Freeness of actions of finite groups on C*-algebras: Part 1

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A rough outline

- Group actions on spaces.
- Group actions on C*-algebras.
- Free actions on spaces.
- Crossed product C*-algebras.
- The Rokhlin property.
- Pointwise outerness.
- The tracial Rokhlin property.
- Applications of the tracial Rokhlin property.

(This goes beyond the end of the first talk.)

Reference

This talk, and the next, is based on part of the survey article:

N. C. Phillips, *Freeness of actions of finite groups on C*-algebras*, preprint (arXiv: 0902.4891 [math.OA]).

A slightly revised version is to appear in the proceedings of a conference in Leiden (Summer 2008).

However, there are few proofs in the survey; instead, it refers to the original papers.

Group actions on spaces

This talk is about noncommutative dynamics. For (partial) motivation, let's look at the commutative situation.

Let G be a group and let X be a set. Then an action of G on X is a map $(g, x) \mapsto gx$ from $G \times X$ to X such that:

- $1 \cdot x = x$ for all $x \in X$.
- $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

If G and X have topologies, then $(g, x) \mapsto gx$ is required to be (jointly) continuous.

In this talk, groups will usually be discrete (often finite), so continuity means that $x \mapsto gx$ is continuous for all $g \in G$. Since the action of g^{-1} is also continuous, this map is in fact a homeomorphism.

Examples of group actions on spaces

- Any group G has a trivial action on any space X, given by $gx = x$ for all $g \in G$ and $x \in X$.
- Any group G acts on itself by (left) translation: gh is the usual product of g and h.
- Similarly, if G is a group and H is a (closed) subgroup (not necessarily normal), then G has a translation action on G/H.
- The finite cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ acts on the circle S^1 by rotation: the standard generator acts as multiplication by $e^{2\pi i/n}$.
- $\mathbb{Z}$ acts on S^1 by rotation: fix $\theta \in \mathbb{R}$, and let the generator act via the homeomorphism $\zeta \mapsto e^{2\pi i\theta}\zeta$.
- $\mathbb{Z}_2$ acts on S^1 via the order two homeomorphism $\zeta \mapsto \overline{\zeta}$.
- $\mathbb{Z}_2$ acts on S^n via the order two homeomorphism $x \mapsto -x$.
- Time evolution is an action of $\mathbb{R}$, and discrete time evolution is an action of $\mathbb{Z}$.

Group actions on C*-algebras

Today, all C*-algebras will be separable and unital, except for a few obvious exceptions. (We won't need this right away.)

Let G be a group and let A be a C*-algebra. Then an action of G on A is a homomorphism $g \mapsto \alpha_g$ from G to $\mathrm{Aut}(A)$, such that for every $a \in A$, the function $g \mapsto \alpha_g(a)$ is continuous from G to A.

Every action of G on a compact Hausdorff space X gives an action $\alpha\colon G \to \mathrm{Aut}(C(X))$, given by the formula $\alpha_g(f)(x) = f(g^{-1}x)$ for $g \in G$, $f \in C(X)$, and $x \in X$. Furthermore, every action of G on $C(X)$ comes this way from an action of G on X.

All the examples of actions on spaces above give actions on commutative C*-algebras.

Suppose $g \mapsto u_g$ is a (continuous) homomorphism from G to the unitary group of a C*-algebra A. Then $\alpha_g(a) = u_g a u_g^*$ defines an action of G on A. (We write $\alpha_g = \mathrm{Ad}(u_g)$.) This is an *inner* action.

Product type actions

We describe a particular "product type action". Let $A_n = (M_2)^{\otimes n}$, the tensor product of n copies of the algebra M_2 of 2×2 matrices. Thus $A_n \cong M_{2^n}$. Define

$$\varphi_n\colon A_n \to A_{n+1} = A_n \otimes M_2$$

by $\varphi_n(a) = a \otimes 1$. Let A be the (completed) direct limit $\varinjlim_n A_n$. (This is just the 2^∞ UHF algebra.) Define a unitary $v \in M_2$ by

$$v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define $u_n \in A_n$ by $u_n = v^{\otimes n}$. Define $\alpha_n \in \mathrm{Aut}(A_n)$ by $\alpha_n = \mathrm{Ad}(u_n)$. Then α_n is an inner automorphism of order 2. Using $u_{n+1} = u_n \otimes v$, one can easily check that $\varphi_n \circ \alpha_n = \alpha_{n+1} \circ \varphi_n$ for all n, and it follows that the α_n determine an order 2 automorphism α of A. Thus, we have an action of $\mathbb{Z}_2$ on A. This action is not inner, although it is "approximately inner".

