hold, even when they are simple.

## First lemma

The proof of the theorem on simplicity needs several lemmas, which are special cases of the corresponding lemmas in the paper of Archbold and Spielberg.

**Lemma**

Let $A$ be a C*-algebra, let $B \subset A$ be a subalgebra, and let $\omega$ be a state on $A$ such that $\omega|_B$ is multiplicative. Then for all $a \in A$ and $b \in B$, we have $\omega(ab) = \omega(a)\omega(b)$ and $\omega(ba) = \omega(b)\omega(a)$.

This is also a special case of a result of Choi.



## Proof of the first lemma

We prove $\omega(ab) = \omega(a)\omega(b)$. The other equation will follow by using adjoints and the relation $\omega(c^*) = \overline{\omega(c)}$.

If $A$ is not unital, then $\omega$ extends to a state on the unitization $A^+$. Thus, we may assume that $A$ is unital. Also, if $\omega$ is multiplicative on $B$, one easily checks that $\omega$ is multiplicative on $B + \mathbb{C} \cdot 1$. Thus, we may assume that $1 \in B$.



## Proof of the first lemma (continued)

We recall from the Cauchy-Schwarz inequality that $|\omega(x^* y)|^2 \leq \omega(y^* y)\omega(x^* x)$. Replacing $x$ by $x^*$, we get $|\omega(xy)|^2 \leq \omega(y^* y)\omega(x x^*)$. Now let $a \in A$ and $b \in B$. Then

$$\begin{aligned} |\omega(ab) - \omega(a)\omega(b)|^2 &= \big|\omega\big(a(b - \omega(b) \cdot 1\big)\big|^2 \\ &\leq \omega\big((b - \omega(b) \cdot 1)^*(b - \omega(b) \cdot 1)\big)\omega(aa^*). \end{aligned}$$

Since $\omega$ is multiplicative on $B$, we have

$$\omega\big((b - \omega(b) \cdot 1)^*(b - \omega(b) \cdot 1)\big) = \omega\big((b - \omega(b) \cdot 1)^*\big)\omega\big(b - \omega(b) \cdot 1\big) = 0.$$

So $|\omega(ab) - \omega(a)\omega(b)|^2 = 0$. This completes the proof.



## Second lemma

**Lemma**

Let a discrete group $G$ act on a locally compact space $X$. Let $x \in X$, let $g \in G$, and assume that $gx \neq x$. Let $\mathrm{ev}_x \colon C_0(X) \to \mathbb{C}$ be the evaluation map $\mathrm{ev}_x(f) = f(x)$ for all $f \in C_0(X)$, and let $\omega$ be a state on $C^*_{\mathrm{r}}(G, X)$ which extends $\mathrm{ev}_x$. Then $\omega(f u_g) = 0$ for all $f \in C_0(X)$.

For the proof, let $\alpha \colon G \to \mathrm{Aut}(C_0(X))$ be $\alpha_g(f)(x) = f(g^{-1}x)$ for $f \in C_0(X)$, $g \in G$, and $x \in X$. Choose $f_0 \in C_0(X)$ such that $f_0(x) = 1$ and $f_0(gx) = 0$. Applying the first lemma to $\omega$ at the second and fourth steps, and using $\omega(f_0) = 1$ at the first step, we have

$$\begin{aligned} \omega(f u_g) &= \omega(f_0)\omega(f u_g) = \omega(f_0 f u_g) = \omega\big(f u_g \alpha_g^{-1}(f_0)\big) \\ &= \omega(f u_g)\omega(\alpha_g^{-1}(f_0)) = \omega(f u_g) f_0(gx) = 0. \end{aligned}$$

This completes the proof.

## Reminder: The conditional expectation

Recall the conditional expectation:
Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group $G$ on a C*-algebra $A$. Then the conditional expectation $E\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ does the following: if $a = \sum_{g \in G} a_g u_g \in C^*_{\mathrm{r}}(G, A, \alpha)$ with $a_g = 0$ for all but finitely many $g$, then $E(a) = a_1$. Moreover, $E$ has the following properties:

1. $E(E(b)) = E(b)$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
2. If $b \geq 0$ then $E(b) \geq 0$.
3. $\|E(b)\| \leq \|b\|$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
4. If $a \in A$ and $b \in C^*_{\mathrm{r}}(G, A, \alpha)$, then $E(ab) = aE(b)$ and $E(ba) = E(b)a$.
5. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $E(a^*a) = 0$, then $a = 0$.



## Proof of the simplicity theorem

Let $I \subset C^*_{\mathrm{r}}(G, X)$ be a nonzero closed ideal.

First suppose $I \cap C_0(X) = 0$. Choose $a \in I$ with $a \neq 0$. Let

$$E\colon C^*_{\mathrm{r}}(G, X) \to C_0(X)$$

be the standard conditional expectation. Then $E(a^*a) \neq 0$ because $E$ is faithful. Choose $b \in C_{\mathrm{c}}(G,\, C_0(X),\, \alpha)$ such that $\|b - a^*a\| < \frac{1}{4}\|E(a^*a)\|$. We can write $b = \sum_{g \in S} b_g u_g$ for some finite set $S \subset G$ and with $b_g \in C_0(X)$ for $g \in S$. Without loss of generality $1 \in S$. Since $E(a^*a)$ is a positive element of $C_0(X)$, there is $x_0 \in X$ such that $E(a^*a)(x_0) = \|E(a^*a)\|$. Essential freeness implies that

$$\{x \in X \colon gx \neq x \text{ for all } g \in S \setminus \{1\}\}$$

is dense in $X$. In particular, there is $x \in X$ so close to $x_0$ that $E(a^*a)(x) > \frac{3}{4}\|E(a^*a)\|$, and also satisfying $gx \neq x$ for all $g \in S$.



## Proof of the simplicity theorem (continued)

The set $C_0(X) + I$ is a C*-subalgebra of $C^*_{\mathrm{r}}(G, X)$. Let $\omega_0\colon C_0(X) + I \to \mathbb{C}$ be the following composition:

$$C_0(X) + I \longrightarrow (C_0(X) + I)/I \xrightarrow{\cong} C_0(X)/(C_0(X) \cap I) = C_0(X) \xrightarrow{\mathrm{ev}_x} \mathbb{C}.$$

Then $\omega_0$ is a homomorphism. Use the Hahn-Banach Theorem in the usual way to get a state $\omega\colon C^*_{\mathrm{r}}(G, X) \to \mathbb{C}$ which extends $\omega_0$. Note that $\omega(a^*a) = 0$.

