Intersection cohomology and quantum cohomology of conical symplectic resolutions

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Abstract. For any conical symplectic resolution, we give a conjecture relating the intersection cohomology of the singular cone to the quantum cohomology of its resolution. We prove this conjecture for hypertoric varieties, recovering the ring structure on hypertoric intersection cohomology that was originally constructed by Braden and the second author.

1. Introduction

Let $\tilde{X}$ be a conical symplectic resolution of X; examples include the Springer resolution, Hilbert schemes of points on ALE spaces, quiver varieties, hypertoric varieties, and transverse slices to Schubert varieties in the affine Grassmannian. The purpose of this paper is to state a conjectural relationship between the intersection cohomology of X and the quantum cohomology of $\tilde{X}$ (Conjecture 2.5), and to prove this conjecture for hypertoric varieties (Theorem 3.15).

Before describing the conjecture itself, we say a few words about the significance of the two sides. Intersection cohomology groups of quiver varieties were shown by Nakajima to coincide with multiplicity spaces of simple modules in standard modules over a specialized quantum loop algebra [Nak01, 3.3.2 & 14.3.10]. The equivariant intersection cohomology of a hypertoric variety is isomorphic to the Orlik-Terao algebra of a hyperplane arrangement [BP09, 4.5], which has been the subject of much recent study [Ter02, PS06, ST09, Sch11, VLR13, DGT14, Le14, Liu]. The equivariant intersection cohomology groups of slices in the affine Grassmannian, with the equivariant parameters specialized to generic values, are isomorphic via the geometric Satake correspondence to weight spaces of simple representations for the Langlands dual group [Gin, 3.11 & 5.2].

On the quantum cohomology side, Okounkov and Pandharipande studied the Hilbert scheme of points in the plane [OP10], and Maulik and Oblomkov studied more generally the Hilbert scheme of points on an ALE space of type A [MO09]. Braverman, Maulik, and Okounkov computed the quantum cohomology of the Springer resolution [BMO11] and gave some indication of how to proceed for arbitrary conical symplectic resolutions. This program was carried out for quiver varieties by Maulik and Okounkov [MO], who relate their quantum cohomology to the representation theory of the Yangian, and for hypertoric varieties by Shenfeld and the first

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author [MS13]. This last paper gives an explicit generators-and-relations presentation of the hypertoric quantum cohomology ring, which is a large part of what makes the hypertoric case of our conjecture more tractable than the others.

Our conjecture very roughly says that the intersection cohomology of X is isomorphic to the quantum cohomology of $\tilde{X}$ specialized at $q = 1$. Of course, this cannot quite be correct as stated. The first problem is that quantum cohomology is an algebra over power series, not polynomials, so it does not make sense to set q equal to 1. We address this problem simply by working with the subalgebra of quantum cohomology generated by ordinary cohomology and polynomials in q, which is tautologically an algebra over polynomials in q. The second problem is that we would expect any specialization of the quantum parameters to have the same dimension as the cohomology of $\tilde{X}$, and the intersection cohomology of X is strictly smaller than that. Indeed, the actual statement involves taking a quotient of this specialization by the annihilator of $\hbar$, where $\hbar$ is the equivariant parameter for the conical action of the multiplicative group. A precise formulation of the conjecture appears in Section 2. We work out the example of $T^*\mathbb{P}^1$ in explicit detail, and give a heuristic reason why we would expect the conjecture to hold in general (Example 2.8).

One of the interesting consequences of our conjecture would be that the intersection cohomology of X inherits a ring structure from the quantum cohomology of $\tilde{X}$. As mentioned above, the intersection cohomology of a hypertoric variety is already known to have a natural ring structure by work of Braden and the second author [BP09]. However, the techniques in that paper were very combinatorial, and it was never adequately explained why such a structure should exist. We regard the proof of our conjecture for hypertoric varieties as an explanation of where this mysterious ring structure comes from. See Section 3.5 for a more detailed discussion of the relationship between our results and those of [BP09]. For other conical symplectic resolutions, our (conjectural) ring structure on the intersection cohomology of X appears to be new. In particular, when X is a slice in the affine Grassmannian, our conjecture posits the existence of a natural ring structure on a weight space of an irreducible representation of the Langlands dual group. This may be related to the ring structure on an entire irreducible representation constructed by Feigin, Frenkel, and Rybnikov [FFR10] (Remark 2.10).

Section 2 is devoted to the statement of our conjecture, while the remainder of the paper is dedicated to the proof in the hypertoric case. The proof involves two technical results about Orlik-Terao algebras that we believe may be of independent interest, and we therefore placed them in an appendix that can be read independently from the rest of the paper.

2. Statement of the conjecture

The purpose of this section is to make the necessary definitions to state our conjecture.

2.1 Conical symplectic resolutions

Let $(\tilde{X}, \omega)$ be a symplectic variety equipped with an action of $\mathbb{C}^\times$, and let $X = \operatorname{Spec} \mathbb{C}[\tilde{X}]$. We say that $\tilde{X}$ is a **conical symplectic resolution** if $\mathbb{C}^\times$ acts on ω with positive weight, $\mathbb{C}[\tilde{X}]$ is non-negatively graded with only the constants in degree zero, and the natural map from $\tilde{X}$ to X is a projective resolution of singularities. Examples of conical symplectic resolutions include the following:

- $\tilde{X}$ is a crepant resolution of $X = \mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $\mathrm{SL}(2;\mathbb{C})$. The action of $\mathbb{C}^\times$ is induced by the inverse of the diagonal action on $\mathbb{C}^2$.
- $\tilde{X}$ is the Hilbert scheme of a fixed number of points on the crepant resolution of $\mathbb{C}^2/\Gamma$, and X is the symmetric variety of unordered collections of points on the singular space.
- $\tilde{X}$ and X are hypertoric varieties (Section 3).
- $\tilde{X} = T^*(G/P)$ for a reductive algebraic group G and a parabolic subgroup P, and X is the affinization of this variety. (If G is of type A, then X is isomorphic to the closure of a nilpotent orbit in the Lie algebra of G.) The action of $\mathbb{C}^\times$ is the inverse scaling action on the cotangent fibers.
- X is a transverse slice between Schubert varieties in the affine Grassmannian, and $\tilde{X}$ is a resolution constructed from a convolution variety (Remark 2.10).
- $\tilde{X}$ and X are Nakajima quiver varieties [Nak94, Nak98].

REMARK 2.1. The last class of examples overlaps significantly with each of the others. The first two classes are special cases of quiver varieties, where the underlying graph of the quiver is the extended Dynkin diagram corresponding to Γ. A hypertoric variety associated to a cographical arrangement is a quiver variety for the corresponding graph, but not all hypertoric varieties are of this form. If G has type A, then $T^*(G/P)$ is a quiver variety of type A, as are slices in the affine Grassmannian for G (but neither of these statements holds in other types).

2.2 BBD decomposition

Let G be a reductive algebraic group acting on $\tilde{X}$ via Hamiltonian symplectomorphisms that commute with the action of $\mathbb{C}^\times$, and let $\mathbf{G} = G \times \mathbb{C}^\times$. Let $Z := \tilde{X} \otimes_X \tilde{X}$ be the **Steinberg variety**, and let $Z_0, Z_1, \ldots, Z_r$ be its irreducible components, with Z_0 being the diagonal copy of $\tilde{X}$. Let

$$H := H_{BM}^{2 \dim X}(Z; \mathbb{C}) = \mathbb{C}\{[Z_0], [Z_1], \ldots, [Z_r]\}$$

be the top degree Borel-Moore homology group of Z. Then H is an algebra under convolution with unit $[Z_0]$ [CG97, 2.7.41], and it acts on $H_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})$. Explicitly, the action of $[Z_i]$ is the graded $H_{\mathbf{G}}^*(*; \mathbb{C})$-linear endomorphism L_i of $H_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})$ given by pulling and pushing along the

two projections from Z_i to $\tilde{X}$. The following results follow from [CG97, § 8.9]; the main tool in the proof is the Beilinson-Bernstein-Deligne decomposition theorem, applied to the map $\tilde{X} \to X$.

THEOREM 2.2. *For each pair (S, χ) consisting of a symplectic leaf of X and a local system χ on S, there is a vector space $V_{(S,\chi)}$ such that the following statements hold.*

(i) *The convolution algebra H is semisimple with*

$$H \cong \bigoplus_{(S,\chi)} \mathrm{End}\big(V_{(S,\chi)}\big).$$

(ii) *Let $\mathring{X}$ be the dense symplectic leaf and triv the trivial local system on $\mathring{X}$. Then $V_{(\mathring{X},\mathrm{triv})} \cong \mathbb{C}$.*

(iii) *There is a canonical isomorphism*

$$IH^*_{\mathbf{G}}(X; \mathbb{C}) \cong \mathrm{Hom}_H\big(V_{(\mathring{X},\mathrm{triv})}, H^*_{\mathbf{G}}(\tilde{X}; \mathbb{C})\big).$$

(iv) *The kernel of the map from H to $\mathrm{End}\big(V_{(\mathring{X},\mathrm{triv})}\big)$ is equal to $\mathbb{C}\{[Z_1], \ldots, [Z_r]\}$.*

From this we may deduce the following corollary.

COROLLARY 2.3. *There is a canonical decomposition of graded $H^*_{\mathbf{G}}(; \mathbb{C})$-modules*

$$H^*_{\mathbf{G}}(\tilde{X}; \mathbb{C}) \cong \bigcap_{i=1}^{r} \mathrm{Ker}(L_i) \oplus \sum_{i=1}^{r} \mathrm{Im}(L_i),$$

and the first summand is canonically isomorphic to $IH^*_{\mathbf{G}}(X; \mathbb{C})$.