General product type actions

We write the automorphism above as

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

In general, one can use an arbitrary group, one need not choose the same unitary representation in each tensor factor, and the tensor factors need not all be the same size.

More examples of product type actions

We will later use the following two additional examples:

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_3,$$

and

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}(\mathrm{diag}(-1, 1, 1, \ldots, 1)) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_{2^n+1}.$$

In the second one, there are supposed to be 2^n ones on the diagonal, giving a $(2^n + 1) \times (2^n + 1)$ matrix.

Irrational rotation algebras

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Recall the irrational rotation algebra A_θ, the (simple, and unique) C*-algebra generated by two unitaries u and v satisfying the commutation relation $vu = e^{2\pi i \theta} uv$.

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on A_θ by sending the matrix

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$$

to the automorphism determined by

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}}$$

and

$$\alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$

The algebra A_θ is often considered to be a noncommutative analog of the torus $S^1 \times S^1$ and this action is the analog of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$.

Irrational rotation algebras (continued)

The group $\mathrm{SL}_2(\mathbb{Z})$ has finite subgroups of orders 2, 3, 4, and 6. Restriction gives actions of these groups on the irrational rotation algebras.

Generators:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ (for } \mathbb{Z}_2\text{)}, \qquad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \text{ (for } \mathbb{Z}_3\text{)},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ (for } \mathbb{Z}_4\text{)}, \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \text{ (for } \mathbb{Z}_6\text{)}.$$

In terms of generators of A_θ, and omitting the scalar factors (which are not necessary when one restricts to these subgroups), the action of $\mathbb{Z}_2$ is generated by

$$u \mapsto u^* \quad \text{and} \quad v \mapsto v^*,$$

and the action of $\mathbb{Z}_4$ is generated by

$$u \mapsto v \quad \text{and} \quad v \mapsto u^*.$$

Irrational rotation algebras (continued)

A_θ is generated by unitaries u and v such that $vu = e^{2\pi i\theta} uv$.

There is a "gauge action" on A_θ, which multiplies the generators by scalars of absolute value 1. In particular, the group $\mathbb{Z}_n$ acts on A_θ by sending a generator of the group to the automorphism determined by

$$u \mapsto e^{2\pi i/n} u \quad \text{and} \quad v \mapsto v.$$

One can also use

$$u \mapsto u \quad \text{and} \quad v \mapsto e^{2\pi i/n} v.$$

Tensor products

Assume (for convenience) that A is nuclear and unital. Then there is an action of $\mathbb{Z}_2$ on $A \otimes A$ generated by the "tensor flip" $a \otimes b \mapsto b \otimes a$.

Similarly, the symmetric group S_n acts on $A^{\otimes n}$, but we will stick to the case $n = 2$ above.

The tensor flip on the 2^∞ UHF algebra A turns out to be conjugate to a product type action, namely

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{on} \quad \bigotimes_{n=1}^{\infty} M_4.$$

Another interesting example is gotten by taking A to be the Jiang-Su algebra Z. The algebra Z is simple, separable, unital, and nuclear. It has no nontrivial projections, its Elliott invariant is the same as for $\mathbb{C}$, and $Z \otimes Z \cong Z$.

Freeness

Definition

An action of the group G on the space X is *free* if whenever $g \in G \setminus \{1\}$, then $gx \neq x$ for all $x \in X$.

Assuming G has more than one element:

- The trivial action is never free.
- The action of G on itself by translation is free.
- The action of G on G/H by translation is free only if $H = \{1\}$, in which case we have the action of G on itself by translation.

