

Resolving toric varieties with Nash blow-ups

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1 Introduction

Let X be a variety over an algebraically closed field K . Its *Nash blow-up* is a variety over K with a projective morphism to X , which is an isomorphism over the smooth locus. Roughly speaking, it parametrizes all limits of tangent planes to X (a precise definition is given in §2 below). The Nash blow-up of a singular X is not always smooth but seems, in some sense, to be less singular than X . Strictly speaking this is false, for in characteristic $p > 0$, as explained by Nobile [14], the plane curve $x^p - y^q = 0$ is its own Nash blow-up for any $q > 0$. In this and other ways the ordinary Nash blow-up proves intractable.

However, let the *normalized Nash blow-up* be the normalization of the Nash blow-up. Then, of course, the normalized Nash blow-up of every curve is smooth. The normalized Nash blow-up of a surface can be singular, but Hironaka [10] and Spivakovsky [7, 16] have shown that every surface becomes smooth after finitely many normalized Nash blow-ups. Thus we are drawn to ask the following.

(1.1) Question. *Is every variety desingularized by finitely many normalized Nash blow-ups?*

According to Spivakovsky [16], Nash asked Hironaka this question in the early 1960s. An affirmative answer would give a canonical procedure for desingularizing an arbitrary variety. The answer to this question is not known and is surely difficult. In this paper we address a more narrow question.

(1.2) Question. *Is every toric variety desingularized by finitely many normalized Nash blow-ups?*

We do not answer this question conclusively either. But we do exhibit abundant evidence supporting an affirmative answer. Using the “toric dictionary,” which translates every problem in toric geometry into a problem on convex polyhedra, we convert Question (1.2) into a problem amenable to computer calculation. Then we carry out this calculation for thousands of examples. In every case, finitely many Nash blow-ups produce a smooth toric variety.

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Summary of the paper. In §2 we introduce the toric dictionary and, following Gonzalez-Sprinberg [6], translate the question into convex geometry. In §3 we summarize the effect of the iterated Nash blow-up, in the toric case, using the notion of a *resolution tree*. In §4 we spell out what happens in the 2-dimensional toric case in terms of continued fractions. In §5 we digress briefly on the classification of quasi-smooth affine toric varieties, those corresponding to simplicial cones in the toric dictionary. Then in §6 we give an account of our computer investigations.

Notation and conventions. We slightly abuse terminology in two ways. First, as we are concerned with the normalized Nash blow-up throughout, we refer to it simply as the *Nash blow-up*. Second, as we are concerned with rational polyhedra and rational polyhedral cones throughout, we refer to them simply as *polyhedra* and *cones*. We denote the natural numbers, including 0, by \mathbb{Z}_+ , and we likewise denote the nonnegative rational numbers, including 0, by \mathbb{Q}_+ . We denote the span of v_1, \dots, v_k with coefficients in S by $S\langle v_1, \dots, v_k \rangle$. Thus, for example, the first quadrant in \mathbb{Q}^2 is denoted $\mathbb{Q}_+\langle e_1, e_2 \rangle$.

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2 Equivalence to a combinatorial problem

Nash blow-ups. Let $X \subset \mathbb{P}^n$ be a quasiprojective variety of dimension d over an algebraically closed field K . The *Gauss map* is the rational map $X \dashrightarrow \text{Gr}(d+1, n+1)$ taking a smooth point to its tangent plane. The *Nash blow-up* of X is defined to be the closure of the graph of the Gauss map. The *normalized Nash blow-up* of X is the normalization of the Nash blow-up of X . Gonzalez-Sprinberg's and Spivakovsky's results are concerned with this variant, as are ours. Consequently, we shall abuse terminology by referring to a normalized Nash blow-up simply as a Nash blow-up.

Remarks:

(1) As defined, the Nash blow-up appears to depend on the projective embedding of X , but it can be reformulated in terms of Kähler differentials and hence depends only on X (and makes sense even if X is not quasiprojective) [7].

(2) Since the normalization of a variety over K is a finite morphism [8, I 3.9A] and the pullback of an ample bundle by a finite morphism is ample [12, 1.7.7], the normalization is a projective morphism. Hence the natural morphism from the (normalized) Nash blow-up of X to X is projective.