Proof. From part (1) of Theorem 2.2, we have

$$H^*_{\mathbf{G}}(\tilde{X}; \mathbb{C}) \cong \bigoplus_{(S,\chi)} \mathrm{Hom}_H\big(V_{(S,\chi)}, H^*_{\mathbf{G}}(\tilde{X}; \mathbb{C})\big) \otimes V_{(S,\chi)}.$$

From parts (2) and (3), the summand corresponding to the pair $(\mathring{X}, \mathrm{triv})$ is canonically isomorphic to $IH^*_{\mathbf{G}}(X; \mathbb{C}) \otimes \mathbb{C} \cong IH^*_{\mathbf{G}}(X; \mathbb{C})$. From part (4), the complementary summand is equal to $\mathbb{C}\{[Z_1], \ldots, [Z_r]\} \cdot H^*_{\mathbf{G}}(\tilde{X}; \mathbb{C}) = \sum \mathrm{Im}(L_i)$. $\qquad\square$

2.3 Quantum cohomology

Let $C \subset H_2(\tilde{X}; \mathbb{Z})/H_2(\tilde{X}; \mathbb{Z})_{\mathrm{torsion}}$ be the semigroup of effective curve classes. Let

$$\Lambda := \mathbb{C}[C] = \mathbb{C}\{q^\beta \mid \beta \in C\}$$

be the semigroup ring of C, and let $\hat{\Lambda}$ be the completion of Λ at the augmentation ideal.

Assume that we are given a class $\kappa \in H^2(\tilde{X}; \mathbb{Z}/2\mathbb{Z})$ with the property that the restriction of κ to any smooth Lagrangian subvariety of $\tilde{X}$ is equal to its second Stiefel-Whitney class. If $\tilde{X}$ is a cotangent bundle, this condition uniquely determines κ. If $\tilde{X}$ is a Hamiltonian reduction of a symplectic vector space by the linear action of a reductive group, then there is a natural choice for κ [BLPW, § 2.4]. These two cases cover all but the fifth class of examples in Section 2.1.[2]

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[2]It is not known whether such a class κ exists in general, or whether there is always a canonical choice.

Let $QH_{\mathbf{G}}^*(\tilde{X};\mathbb{C})$ denote the $\mathbf{G}$-equivariant quantum cohomology ring of $\tilde{X}$, modified in the sense of [MO, §1.2.5]. More precisely, the element q^β in our ring corresponds to the element $(-1)^{(\beta,\kappa)}q^\beta$ in the usual quantum cohomology ring. As a graded vector space, we have

$$QH_{\mathbf{G}}^*(\tilde{X};\mathbb{C}) := H_{\mathbf{G}}^*(\tilde{X};\mathbb{C}) \otimes_{\mathbb{C}} \hat{\Lambda},$$

where $\hat{\Lambda}$ lies in degree zero. Let $QH_{\mathbf{G}}^*(\tilde{X};\mathbb{C})_{\mathrm{pol}} \subset QH_{\mathbf{G}}^*(\tilde{X};\mathbb{C})$ be the Λ-subalgebra generated by the subspace $H_{\mathbf{G}}^*(\tilde{X};\mathbb{C}) \otimes_{\mathbb{C}} \Lambda$.

Consider the maximal ideal

$$\mathfrak{m} := \left\langle 1 - q^\beta \mid \beta \in C \right\rangle \subset \Lambda.$$

REMARK 2.4. Philosophically, $\mathfrak{m}$ should be the "worst possible" maximal ideal, in that the formula for modified quantum multiplication for the Springer resolution [BMO11, 1.1], hypertoric varieties [MS13, 4.2], and quiver varieties [MO, 1.3.2] all involve rational functions with denominators of the form $1 - q^\beta$ for some effective curve class β.

Consider the ring

$$R_{\mathbf{G}}(\tilde{X}) := QH_{\mathbf{G}}^*(\tilde{X};\mathbb{C})_{\mathrm{pol}} \otimes_\Lambda \Lambda/\mathfrak{m},$$

which is a graded algebra over $\mathbb{C}[\hbar] = H_{\mathbb{C}^\times}^*(*;\mathbb{C})$. Let

$$R_{\mathbf{G}}'(\tilde{X}) := R_{\mathbf{G}}(\tilde{X})/\operatorname{Ann}(\hbar).$$

We are now prepared to state our main conjecture.

CONJECTURE 2.5. *Consider the natural map $\psi_{\mathbf{G}} : H_{\mathbf{G}}^*(\tilde{X};\mathbb{C}) \to R_{\mathbf{G}}'(\tilde{X})$ of graded $\mathbb{C}[\hbar]$-modules given by the composition*

$$H_{\mathbf{G}}^*(\tilde{X};\mathbb{C}) \hookrightarrow QH_{\mathbf{G}}^*(\tilde{X};\mathbb{C})_{\mathrm{pol}} \twoheadrightarrow R_{\mathbf{G}}'(\tilde{X}).$$

(i) *The map $\psi_{\mathbf{G}}$ is surjective.*

(ii) *The kernel of $\psi_{\mathbf{G}}$ is equal to $\sum_{i=1}^r \operatorname{Im}(L_i)$.*

PROPOSITION 2.6. *If Conjecture 2.5 holds, then $\psi_{\mathbf{G}}$ descends to isomorphism of graded $H_{\mathbf{G}}^*(*;\mathbb{C})$-modules from $IH_{\mathbf{G}}^*(X;\mathbb{C})$ to $R_{\mathbf{G}}'(\tilde{X})$.*

Proof. By Corollary 2.3, we have a canonical isomorphism $IH_{\mathbf{G}}^*(X;\mathbb{C}) \cong H_{\mathbf{G}}^*(\tilde{X};\mathbb{C})/\sum \operatorname{Im}(L_i)$. $\square$

REMARK 2.7. Since $H^*(\tilde{X};\mathbb{C})$ and $H_{\mathbf{G}}^*(*;\mathbb{C})$ both vanish in odd degree [BPW, 2.5], the Leray-Serre spectral sequence for the fibration $X_{\mathbb{C}^\times} \hookrightarrow X_{\mathbf{G}} \to BG$ tells us that $H_{\mathbf{G}}^*(\tilde{X};\mathbb{C})$ is a free module over $H_G^*(*;\mathbb{C})$ and $H_{\mathbb{C}^\times}^*(\tilde{X};\mathbb{C}) \cong H_{\mathbf{G}}^*(\tilde{X};\mathbb{C}) \otimes_{H_{\mathbf{G}}^*(*;\mathbb{C})} \mathbb{C}[\hbar]$. Similar statements hold for quantum cohomology of $\tilde{X}$ and intersection cohomology of X. For this reason, if Conjecture 2.5 holds for $\psi_{\mathbf{G}}$, then it also holds for $\psi_{\mathbb{C}^\times}$.

EXAMPLE 2.8. Consider $T^*\mathbb{P}^1$, equipped with the inverse scaling action of $\mathbb{C}^\times$ on the fibers and the natural action of a maximal torus $T \subset \mathrm{PGL}(2)$. We have $H_{\mathbf{T}}^*(T^*\mathbb{P}^1; \mathbb{C}) = \mathbb{C}[x, y, \hbar]/\langle xy \rangle$, where $x = [T_0^*\mathbb{P}^1]$ and $y = [T_\infty^*\mathbb{P}^1]$.

In quantum cohomology, we have

$$x * y = \frac{q}{1-q}\hbar L_1(y).$$

Here $L_1(y) = [\mathbb{P}^1] = \hbar - x - y$, but it will not be necessary to know this for the discussion that follows. We have presentations

$$QH_{\mathbf{T}}^*(T^*\mathbb{P}^1; \mathbb{C}) = \mathbb{C}[x, y, \hbar][[q]] \Big/ \left\langle xy - \frac{q}{1-q}\hbar L_1(y) \right\rangle$$

and

$$QH_{\mathbf{T}}^*(T^*\mathbb{P}^1; \mathbb{C})_{\mathrm{pol}} = \mathbb{C}[x, y, \hbar, q] \Big/ \left\langle (1-q)xy - q\hbar L_1(y) \right\rangle.$$

Setting $q = 1$ gives us

$$R_{\mathbf{T}}(T^*\mathbb{P}^1) \cong \mathbb{C}[x, y, \hbar, q]/\langle \hbar L_1(y) \rangle,$$

and killing the annihilator of $\hbar$ gives us

$$R_{\mathbf{T}}'(T^*\mathbb{P}^1) \cong \mathbb{C}[x, y, \hbar, q]/\langle L_1(y) \rangle.$$

Of course, it is not the case that $\psi_{\mathbf{T}}$ takes the class in $H_{\mathbf{T}}^*(T^*\mathbb{P}^1; \mathbb{C})$ represented by an arbitrary polynomial $f(x, y, \hbar)$ to the class in $R_{\mathbf{T}}'(T^*\mathbb{P}^1)$ represented by the same polynomial; this would not be well-defined. However, $\psi_{\mathbf{T}}$ does behave this way on linear polynomials, and this (along with $H_{\mathbf{T}}^*(*; \mathbb{C})$-linearity) is enough to conclude both that $\psi_{\mathbf{T}}$ is surjective and that the image of L_1 is contained in the kernel. The fact that the image of L_1 is equal to the kernel can be concluded by counting dimensions.