Freeness (continued)

- Rotation by $e^{2\pi i/n}$ generates a free action of $\mathbb{Z}_n$ on S^1.
- Rotation by $e^{2\pi i\theta}$ generates a free action of $\mathbb{Z}$ on S^1 if and only if θ is irrational.
- The order 2 homeomorphism $\zeta \mapsto \overline{\zeta}$ generates an action of $\mathbb{Z}_2$ on S^1 which is not free.
- The order 2 homeomorphism $x \mapsto -x$ generates a free action of $\mathbb{Z}_2$ on S^n.

What is good about freeness?

- Free actions are "regular". For example, if a finite group G acts freely on a manifold X via diffeomorphisms, then the quotient space X/G (obtained by identifying each orbit $\{gx\colon g \in G\}$ to a point) is again a manifold.
- "Classification": Free actions of finite groups on path connected spaces come from quotient groups of the fundamental group of the quotient space.
- Algebraic topology: If G is compact and acts freely on X, then the equivariant K-theory $K_G^*(X)$ is isomorphic to the ordinary K-theory $K^*(X/G)$.
- Behavior of crossed products.

Crossed products

Let G be a locally compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on a C*-algebra A. There is a *crossed product C*-algebra* $C^*(G, A, \alpha)$, which is a kind of generalization of the group C*-algebra $C^*(G)$. Crossed products are quite important in the theory of C*-algebras.

One motivation: Suppose G is a semidirect product $N \rtimes H$. The action of H on N gives an action $\alpha\colon H \to \mathrm{Aut}(C^*(N))$, and one has $C^*(G) \cong C^*(H, \, C^*(N), \, \alpha)$. Thus, crossed products appear even if one is only interested in group C*-algebras and unitary representations of groups.

Another motivation: The noncommutative version of X/G is the fixed point algebra A^G. In particular, for compact G, one can check that $C(X/G) \cong C(X)^G$. For noncompact groups, often X/G is very far from Hausdorff and A^G is far too small. The crossed product provides a much more generally useful algebra, which is the "right" substitute for the fixed point algebra when the action is free.

Crossed products by finite groups

Let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on a C*-algebra A. As a vector space, $C^*(G, A, \alpha)$ is the group ring $A[G]$, consisting of all finite formal linear combinations of elements in G with coefficients in A. The multiplication and adjoint are given by

$$(a \cdot g)(b \cdot h) = (a[gbg^{-1}]) \cdot (gh) = (a\alpha_g(b)) \cdot (gh)$$

and

$$(a \cdot g)^* = \alpha_g^{-1}(a^*) \cdot g^{-1}$$

for $a, b \in A$ and $g, h \in G$, extended linearly. There is a unique norm which makes this a C*-algebra.

If G is discrete but not finite, $C^*(G, A, \alpha)$ is the completion of $A[G]$ in a suitable norm. (Before completion, we have the *skew group ring*.)

Examples of crossed products

Despite their importance, crossed products are often very hard to compute. We give some examples, which can be obtained by direct computations (which are mostly not obvious) or from general theory.

First, the trivial action of G on a one point space gives the crossed product $C^*(G)$, the usual group C*-algebra. This is essentially by definition.

In the examples below, note the appearance of the orbit space X/G.

Some crossed products

- The trivial action of G on X gives the crossed product $C(X, C^*(G))$. There is an analogous result for the trivial action on any C*-algebra.
- The action of G on itself by translation gives the crossed product $K(L^2(G))$.
- The action of G on G/H by translation gives the crossed product $K(L^2(G/H)) \otimes C^*(H)$. (Observe the effect of the failure of the action to be free.)
- Rotation by $e^{2\pi i/n}$ generates a free action of $\mathbb{Z}_n$ on S^1, and the crossed product is $C(S^1/\mathbb{Z}_n, M_n) \cong C(S^1, M_n)$.
- Rotation by $e^{2\pi i\theta}$, for θ irrational, gives the crossed product A_θ.
- The order 2 homeomorphism $\zeta \mapsto \overline{\zeta}$ gives

$$\{f : [-1, 1] \to M_2 : f \text{ is continuous and } f(1), f(-1) \text{ are diagonal}\}.$$

- The order 2 homeomorphism $x \mapsto -x$ generates a free action of $\mathbb{Z}_2$ on S^n. Its crossed product is the section algebra of a locally trivial bundle over $\mathbb{R}P^n$ with fiber M_2, a twisted version of $C(S^n/\mathbb{Z}_2, M_2)$.