(3) Clearly the Nash blow-up of a smooth variety is itself, and the Nash blow-up of a product is a product.

(4) If $X \subset \mathbb{A}^d$ is an affine variety, we may consider the analogous construction using the Gauss map $X \dashrightarrow \text{Gr}(d, n)$, but this produces exactly the same thing, since the morphism $X \times \text{Gr}(d, n) \rightarrow X \times \text{Gr}(d+1, n+1)$ given by $(x, V) \mapsto (x, K\langle(1 \times x)\rangle \oplus (0 \times V))$ is a closed embedding.

Toric varieties. We review here some standard definitions and facts about toric varieties. For proofs, we refer the reader to Ewald [3], Fulton [4], Miller & Sturmfels [13], and Thaddeus [17].

A *polyhedron* in \mathbb{Q}^d is a subset P defined by finitely many weak affine inequalities, say $\sum_{j=1}^d a_{ij}x_j \geq b_i$. It is a *polyhedral cone* if the inequalities are linear, that is, all $b_i = 0$. For simplicity we refer to polyhedral cones simply as *cones*. We also assume that all polyhedra are *rational*, meaning that all a_{ij} and all b_i are rational. A polyhedron is *proper* if it contains no affine linear subspace besides a point, and is contained in no affine linear subspace besides \mathbb{Q}^d .

A *face* F of P is the locus where equality holds in some fixed subset of the inequalities above. It is a *facet* if its affine linear span has codimension 1 in that of P . It is a *vertex* if it is a point. A proper cone in \mathbb{Q}^d is *simplicial* if it has exactly d facets. For any face $F \subset P$, the *localization* P_F is the cone generated, as a semigroup, by the Minkowski difference $P - F$.

For $t \neq 0$, let $tP = \{tv \mid v \in P\}$; however, for $t = 0$, by convention let $0P$ denote the *cone at infinity* defined by the same inequalities as P , except with constant terms set to zero. The reason for this convention is that $\{(t, v) \in \mathbb{Q}_+ \times \mathbb{Q}^d \mid v \in tP\}$ is then a cone in \mathbb{Q}^{d+1} , defined by the inequalities $\sum_{j=1}^d a_{ij}x_j \geq b_i x_0$ and $x_0 \geq 0$. It is called the *cone over* P and denoted $C(P)$.

A *torus* is a product of finitely many copies of the multiplicative group of K . A *toric variety* is a normal variety on which a torus acts with finitely many orbits. There is a one-to-one correspondence between polyhedra with integer vertices and toric varieties that are projective over an affine, equipped with a lifting of the torus action to $\mathcal{O}(1)$. It is given as follows. For a polyhedron $P \subset \mathbb{Q}^d$, the semigroup algebra $K[C(P) \cap \mathbb{Z}^{d+1}]$ is graded by the 0th coordinate. Let $X(P) = \text{Proj } K[C(P) \cap \mathbb{Z}^{d+1}]$. This is a quasiprojective variety acted on by the torus $T = \text{Spec } K[\mathbb{Z}^d]$. For example, if P is already a cone, then $C(P) = \mathbb{Q}_+ \times P$ and $X(P) = \text{Proj } K[P \cap \mathbb{Z}^d][x_0] = \text{Spec } K[P \cap \mathbb{Z}^d]$, the affine toric variety usually associated to a cone. In general, $X(P)$ is projective over the affine $X(0P)$, because $C(P) \cap (0 \times \mathbb{Q}^d) = 0P$.

Further remarks:

(5) A polyhedron P is proper if and only if (a) the torus action on $X(P)$ is effective, and (b) $X(P)$ is not a direct product of a toric variety with a torus. So in light of remark (3), there is no loss of generality, for the purposes of Nash blowing-up, in assuming that P is proper.

(6) Any toric variety has a natural cover by open affine toric subvarieties. Indeed, $X(P)$ is covered by the affine varieties $X(P_F)$, where F runs over the faces of P . If P is proper, just the vertices are sufficient.