REMARK 2.9. The quantum correction to multiplication by a divisor is conjecturally given by linear combinations of operators of the form

$$\frac{q^\beta}{1 - q^\beta}\hbar L_\beta,$$

where $\beta \in H_2(\tilde{X}; \mathbb{C})$ is one of finitely many "roots" of $\tilde{X}$, and L_β is a linear combination of $L_1, \ldots, L_r$. This conjecture is proved for the Springer resolution [BMO11], as well as for quiver varieties (modulo Kirwan surjectivity in degree 2) [MO]. In these situations, quantum multiplication is naturally defined on the vector space $QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{loc}} := H_{\mathbf{G}}^*(\tilde{X}; \mathbb{C}) \otimes_{\mathbb{C}} \Lambda_{\mathrm{loc}}$, where Λ_{loc} is given by adjoining inverses to the classes $1 - q^\beta$. We may regard $\mathrm{Spec}\, QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{loc}}$ as an affine subvariety of

$$\mathrm{Spec}\ \mathrm{Sym}_{H_{\mathbf{G}}^*(*; \mathbb{C})}\big(H_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})\big) \times \mathrm{Spec}\, \Lambda_{\mathrm{loc}} = H_*^{\mathbf{G}}(\tilde{X}; \mathbb{C}) \times \mathrm{Spec}\, \Lambda_{\mathrm{loc}},$$

and $\mathrm{Spec}\, QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{pol}}$ is then the closure of $\mathrm{Spec}\, QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{loc}}$ inside of $H_*^{\mathbf{G}}(\tilde{X}; \mathbb{C}) \times \mathrm{Spec}\, \Lambda$. If $\mathrm{Spec}\, QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{pol}}$ is flat over $\mathrm{Spec}\, \Lambda$ and all of the roots are primitive, then it follows that

the image of every L_β is contained in the kernel of $\psi_{\mathbf{G}}$. This gives us a strategy for proving one of the two inclusions needed for part (2) of Conjecture 2.5 in a number of other cases.

REMARK 2.10. The intersection cohomology $IH^*_{\mathbf{G}}(X; \mathbb{C})$ is a priori only a graded $H^*_{\mathbf{G}}(*; \mathbb{C})$-module, while $R'_{\mathbf{G}}(\tilde{X})$ is an algebra. One of the interesting consequences of Conjecture 2.5 and Proposition 2.6 is that it would endow $IH^*_{\mathbf{G}}(X; \mathbb{C})$ with an algebra structure. In the case of hypertoric varieties, the module $IH^*_T(\tilde{X}; \mathbb{C})$ was given an algebra structure by Braden and the second author [BP09] via completely different means, and this coincides with the algebra structure that we obtain in this paper after setting $\hbar$ to zero (Proposition 3.16).

Another intriguing class of examples is the fifth one mentioned in Section 2.1. Fix a simple, simply laced algebraic group G with maximal torus $T \subset G$. Let Gr be the affine Grassmannian for G, and for any dominant coweight $\lambda \in \mathrm{Hom}(\mathbb{C}^\times, T)$, consider the Schubert variety $\mathrm{Gr}^\lambda \subset \mathrm{Gr}$. Fix dominant coweights $\lambda \geqslant \mu$, and let X be a normal slice to Gr^μ inside of Gr^λ. Using the geometric Satake correspondence [Gin, MV07], Ginzburg produces an isomorphism between a quotient of $IH^*_T(X; \mathbb{C})$ (obtained by choosing generic values for the equivariant parameters) and the μ weight space of the irreducible representation $V(\lambda)$ of G^L [Gin, 3.11 & 5.2]. If λ is a sum of minuscule coweights (for example, if G is of type A), then X admits a conical symplectic resolution [KWWY14, 2.9]. Thus our Conjecture 2.5 and Proposition 2.6 would endow the weight space $V(\lambda)_\mu$ with a ring structure.

In [FFR10], Feigin, Frenkel, and Rybnikov define a ring structure on the whole representation $V(\lambda)$ using a *quantum shift of argument subalgebra*. We believe that the above considerations may lead to a geometric explanation of their results.

3. Hypertoric varieties

In this section we prove Conjecture 2.5 for hypertoric varieties.

3.1 Definitions

We begin by reviewing the constructive definition of a projective hypertoric variety, which was first introduced in [BD00]. An intrinsic approach to hypertoric varieties can be found in [AP].

Fix a finite-rank lattice N, along with a list of (not necessarily distinct) nonzero primitive vectors $a_1, \ldots, a_n \in N$ and integers $\theta_1, \ldots, \theta_n$. Consider the hyperplanes

$$H_i := \{x \in N^\vee_{\mathbb{R}} \mid \langle a_i, x \rangle + \theta_i = 0\}$$

along with the associated half-spaces

$$H_i^+ := \{x \in N^\vee_{\mathbb{R}} \mid \langle a_i, x \rangle + \theta_i \geqslant 0\} \qquad \text{and} \qquad H_i^- := \{x \in N^\vee_{\mathbb{R}} \mid \langle a_i, x \rangle + \theta_i \leqslant 0\}.$$

We make the following assumptions on our data:

– **Full rank:** The lattice N is spanned by $\{a_1, \ldots, a_n\}$.

- **No co-loops:** For all i, the lattice N is spanned by $\{a_1, \ldots, a_n\} \smallsetminus \{a_i\}$.

- **Unimodular:** For any $S \subset [n]$, if $\{a_i \mid i \in S\}$ spans $N_{\mathbb{Q}}$ over $\mathbb{Q}$, then it spans N over $\mathbb{Z}$.

- **Simple:** For any $S \subset [n]$, $\operatorname{codim} \bigcap_{i \in S} H_i = |S|$ (note that the empty set has every codimension).

Consider the short exact sequence

$$0 \longrightarrow P \overset{\iota}{\longrightarrow} \mathbb{Z}^n \overset{\pi}{\longrightarrow} N \longrightarrow 0,$$

where π takes the ith coordinate vector to a_i and $P := \ker(\pi)$. Dualizing and then taking homomorphisms into $\mathbb{C}^\times$, we obtain an exact sequence of tori

$$1 \to K \to T^n \to T \to 1.$$

The torus T^n acts symplectically on $T^*\mathbb{C}^n$ with moment map

$$\mu_n : T^*\mathbb{C}^n \to \operatorname{Lie}(T^n)^\vee \cong \mathbb{C}^n$$

given by the formula

$$\mu_n(z_1, w_1, \ldots, z_n, w_n) = (z_1 w_1, \ldots, z_n w_n).$$

Composing with $\iota^\vee : \mathbb{C}^n \to P_{\mathbb{C}}^\vee$, we obtain a moment map

$$\mu_K : T^*\mathbb{C}^n \to P_{\mathbb{C}}^\vee$$

for the action of K. The element $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{Z}^n \cong \operatorname{Hom}(T^n, \mathbb{C}^\times)$ is a character of T^n, which we also regard as a character of K by restriction. Consider the symplectic quotients

$$X := \mu_K^{-1}(0) /\!\!/_0 K = \operatorname{Spec} \mathbb{C}[\mu_K^{-1}(0)]^K,$$

and

$$\tilde{X} := \mu_K^{-1}(0) /\!\!/_\theta K = \operatorname{Proj}\left(\mathbb{C}[\mu_K^{-1}(0)] \otimes \mathbb{C}[t]\right)^K,$$

where K acts on t via the character θ. The assumptions of simplicity and unimodularity imply that the natural map from $\tilde{X}$ to X is a projective symplectic resolution [BD00, 3.2 & 3.3].

The action of $\mathbb{C}^\times$ on $T^*\mathbb{C}^n$ via inverse scaling of the cotangent fibers descends to an action on $\tilde{X}$, and the symplectic form has weight 1 with respect to this action. The assumption of no co-loops implies that $\mathbb{C}[X]^{\mathbb{C}^\times} = \mathbb{C}$, and therefore that $\tilde{X}$ is a conical symplectic resolution of X. The Hamiltonian action of T^n on $T^*\mathbb{C}^n$ induces an action on $\tilde{X}$, and this descends to an effective Hamiltonian action of T that commutes with the action of $\mathbb{C}^\times$. Let $\mathbf{T} = T \times \mathbb{C}^\times$.

3.2 Cohomology

We next review some basic facts about the cohomology of hypertoric varieties. A minimal set $C \subset [n]$ such that $\bigcap_{i \in C} H_i = \emptyset$ is called a **circuit**. If C is a circuit, then there exists a unique

decomposition

$$C = C^+ \sqcup C^- \qquad \text{such that} \qquad \bigcap_{i \in C^+} H_i^+ \cap \bigcap_{i \in C^-} H_i^- = \emptyset.$$

Let $A := \operatorname{Sym} N^\vee \cong H_T^*(*; \mathbb{Z})$ be the T-equivariant cohomology ring of a point. The **T**-equivariant cohomology ring of $\tilde{X}$ was computed by Harada and the second author [HP04, 4.4], building on the T-equivariant computation in [Kon99].