We now have, using the second lemma at the fifth step,

$$\begin{aligned}
\tfrac{1}{4}\|E(a^*a)\| > \|b - a^*a\| &\geq |\omega(b - a^*a)| = |\omega(b)| = \left|\textstyle\sum_{g \in S} \omega(b_g u_g)\right| \\
&= |\omega(b_1)| = |\omega_0(b_1)| = |b_1(x)| \geq E(a^*a)(x) - \|E(a^*a) - b_1\| \\
&\geq E(a^*a)(x) - \|a^*a - b\| > \tfrac{3}{4}\|E(a^*a)\| - \tfrac{1}{4}\|E(a^*a)\| = \tfrac{1}{2}\|E(a^*a)\|.
\end{aligned}$$

This contradiction shows that $I \cap C_0(X) \neq 0$.



## Proof of the simplicity theorem (continued)

Since $I \cap C_0(X)$ is an ideal in $C_0(X)$, it has the form $C_0(U)$ for some nonempty open set $U \subset X$. We claim that $U$ is $G$-invariant. Let $g \in G$ and let $f \in C_0(U)$. Let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for $C_0(X)$. Then the elements $e_\lambda u_g$ are in $C^*_{\mathrm{r}}(G, X)$, and we have $(e_\lambda u_g) f (e_\lambda u_g)^* = e_\lambda \alpha_g(f) e_\lambda$, which converges to $\alpha_g(f)$. We also have $(e_\lambda u_g) f (e_\lambda u_g)^* \in I \cap C_0(X)$, since $I$ is an ideal. So $\alpha_g(C_0(U)) \subset C_0(U)$ for all $g \in G$, and the claim follows.

Since $U$ is open, invariant, and nonempty, we have $U = X$. One easily checks that an approximate identity for $C_0(X)$ is also an approximate identity for $C^*_{\mathrm{r}}(G, X)$, so $I = C^*_{\mathrm{r}}(G, X)$, as desired. This completes the proof.

## Kishimoto's result

The following theorem, originally due to Kishimoto, also follows from the full Archbold-Spielberg theorem.

### Theorem

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group $G$ on a simple C*-algebra $A$. Suppose that $\alpha_g$ is outer for every $g \in G \setminus \{1\}$. Then $C^*_{\mathrm{r}}(G, A, \alpha)$ is simple.



## The main theorem on minimal homeomorphisms

Definitions (with explanation) and the main corollary are below.

### Theorem (with H. Lin; to appear)

Let $X$ be an infinite compact metric space with finite covering dimension, and let $h \colon X \to X$ be a minimal homeomorphism. Suppose that $\rho(K_0(C^*(\mathbb{Z}, X, h)))$ is dense in $\mathrm{Aff}(T(C^*(\mathbb{Z}, X, h)))$. Then $C^*(\mathbb{Z}, X, h)$ is a simple unital C*-algebra with tracial rank zero which satisfies the Universal Coefficient Theorem.

We do not give a full proof here. Instead, we assume $X$ is the Cantor set. The full proof would require three lectures about recursive subhomogeneous algebras and their direct limits, as well as several lectures on technical computations involving KK-theory.

Note: Some of the discussion here is not in the notes, and some of the results are in a slightly different order.



## The main corollary

### Corollary (with H. Lin; to appear)

Let $X$ be an infinite compact metric space with finite covering dimension, and let $h \colon X \to X$ be a minimal homeomorphism. Suppose that $\rho(K_0(C^*(\mathbb{Z}, X, h)))$ is dense in $\mathrm{Aff}(T(C^*(\mathbb{Z}, X, h)))$. Then $C^*(\mathbb{Z}, X, h)$ is a simple AH algebra with no dimension growth and with real rank zero.

An AH algebra is a direct limit $\varinjlim A_n$, in which each $A_n$ is a finite direct sum of C*-algebras of the form $C(Y, M_r)$, for varying compact metric spaces $Y$ and positive integers $r$. "No dimension growth" means that there is a finite upper bound on the (covering) dimensions of the spaces $X$ which occur throughout in the direct system.

The theorem also implies that $C^*(\mathbb{Z}, X, h)$ is classifiable in the sense of the Elliott program. (This is used in the proof of the corollary.)



## Covering dimension

The covering dimension $\dim(X)$ is defined for compact metric spaces $X$ (and more generally). Rather than giving the (somewhat complicated) definition, here are examples:

- The Cantor set has covering dimension zero.
- If $X$ is a compact manifold with dimension $d$ as a manifold, then $\dim(X) = d$.
- If $X$ is a finite CW complex, then $\dim(X)$ is the dimension of the highest dimensional cell.
- $\dim\left([0,1]^{\mathbb{Z}}\right) = \infty$.

## The map from $K_0$ to the affine functions on the tracial state space

For any unital C*-algebra $A$, we let $T(A)$ be the tracial state space of $A$. For any compact convex set $\Delta$, we let $\mathrm{Aff}(\Delta)$ be the space of real valued continuous affine functions on $\Delta$, with the supremum norm. Further let $\rho = \rho_A \colon K_0(A) \to \mathrm{Aff}(T(A))$ be the homomorphism determined by $\rho([p])(\tau) = \tau(p)$ for $\tau \in T(A)$ and $p$ a projection in some matrix algebra over $A$.

In the situation at hand, $T(C^*(\mathbb{Z}, X, h))$ is affinely homeomorphic to the simplex of $h$-invariant Borel probability measures on $X$. (See the end of the lecture.)

There is machinery available to compute the range of $\rho$ in the above theorem without computing $C^*(\mathbb{Z}, X, h)$. See, for example, Ruy Exel's Ph.D. thesis. (Reference to the published version in the notes.)



## The definition of tracial rank zero

We use the notation $[a, b]$ for the commutator $ab - ba$.

**Definition**

Let $A$ be a simple unital C*-algebra. Then $A$ has tracial rank zero if for every finite subset $F \subset A$, every $\varepsilon > 0$, and every nonzero positive element $c \in A$, there exists a projection $p \in A$ and a unital finite dimensional subalgebra $D \subset pAp$ such that:

1. $\|[a, p]\| < \varepsilon$ for all $a \in F$.
2. $\mathrm{dist}(pap, D) < \varepsilon$ for all $a \in F$.
3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{cAc}$.

(This is equivalent to the original definition for simple C*-algebras. See the notes.)

The condition was originally called "tracially AF".



## Restricting to the Cantor set

The notes contain a complete proof of the main theorem when $X$ is the Cantor set, using the same methods as in the proof of the full theorem.

The restriction to the Cantor set simplifies the argument by avoiding recursive subhomogeneous C*-algebras and some K-theory computations.