(7) Define two cones to be *equivalent* if an element of $GL(d, \mathbb{Z})$ takes one to the other. Then equivalent cones clearly lead to isomorphic toric varieties, with the torus action adjusted by the appropriate automorphism of T .

(8) An affine toric variety $X(C)$, with C proper, is smooth if and only if it is isomorphic to \mathbb{A}^d , or equivalently, if C is equivalent to the orthant $\mathbb{Q}_+\langle e_1, \dots, e_d \rangle$.

Nash blow-ups of toric varieties. Now let C be a cone in \mathbb{Q}^d . Let H be the Hilbert basis of the semigroup $C \cap \mathbb{Z}^d$, that is, the set of indecomposable nonzero elements in the semigroup. This is the unique minimal set of generators of $C \cap \mathbb{Z}^d$. By Gordan's lemma [4, §1.2, Prop. 1] H is a finite set, say with n elements. Let M be the $n \times d$ integer matrix whose rows are the elements of H .

Let $S = \{h_1 + \dots + h_d \mid h_i \in H \text{ linearly independent}\}$. Since S is finite, its convex hull is a compact polyhedron $\text{Hull } S$. Hence the Minkowski sum $C + \text{Hull } S$ is a polyhedron whose

cone at infinity is C .

The following result is proved (in the language of fans) by Gonzalez-Sprinberg [6].

(2.1) Theorem. *The Nash blow-up of $X(C)$ is $X(C + \text{Hull } S)$.*

Proof. Without loss of generality C may be assumed proper. In this case $X(C)$ has a unique T -fixed point q .

Let $X = X(C)$. The Nash blow-up of X is plainly a toric variety, projective over X . It is therefore $X(P)$ for some polyhedron P with $0P = C$.

Such a polyhedron is uniquely determined by its cone at infinity C and its vertices v_i . Indeed, the cone over P is $C(P) = \mathbb{Q}_+\langle 0 \times C, 1 \times v_i \rangle$, and $P = C(P) \cap (1 \times \mathbb{Q}^d)$. So it suffices to show that, at the fixed points of the torus action on the Nash blow-up, the weights of the torus action on $\mathcal{O}(1)$ are exactly the coordinates of the vertices of $C + \text{Hull } S$.

Our choice of an embedding $X \subset \mathbb{A}^n$ will be the following canonical one. The surjection $\mathbb{Z}_+^n \rightarrow C \cap \mathbb{Z}^d$ sending the standard basis vectors to the rows of M induces a surjection of algebras $K[\mathbb{Z}_+^n] \rightarrow K[C \cap \mathbb{Z}^d]$. The corresponding morphism $\text{Spec } K[C \cap \mathbb{Z}^d] \rightarrow \text{Spec } K[\mathbb{Z}_+^n]$ is the desired embedding.

Let p be the basepoint of X : the point so that for every monomial $f \in K[C \cap \mathbb{Z}^d]$, $f(p) = 1$. The homomorphisms of algebras

$$K[\mathbb{Z}_+^n] \rightarrow K[C \cap \mathbb{Z}^d] \rightarrow K[\mathbb{Z}^d] \rightarrow K,$$

where the last map sends every monomial to 1, correspond to the inclusions of schemes

$$\mathbb{A}^n \supset X \supset T \supset \{p\}.$$

By remark (4), we may consider the affine version of the Gauss map for this embedding. This is a rational map $G : X \dashrightarrow \text{Gr}(d, n)$. We claim that $G(p)$ is the span of the columns of M . Indeed, in terms of variables x_1, \dots, x_n and y_1, \dots, y_d , the homomorphism $K[\mathbb{Z}_+^n] \rightarrow K[\mathbb{Z}^d]$ is given by $x_i \mapsto \prod_j y_j^{m_{ij}}$. The parametric curve $y_j = 1 + t\delta_{ij}$ in T therefore maps to $x_i = (1 + t)^{m_{ij}}$ in \mathbb{A}^n , so its derivative with respect to t at 0 is (m_{1j}, \dots, m_{nj}) , the j th column of M .