THEOREM 3.1. *The ring $H_{\mathbf{T}}^*(\tilde{X}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}[u_1, \ldots, u_n, \hbar]/J_0$, where J_0 is the ideal generated by*

$$\prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} (\hbar - u_j)$$

for each circuit $C \subset [n]$. We have $\deg(u_i) = \deg(\hbar) = 2$ for all i, and the A-algebra structure is given by the natural inclusion

$$\pi^\vee : N^\vee \to \mathbb{Z}^n \cong \mathbb{Z}\{u_1, \ldots, u_n\}.$$

COROLLARY 3.2. *We have a canonical isomorphism $H_2(\tilde{X}; \mathbb{Z}) \cong P$.*

Proof. Since the generators of J all have degree at least 4, we have

$$H_{\mathbf{T}}^2(\tilde{X}; \mathbb{Z}) \cong \mathbb{Z}\{u_1, \ldots, u_n, \hbar\} = \mathbb{Z}^n \oplus \mathbb{Z},$$

and therefore $H^2(\tilde{X}; \mathbb{Z}) \cong \mathbb{Z}^n/N^\vee \cong P^\vee$. Dualizing, we obtain our result. $\qquad \square$

For any set $S \subset [n]$, let $u_S := \prod_{i \in S} u_i$. The set S is called **independent** if it contains no circuits.

COROLLARY 3.3. *The ring $H_{\mathbf{T}}^*(\tilde{X}; \mathbb{Z})$ is spanned over $A[\hbar]$ by monomials of the form u_S, where $S \subset [n]$ is independent.*

Proof. It is sufficient to prove that $H_T^*(\tilde{X}; \mathbb{Z})$ is spanned over A by monomials of the form u_S, where $S \subset [n]$ is independent. This is shown in the appendix (Lemma A.4). $\qquad \square$

3.3 Quantum cohomology

We continue by describing the various versions of the quantum cohomology ring of $\tilde{X}$. For any circuit $C \subset [n]$, unimodularity implies that

$$\sum_{i \in C^+} a_i - \sum_{j \in C^-} a_j = 0.$$

Let $C_i = 1$ if $i \in C^+$, -1 if $i \in C^-$, and 0 otherwise, and consider the element

$$\beta_C := \sum_{i=1}^n C_i e_i \in \ker(\mathbb{Z}^n \to N) = P \cong H_2(\tilde{X}; \mathbb{Z}).$$

To compute $QH_{\mathbf{T}}^*(\tilde{X}; \mathbb{C})_{\text{pol}}$, we need the following lemma, which is implicit in [MS13].

LEMMA 3.4. *For any independent subset $S \subset [n]$, the quantum product of $\{u_i \mid i \in S\}$ is equal to the ordinary product u_S.*

Proof. We proceed by induction on the size of S. Let j be the maximal element of S, and let $\bar{S} = S \smallsetminus \{j\}$. By our inductive hypothesis, the quantum product of the elements $\{u_i \mid i \in S\}$ is equal to the quantum product of u_j with $u_{\bar{S}}$. By [MS13, 4.2], we have[3]

$$u_j \cdot u_{\bar{S}} = u_S + \hbar \sum_C C_j \frac{q^{\beta_C}}{1 - q^{\beta_C}} L_C(u_{\bar{S}}),$$

where L_C is a certain linear combination of $L_1, \ldots, L_r$. Thus it is sufficient to show that $L_C(u_{\bar{S}}) = 0$ for all circuits C containing j.

Let $\mu : \tilde{X} \to N_{\mathbb{C}}^\vee$ be the moment map induced by μ_n for the action of T on $\tilde{X}$. The operator L_C is given by a correspondence $Z_C \subset Z = \tilde{X} \times_X \tilde{X}$ that lies over the locus

$$H_C := \bigcap_{i \in C} H_i^{\mathbb{C}} \subset N_{\mathbb{C}}^\vee.$$

On the other hand, the element $u_{\bar{S}}$ may be represented by a cycle that lies over $H_{\bar{S}}$, thus $L_C(u_{\bar{S}})$ may be represented by a cycle that lies over $H_{C \cup \bar{S}} \subset H_S$. Since S is independent, we have

$$\operatorname{codim} H_S = |S| > |\bar{S}| = \frac{1}{2} \deg u_{\bar{S}} = \frac{1}{2} \deg L_C(u_{\bar{S}}),$$

which implies that $L_C(u_{\bar{S}}) = 0$. $\qquad\square$

THEOREM 3.5. *The ring $QH_{\mathbf{T}}^*(\tilde{X}; \mathbb{C})_{\mathrm{pol}}$ is isomorphic to $\Lambda[u_1, \ldots, u_n, \hbar]/J$, where J is the ideal generated by*

$$\prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} (u_j - \hbar) \quad - \quad q^{\beta_C} \prod_{i \in C^+} (u_i - \hbar) \cdot \prod_{j \in C^-} u_j$$

for each circuit $C \subset [n]$. The $A_{\mathbb{C}}$-algebra structure is as in Theorem 3.1.

Proof. The fact that $QH_{\mathbf{T}}^*(\tilde{X}; \mathbb{C})_{\mathrm{pol}}$ is generated over Λ by $H_{\mathbf{T}}^2(\tilde{X}; \mathbb{C})$ follows from Corollary 3.3 and Lemma 3.4. The rest of the theorem appears in [MS13, 1.1]. $\qquad\square$

COROLLARY 3.6. *The ring $R_{\mathbf{T}}(\tilde{X})$ is isomorphic to $\mathbb{C}[u_1, \ldots, u_n, \hbar]/J_1$, where J_1 is the ideal generated by*

$$\prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} (u_j - \hbar) \quad - \quad \prod_{i \in C^+} (u_i - \hbar) \cdot \prod_{j \in C^-} u_j$$

for each circuit $C \subset [n]$. The ring $R_{\mathbf{T}}'(\tilde{X}) := R_{\mathbf{T}}(\tilde{X})/\operatorname{Ann}(\hbar)$ is isomorphic to $\mathbb{C}[u_1, \ldots, u_n, \hbar]/J_1'$, where J_1' is the ideal generated by

$$f_C := \hbar^{-1} \left(\prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} (u_j - \hbar) \quad - \quad \prod_{i \in C^+} (u_i - \hbar) \cdot \prod_{j \in C^-} u_j \right)$$

---

[3]The formula in [MS13] has q^{β_C} replaced with $(-1)^{|C|} q^{\beta_C}$ because that paper uses the unmodified quantum product.

for each circuit $C \subset [n]$.

PROPOSITION 3.7. *The ring $R'_{\mathbf{T}}(\tilde{X})$ is spanned over $A_{\mathbb{C}}[\hbar]$ by monomials of the form u_S, where $S \subset [n]$ is independent.*

Proof. It is sufficient to prove that $R'_{\mathbf{T}}(\tilde{X})/\langle\hbar\rangle$ is spanned over $A_{\mathbb{C}}$ by monomials of the form u_S, where $S \subset [n]$ is independent. This is shown in the appendix (Theorem A.3 and Remark A.8). $\square$

COROLLARY 3.8. *The map $\psi_{\mathbf{T}}$ from Conjecture 2.5 is surjective.*

Proof. Lemma 3.4 tells us that $\psi_{\mathbf{T}}$ takes the image of u_S in $QH^*_{\mathbf{T}}(\tilde{X};\mathbb{C})_{\mathrm{pol}}$ to the image of u_S in $R'_{\mathbf{T}}(\tilde{X})$ for all independent $S \subset [n]$. By Proposition 3.7, this implies that $\psi_{\mathbf{T}}$ is surjective. $\square$

3.4 The kernel of $\psi_{\mathbf{T}}$

In this section we prove the second half of Conjecture 2.5 for hypertoric varieties. Let

$$ U \quad := \quad \mathrm{Ker}(\psi_{\mathbf{T}}) \quad \subset \quad H^*_{\mathbf{T}}(\tilde{X};\mathbb{C}) \quad \supset \quad \mathrm{Im}(L_1) + \ldots + \mathrm{Im}(L_r) \quad =: \quad V; $$

the conjecture says that $U = V$.

For any circuit C, let $\overline{C}$ be the set obtained from C by deleting the maximal element $j_{\max} \in C$, and consider the graded vector subspace

$$ W := A_{\mathbb{C}}[\hbar] \cdot \big\{ u_S f_C \ \big| \ C \text{ is a circuit, } S \cap C = \emptyset, \text{ and } S \cup \overline{C} \text{ is independent} \big\}. $$

LEMMA 3.9. $W \subset U$.

Proof. Each term of $u_S f_C$ is equal to plus or minus a power of $\hbar$ times a square-free monomial of independent support. By Lemma 3.4, such a monomial is taken to itself by $\psi_{\mathbf{T}}$. This means that $\psi_{\mathbf{T}}$ takes $u_S f_C$ to itself, and $u_S f_C$ represents the zero element of $R'_{\mathbf{T}}(\tilde{X})$ by Corollary 3.6. $\square$

LEMMA 3.10. $W \subset V$.