However, for the Cantor set, there is an older and shorter proof, of which the main part is due to Putnam. This proof gives directly the result that $C^*(\mathbb{Z}, X, h)$ is a direct limit of finite direct sums of C*-algebras of the form $C(S^1, M_r)$ (for varying $r$).

One can get this result for the Cantor set by combining the main theorem with known classification results, so Putnam's argument doesn't give any more in the end.



## Other known results: Minimal diffeomorphisms

Let $X$ be a compact manifold, and let $h \colon X \to X$ be a minimal diffeomorphism. Then $C^*(\mathbb{Z}, X, h)$ is a direct limit, with no dimension growth, of recursive subhomogeneous C*-algebras.

There is no condition on $\rho(K_0(C^*(\mathbb{Z}, X, h))) \subset \mathrm{Aff}(T(C^*(\mathbb{Z}, X, h)))$. In particular, $C^*(\mathbb{Z}, X, h)$ sometimes has no nontrivial projections.

This is joint with Qing Lin (unpublished). It uses Putnam's methods, but is very long. As a corollary, $C^*(\mathbb{Z}, X, h)$ has stable rank one. Applicable classification results are just now starting to appear.

It should be possible to generalize to minimal homeomorphisms of finite dimensional compact metric spaces.

## Other known results: Infinite dimensional spaces

The condition $\dim(X) < \infty$ is needed. Giol and Kerr have an example of a minimal homeomorphism $h$ of an infinite dimensional compact metric space $X$ such that $C^*(\mathbb{Z}, X, h)$ does not have stable rank one.

However, probably $h$ having "mean dimension zero" is enough. Uniquely ergodic implies mean dimension zero, and $\dim(X) < \infty$ implies mean dimension zero.



## Other known results: Actions of $\mathbb{Z}^d$, groups which are not discrete, and actions on simple C*-algebras

Let $X$ be the Cantor set, and suppose $\mathbb{Z}^d$ acts freely and minimally on $X$. Then $C^*(\mathbb{Z}^d, X)$ has stable rank one, real rank zero, and the order on projections over $C^*(\mathbb{Z}^d, X)$ is determined by traces.

All these follow from tracial rank zero, but do not imply tracial rank zero. It should be true that $C^*(\mathbb{Z}^d, X)$ has tracial rank zero, but this is still open.

Let $X$ be a finite dimensional compact metric space, and suppose $\mathbb{Z}^d$ acts freely and minimally on $X$. Then $C^*(\mathbb{Z}^d, X)$ has strict comparison of positive elements (the Cuntz semigroup analog of the order on projections over $C^*(\mathbb{Z}^d, X)$ being determined by traces).

There is work in progress which will probably give positive results about $C^*(\mathbb{R}, X)$ when $\dim(X) < \infty$ and the action is free and minimal.

There is also a collection of related results on crossed products of simple C*-algebras by actions of $\mathbb{Z}$ and of finite groups which have the tracial Rokhlin property, and generalizations.



## Lemma 1

The proof of the main theorem, even for just the Cantor set, requires a number of lemmas. The first essentially says one can use AF algebras in place of finite dimensional algebras in the definition of tracial rank zero. The proof is omitted; see the notes.

**Lemma**

Let $A$ be a simple unital C*-algebra. Suppose that for every finite subset $F \subset A$, every $\varepsilon > 0$, and every nonzero positive element $c \in A$, there exists a projection $p \in A$ and a unital AF subalgebra $B \subset A$ with $p \in B$ such that:

1. $\|[a, p]\| < \varepsilon$ for all $a \in F$.
2. $\mathrm{dist}(pap, pBp) < \varepsilon$ for all $a \in F$.
3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{cAc}$.

Then $A$ has tracial rank zero.



## Lemma 2

We need only consider finite subsets of a generating set:

**Lemma**

Let $A$ be a unital C*-algebra, and let $S \subset A$ be a subset which generates $A$ as a C*-algebra. Assume that the condition of the previous lemma holds for all finite subsets $F \subset S$. Then $A$ has tracial rank zero.

The proof is an exercise.

## $C^*(\mathbb{Z}, X, h)_Y$

A key point, in this proof and others, is the construction of a "large" and more tractable subalgebra of the crossed product.

**Notation**

Let $X$ be a compact metric space, and let $h\colon X \to X$ be a homeomorphism. In the transformation group C*-algebra $C^*(\mathbb{Z}, X, h)$, we write $u$ for the standard unitary representing the generator of $\mathbb{Z}$. For a closed subset $Y \subset X$, we define the C*-subalgebra $C^*(\mathbb{Z}, X, h)_Y$ to be

$$C^*(\mathbb{Z}, X, h)_Y = C^*(C(X),\ uC_0(X \setminus Y)) \subset C^*(\mathbb{Z}, X, h).$$



## $C^*(\mathbb{Z}, X, h)_Y$ is the C*-algebra of a groupoid

Although we will not use formally groupoids in these notes, it should be pointed out that $C^*(\mathbb{Z}, X, h)_Y$ is the C*-algebra of a subgroupoid of the transformation group groupoid $\mathbb{Z} \ltimes X$ made from the action of $\mathbb{Z}$ on $X$ generated by $h$. Informally, we "break" every orbit each time it goes through $Y$. More formally, for $n < 0$ the pair $(n, x)$ is in the subgroupoid only if all of $h^n(x)$, $h^{n+1}(x)$, $\ldots$, $h^{-1}(x)$ are in $X \setminus Y$, and for $n > 0$ the pair $(n, x)$ is in the subgroupoid only if all of $x$, $h(x)$, $\ldots$, $h^{n-1}(x)$ are in $X \setminus Y$.

For actions of $\mathbb{Z}^d$, it appears to be necessary to use subalgebras of the crossed product for which the only nice description is in terms of subgroupoids of the transformation group groupoid.



## Lemma 3

The following lemma is due to Putnam.

**Lemma**

Let $X$ be the Cantor set, and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a nonempty compact open subset. Then $C^*(\mathbb{Z}, X, h)_Y$ is an AF algebra.



## Proof of Lemma 3

The proof depends on the construction of Rokhlin towers, which is a crucial element of many structure results for crossed products.

We first claim that there is $N \in \mathbb{Z}_{>0}$ such that $\bigcup_{n=1}^{N} h^{-n}(Y) = X$. Set $U = \bigcup_{n=1}^{\infty} h^{-n}(Y)$, which is a nonempty open subset of $X$ such that $U \subset h(U)$. Then $Z = X \setminus \bigcup_{n=1}^{\infty} h^{-n}(Y)$ is a closed subset of $X$ such that $h(Z) \subset Z$, and $Z \neq X$. Therefore $Z = \varnothing$. So $U = X$, and the claim now follows from compactness of $X$.