The coordinates of the Plücker embedding $\text{Gr}(d, n) \rightarrow \mathbb{P}\Lambda^d K^n$ are indexed by d -element subsets $I \subset \{1, \dots, n\}$. This embedding is T -equivariant for the induced linear action of T on $\mathbb{P}\Lambda^d K^n$. The I th Plücker coordinate of $G(p)$ is the I th minor of M . Hence $G(p)$ is contained in the linear subspace of $\mathbb{P}\Lambda^d K^n$ spanned by those coordinates I for which the I th minor of M is nonzero. Since the T -action on $\mathbb{P}\Lambda^d K^n$ is diagonal, the entire closure of the orbit of $G(p)$ must be contained in this subspace. Hence any fixed point in the closure of this orbit must be the I th coordinate axis e_I for some I as above. If $I = \{i_1, \dots, i_d\}$, then the nonvanishing of the I th minor is equivalent to the linear independence of $h_{i_1}, \dots, h_{i_d} \in H$, and the fiber of $\mathcal{O}(1)$ at this point is acted on with weight $h_{i_1} + \dots + h_{i_d}$. That is, the weights at fixed points in this subspace are exactly the elements of S .

The closure of the graph of the Gauss map is clearly contained in $X \times \overline{Tp}$, so its T -fixed points must be of the form $q \times e_I$, where q is the unique fixed point in X , and e_I is as above. The weights of $\mathcal{O}(1)$ at these points must therefore belong to S . The same is true for the normalization, since $\mathcal{O}(1)$ pulls back to an ample bundle there.

Consequently, P is a polyhedron with $0P = C$ and with vertices contained in S . Therefore $P \subset C + \text{Hull } S$.

To establish equality, it suffices to show that every vertex in $C + \text{Hull } S$ is the weight of some fixed point in the Nash blow-up. For every vertex v_I of $C + \text{Hull } S$, there is a linear functional f on \mathbb{Q}^d whose restriction to $C + \text{Hull } S$ takes on its minimum only at v_I . Hence its restriction to $\text{Hull } S$ takes on its minimum only at v_I , and its restriction to C takes on its minimum only at 0. The corresponding 1-parameter subgroup $\lambda(t) : K^\times \rightarrow T$ therefore satisfies $\lim_{t \rightarrow 0} \lambda(t) \cdot G(p) = e_I$ and $\lim_{t \rightarrow 0} \lambda(t) \cdot p = q$. Hence $q \times e_I \in \overline{T \cdot (p \times G(p))}$, the closure of the graph of the Gauss map. A point in the normalization lying over $q \times e_I$ is acted on with the same weight. This completes the proof. \square

3 Resolution trees

We wish to consider whether a toric variety is desingularized by a finite sequence of Nash blow-ups. The Nash blow-up is a local construction: that is, the Nash blow-ups of an open cover furnish an open cover of the Nash blow-up. Hence it suffices to consider an affine toric variety $X(C)$. The Nash blow-up of $X(C)$ is $X(C + \text{Hull } S)$; by remark (6), an open cover of this consists of the affines $X((C + \text{Hull } S)_v)$, where v runs over the vertices of $C + \text{Hull } S$. By remark (8), $X(C + \text{Hull } S)$ is smooth if and only if each localization $(C + \text{Hull } S)_v$ is equivalent to the orthant under the action of $GL(d, \mathbb{Z})$. If not, the Nash blow-up can be repeated by applying the theorem to each cone $(C + \text{Hull } S)_v$.

In other words, the process of iterating Nash blow-ups of $X(C)$ corresponds, via the toric dictionary, to the following algorithm in convex geometry:

- (1) Given the cone C , find the Hilbert basis H of $C \cap \mathbb{Z}^d$.
- (2) Find $S = \{h_1 + \cdots + h_d \mid h_i \in H \text{ linearly independent}\}$.
- (3) Find the convex hull $\text{Hull } S$ (i.e. list its vertices, or list the inequalities defining it).
- (4) Find the Minkowski sum $C + \text{Hull } S$ (i.e. list its vertices and cone at infinity, or list the inequalities defining it).
- (5) Find the localization $C' = (C + \text{Hull } S)_v$ of this Minkowski sum at each vertex v .
- (6) Determine whether each such C' is equivalent to the orthant. If so, stop; if not, apply the entire algorithm to C' .