Proof. Fix a circuit C, and assume that $j_{\max} \in C^+$. Let

$$ g_C \quad := \quad \prod_{i \in C^+ \setminus \{j_{\max}\}} u_i \ \cdot \prod_{j \in C^-} (u_j - \hbar) \quad \in \quad H^*_{\mathbf{T}}(\tilde{X};\mathbb{C}) \quad \subset \quad QH^*_{\mathbf{T}}(\tilde{X};\mathbb{C})_{\mathrm{pol}}. $$

(Note that, by Lemma 3.4, this product has no quantum correction.) By Theorem 3.5, we have

$$ u_{j_{\max}} g_C \quad = \quad \prod_{i \in C^+} u_i \ \cdot \prod_{j \in C^-} (u_j - \hbar) \quad = \quad q^{\beta_C} \prod_{i \in C^+} (u_i - \hbar) \ \cdot \prod_{j \in C^-} u_j \quad \in \quad QH^*_{\mathbf{T}}(\tilde{X};\mathbb{C})_{\mathrm{pol}}. $$

By definition of f_C, we have

$$ \begin{aligned} q^{\beta_C}\hbar f_C &= q^{\beta_C} \prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} (u_j - \hbar) \quad - \quad q^{\beta_C} \prod_{i \in C^+} (u_i - \hbar) \cdot \prod_{j \in C^-} u_j \\ &= q^{\beta_C} u_{j_{\max}} g_C - u_{j_{\max}} g_C \\ &= \big(q^{\beta_C} - 1 \big) u_{j_{\max}} g_C \quad \in \quad QH^*_{\mathbf{T}}(\tilde{X};\mathbb{C})_{\mathrm{pol}}. \end{aligned} $$

Fix a set S such that $S \cap C = \emptyset$ and $S \cup \overline{C}$ is independent. Multiplying both sides of the above equation by u_S, we obtain

$$q^{\beta_C} \hbar u_S f_C = \left(q^{\beta_C} - 1 \right) u_{j_{\max}} u_S g_C.$$

Since the classical product of $u_{j_{\max}}$ with g_C vanishes and $u_S g_C \in H_{\mathbf{T}}^*(\tilde{X}; \mathbb{C})$ by Lemma 3.4, we have [MS13, 4.2]

$$u_{j_{\max}} \cdot u_S g_C = \hbar \sum_D D_{j_{\max}} \frac{q^{\beta_D}}{1 - q^{\beta_D}} L_D(u_S g_C),$$

where D ranges over all circuits.

Let $\mathbb{C}(\Lambda)$ be the field of fractions of Λ, and let $QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{rat}}$ be the ring generated by $H_{\mathbf{G}}^*(\tilde{X}; \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}(\Lambda)$ under the quantum product. It follows easily from [MS13, 4.2] that in fact $QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{rat}} = H_{\mathbf{G}}^*(\tilde{X}; \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}(\Lambda)$ as a vector space, and that $QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{pol}} \subset QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{rat}}$. We may think of elements of $QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{rat}}$ as meromorphic sections of the vector bundle with fiber $H_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})$ over $\operatorname{Spec}\Lambda$. In particular they have well-defined loci of poles.

We can now combine the two above equations to obtain

$$\frac{q^{\beta_C}}{q^{\beta_C} - 1} \hbar u_S f_C = \hbar \sum_D D_{j_{\max}} \frac{q^{\beta_D}}{1 - q^{\beta_D}} L_D(u_S g_C) \in QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{rat}}.$$

Since the left-hand side has poles only at $q^{\beta_C} = 1$, so does the right-hand side. We conclude that all summands such that $D \neq C$ vanish, and we are left with

$$\frac{q^{\beta_C}}{q^{\beta_C} - 1} \hbar u_S f_C = \hbar \frac{q^{\beta_C}}{1 - q^{\beta_C}} L_C(u_S g_C).$$

Dividing by $\frac{q^{\beta_C}}{1 - q^{\beta_C}}$, we have

$$\hbar u_S f_C = -\hbar L_C(u_S g_C).$$

A priori, this equation lives in $QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{rat}}$. However, it is clear that both sides live in the subspace $QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{pol}} \subset QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{rat}}$. Furthermore, since $u_S f_C$ is a sum of powers of $\hbar$ times independent square-free monomials, Lemma 3.4 tells us that $u_S f_C$ lies in $H_{\mathbf{T}}^*(\tilde{X}; \mathbb{C}) \subset QH_{\mathbf{G}}^*(\tilde{X}; \mathbb{C})_{\mathrm{pol}}$. Since $H_{\mathbf{T}}^*(\tilde{X}; \mathbb{C})$ is a free module over $\mathbb{C}[\hbar]$, we may divide by $\hbar$ to obtain

$$u_S f_C = -L_C(u_S g_C) \in H_{\mathbf{T}}^*(\tilde{X}; \mathbb{C}).$$

Thus we see that $u_S f_C$ is in the image of L_C, and is therefore in the span of the images of $L_1, \ldots, L_r$. A similar argument can be applied if $j_{\max} \in C^-$. $\square$

For any $\mathbb{N}$-graded vector space $Y = \bigoplus Y^k$ with finite-dimensional graded pieces, let

$$\operatorname{Hilb}(Y; t) := \sum_{k=0}^{\infty} \dim Y^k t^k \in \mathbb{N}[[t]].$$

LEMMA 3.11. $\operatorname{Hilb}(U; t) = \operatorname{Hilb}(V; t)$.

Proof. Since $\psi_{\mathbf{T}}$ is surjective (Corollary 3.8), we have $R_{\mathbf{T}}'(\tilde{X}) \cong H_{\mathbf{T}}^*(\tilde{X}; \mathbb{C})/U$. By Corollary 2.3,

we also have $IH^*_{\mathbf{T}}(X;\mathbb{C}) \cong H^*_{\mathbf{T}}(\tilde{X};\mathbb{C})/V$. Thus the statement that $\mathrm{Hilb}(U;t) = \mathrm{Hilb}(V;t)$ is equivalent to the statement that $\mathrm{Hilb}(R'_{\mathbf{T}}(\tilde{X});t) = \mathrm{Hilb}(IH^*_{\mathbf{T}}(X;\mathbb{C});t)$.

By Remark A.8 and Theorem A.9, we have

$$\mathrm{Hilb}(R'_{\mathbf{T}}(\tilde{X});t) = \mathrm{Hilb}(OT_\hbar;t) = (1-t)^{-1}\,\mathrm{Hilb}(OT;t).$$

By Proposition A.2, $\mathrm{Hilb}(OT;t)$ is equal to $(1-t)^{-\mathrm{rk}\,N}$ times the h-polynomial of the broken circuit complex of the matroid represented by the vectors $a_1, \ldots, a_n$. On the other hand, we have

$$\mathrm{Hilb}(IH^*_{\mathbf{T}}(X;\mathbb{C});t) = (1-t)^{-\mathrm{rk}\,N-1}\,\mathrm{Hilb}(IH^*(X;\mathbb{C});t),$$

and $\mathrm{Hilb}(IH^*(X;\mathbb{C});t)$ is itself equal to the h-polynomial of the broken circuit complex [PW07, 4.3]. Thus $\mathrm{Hilb}(U;t) = \mathrm{Hilb}(V;t)$. $\qquad\square$

Let $V_0 = V \otimes_{\mathbb{C}[\hbar]} \mathbb{C}$, and let W_0 be the image of $W \subset V$ in V_0. More concretely, V_0 is the complement of $IH^*_T(X;\mathbb{C})$ in

$$H^*_T(\tilde{X};\mathbb{C}) \cong \mathbb{C}[u_1,\ldots,u_n]/\langle u_C \mid C \text{ a circuit}\rangle,$$

and W_0 is the $A_{\mathbb{C}}$-submodule of $H^*_T(\tilde{X};\mathbb{C})$ spanned by $\{u_S f_{C,0}\}$, where $f_{C,0}$ is obtained from f_C by setting $\hbar$ equal to zero.

LEMMA 3.12. *Let C be a circuit and let S be a set disjoint from C. For any collections of non-negative integers $\underline{d} = (d_i \mid i \in S)$ and $\underline{e} = (e_j \mid j \in \overline{C})$, we have*

$$f(S,C,\underline{d},\underline{e}) \;:=\; u_S f_C \cdot \prod_{i \in S} u_i^{d_i} \cdot \prod_{j \in \overline{C}} (C_j u_j - C_{j_{\max}} u_{j_{\max}})^{e_j} \;\in\; W_0.$$

Proof. We proceed via a double induction. First, we fix C and induct downward on the size of S. If $|S| > \mathrm{rk}\,N - |\overline{C}|$, then every term of $u_S f_C$ contains a monomial supported on a dependent set, so $u_S f_C = 0$. Thus we will fix S and assume that the lemma holds for all sets $S' \supsetneq S$ disjoint from C. By the same reasoning, we may assume that $S \cup \overline{C}$ is independent.

Second, we induct upward on the exponents. The base case is where $d_i = 0 = e_j$ for all i and j, in which case $f(S,C,\underline{d},\underline{e}) = u_S f_C \in W_0$ by definition of W_0. Thus we may fix $\underline{d}$ and $\underline{e}$ such that $f(S,C,\underline{d},\underline{e}) \in W_0$ and prove that for all $i \in S$ and $j \in \overline{C}$, we have $u_i f(S,C,\underline{d},\underline{e}) \in W_0$ and $(C_j u_j - C_{j_{\max}} u_{j_{\max}}) f(S,C,\underline{d},\underline{e}) \in W_0$.

Let $i \in S$ be given. Since $S \cup \overline{C}$ is independent, there exists $x \in N^\vee$ such that $\pi^\vee(x) = \sum \gamma_k u_k$ with $\gamma_i = 1$ and $\gamma_k = 0$ for all $k \in S \cup C \smallsetminus \{i\}$. Then

$$\pi^\vee(x) \cdot f(S,C,\underline{d},\underline{e}) = \sum_{k=1}^n \gamma_k u_k \cdot f(S,C,\underline{d},\underline{e})$$
$$= u_i \cdot f(S,C,\underline{d},\underline{e}) + \sum_{k \notin S \cup C} \gamma_k u_k \cdot f(S,C,\underline{d},\underline{e})$$
$$= u_i \cdot f(S,C,\underline{d},\underline{e}) + \sum_{k \notin S \cup C} \gamma_k \cdot f(S \cup \{k\},C,\underline{d},\underline{e}).$$

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Our first inductive hypothesis tells us that $f(S \cup \{k\}, C, \underline{d}, \underline{e}) \in W_0$, and W_0 is by definition closed under multiplication by elements of A, so we also have $\pi^\vee(x) \cdot f(S, C, \underline{d}, \underline{e}) \in W_0$. This implies that $u_i \cdot f(S, C, \underline{d}, \underline{e}) \in W_0$.