It follows that for each fixed $y \in Y$, the sequence of iterates $h(y), h^2(y), \ldots$ of $y$ under $h$ must return to $Y$ in at most $N$ steps. Define the *first return time* $r(y)$ to be

$$r(y) = \min\{n \geq 1\colon h^n(y) \in Y\} \leq N.$$

## Proof of Lemma 3 (continued)

Let $n(0) < n(1) < \cdots < n(l) \leq N$ be the values of $r$. Set

$$Y_k = \{y \in Y : r(y) = n(k)\}.$$

Then the sets $Y_k$ are compact, open, and partition $Y$, and the sets $h^j(Y_k)$, for $1 \leq j \leq n(k)$, partition $X$:

$$Y = \coprod_{k=0}^{l} Y_k \quad \text{and} \quad X = \coprod_{k=0}^{l} \coprod_{j=1}^{n(k)} h^j(Y_k).$$

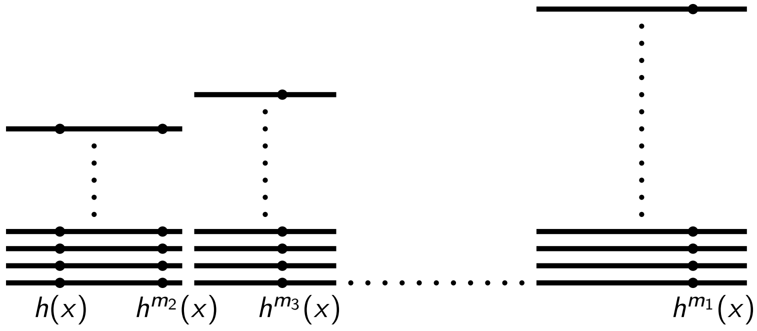
Each finite sequence $h(Y_k), h^2(Y_k), \ldots, h^{n(k)}(Y_k)$ is a *Rokhlin tower* with base $h(Y_k)$ and height $n(k)$. (It is more common to let the power of $h$ run from 0 to $n(k) - 1$. The choice made here, effectively taking the base of the collection of Rokhlin towers to be $h(Y)$ rather than $Y$, is more convenient for use with our definition of $C^*(\mathbb{Z}, X, h)_Y$.) Further set $X_k = \bigcup_{j=1}^{n(k)} h^j(Y_k)$. The sets $X_k$ then also partition $X$.



## Rokhlin towers, with part of an orbit

We take $x \in Y$, in fact, $x \in Y_0$.

The bases of the towers are $h(Y_0), h(Y_1), \ldots, h(Y_l)$, and the heights are $n(0), n(1), \ldots, n(l)$. The tower over $h(Y_k)$ corresponds to a summand of $C^*(\mathbb{Z}, X, h)_Y$ isomorphic to $M_{n(k)} \otimes C(Y_k)$.




$$m_1 = n(0) + 1, \quad m_2 = n(0) + n(l) + 1, \quad m_3 = n(0) + n(l) + n(1) + 1$$



## Proof of Lemma 3 (continued)

Define $p_k \in C(X) \subset C^*(\mathbb{Z}, X, h)_Y$ by $p_k = \chi_{X_k}$. Then $p_k$ trivially commutes with every element of $C(X)$. Moreover, suppose $f \in C(X)$ vanishes on $Y$. Since $Y_k$, $h^{n(k)}(Y_k) \subset Y$, we have

$$\chi_{Y_k} f = 0 \quad \text{and} \quad \chi_{h^{n(k)}(Y_k)} f = 0$$

Use the action of $h$ on the levels of the tower at the first step, and the equations above at the second step, to get

$$p_k u f = u\big(p_k - \chi_{h^{n(k)}(Y_k)} + \chi_{Y_k}\big) f = u p_k f = u f p_k.$$

It follows that $p_k$ commutes with all elements of $C^*(\mathbb{Z}, X, h)_Y$. So it suffices to prove that $p_k C^*(\mathbb{Z}, X, h)_Y p_k$ is AF for each $k$.



## Proof of Lemma 3 (continued)

Now $p_k C^*(\mathbb{Z}, X, h)_Y p_k$ is the C*-algebra generated by $C(X_k)$ and

$$u(\chi_{X \setminus Y}) p_k = u(\chi_{X_k \setminus h^{n(k)}(Y_k)}) = \sum_{j=1}^{n(k)-1} u(\chi_{h^j(Y_k)})$$
$$= \sum_{j=1}^{n(k)-1} (\chi_{h^{j+1}(Y_k)}) u(\chi_{h^j(Y_k)}).$$

One can now check, although it is a bit tedious to write out the details, that there is an isomorphism $\psi_k : p_k C^*(\mathbb{Z}, X, h)_Y p_k \to M_{n(k)} \otimes C(Y_k)$ such that for $f \in C(X_k)$ we have

$$\psi_k(f) = \mathrm{diag}\big(f \circ h|_{Y_k}, \, f \circ h^2|_{Y_k}, \, \ldots, \, f \circ h^{n(k)}|_{Y_k}\big)$$

and for $1 \leq j \leq n(k) - 1$ we have

$$\psi_k\big(\chi_{h^{j+1}(Y_k)} u \chi_{h^j(Y_k)}\big) = e_{j+1,j} \otimes 1.$$

The algebra $M_{n(k)} \otimes C(Y_k)$ is AF because $Y_k$ is totally disconnected. This completes the proof.

## Lemma 4

### Notation

Let $A$ be a C*-algebra, and let $p, q \in A$ be projections. We write $p \sim q$ to mean that $p$ and $q$ are Murray-von Neumann equivalent in $A$, that is, there exists $v \in A$ such that $v^* v = p$ and $v v^* = q$. We write $p \precsim q$ if $p$ is Murray-von Neumann equivalent to a subprojection of $q$.

### Lemma

Let $X$ be the Cantor set, and let $h \colon X \to X$ be a minimal homeomorphism. Let $c \in C^*(\mathbb{Z}, X, h)$ be a nonzero positive element. Then there exists a nonzero projection $p \in C(X)$ such that $p$ is Murray-von Neumann equivalent in $C^*(\mathbb{Z}, X, h)$ to a projection in $\overline{c C^*(\mathbb{Z}, X, h) c}$.

This shows that we can control the size of the leftover in the definition of tracial rank zero using projections in $C(X)$ instead of in the crossed product. This is a big advantage.