Crossed products by inner actions

Recall the inner action $\alpha_g = \text{Ad}(u_g)$ for a continuous homomorphism $g \mapsto u_g$ from G to the unitary group of a C*-algebra A. It is easy to show that the crossed product is the same as for the trivial action, in a canonical way.

For G discrete, the isomorphism sends $a \cdot g$ to $au_g \cdot g$. (Check that this defines an isomorphism of the skew group rings!)

What is a free action on a C*-algebra?

To keep things simple, we assume the group is finite. Even then, there are several answers, and, with the current state of knowledge, some of them require conditions on the algebra or the group.

By name, in rough decreasing order of strength:

- Free action on the primitive ideal space.
- The Rokhlin property.
- The tracial Rokhlin property.
- Pointwise outerness (each α_g, for $g \neq 1$, is not of the form $\text{Ad}(u)$ for any unitary u).
- Full (strong) Connes spectrum.
- Saturation (full (strong) Arveson spectrum).

If the algebra is simple (the case I will concentrate on), the primitive ideal space consists of just one point, so free action on it is impossible. I will concentrate on the next two conditions, with some discussion of pointwise outerness. I omit the definition of the (strong) Arveson spectrum and (strong) Connes spectrum.

Motivation for the Rokhlin property

Let X be the Cantor set, let G be a finite group, and let G act freely on X.

Fix $x_0 \in X$. Then the points gx_0, for $g \in G$, are all distinct, so by continuity and total disconnectedness of the space, there is a compact open set $K \subset X$ such that $x_0 \in K$ and the sets gK, for $g \in G$, are all disjoint.

By repeating this process, one can find a compact open set $L \subset X$ such that the sets $L_g = gL$, for $g \in G$, are all disjoint, and such that their union is X.

The Rokhlin property

Definition

Let A be a separable unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *Rokhlin property* if for every finite set $F \subset A$, and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

For C*-algebras, this goes back to about 1980, and is adapted from earlier work on von Neumann algebras.

The Rokhlin property is good for understanding the structure of group actions.

The Rokhlin property (continued)

The conditions in the definition:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

The projections e_g are the analogs of the characteristic functions of the compact open sets gL from the Cantor set example.

Condition (1) is an approximate version of $gL_h = L_{gh}$. (Recall that $L_g = gL$.)

Condition (3) is the requirement that X be the disjoint union of the sets L_g.

Condition (2) is vacuous for a commutative C*-algebra. In the noncommutative case, without it the inner action of $\mathbb{Z}_2$ on M_2 generated by $\mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ would have the Rokhlin property.

An example

Recall the product type action of $\mathbb{Z}_2$ generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(v) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2$$

with $v = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We show that this action has the Rokhlin property.

Define projections $p_0, p_1 \in M_2$ by

$$p_0 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad p_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then $v p_0 v^* = p_1$, $v p_1 v^* = p_0$, and $p_0 + p_1 = 1$.

The action is generated by $\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(v)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, $v p_0 v^* = p_1$, $v p_1 v^* = p_0$, and $p_0 + p_1 = 1$.

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections e_g such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Since the union of the subalgebras $(M_2)^{\otimes n} = A_n$ is dense, we can assume $F \subset A_n$ for some n.

For $g = 0, 1$ take

$$e_g = 1_{A_n} \otimes p_g \in A_n \otimes M_2 = A_{n+1} \subset A.$$

Clearly $e_0 + e_1 = 1$. Check that $\alpha(e_0) = e_1$ and $\alpha(e_1) = e_0$. Also, e_0 and e_1 actually commute with everything in F, since the nontrival parts are in different tensor factors.

Crossed products by actions with the Rokhlin property

Theorem

Let A be a unital AF algebra. Let G be a finite group, and let $\alpha \colon G \to \text{Aut}(A)$ have the Rokhlin property. Then $C^*(G, A, \alpha)$ is AF.