Let $j \in \overline{C}$ be given. Since $S \cup \overline{C}$ is independent, there exists $x \in N^\vee$ such that $\pi^\vee(x) = \sum \gamma_k u_k$ with $\gamma_j = C_j$, $\gamma_{j_{\max}} = -C_{j_{\max}}$, and $\gamma_k = 0$ for all $k \in S \cup C \smallsetminus \{j, j_{\max}\}$. Then

$$
\begin{aligned}
\pi^\vee(x) \cdot f(S, C, \underline{d}, \underline{e}) &= \sum_{k=1}^n \gamma_k u_k \cdot f(S, C, \underline{d}, \underline{e}) \\
&= (C_j u_i - C_{j_{\max}} u_{j_{\max}}) \cdot f(S, C, \underline{d}, \underline{e}) + \sum_{k \notin S \cup C} \gamma_k u_k \cdot f(S, C, \underline{d}, \underline{e}) \\
&= (C_j u_i - C_{j_{\max}} u_{j_{\max}}) \cdot f(S, C, \underline{d}, \underline{e}) + \sum_{k \notin S \cup C} \gamma_k \cdot f(S \cup \{k\}, C, \underline{d}, \underline{e}).
\end{aligned}
$$

By the same reasoning as above, this implies that $(C_j u_i - C_{j_{\max}} u_{j_{\max}}) \cdot f(S, C, \underline{d}, \underline{e}) \in W_0$. $\square$

LEMMA 3.13. $V = W$.

Proof. We will start by proving that $\mathrm{Hilb}(W_0; t) = \mathrm{Hilb}(V_0; t)$. Consider the degree-lexicographic monomial order on $H_T^*(\tilde{X}; \mathbb{C})$ with $u_1 > u_2 > \ldots > u_n$. Given C, S, $\underline{d}$, and $\underline{e}$ as in Lemma 3.12, The initial term of $f(S, C, \underline{d}, \underline{e})$ with respect to this order is $\pm u_{S \cup \overline{C}} \prod_S u_i^{d_i} \prod_{\overline{C}} u_j^{e_j}$. These monomials span the kernel of the projection

$$
H_T^*(\tilde{X}; \mathbb{C}) \cong \mathbb{C}[u_1, \ldots, u_n] / \langle u_C \mid C \text{ a circuit} \rangle \to \mathbb{C}[u_1, \ldots, u_n] / \langle u_{\overline{C}} \mid C \text{ a circuit} \rangle =: SR_{\mathrm{bc}}, \,^{[4]}
$$

thus Lemma 3.12 tells us that $\mathrm{in}(W_0)$ contains this kernel. We therefore have

$$
\begin{aligned}
\mathrm{Hilb}(H_T^*(\tilde{X}; \mathbb{C})/W_0; t) &= \mathrm{Hilb}(H_T^*(\tilde{X}; \mathbb{C})/\mathrm{in}(W_0); t) \\
&\leqslant \mathrm{Hilb}(SR_{\mathrm{bc}}; t) \\
&= \mathrm{Hilb}(IH_T^*(X; \mathbb{C})) \quad \text{by [PW07, 4.3]} \\
&= \mathrm{Hilb}(H_T^*(\tilde{X}; \mathbb{C})/V_0; t).
\end{aligned}
$$

Since $W_0 \subset V_0$, this implies that $W_0 = V_0$.

We would like to use this to conclude that $W = V$. Suppose not, and let $v \in V$ be a homogeneous element of minimal degree that is not contained in W. Let v_0 be the image of v in V_0. Since $W_0 = V_0$, there exists a homogeneous $w \in W$ such that $w_0 = v_0$. This means that $v - w$ is in the kernel of the projection from V to V_0, so there exists a homogeneous $v' \in V$ with $v - w = \hbar v'$. By minimality of the degree of v, we have $v' \in W$, and therefore $v = \hbar v' + w \in W$, which is a contradiction. $\square$

COROLLARY 3.14. $U = V$.

Proof. Lemmas 3.9 and 3.13 imply that $V \subset U$, and they have the same Hilbert series by Lemma 3.11, so they must be equal. $\square$

---

[4] This is the Stanley-Reisner ring of the broken circuit complex; see Section A.1.

THEOREM 3.15. *Conjecture 2.5 holds for hypertoric varieties.*

Proof. The first part of the conjecture is Corollary 3.8, while the second is Corollary 3.14. ☐

3.5 Comparison with previous work

Let $OT := R'_{\mathbf{T}}(\tilde{X})/\langle \hbar \rangle$; this algebra is called the **Orlik-Terao algebra**. In an earlier paper, Braden and the second author showed that $IH^*_T(X;\mathbb{C})$ is canonically isomorphic to OT [BP09, 4.5]. The first thing we want to establish is that the isomorphism in this paper is the same as the isomorphism in that paper.

PROPOSITION 3.16. *The isomorphism from $IH^*_T(X;\mathbb{C})$ to OT induced by $\psi_{\mathbf{T}}$ (after setting $\hbar$ equal to zero) coincides with the isomorphism in [BP09].*

Proof. Let $F \subset [n]$ be a flat of the matroid represented by the vectors $a_1, \ldots, a_n$. Working only with the vectors $\{a_i \mid i \in F\}$, we obtain an algebra OT_F which is isomorphic to the quotient of OT by the ideal generated by $\{u_i \mid i \notin F\}$. We also obtain a hypertoric variety X_F which is a "normal slice" to a stratum of X; in particular, this means that an analytic neighborhood of the cone point of X_F admits a normally nonsingular inclusion into X [PW07, 2.5]. Since X_F is a cone, we may assume that our analytic neighborhood is equivariantly (with respect to the maximal compact subtorus of T_F) homeomorphic to X_F itself. This inclusion is easily seen to be equivariant with respect to the maximal compact subtorus of $T_F \subset T$, and therefore induces a map

$$IH^*_T(X;\mathbb{C}) \to IH^*_{T_F}(X;\mathbb{C}) \to IH^*_{T_F}(X_F;\mathbb{C}).$$

The isomorphisms in [BP09] are the unique isomorphisms such that the diagrams

$$\begin{array}{ccc}
IH^*_T(X;\mathbb{C}) & \longrightarrow & OT \\
\downarrow & & \downarrow \\
IH^*_{T_F}(X_F;\mathbb{C}) & \longrightarrow & OT_F
\end{array}$$

commute for all F. Thus it is sufficient to show that this diagram commutes using the isomorphisms constructed in this paper for the horizontal arrows. This follows from the fact that the inclusion of X_F into X lifts to an inclusion of $\tilde{X}_F$ into $\tilde{X}$ [PW07, 2.5], and the induced map from $H^*_T(\tilde{X};\mathbb{C})$ to $H^*_{T_F}(\tilde{X}_F;\mathbb{C})$ is given by setting u_i to zero for all $i \notin F$. ☐

We conclude by discussing some of the advantages and disadvantages of the two approaches. The main advantage of [BP09] is that the ring structure is defined at a higher categorical level: it is shown there that the intersection cohomology sheaf IC_X admits the structure of a ring object in the equivariant derived category of constructible sheaves on X, and that the isomorphism from $IH^*_T(X;\mathbb{C})$ to OT is compatible with this structure.

On the other hand, there are two advantages to the approach we take in this paper. The first

is that we work $\mathbf{T}$-equivariantly rather than T-equivariantly. This may not seem like a big deal, but it is not so easy to modify the techniques of [BP09] to account for the extra $\mathbb{C}^{\times}$-action. Any attempt in this direction would have to begin with a proof of Theorem A.9.

The second, and more significant, advantage of our approach is that the isomorphism in [BP09] comes out of nowhere: one simply shows that the ring OT has the same Hilbert series and functorial properties as $IH_T^*(X;\mathbb{C})$, and that these functorial properties are sufficiently rigid to ensure that the two groups are canonically isomorphic. In contrast, the isomorphism in this paper is induced by the natural map $\psi_{\mathbf{T}}$, and can be (at least conjecturally) generalized to arbitrary conical symplectic resolutions.

Appendix A. The Orlik-Terao algebra

In this paper we have required two technical results about the Orlik-Terao algebra of a collection of vectors (Theorems A.3 and A.9). Since we believe that these two statements may be of general interest in the theory of hyperplane arrangements, we put them in an appendix which may be read independently from the rest of the paper.

A.1 A spanning set

Let k be a field of characteristic zero. Let V be a vector space over k, and let $a_1, \ldots, a_n$ be nonzero linear functions on V that span V^*. Let $I \subset k[u_1, \ldots, u_n]$ be the kernel of the map taking u_i to a_i^{-1}. The graded k-algebra $OT := k[u_1, \ldots, u_n]/I$ is called the **Orlik-Terao** algebra.

For any subset $S \subset [n]$, let $u_S := \prod_{i \in S} u_i$. A set $C \subset [n]$ is called **dependent** if there exist constants $\{\eta_i \mid i \in C\}$, not all zero, such that $\sum \eta_i a_i = 0$. In this case, we have a nontrivial element

$$f_{C,0} := \sum_{i \in C} \eta_i u_{C \smallsetminus \{i\}} \in I.$$

This notation is somewhat sloppy, as $f_{C,0}$ depends not only on C, but also on the constants η_i. However, if C is a **circuit** (a minimal dependent set), then the constants are determined up to a global nonzero scalar, thus the same is true for $f_{C,0}$.