## What we have so far

Combining our results so far, we have reduced our problem to proving the following:

Let $X$ be the Cantor set, and let $h \colon X \to X$ be a minimal homeomorphism. Let $S = C(X) \cup \{u\}$ (clearly a generating set for $C^*(\mathbb{Z}, X, h)$). Then for every finite subset $F \subset C(X)$, every $\varepsilon > 0$, and every nonempty open set $U \subset X$, there exists a compact open set $Y \subset X$ and a projection $p \in C^*(\mathbb{Z}, X, h)_Y$ such that:

(1) $\| pa - ap \| < \varepsilon$ for all $a \in F \cup \{u\}$.
(2) $pap \in p C^*(\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{u\}$.
(3) There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

The point is that $C^*(\mathbb{Z}, X, h)_Y$ is an AF algebra, and (3) says, in view of Lemma 4, that $1 - p$ is "small".



## Proof of Lemma 4

Let $E \colon C^*(\mathbb{Z}, X, h) \to C(X)$ be the standard conditional expectation. Then $E(c)$ is a nonzero positive element of $C(X)$. Choose a nonempty compact open subset $K_0 \subset X$ and $\delta > 0$ such that the function $E(c)$ satisfies $E(c)(x) > 4\delta$ for all $x \in K_0$. Choose a finite sum $b = \sum_{n=-N}^{N} b_n u^n \in C^*(\mathbb{Z}, X, h)$ such that $\| b - c \| < \delta$. Since the action of $\mathbb{Z}$ induced by $h$ is free, there is a nonempty compact open subset $K \subset K_0$ such that the sets

$$h^{-N}(K), \; h^{-N+1}(K), \; \ldots, \; h^{N}(K)$$

are disjoint. Set $p = \chi_K \in C(X)$. For $n \in \{-N, \, -N+1, \, \ldots, \, N\} \setminus \{0\}$, the disjointness condition implies that $p u^n p = 0$. Therefore

$$pbp = p b_0 p = p E(b) p.$$



## Proof of Lemma 4 (continued)

Using this equation at the first step, we get

$$\| pcp - pE(c)p \| \leq \| pcp - pbp \| + \| pE(b)p - pE(c)p \| \leq 2 \| c - b \| < 2\delta. \quad (1)$$

Since $K \subset K_0$, the function $pE(c)p$ is invertible in $pC(X)p$. In the following, we take inverses in $pC^*(\mathbb{Z}, X, h)p$. Then, in fact, $\| [pE(c)p]^{-1} \| < \frac{1}{4} \delta^{-1}$. The estimate (1) now implies that $pcp$ is invertible in $pC^*(\mathbb{Z}, X, h)p$. Let $a = (pcp)^{-1/2}$, calculated in $pC^*(\mathbb{Z}, X, h)p$. Set $v = apc^{1/2}$. Then

$$v v^* = apcpa = (pcp)^{-1/2} (pcp) (pcp)^{-1/2} = p$$

and

$$v^* v = c^{1/2} p a^2 p c^{1/2} \in \overline{c C^*(\mathbb{Z}, X, h) c}.$$

This completes the proof.

## Lemma 5

### Lemma

Let $X$ be the Cantor set, and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a nonempty compact open subset. Let $N \in \mathbb{Z}_{>0}$, and suppose that $Y, h(Y), \ldots, h^N(Y)$ are disjoint. Then the projections $\chi_Y$ and $\chi_{h^N(Y)}$ are Murray-von Neumann equivalent in $C^*(\mathbb{Z}, X, h)_Y$.

### Proof.

First, observe that if $Z \subset X$ is a compact open subset such that $Y \cap Z = \varnothing$, then $v = u\chi_Z \in C^*(\mathbb{Z}, X, h)_Y$ and satisfies $v^*v = \chi_Z$ and $vv^* = \chi_{h(Z)}$. Thus $\chi_Z \sim \chi_{h(Z)}$ in $C^*(\mathbb{Z}, X, h)_Y$.

An induction argument now shows that $\chi_{h(Y)} \sim \chi_{h^N(Y)}$ in $C^*(\mathbb{Z}, X, h)_Y$. Also, $\chi_{X \setminus Y} \sim \chi_{X \setminus h(Y)}$ in $C^*(\mathbb{Z}, X, h)_Y$. Since $C^*(\mathbb{Z}, X, h)_Y$ is an AF algebra, it follows that $\chi_Y \sim \chi_{h(Y)}$. The result follows by transitivity. □



## Lemma 6

### Lemma

Let $X$ be the Cantor set, and let $h\colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Then for any $\varepsilon > 0$, any nonempty open set $U \subset X$, and any finite subset $F \subset C(X)$, there is a compact open set $Y \subset X$ containing $y$ and a projection $p \in C^*(\mathbb{Z}, X, h)_Y$ such that:

(1) $\|pa - ap\| < \varepsilon$ for all $a \in F \cup \{u\}$.

(2) $pap \in pC^*(\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{u\}$.

(3) There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.



## Lemma 6 finishes the proof

Recall that we reduced our problem to proving the following:

Let $X$ be the Cantor set, and let $h\colon X \to X$ be a minimal homeomorphism. Let $S = C(X) \cup \{u\}$ Then for every finite subset $F \subset S$, every $\varepsilon > 0$, and every nonempty open set $U \subset X$, there exists compact open set $Y \subset X$ and a projection $p \in C^*(\mathbb{Z}, X, h)_Y$ such that:

(1) $\|pa - ap\| < \varepsilon$ for all $a \in F \cup \{u\}$.

(2) $pap \in pC^*(\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{u\}$.

(3) There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

So Lemma 6 finishes the proof.



## Comment on the proof of Lemma 6: Berg's technique

The basic idea behind the proof of Lemma 6 is Berg's technique. It was originally used to prove that the direct sum of the bilateral shift and a finite cyclic shift is approximately unitarily equivalent, to within an error depending on the length of the finite cyclic shift, to the bilateral shift. The error goes to zero as the length of the finite cyclic shift goes to infinity.

We effectively do this in reverse: the bilateral shift (in this case, the standard generating unitary of the crossed product by $\mathbb{Z}$) can be approximated by something which has a (long) finite cyclic shift as a direct summand.

In this proof, we apply Berg's technique "over" a small subset of the space $X$. For some other related results on crossed products, the methods of proof known so far require an application "over" subsets which fill up most of $X$.

For actions of more general groups on compact metric spaces, to get the strongest expected results one needs a generalization or replacement, so far not found.

## Proof of Lemma 6

Let $d$ be the metric on $X$.