Because each cone may give rise to several more in step (5), the algorithm branches. This can be expressed in terms of a graph as follows. Define the *Nash blow-up of a cone* C to be the finite set of cones of the form $(C + \text{Hull } S)_v$, where S is as in (2), and v runs over the vertices of $C + \text{Hull } S$. Then define the *resolution tree* of C or $X(C)$ to be the unique rooted tree, with nodes labeled by cones in \mathbb{Q}^d , whose root is labeled by C , and where for every node, say labeled by C' :

- (a) if C' is equivalent to the orthant, there are no edges beginning at C' (that is, C' is a leaf);
- (b) otherwise, the edges beginning at C' connect it to nodes labeled by the cones $(C' + \text{Hull } S')_{v'}$ appearing in its Nash blow-up.

It is then clear that $X(C)$ is desingularized by a finite number of Nash blow-ups if and only if its resolution tree is finite. It is equally clear that the latter property is amenable to computer investigation, using the algorithm above. We will report on this presently, but first,

we explain how, in the 2-dimensional case, the situation can be completely understood.

4 The 2-dimensional case

Gonzalez-Sprinberg showed [5, 6] that toric surfaces are desingularized by a finite sequence of (normalized) Nash blow-ups. This was later extended to arbitrary surfaces by Hironaka [10] and Spivakovsky [7, 16]. In this section, we give an alternative proof of Gonzalez-Sprinberg's original result, emphasizing the role of Hirzebruch-Jung continued fractions. We begin by defining them and recalling their basic properties.

For integers a_1, a_2, \dots , let

$$[a_1, \dots, a_i] = a_1 - \frac{1}{a_2 - \frac{1}{\dots - \frac{1}{a_i}}}.$$

We assume implicitly throughout that no denominator is zero; this is the case, for example, when $a_i > 1$ for $i > 1$.

Set $p_{-1} = 0$ and $q_0 = 0$; set $p_0 = 1$ and $q_1 = 1$. Then recursively let

$$p_i = a_i p_{i-1} - p_{i-2}, \quad q_i = a_i q_{i-1} - q_{i-2}$$

for greater values of i .

(4.1) Proposition. *For p_i, q_i as above, $[a_1, \dots, a_i] = p_i/q_i$.*

Proof. Using induction on i , we will prove the more general statement where the a_i are merely rational. The case $i = 1$ is trivial. For $i > 1$, assume the statement holds for continued fractions of length $i - 1$, and consider $[a_1, \dots, a_{i-2}, a_{i-1} - 1/a_i]$. Let P_j, Q_j be the numbers defined as above for this continued fraction. Then $P_j = p_j$ and $Q_j = q_j$ for $j < i - 1$, and

$$\begin{aligned} [a_1, \dots, a_i] &= [a_1, \dots, a_{i-2}, a_{i-1} - 1/a_i] \\ &= \frac{P_{i-1}}{Q_{i-1}} \\ &= \frac{(a_{i-1} - 1/a_i)p_{i-2} - p_{i-3}}{(a_{i-1} - 1/a_i)q_{i-2} - q_{i-3}} \\ &= \frac{(a_{i-1}a_i - 1)p_{i-2} - a_i p_{i-3}}{(a_{i-1}a_i - 1)q_{i-2} - a_i q_{i-3}} \\ &= \frac{a_i p_{i-1} - p_{i-2}}{a_i q_{i-1} - q_{i-2}} \\ &= \frac{p_i}{q_i}. \quad \square \end{aligned}$$

(4.2) Proposition. For $i > 0$, $p_{i-1}q_i - p_iq_{i-1} = 1$.