Crossed products by actions of finite groups with the Rokhlin property preserve many other structural properties of C*-algebras. See, for example, joint work with Osaka.

The basic idea: Let $e_g \in A$, for $g \in G$, be Rokhlin projections. Let $u_g \in C^*(G, A, \alpha)$ be the canonical unitary implementing the automorphism α_g. Then $w_{g,h} = u_{gh^{-1}} e_h$ defines an approximate system of matrix units in $C^*(G, A, \alpha)$. Let $(v_{g,h})_{g,h \in G}$ be a nearby true system of matrix units. Using the homomorphism $M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ given by $v_{g,h} \otimes d \mapsto v_{g,1} d v_{1,h}$, one can approximate $C^*(G, A, \alpha)$ by matrix algebras over corners of A.

Crossed products by actions with the Rokhlin property (continued)

To prove the theorem, we prove that for every finite set $S \subset C^*(G, A, \alpha)$ and every $\varepsilon > 0$, there is an AF subalgebra $D \subset C^*(G, A, \alpha)$ such that every element of S is within ε of an element of D. It suffices to consider a finite set of the form $S = F \cup \{u_g \colon g \in G\}$, where F is a finite subset of the unit ball of A and $u_g \in C^*(G, A, \alpha)$ is the canonical unitary implementing the automorphism α_g. So let $F \subset A$ be a finite subset with $\|a\| \leq 1$ for all $a \in F$ and let $\varepsilon > 0$.

Set $n = \text{card}(G)$, and set $\varepsilon_0 = \varepsilon/(4n)$.

Crossed products by actions with the Rokhlin property (continued)

Choose $\delta > 0$ such that a system of δ-approximate $n \times n$ matrix units, in which the diagonal approximate matrix units are orthogonal projections summing to 1, can be approximated within ε_0 by a true system of matrix units, with the given diagonal matrix units.

More precisely, choose $\delta > 0$ such that, whenever $(e_{j,k})_{1 \leq j,k \leq n}$ is a system of matrix units for M_n, whenever B is a unital C*-algebra, and whenever $w_{j,k}$, for $1 \leq j, k \leq n$, are elements of B such that $\|w_{j,k}^* - w_{k,j}\| < \delta$ for $1 \leq j, k \leq n$, such that $\|w_{j_1,k_1} w_{j_2,k_2} - \delta_{j_2,k_1} w_{j_1,k_2}\| < \delta$ for $1 \leq j_1, j_2, k_1, k_2 \leq n$, and such that the $w_{j,j}$ are orthogonal projections with $\sum_{j=1}^{n} w_{j,j} = 1$, then there exists a unital homomorphism $\varphi \colon M_n \to B$ such that $\varphi(e_{j,j}) = w_{j,j}$ for $1 \leq j \leq n$ and $\|\varphi(e_{j,k}) - w_{j,k}\| < \varepsilon_0$ for $1 \leq j, k \leq n$.

Also require $\delta \leq \varepsilon/[2n(n+1)]$.

Crossed products by actions with the Rokhlin property (continued)

Apply the Rokhlin property to α with F as given and with δ in place of ε, obtaining projections $e_g \in A$ for $g \in G$.

Define $w_{g,h} = u_{gh^{-1}} e_h$ for $g, h \in G$.

We claim that the $w_{g,h}$ form a δ-approximate system of $n \times n$ matrix units in $C^*(G, A, \alpha)$. We estimate:

$$\begin{aligned} \|w_{g,h}^* - w_{h,g}\| &= \|e_h u_{gh^{-1}}^* - u_{hg^{-1}} e_g\| \\ &= \|u_{gh^{-1}} e_h u_{gh^{-1}}^* - e_g\| = \|\alpha_{gh^{-1}}(e_h) - e_g\| < \delta. \end{aligned}$$

Crossed products by actions with the Rokhlin property (continued)

Recall that $w_{g,h} = u_{gh^{-1}} e_h$ for $g, h \in G$.

We proved $\|w_{g,h}^* - w_{h,g}\| < \delta$.