Choose $N_0 \in \mathbb{Z}_{>0}$ so large that $4\pi/N_0 < \varepsilon$.

Choose $\delta_0 > 0$ with $\delta_0 < \frac{1}{2}\varepsilon$ and so small that $d(x_1, x_2) < 4\delta_0$ implies $|f(x_1) - f(x_2)| < \frac{1}{4}\varepsilon$ for all $f \in F$.

Choose $\delta > 0$ with $\delta \leq \delta_0$ and such that whenever $d(x_1, x_2) < \delta$ and $0 \leq k \leq N_0$, then $d(h^{-k}(x_1),\, h^{-k}(x_2)) < \delta_0$.

Since $h$ is minimal, there is $N > N_0 + 1$ such that $d(h^N(y),\, y) < \delta$.



## Proof of Lemma 6 (continued)

Choose $N + N_0 + 1$ disjoint nonempty open subsets $U_{-N_0}, U_{-N_0+1}, \ldots, U_N \subset U$. Using minimality again, choose $r_{-N_0}, r_{-N_0+1}, \ldots, r_N \in \mathbb{Z}$ such that $h^{r_l}(y) \in U_l$ for $-N_0 \leq l \leq N$. Since $h$ is free, there is a compact open set $Y \subset X$ containing $y$ such that

$$h^{-N_0}(Y),\ h^{-N_0+1}(Y),\ \ldots,\ Y,\ h(Y),\ \ldots,\ h^N(Y)$$

are disjoint and all have diameter less than $\delta$. We may also require that $h^{r_l}(Y) \subset U_l$ for $-N_0 \leq l \leq N$.

Set $q_0 = \chi_Y$. For $-N_0 \leq n \leq N$ set

$$T_n = h^n(Y) \quad \text{and} \quad q_n = u^n q_0 u^{-n} = \chi_{h^n(Y)}.$$

Then the $q_n$ are mutually orthogonal projections in $C(X)$.



## Proof of Lemma 6 (continued)

We now have a sequence of projections, in principle going to infinity in both directions:

$$\ldots,\ q_{-N_0},\ \ldots,\ q_{-1},\ q_0,\ q_1,\ \ldots,\ q_{N-N_0},\ \ldots,\ q_{N-1},\ q_N,\ \ldots.$$

The ones shown are orthogonal, and conjugation by $u$ is the shift. The projections $q_0$ and $q_N$ are the characteristic functions of compact open sets which are disjoint but close to each other, and similarly for the pairs $q_{-1}$ and $q_{N-1}$ down to $q_{-N_0}$ and $q_{N-N_0}$. We are now going to use Berg's technique to splice this sequence along the pairs of indices $(-N_0,\, N - N_0)$ through $(0, N)$, obtaining a loop of length $N$ on which conjugation by $u$ is approximately the cyclic shift.



## Proof of Lemma 6 (continued)

Lemma 5 provides a partial isometry $w \in C^*(\mathbb{Z}, X, h)_Y$ such that $w^*w = q_0$ and $ww^* = q_N$. For $t \in [0, 1]$ define

$$v(t) = \cos(\pi t/2)(q_0 + q_N) + \sin(\pi t/2)(w - w^*).$$

Then $v(t)$ is a unitary in the corner

$$(q_0 + q_N)C^*(\mathbb{Z}, X, h)_Y(q_0 + q_N)$$

whose matrix with respect to the obvious block decomposition is

$$v(t) = \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

For $0 \leq k \leq N_0$ define $z_k = u^{-k} v(k/N_0) u^k$.

## Proof of Lemma 6 (continued)

We claim that $z_k \in C^*(\mathbb{Z}, X, h)_Y$ for $0 \leq k \leq N_0$. For $k = 0$ this is true by construction. For $1 \leq k \leq N_0$, with

$$a_k = q_0 u^k = (uq_{-1})(uq_{-2}) \cdots (uq_{-k}) \in C^*(\mathbb{Z}, X, h)_Y$$

and

$$b_k = q_N u^k = (uq_{N-1})(uq_{N-2}) \cdots (uq_{N-k}) \in C^*(\mathbb{Z}, X, h)_Y$$

(because $N_0 < N$), and using $T_{-k} \cap T_{N-k} = T_0 \cap T_N = \varnothing$, we can write

$$z_k = (a_k + b_k)^* v(k/N_0)(a_k + b_k) \in C^*(\mathbb{Z}, X, h)_Y.$$

Therefore $z_k$ is a unitary in the corner

$$(q_{-k} + q_{N-k}) C^*(\mathbb{Z}, X, h)_Y (q_{-k} + q_{N-k}).$$



## Proof of Lemma 6 (continued)

Moreover, adding estimates on the differences of the matrix entries at the second step,

$$\|uz_{k+1}u^* - z_k\| = \|v(k/N_0) - v((k-1)/N_0)\| \leq 2\pi/N_0 < \tfrac{1}{2}\varepsilon.$$

Now define $e_n = q_n$ for $0 \leq n \leq N - N_0$, and for $N - N_0 \leq n \leq N$ write $k = N - n$ and set $e_n = z_k q_{-k} z_k^*$. The two definitions for $n = N - N_0$ agree because $z_{N_0} q_{-N_0} z_{N_0}^* = q_{N-N_0}$, and moreover $e_N = e_0$. Therefore $ue_{n-1}u^* = e_n$ for $1 \leq n \leq N - N_0$, and also $ue_N u^* = e_1$, while for $N - N_0 < n \leq N$ we have

$$\|ue_{n-1}u^* - e_n\| \leq 2\|uz_{N-n+1}u^* - z_{N-n}\| < \varepsilon.$$

Also, clearly $e_n \in C^*(\mathbb{Z}, X, h)_Y$ for all $n$.



## Proof of Lemma 6 (continued)

Set $e = \sum_{n=1}^{N} e_n$ and $p = 1 - e$. We verify that $p$ satisfies (1) through (3):

1. $\|pa - ap\| < \varepsilon$ for all $a \in F \cup \{u\}$.
2. $pap \in pC^*(\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{u\}$.
3. There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

First,

$$p - upu^* = ueu^* - e = \sum_{n=N_0+1}^{N} (ue_{n-1}u^* - e_n).$$

The terms in the sum are orthogonal and have norm less than $\varepsilon$, so $\|upu^* - p\| < \varepsilon$. Furthermore, since $p \leq 1 - q_0 = 1 - \chi_Y$, we get $pup \in C^*(\mathbb{Z}, X, h)_Y$. This is (1) and (2) for the element $u \in F \cup \{u\}$.