Proof. Again use induction on i . The case $i = 1$ is trivial. For $i > 1$, by the induction hypothesis,

$$\begin{aligned} p_{i-1}q_i - p_iq_{i-1} &= p_{i-1}(a_iq_{i-1} - q_{i-2}) - (a_ip_{i-1} - p_{i-2})q_{i-1} \\ &= -p_{i-1}q_{i-2} + p_{i-2}q_{i-1} \\ &= 1. \quad \square \end{aligned}$$

(4.3) Corollary. The fraction p_i/q_i is in lowest terms. \square

(4.4) Proposition. For $i < j$, the denominator of $[a_{i+1}, \dots, a_j]$ as a fraction in lowest terms is $p_iq_j - p_jq_i$.

Proof. The case $i = j - 1$ is covered by the previous proposition. Now proceed by descending induction on i . Let $[a_{i+1}, \dots, a_j] = N_i/D_i$ in lowest terms, so that $D_{j-1} = 1$ in particular. Take $N_j = 1, D_j = 0$ by convention. Then for all $i < j$ we have

$$\begin{aligned} [a_{i+1}, \dots, a_j] &= a_{i+1} - 1/[a_{i+2}, \dots, a_j] \\ &= a_{i+1} - \frac{D_{i+1}}{N_{i+1}} \\ &= \frac{a_{i+1}N_{i+1} - D_{i+1}}{N_{i+1}}, \end{aligned}$$

which is also in lowest terms. Hence $D_i = N_{i+1}$, and the D_i satisfy the descending recursion $D_i = a_{i+2}D_{i+1} - D_{i+2}$ with initial conditions $D_0 = 0, D_1 = 1$. The same holds for

$$\begin{aligned} p_iq_j - p_jq_i &= (a_{i+2}p_{i+1} - p_{i+2})q_j - p_j(a_{i+2}q_{i+1} - q_{i+2}) \\ &= a_{i+2}(p_{i+1}q_j - p_jq_{i+1}) - (p_{i+2}q_j - p_jq_{i+2}), \end{aligned}$$

which completes the proof. \square

(4.5) Proposition. For any rational x , there exists a unique finite sequence of integers a_1, \dots, a_k with $a_i > 1$ for $i > 1$ such that $x = [a_1, \dots, a_k]$.

Proof. For any such sequence and for any $i > 1$, we have $[a_i, \dots, a_k] > 1$ by descending induction on i . If $x = [a_1, \dots, a_k]$, then $x = a_1 - 1/[a_2, \dots, a_k]$, so $a_1 = [x]$ is uniquely determined. Then $1/(a_1 - x) = [a_2, \dots, a_k]$, and hence a_2 is uniquely determined too. By induction, all the a_i are uniquely determined.

As for existence, this can be established by iterating three operations: round up, subtract, and invert. That is, given $x_1 = x$, let $a_1 = [x_1]$, let $b_1 = a_1 - x_1$, and let $x_2 = 1/b_1$. Recursively, given x_i , let $a_i = [x_i]$, let $b_i = a_i - x_i$, and let $x_{i+1} = 1/b_i$. If $x_i = n_i/d_i$ is in lowest terms, then $x_{i+1} = d_i/(a_id_i - n_i)$ is also in lowest terms, so $n_{i+1} = d_i$. Since $x_i > 1$ for $i > 1$, the sequence of d_i must be nonnegative and strictly decreasing, so eventually some $d_i = 1$ (whereupon x_{i+1} is undefined and the sequence ends). It is then easy to verify that $x = [a_1, \dots, a_k]$. \square

(4.6) Corollary. *If $a_i > 1$ for $i > 1$, then for $1 < i \leq j$, the denominator of $[a_i, \dots, a_j]$ is strictly less than that of $[a_1, \dots, a_j]$.*

Proof. The sequence of denominators is the strictly decreasing sequence d_i appearing in the proof of the previous proposition. \square

(4.7) Proposition. *If $a_i > 1$ for $i > 1$, then for all $i < j$, the denominator of $[a_1, \dots, a_i]$ is strictly less than that of $[a_1, \dots, a_j]$.*

Proof. The denominators are exactly the q_i , so this is equivalent to showing the q_i are strictly increasing, which is proved by induction on i : if $q_{i-1} - q_{i-2} > 0$, then $q_i - q_{i-1} = a_i q_{i-1} - q_{i-2} - q_{i-1} = (a_i - 1)q_{i-1} + q_{i-1} - q_{i-2} > 0$. \square

(4.8) Corollary. *If $a_i > 1$ for $i > 1$, then for all $1 < i < j < k$, the denominator of $[a_i, \dots, a_j]$ is strictly less than that of $[a_1, \dots, a_k]$.*

Proof. Combine the last two results. \square

Now let C be a proper cone in \mathbb{Q}^2 . It can be placed in a standard form as follows.