Note that if our collection of vectors is unimodular and C is a circuit, then we may take $\eta_i = \pm 1$ for all i, and then $f_{C,0}$ will be the polynomial obtained from the polynomial f_C of Corollary 3.6 (and also of Section A.2) by setting $\hbar$ equal to zero; this explains our funny notation.

The following result is proved in [PS06, Theorem 4].

THEOREM A.1. *The set $\{f_{C,0} \mid C$ a circuit$\}$ is a universal Gröbner basis for I.*

For any circuit C, let $\overline{C}$ be the set obtained from C by deleting the maximal element. Let

$$SR_{\mathrm{ind}} := k[u_1, \ldots, u_n]/\langle u_C \mid C \text{ a circuit}\rangle \qquad \text{and} \qquad SR_{\mathrm{bc}} := k[u_1, \ldots, u_n]/\langle u_{\overline{C}} \mid C \text{ a circuit}\rangle.$$

These algebras are called the **Stanley-Reisner rings** of the independence complex and the

broken circuit complex, respectively. Note that $u_{\overline{C}}$ is (up to scale) the initial term of $f_{C,0}$, hence Theorem A.1 says exactly that SR_{bc} is a flat degeneration of OT.

Consider the map $\mathrm{Sym}(V) \to k[u_1, \ldots, u_n]$ taking $v \in V$ to $\sum a_i(v)u_i$. This makes the algebras OT, SR_{ind}, and SR_{bc} into graded $\mathrm{Sym}(V)$-algebras. The following result is proved in [PS06, Propositions 1 & 7].

PROPOSITION A.2. *The rings OT, SR_{ind}, and SR_{bc} are free as graded $\mathrm{Sym}(V)$-modules. The graded rank of SR_{ind} is given by the h-numbers of the independence complex, while the graded ranks of OT and SR_{bc} are given by the h-numbers of the broken circuit complex.*

The main result of this subsection is the following.

THEOREM A.3. *The ring OT is spanned over $\mathrm{Sym}(V)$ by elements of the form u_S where $S \subset [n]$ is independent.*

We begin by proving the analogous statement for SR_{ind} and SR_{bc}, which will be used in the proof of Theorem A.3.

LEMMA A.4. *The rings SR_{ind} and SR_{bc} are spanned over $\mathrm{Sym}(V)$ by elements of the form u_S where $S \subset [n]$ is independent.*[5]

Proof. First note that SR_{bc} is a quotient of SR_{ind}, so it is sufficient to prove the lemma only for SR_{ind}. Since u_S vanishes whenever S contains a circuit, it is sufficient to prove that SR_{ind} is spanned over $\mathrm{Sym}(V)$ by square-free monomials. This is equivalent to showing that $SR_{\mathrm{ind}} \otimes_{\mathrm{Sym}(V)} k$ is spanned over k by square-free monomials.

Consider an arbitrary monomial u^σ for some $\sigma \in \mathbb{N}^n$ with independent support. This means that there exists a set $B \subset [n]$ containing the support of σ such that $\{a_i \mid i \in B\}$ is a basis for V^*. If $\sigma_i \leqslant 1$ for all i, then we are already done, so let us suppose that there exists an index $i \in [n]$ for which $\sigma_i > 1$. Consider the element $v \in V$ that pairs to 1 with a_i and to 0 with a_j for all $j \in B \smallsetminus \{i\}$, and let $u_v := v \cdot 1 \in SR_{\mathrm{ind}}$. By replacing u_i with $u_i - u_v$ (which has the same image in $SR_{\mathrm{ind}} \otimes_{\mathrm{Sym}(V)} k$), we replace u^σ with a sum of monomials of the form u^τ, where $\tau_i = \sigma_i - 1$, $\tau_j = \sigma_j$ for all $j \in B \smallsetminus \{i\}$, and $\tau_k \leqslant 1$ for all $k \notin B$. Applying this procedure recursively, we may express the image of u^σ as a sum of square-free monomials. $\qquad\square$

Given a subset $S \subset [n]$, let $\langle S \rangle$ be the set of all i such that a_i is contained in the k-linear span of $\{a_j \mid j \in S\}$. We always have $S \subset \langle S \rangle$; if $\langle S \rangle = S$, then S is called a **flat**. Given any flat F, let V_F be the quotient of V by the elements that vanish on a_i for all $i \in F$. Then we can regard $\{a_i \mid i \in F\}$ as a set of linear functionals on V_F that span V_F^*, which allows us to define the $\mathrm{Sym}(V_F)$-algebras $(SR_{\mathrm{bc}})_F$ and OT_F. When $F = [n]$, we have $V_F = V$, $(SR_{\mathrm{bc}})_F = SR_{\mathrm{bc}}$, and $OT_F = OT$.

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[5]The proof of this lemma does not require k to be a field; in particular, it holds over the integers.

We have canonical maps

$$\mu_F : SR_{\mathrm{bc}} \to (SR_{\mathrm{bc}})_F \qquad \text{and} \qquad \nu_F : OT \to OT_F$$

given by setting the variables not in F to zero, as well as sections

$$\alpha_F : (SR_{\mathrm{bc}})_F \to SR_{\mathrm{bc}} \qquad \text{and} \qquad \beta_F : OT_F \to OT$$

taking u_i to u_i for all $i \in F$. The following result is proved in [BP09, 3.12].

THEOREM A.5. *We may choose a* $\mathrm{Sym}(V)$*-module isomorphism* $\varphi : SR_{\mathrm{bc}} \to OT$ *and a* $\mathrm{Sym}(V_F)$*-module isomorphism* $\varphi_F : (SR_{\mathrm{bc}})_F \to OT_F$ *for every flat* F *such that the diagram*

$$
\begin{array}{ccc}
SR_{\mathrm{bc}} & \xrightarrow{\ \varphi\ } & OT \\
{\scriptstyle \mu_F} \downarrow & & \downarrow {\scriptstyle \nu_F} \\
(SR_{\mathrm{bc}})_F & \xrightarrow{\ \varphi_F\ } & OT_F
\end{array}
$$

commutes. Furthermore, these choices are unique if we require that $\varphi(1) = 1$.

LEMMA A.6. *Let* $S \subset [n]$ *be independent. There exist constants* $c_{S'} \in k$ *for each independent set* $S' \subset [n]$ *with* $\langle S' \rangle = \langle S \rangle$ *such that*

$$\varphi(u_S) = \sum_{S'} c_{S'} u_{S'} \in OT.$$

Proof. Start by choosing any constants c_σ such that $\varphi(u_S) = \sum_\sigma c_\sigma u^\sigma \in OT$, where the sum runs over $\sigma \in \mathbb{N}^n$. We will show that we can kill all those c_σ with $S \not\subset \langle \mathrm{Supp}(\sigma) \rangle$ without changing the class that it represents in OT. Since S is independent, this will imply that $u^\sigma = u_{S'}$ for some independent set S' with $\langle S' \rangle = \langle S \rangle$.

Suppose that F is a flat that does not contain S. Then $\mu_F(u_S) = 0$, so

$$\nu_F \circ \varphi(u_S) = \varphi_F \circ \mu_F(u_S) = \varphi_F(0) = 0.$$

This means that

$$\sum_{\langle \sigma \rangle \subset F} c_\sigma u^\sigma = 0 \in OT_F.$$

Applying β_F, this implies that

$$\sum_{\langle \sigma \rangle \subset F} c_\sigma u^\sigma = 0 \in OT.$$

Hence we may assume that $c_\sigma = 0$ for all σ such that $\langle \mathrm{Supp}(\sigma) \rangle \subset F$. Since we chose F to be an arbitrary flat not containing S, this means that we can assume $c_\sigma = 0$ for all σ such that $S \not\subset \langle \mathrm{Supp}(\sigma) \rangle$. $\qquad \square$

Proof of Theorem A.3: By Lemma A.4, it is sufficient to show that, for all independent $S \subset [n]$, $\phi(u_S)$ may be expressed as a linear combination of elements of the form $u_{S'}$ where

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$S' \subset [n]$ is independent. This is exactly the content of Lemma A.6. □

A.2 A flat deformation (in the unimodular case)

As in Section 3.1, fix a finite-rank lattice N, along with a list of (not necessarily distinct) nonzero primitive vectors $a_1, \dots, a_n \in N$ that span N. Let $V = N \otimes_{\mathbb{Z}} k$, and consider the associated Orlik-Terao algebra OT. Again as in Section 3.1, we assume that our collection of vectors is unimodular. This implies that, for any circuit C, there is a decomposition $C = C^+ \sqcup C^-$ (unique up to swapping C^+ and C^-) such that $\sum_{i \in C^+} a_i - \sum_{j \in C^-} a_j = 0$. In other words, we may always take the constants η_i from the previous section to be ± 1. We define a **signed circuit** to be a circuit equipped with a choice of decomposition.

REMARK A.7. In Section 3.1, we chose a simple affine hyperplane arrangement, and used this to choose a distinguished signed circuit for each circuit. Here we have no such affine arrangement, and there is no distinuguished choice.