## Proof of Lemma 6 (continued)

Next, let $f \in F$. The sets $T_0, T_1, \ldots, T_N$ all have diameter less than $\delta$. We have $d(h^N(y), y) < \delta$, so the choice of $\delta$ implies that $d(h^n(y), h^{n-N}(y)) < \delta_0$ for $N - N_0 \leq n \leq N$. Also, $T_{n-N} = h^{n-N}(T_0)$ has diameter less than $\delta$. Therefore $T_{n-N} \cup T_n$ has diameter less than $2\delta + \delta_0 \leq 3\delta_0$. Since $f$ varies by at most $\frac{1}{4}\varepsilon$ on any set with diameter less than $4\delta_0$, and since the sets

$$S_1 = T_1, \ S_2 = T_2, \ \ldots, \ S_{N-N_0-1} = T_{N-N_0-1},$$

$$S_{N-N_0} = T_{N-N_0} \cup T_{-N_0}, \ S_{N-N_0+1} = T_{N-N_0+1} \cup T_{-N_0+1}, \ \ldots,$$

$$\ldots, \ S_N = T_N \cup T_0$$

are disjoint, there is $g \in C(X)$ which is constant on each of these sets and satisfies $\|f - g\| < \frac{1}{2}\varepsilon$.

## Proof of Lemma 6 (continued)

Let the values of $g$ on these sets be $\lambda_1$ on $S_1$ through $\lambda_N$ on $S_N$. Then $ge_n = e_n g = \lambda_n e_n$ for $0 \leq n \leq N - N_0$. For $N - N_0 < n \leq N$ we use $e_n \in (q_{n-N} + q_n) C^*(\mathbb{Z}, X, h)_Y (q_{n-N} + q_n)$ to get,

$$ge_n = g(q_{n-N} + q_n)e_n = \lambda_n (q_{n-N} + q_n) e_n = e_n (q_{n-N} + q_n) g = e_n g.$$

Since $\|f - g\| < \frac{1}{2}\varepsilon$ and $ge = eg$, it follows that

$$\|pf - fp\| = \|fe - ef\| < \varepsilon.$$

This is (1) for $f$. That $pfp \in C^*(\mathbb{Z}, X, h)_Y$ follows from the fact that $f$ and $p$ are in this subalgebra. So we also have (2) for $f$.



## Proof of Lemma 6 (continued)

It remains only to verify (3). Using $h^{r_l}(Y) \subset U_l$ for $-N_0 \leq l \leq N$ at the third step, and with $Z = \bigcup_{l=-N_0}^{N} h^{r_l}(Y) \subset U$, we get (with Murray-von Neumann equivalence in $C^*(\mathbb{Z}, X, h)$)

$$1 - p = e \leq \sum_{l=-N_0}^{N} q_l \sim \sum_{l=-N_0}^{N} \chi_{h^{r_l}(Y)} = \chi_Z.$$

This completes the proof.



## Some comments on the general case

In the general case, one major complication is that one can't choose the Rokhlin towers to consist of compact open sets. It turns out that one must take $Y$ to be closed with nonempty interior, and replace the sets $Y_k$ by their closures. Then they are no longer disjoint. The algebra $C^*(\mathbb{Z}, X, h)_Y$ is now a very complicated subalgebra of $\bigoplus_{k=0}^{l} M_{n(k)} \otimes C(Y_k)$. It is what is known as a "recursive subhomogeneous algebra". Such algebras are generally not AF, and may have few or no nontrivial projections. The hypothesis on the range of $\rho$ (which was not used above, although it is automatic when $X$ is the Cantor set) must be used to produce sufficiently many nonzero projections and approximating finite dimensional subalgebras.



## Some comments on the general case (continued)

Here are a few more details. The algebra $C^*(\mathbb{Z}, X, h)_Y$ is not AF. So we choose closed sets $Y_n$ with

$$Y_0 \supset Y_1 \supset Y_2 \supset \cdots \quad \text{and} \quad \bigcap_{n=0}^{\infty} Y_n = \{y_0\}$$

for a suitable $y_0 \in X$, and such that $\mathrm{int}(Y_n) \neq \varnothing$ for all $n$. Then

$$C^*(\mathbb{Z}, X, h)_{Y_0} \subset C^*(\mathbb{Z}, X, h)_{Y_1} \subset C^*(\mathbb{Z}, X, h)_{Y_2} \subset \cdots$$

and

$$\overline{\bigcup_{n=0}^{\infty} C^*(\mathbb{Z}, X, h)_{Y_n}} = C^*(\mathbb{Z}, X, h)_{\{y_0\}}.$$

## Some comments on the general case (continued)

$C^*(\mathbb{Z}, X, h)_{\{y_0\}}$ is simple and has the same tracial states and $K_0$-group as $C^*(\mathbb{Z}, X, h)$. (These facts require proof. The one about $K_0$ is hard, but had already been done by Putnam.) Using this, and some results on dymanics (essentially "mean dimension zero"), the theory of direct limits of recursive subhomogeneous algebras can be used to prove that $C^*(\mathbb{Z}, X, h)_{\{y_0\}}$ has tracial rank zero. This turns out to be a sufficient substitute for $C^*(\mathbb{Z}, X, h)_Y$ being an AF algebra.



## The tracial state space of $C^*(\mathbb{Z}, X, h)_{\{y\}}$

We describe the proof of one of the pieces needed.

**Lemma**

Let $X$ be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Then the restriction map $T(C^*(\mathbb{Z}, X, h)) \to T\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big)$ is a bijection.

The proof follows from the following two results. The first is well known, and holds in much greater generality. The proofs are roughly the same, but the proof of the second is more complicated, since the algebra is smaller.



## Tracial states and invariant measures

**Proposition**

Let $X$ be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism. Then the restriction map $T(C^*(\mathbb{Z}, X, h)) \to T(C(X))$ is a bijection from $T(C^*(\mathbb{Z}, X, h))$ to the set of $h$-invariant Borel probability measures on $X$. Moreover, if $E\colon C^*(\mathbb{Z}, X, h) \to C(X)$ is the standard conditional expectation, and $\mu$ is an $h$-invariant Borel probability measure on $X$, then the tracial state $\tau_\mu$ on $C^*(\mathbb{Z}, X, h)$ which restricts to $\mu$ is given by the formula $\tau_\mu(a) = \int_X E(a)\, d\mu$ for $a \in C^*(\mathbb{Z}, X, h)$.



## Proof of the proposition

It is immediate that $\tau_\mu$ is positive, and that $\tau_\mu(1) = 1$. So $\tau_\mu$ is a state.