(4.9) Proposition. *There exists an element of $SL(2, \mathbb{Z})$ taking C to $\mathbb{Q}_+ \langle (1, 0), (p, q) \rangle$ with $0 \leq p < q$ and p, q coprime; that is, a cone in the first quadrant subtending an angle between 45° and 90° .*

Proof. Any proper cone in \mathbb{Q}^2 has two facets or edges. Let $(a, b) \in \mathbb{Z}^2$ be the smallest nonzero integer point along the clockwise edge. Then a is coprime to b , say $ac + bd = 1$, and $\begin{pmatrix} c & d \\ -b & a \end{pmatrix} \in SL(2, \mathbb{Z})$ takes C to a cone whose clockwise edge is along the positive x -axis and hence is contained in the first and second quadrants. Let (e, f) be the smallest nonzero integer point along the counterclockwise edge. Since $f > 0$, there exists an integer g such that $0 \leq e + gf < f$. Then $\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$ takes this cone to $\mathbb{Q}_+ \langle (1, 0), (e + gf, f) \rangle$, which satisfies the desired properties. \square

In light of the last proposition, we may assume $C = \mathbb{Q}_+ \langle (1, 0), (p, q) \rangle$ for coprime p, q with $0 \leq p < q$. As in §2, the intersection $C \cap \mathbb{Z}^2$ is an additive semigroup with a finite Hilbert basis H . In this simple case, the Hilbert basis may be explicitly described.

(4.10) Proposition. *If $p/q = [a_1, \dots, a_k]$, then $H = \{v_0, \dots, v_k\}$, where $v_i = (p_i, q_i) \in \mathbb{Z}^2$.*

Proof. Since $p_{i-1}q_i - p_iq_{i-1} = 1$, the slopes of the rays through the v_i are strictly increasing, and the lattice points in $\mathbb{Q}_+ \langle v_{i-1}, v_i \rangle$ are all integral linear combinations of v_{i-1} and v_i . This fan of subcones covers the entire cone, so any nonzero indecomposable element must be one of the v_i .

Conversely, since the q_i are strictly increasing, if any v_i can be nontrivially expressed as an integral combination of indecomposable elements, those elements must belong to $\{v_0, \dots, v_{i-1}\}$. But this is absurd, as those elements subtend a smaller cone that does not contain v_i . \square

Now, as in §§2 and 3, let $S = \{v_i + v_j \mid 0 \leq i < j \leq k\}$.