Next, using $e_g e_h = \delta_{g,h} e_h$ at the second step,

$$\begin{aligned} \|w_{g_1,h_1} w_{g_2,h_2} - \delta_{g_2,h_1} w_{g_1,h_2}\| &= \left\|u_{g_1h_1^{-1}} e_{h_1} u_{g_2h_2^{-1}} e_{h_2} - \delta_{g_2,h_1} u_{g_1h_2^{-1}} e_{h_2}\right\| \\ &= \left\|u_{g_1h_1^{-1}} e_{h_1} u_{g_2h_2^{-1}} e_{h_2} - u_{g_1h_1^{-1}g_2h_2^{-1}} e_{h_2g_2^{-1}h_1} e_{h_2}\right\| \\ &= \left\|u_{g_1h_1^{-1}g_2h_2^{-1}} \left(u^*_{g_2h_2^{-1}} e_{h_1} u_{g_2h_2^{-1}} - e_{h_2g_2^{-1}h_1}\right) e_{h_2}\right\| < \delta. \end{aligned}$$

Finally, $\sum_{g\in G} w_{g,g} = \sum_{g\in G} e_g = 1$. This proves the claim that we have a system of δ-approximate matrix units.

Also, $w_{g,g} = u_1 e_g = e_g$, so the $w_{g,g}$ are orthogonal projections which add up to 1.

Crossed products by actions with the Rokhlin property (continued)

Let $(v_{g,h})_{g,h\in G}$ be a system of matrix units for M_n. By the choice of δ, there exists a unital homomorphism $\varphi_0 \colon M_n \to C^*(G, A, \alpha)$ such that $\|\varphi_0(v_{g,h}) - w_{g,h}\| < \varepsilon_0$ for all $g, h \in G$, and $\varphi_0(v_{g,g}) = e_g$ for all $g \in G$. Now define a unital homomorphism $\varphi \colon M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ by $\varphi(v_{g,h} \otimes d) = \varphi_0(v_{g,1}) d \varphi_0(v_{1,h})$ for $g, h \in G$ and $d \in e_1 A e_1$. Corners of AF algebras are AF, and φ is injective, so $D = \varphi(M_n \otimes e_1 A e_1)$ is an AF subalgebra of $C^*(G, A, \alpha)$. We complete the proof by showing that every element of S is within ε of an element of D.

Crossed products by actions with the Rokhlin property (continued)

For $g \in G$ we have $\sum_{h\in G} \varphi_0(v_{gh,h}) \in D$ and

$$\begin{aligned} \left\|u_g - \sum\nolimits_{h\in G} \varphi_0(v_{gh,h})\right\| &\leq \sum_{h\in G} \|u_g e_h - \varphi_0(v_{gh,h})\| \\ &= \sum_{h\in G} \|w_{gh,h} - \varphi_0(v_{gh,h})\| < n\varepsilon_0 \leq \varepsilon. \end{aligned}$$

This gives the approximation of the canonical unitaries in the crossed product.

Crossed products by actions with the Rokhlin property (continued)

Now let $a \in F$.

Recall that $\varphi \colon M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ is defined by $\varphi(v_{g,h} \otimes d) = \varphi_0(v_{g,1}) d \varphi_0(v_{1,h})$ for $g, h \in G$ and $d \in e_1 A e_1$.

The obvious first step in approximating a is to use

$$\sum_{g\in G} e_g a e_g.$$

In fact, and this is perhaps the main trick, one needs in the end to (implicitly) use the approximation

$$\sum_{g\in G} \alpha_g(e_1 \alpha_g^{-1}(a) e_1).$$

This happens because the definition of φ sends $e_{g,h} \otimes d$, for $d \in e_1 A e_1$, to an element obtained by using (approximately) the action of the group elements g and h.