We prove $\tau_\mu(ab) = \tau_\mu(ba)$ for all $a, b \in C^*(\mathbb{Z}, X, h)$. Since $\tau_\mu$ is continuous, and since $C^*(\mathbb{Z}, X, h)$ is the closed linear span of all elements $fu^m$ with $f \in C(X)$ and $m \in \mathbb{Z}$, it suffices to prove $\tau_\mu((fu^m)(gu^n)) = \tau_\mu((gu^n)(fu^m))$ for all $f, g \in C(X)$ and $m, n \in \mathbb{Z}$. For $n \neq -m$, we have

$$\tau_\mu((fu^m)(gu^n)) = \tau_\mu((gu^n)(fu^m)) = 0,$$

while for $n = -m$, we have, using $h$-invariance of $\mu$ at the second step,

$$\begin{aligned}\tau_\mu((fu^m)(gu^n)) &= \tau_\mu\big(f(u^m g u^{-m})\big) = \int_X f\big(g \circ h^{-m}\big)\, d\mu \\ &= \int_X (f \circ h^m) g\, d\mu = \tau_\mu\big(g(u^{-m} f u^m)\big) = \tau_\mu\big((gu^n)(fu^m)\big).\end{aligned}$$

This completes the proof that $\tau_\mu$ is a tracial state on $C^*(\mathbb{Z}, X, h)$.

## Proof of the proposition (continued)

Now let $\tau$ be any tracial state on $C^*(\mathbb{Z}, X, h)$. Let $\mu$ be the Borel probability measure on $X$ determined by $\tau(f) = \int_X f \, d\mu$ for $f \in C(X)$. We complete the proof by showing that $\mu$ is $h$-invariant, and that $\tau = \tau_\mu$.

For the first, for every $f \in C(X)$, we use the trace property at the second step to get

$$\int_X (f \circ h^{-1}) \, d\mu = \tau(ufu^*) = \tau(u^*(uf)) = \tau(f) = \int_X f \, d\mu.$$

Since $f \in C(X)$ is arbitrary, it follows that $\mu$ is $h$-invariant.



## Proof of the proposition (continued)

For the second, since $C^*(\mathbb{Z}, X, h)$ is the closed linear span of all elements $fu^n$ with $f \in C(X)$ and $m \in \mathbb{Z}$, it suffices to prove $\tau(fu^n) = 0$ for $f \in C(X)$ and $n \in \mathbb{Z} \setminus \{0\}$. Since $h^n$ has no fixed points, there is an open cover of $X$ consisting of sets $U$ such that $h^n(U) \cap U = \varnothing$. Choose $g_1, g_2, \ldots, g_m \in C(X)$ which form a partition of unity subordinate to this cover. In particular, the supports of $g_j$ and $g_j \circ h^{-n}$ are disjoint for all $j$. For $1 \leq j \leq m$ we have, using the trace property at the second step, and the relation $u^n g u^{-n} = g \circ h^{-n}$ at the third step,

$$\tau(g_j f u^n) = \tau\big(g_j^{1/2} f u^n g_j^{1/2}\big) = \tau\big(g_j^{1/2} f \big(g_j^{1/2} \circ h^{-n}\big) u^n\big) = \tau(0) = 0.$$

Summing over $j$ gives $\tau(fu^n) = 0$. This completes the proof.



## Tracial states on the subalgebra

**Lemma**

Let $X$ be an infinite compact metric space, and let $h \colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Then the restriction map $T\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big) \to T(C(X))$ is a bijection from $T\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big)$ to the set of $h$-invariant Borel probability measures on $X$.



## Proof of the lemma

Applying the previous proposition and restricting from $C^*(\mathbb{Z}, X, h)$ to $C^*(\mathbb{Z}, X, h)_{\{y\}}$, we see that every $h$-invariant Borel probability measure on $X$ gives a tracial state on $C^*(\mathbb{Z}, X, h)_{\{y\}}$.

Now let $\tau$ be any tracial state on $C^*(\mathbb{Z}, X, h)_{\{y\}}$. Let $\mu$ be the Borel probability measure on $X$ determined by $\tau(f) = \int_X f \, d\mu$ for $f \in C(X)$. As in the proof of the previous proposition, we complete the proof by showing that $\mu$ is $h$-invariant, and that $\tau = \tau_\mu|_{C^*(\mathbb{Z}, X, h)_{\{y\}}}$.

## Proof of the lemma (continued)

For the first, we again show that $\int_X (f \circ h^{-1})\, d\mu = \int_X f\, d\mu$ for every $f \in C(X)$. This is clearly true for constant functions $f$. Therefore, it suffices to consider functions $f$ such that $f(y) = 0$. For such a function $f$, write $f = f_1 f_2^*$ with $f_1 f_2 \in C(X)$ such that $f_1(y) = f_2(y) = 0$. (For example, take $f_1 = f|f|^{-1/2}$ and $f_2 = |f|^{1/2}$.) Then $uf_1,\ uf_2 \in C^*(\mathbb{Z}, X, h)_{\{y\}}$. So

$$f \circ h^{-1} = ufu^* = (uf_1)(uf_2)^* \in C^*(\mathbb{Z}, X, h)_{\{y\}}.$$

We now use the trace property at the second step to get

$$\int_X (f \circ h^{-1})\, d\mu = \tau\big((uf_1)(uf_2)^*\big) = \tau\big((uf_2)^*(uf_1)\big) = \tau(f) = \int_X f\, d\mu.$$

Thus $\mu$ is $h$-invariant.



## Proof of the lemma (continued)

For the second, we first claim that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is the closed linear span of all elements of the form $fu^m$, with $f \in C(X)$ and $m \in \mathbb{Z}$, which actually happen to be in $C^*(\mathbb{Z}, X, h)_{\{y\}}$. The claim follows from three facts: all the generators listed in the definition of $C^*(\mathbb{Z}, X, h)_{\{y\}}$ have this form (using $uf = (f \circ h^{-1})u$), the product of two things of this form again has this form (use $(fu^m)(gu^n) = [f(g \circ h^{-m})]u^{m+n}$, which is again in $C^*(\mathbb{Z}, X, h)_{\{y\}}$ because $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is closed under multiplication), and the adjoint of something of this form again has this form (by a similar argument).

It now suffices to prove that if $fu^n \in C^*(\mathbb{Z}, X, h)_{\{y\}}$ and $n \neq 0$, then $\tau(fu^m) = 0$. The argument is the same as in the proof of the previous Proposition; one merely needs to note that the elements $g_j$ used there are in $C(X) \subset C^*(\mathbb{Z}, X, h)_{\{y\}}$. This completes the proof.