For each signed circuit C, let

$$f_C := \hbar^{-1} \left(\prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} (u_j - \hbar) \ - \ \prod_{i \in C^+} (u_i - \hbar) \cdot \prod_{j \in C^-} u_j \right),$$

and consider the $k[\hbar]$-algebra

$$OT_{\hbar} := k[u_1, \dots, u_n, \hbar]/I_{\hbar},$$

where

$$I_{\hbar} = \langle f_C \mid C \text{ a signed circuit} \rangle.$$

For any $t \in k$, let $I_t \subset k[u_1, \dots, u_n]$ be the ideal obtained from $I_{\hbar}$ by setting $\hbar$ equal to t, and let

$$OT_t := k[u_1, \dots, u_n]/I_t.$$

In particular, we have $I_0 = I$, and therefore OT_0 is equal to the Orlik-Terao algebra OT.

REMARK A.8. If C and C' are opposite signed circuits, then $f_C = -f_{C'}$, thus it is enough to pick one signed circuit for each circuit in the definition of $I_{\hbar}$. In particular, this means that $I_{\hbar}$ coincides with the ideal J_1' defined in Corollary 3.6 when $k = \mathbb{C}$, and therefore $OT_{\hbar}$ coincides with $R_{\mathbf{T}}'(\tilde{X})$. Furthermore, the $\mathrm{Sym}(V)[\hbar]$-algebra structure on $OT_{\hbar}$ coincides with the $A_{\mathbb{C}}[\hbar]$-algebra structure on $R_{\mathbf{T}}'(\tilde{X})$.

THEOREM A.9. *The algebra $OT_{\hbar}$ is a free module over $k[\hbar]$, and is thus a flat deformation of OT_0.*

Proof. Let

$$I_{\hbar}' := \{f \mid \hbar^k f \in I_{\hbar} \text{ for some } k \in \mathbb{N}\} \qquad \text{and} \qquad OT_{\hbar}' := k[u_1, \dots, u_n, \hbar]/I_{\hbar}';$$

then $OT_{\hbar}'$ is a flat deformation of OT_0'. It is clear that $OT_1 = OT_1'$ and that we have a surjection $OT_0 \twoheadrightarrow OT_0'$, and therefore a closed inclusion $\mathrm{Spec}\, OT_0' \subset \mathrm{Spec}\, OT_0$. Theorem A.9 is equivalent

to the statement that this inclusion is an isomorphism. Since $\operatorname{Spec} OT_0$ is reduced and irreducible of dimension $\operatorname{rk} N$, it is sufficient to show that $\dim \operatorname{Spec} OT_0' = \operatorname{rk} N$. Since $OT_\hbar'$ is flat, this is equivalent to showing that $\dim \operatorname{Spec} OT_1 = \operatorname{rk} N$. Since we already know one inequality, we need only show that $\dim \operatorname{Spec} OT_1 \geqslant \operatorname{rk} N$.

Consider the ideal $\tilde{I}_1 \subset k[u_1, v_1, \ldots, u_n, v_n]$ generated by elements of the form

$$\prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} v_j \;-\; \prod_{i \in C^+} v_i \cdot \prod_{j \in C^-} u_j$$

for each signed circuit C, and let $\widetilde{OT}_1 := k[u_1, v_1, \ldots, u_n, v_n]/\tilde{I}_1$. Since $\operatorname{Spec} OT_1$ is cut out of $\operatorname{Spec} \widetilde{OT}_1$ by the n equations $v_i = u_i - 1$ and intersects the regular locus of $\operatorname{Spec} \widetilde{OT}_1$ nontrivially, it is sufficient to show that $\dim \operatorname{Spec} \widetilde{OT}_1 \geqslant \operatorname{rk} N + n$.

Consider the lattice $L := N \oplus \mathbb{Z}^n$ and the elements $r_i = (a_i, e_i) \in L$ and $s_i = (-a_i, e_i) \in L$. Define a map from $\widetilde{OT}_1$ to $k\{q^\ell \mid \ell \in L\}$ by sending u_i to q^{r_i} and v_i to q^{s_i}. Since $\{r_1, s_1, \ldots, r_n, s_n\}$ spans a finite index sublattice of L, the induced map from the torus $T_L := \operatorname{Spec} k\{q^\ell \mid \ell \in L\}$ to $\operatorname{Spec} \widetilde{OT}_1$ is finite-to-one. Since $\dim T_L = \operatorname{rk} N + n$, this completes the proof. $\qquad\square$

REMARK A.10. The quotient

$$AOT_0 := k[u_1, \ldots, u_n] \,\Big/\, I + \langle u_1^2, \ldots, u_n^2 \rangle$$

is called the **Artinian Orlik-Terao algebra**. Moseley [Mos, 4.5] studied the ring

$$AOT_\hbar := k[u_1, \ldots, u_n, \hbar] \,\Big/\, I_\hbar + \langle u_1(u_1 - \hbar), \ldots, u_n(u_n - \hbar) \rangle,$$

and showed that $AOT_\hbar$ is a flat deformation of AOT_0 into the Varchenko-Gelfand algebra. This is the Artinian analogue of Theorem A.9, but neither result follows from the other.

REFERENCES

AP   Matthew Arbo and Nicholas Proudfoot, *Abstract hypertoric varieties*, in preparation.

BD00   Roger Bielawski and Andrew S. Dancer, *The geometry and topology of toric hyperkähler manifolds*, Comm. Anal. Geom. **8** (2000), no. 4, 727–760.

BLPW   Tom Braden, Anthony Licata, Nicholas Proudfoot, and Ben Webster, *Quantizations of conical symplectic resolutions II: category $\mathcal{O}$*, arXiv:1407.0964.

BMO11   Alexander Braverman, Davesh Maulik, and Andrei Okounkov, *Quantum cohomology of the Springer resolution*, Adv. Math. **227** (2011), no. 1, 421–458.

BP09   Tom Braden and Nicholas Proudfoot, *The hypertoric intersection cohomology ring*, Invent. Math. **177** (2009), no. 2, 337–379.

BPW   Tom Braden, Nicholas Proudfoot, and Ben Webster, *Quantizations of conical symplectic resolutions I: local and global structure*, arXiv:1208.3863.

CG97   Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997.

DGT14   Graham Denham, Mehdi Garrousian, and Ştefan O. Tohǎneanu, *Modular decomposition of the Orlik-Terao algebra*, Ann. Comb. **18** (2014), no. 2, 289–312.

FFR10   Boris Feigin, Edward Frenkel, and Leonid Rybnikov, *Opers with irregular singularity and spectra of the shift of argument subalgebra*, Duke Math. J. **155** (2010), no. 2, 337–363.

Gin   Victor Ginzburg, *Perverse sheaves on a Loop group and Langlands' duality*, arXiv:alg-geom/9511007.

HP04   Megumi Harada and Nicholas Proudfoot, *Properties of the residual circle action on a hypertoric variety*, Pacific J. Math. **214** (2004), no. 2, 263–284.

Kon99   Hiroshi Konno, *Equivariant cohomology rings of toric hyperkähler manifolds*, Quaternionic structures in mathematics and physics (Rome, 1999), Univ. Studi Roma "La Sapienza", Rome, 1999, pp. 231–240.

KWWY14   Joel Kamnitzer, Ben Webster, Alex Weekes, and Oded Yacobi, *Yangians and quantizations of slices in the affine Grassmannian*, Algebra Number Theory **8** (2014), no. 4, 857–893.

Le14   Dinh Van Le, *On the Gorensteinness of broken circuit complexes and Orlik-Terao ideals*, J. Combin. Theory Ser. A **123** (2014), 169–185.

Liu   Ricky Ini Liu, *On the commutative quotient of Fomin-Kirillov algebras*, arXiv:1409.4872.

MO   Davesh Maulik and Andrei Okounkov, *Quantum groups and quantum cohomology*, arXiv:1211.1287.

MO09   Davesh Maulik and Alexei Oblomkov, *Quantum cohomology of the Hilbert scheme of points on $\mathcal{A}_n$-resolutions*, J. Amer. Math. Soc. **22** (2009), no. 4, 1055–1091.

Mos   Daniel Moseley, *Equivariant cohomology and the Varchenko-Gelfand filtration*, arXiv:1110.5369.

MS13   Michael B. McBreen and Daniel K. Shenfeld, *Quantum cohomology of hypertoric varieties*, Lett. Math. Phys. **103** (2013), no. 11, 1273–1291.

MV07   I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143.

Nak94   Hiraku Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416.

Nak98   _____, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3, 515–560.

Nak01   _____, *Quiver varieties and finite-dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), no. 1, 145–238 (electronic).

OP10   A. Okounkov and R. Pandharipande, *Quantum cohomology of the Hilbert scheme of points in the plane*, Invent. Math. **179** (2010), no. 3, 523–557.

PS06   Nicholas Proudfoot and David Speyer, *A broken circuit ring*, Beiträge Algebra Geom. **47** (2006), no. 1, 161–166.

PW07   Nicholas Proudfoot and Ben Webster, *Intersection cohomology of hypertoric varieties*, J. Algebraic Geom. **16** (2007), no. 1, 39–63.

Sch11   Hal Schenck, *Resonance varieties via blowups of $\mathbb{P}^2$ and scrolls*, Int. Math. Res. Not. IMRN (2011), no. 20, 4756–4778.

ST09   Hal Schenck and Ştefan O. Tohǎneanu, *The Orlik-Terao algebra and 2-formality*, Math. Res. Lett. **16** (2009), no. 1, 171–182.

Ter02   Hiroaki Terao, *Algebras generated by reciprocals of linear forms*, J. Algebra **250** (2002), no. 2, 549–558.

VLR13   Dinh Van Le and Tim Römer, *Broken circuit complexes and hyperplane arrangements*, J. Algebraic Combin. **38** (2013), no. 4, 989–1016.