Set
$$b = \sum_{g \in G} v_{g,g} \otimes e_1 \alpha_g^{-1}(a) e_1 \in M_n \otimes e_1 A e_1.$$
Using $\|e_g a e_h\| \leq \|[e_g, a]\| + \|a e_g e_h\|$, we get
$$\left\| a - \sum_{g \in G} e_g a e_g \right\| \leq \sum_{g \neq h} \|e_g a e_h\| < n(n-1)\delta.$$
We use this, and the inequalities
$$\|\varphi_0(v_{g,1})e_1 - u_g e_1\| < \varepsilon_0 \quad \text{and} \quad \left\|e_1 \alpha_g^{-1}(a) e_1 - \alpha_g^{-1}(e_g a e_g)\right\| < 2\delta,$$
to get
$$\begin{aligned} \|a - \varphi(b)\| &= \left\| a - \sum_{g \in G} \varphi_0(v_{g,1}) e_1 \alpha_g^{-1}(a) e_1 \varphi_0(v_{1,g}) \right\| \\ &< 2n\varepsilon_0 + \left\| a - \sum_{g \in G} u_g e_1 \alpha_g^{-1}(a) e_1 u_g^* \right\| \\ &< 2n\varepsilon_0 + 2n\delta + \left\| a - \sum_{g \in G} u_g \alpha_g^{-1}(e_g a e_g) u_g^* \right\| \\ &< 2n\varepsilon_0 + 2n\delta + n(n-1)\delta \leq \varepsilon. \end{aligned}$$
This completes the proof of the theorem.

Freeness and the Rokhlin property

A free action of a finite group on the Cantor set has the Rokhlin property.

Free actions on connected spaces don't, since there are no nontrivial projections.

We will consider mostly simple C*-algebras with many projections. Recall that the irrational rotation algebras UHF algebras have real rank zero. If there are not enough projections, the situation is much less well understood, although there has been some recent progress. In the nonsimple case, rather little is known.

Tracial states and the Rokhlin property

Recall that if A is a unital C*-algebra, $p, q \in A$ are projections with $\|p - q\| < 1$, and τ is a tracial state on A, then $\tau(p) = \tau(q)$.

Suppose now A has a unique tracial state. (This is true for both UHF algebras and irrational rotation algebras.) Let G be finite, and let $\alpha \colon G \to \mathrm{Aut}(A)$ have the Rokhlin property. In the definition take $\varepsilon = 1$ and $F = \varnothing$. We get projections e_g such that, in particular:

- $\|\alpha_g(e_1) - e_g\| < 1$ for all $g \in G$.
- $\sum_{g \in G} e_g = 1$.

Since τ is unique, we have $\tau \circ \alpha_g = \tau$ for all $g \in G$. So $\tau(e_g) = \tau(e_1)$. It follows that
$$\tau(e_1) = \frac{1}{\mathrm{card}(G)}.$$

The Rokhlin property and A_θ

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and recall that A_θ is generated by unitaries u and v satisfying $vu = e^{2\pi i \theta} uv$. Further recall the action $\alpha \colon \mathbb{Z}_n \to \mathrm{Aut}(A_\theta)$ generated by
$$u \mapsto e^{2\pi i/n} u \quad \text{and} \quad v \mapsto v.$$

This is a noncommutative version of the free action of $\mathbb{Z}_n$ on $S^1 \times S^1$ given by rotation by $e^{2\pi i/n}$ in the first coordinate. Moreover, A_θ has many projections, like $C(X)$ when X is the Cantor set. So one would hope that α has the Rokhlin property.

The Rokhlin property and A_θ (continued)

In fact, *no* action of *any* nontrivial finite group on A_θ has the Rokhlin property! The reason is that there is no projection $e \in A_\theta$ with $\tau(e) = \frac{1}{n}$, for any $n \geq 2$. (Recall that the tracial state τ defines an isomorphism $\tau_* \colon K_0(A_\theta) \to \mathbb{Z} + \theta\mathbb{Z}$.)

For similar reasons, no action of $\mathbb{Z}_2$ on $D = \bigotimes_{n=1}^{\infty} M_3$ has the Rokhlin property. (The tracial state τ defines an isomorphism $\tau_* \colon K_0(D) \to \mathbb{Z}\left[\frac{1}{3}\right]$.)

There are more subtle obstructions to the Rokhlin property. For example, $M_2 \otimes \bigotimes_{n=1}^{\infty} M_3$ does have projections with trace $\frac{1}{2}$, but there are still no actions of $\mathbb{Z}_2$ with the Rokhlin property. (One at least needs a copy of M_{2^n}) for every n.)

In fact, the Rokhlin property is very rare.