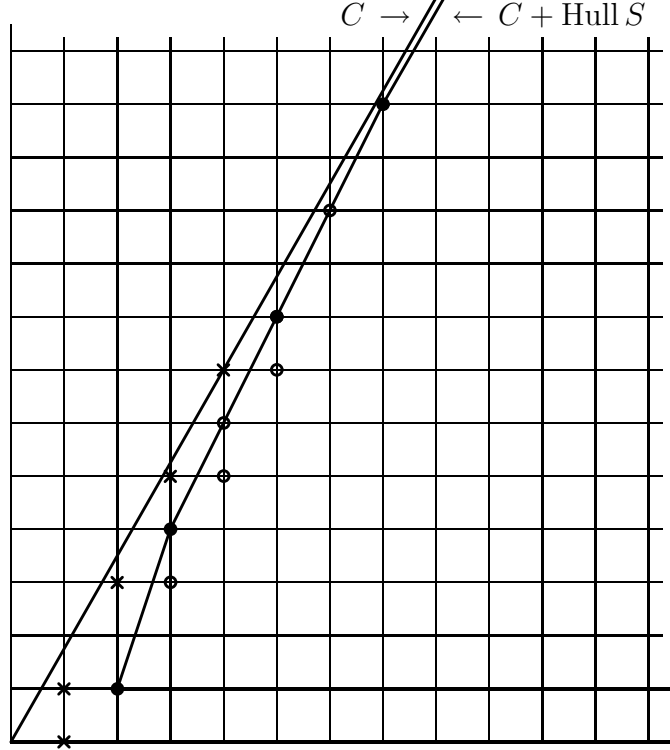


FIGURE 1. The case $p/q = 4/7$: $C = \mathbb{Q}_+ \langle (1, 0), (4, 7) \rangle$,
 $H = \{\times\}$, $S = \{\bullet, \circ\}$, $S' = \{\bullet\}$

(4.11) Proposition. *If $S' = \{v_i + v_{i+1} \mid 0 \leq i < k\}$, then $C + \text{Hull } S' = C + \text{Hull } S$.*

Proof. One inclusion is trivial. For the other, it suffices to show that $v_i + v_j \in C + \text{Hull } S'$ for $0 < i + 1 < j \leq k$. In fact, we will show that $v_i + v_j$ is in the even smaller set $C + \text{Hull}\{v_i + v_{i+1}, v_{j-1} + v_j\}$. This is bounded by three lines, so it suffices to show that $v_i + v_j$ is on the correct side of each.

First, consider the line joining $v_i + v_{i+1}$ and $v_{j-1} + v_j$. To simplify the notation, let $\langle (x, y), (x', y') \rangle = xy' - x'y$, which is positive if and only if (x', y') is counterclockwise from (x, y) . By (4.4), $\langle v_i, v_j \rangle > 0$ for $i < j$, and hence $\langle v_i + v_{i+1}, v_{j-1} + v_j \rangle > 0$ too. For any two points $u_1, u_2 \in \mathbb{Q}^2$, the affine linear functional $f(u) = \langle u_1, u \rangle + \langle u, u_2 \rangle - \langle u_1, u_2 \rangle$ vanishes on the line joining u_1 and u_2 . Let $u_1 = v_i + v_{i+1}$ and $u_2 = v_{j-1} + v_j$; then $f(0) = -\langle v_i + v_{i+1}, v_{j-1} + v_j \rangle < 0$. A brief calculation shows $f(v_i + v_j) = \langle v_i, v_j \rangle - \langle v_i, v_{i+1} \rangle - \langle v_{i+1}, v_{j-1} \rangle - \langle v_{j-1}, v_j \rangle$. By (4.2) $\langle v_i, v_{i+1} \rangle = \langle v_{j-1}, v_j \rangle = 1$, and by (4.4) and (4.8) $\langle v_i, v_j \rangle > \langle v_{i+1}, v_j \rangle > \langle v_{i+1}, v_{j-1} \rangle$, so $\langle v_i, v_j \rangle \geq \langle v_{i+1}, v_{j-1} \rangle + 2$, and $f(v_i + v_j) \geq 0$. Therefore $v_i + v_j$ is on the correct side of the line.

Next, consider the line through $v_i + v_{i+1}$ with slope v_0 . For $v_i + v_j$ to be on the correct side of the line, we need $(v_i + v_j) - (v_i + v_{i+1}) = v_j - v_{i+1}$ to be counterclockwise from v_0 , that is $\langle v_0, v_j \rangle > \langle v_0, v_{i+1} \rangle$. Since $j \geq i + 1$, this follows from (4.4) and (4.7).

The case of the third line is similar. \square

So if i and j are not consecutive, then $v_i + v_j$ is inessential to the shape of $C + \text{Hull } S$.

However, if they are consecutive, then the opposite is true, in the following sense.

(4.12) Proposition. *The boundary of $C + \text{Hull } S$ consists of the line segments joining $v_{i-1} + v_i$ and $v_i + v_{i+1}$ for $0 < i < k$, together with two rays starting at $v_0 + v_1$ and $v_{k-1} + v_k$ and pointing in the directions v_0 and v_k , respectively.*

Proof. Since $v_0 = (1, 0)$, it suffices to show that the slopes of these line segments are positive (or possibly $+\infty$) and weakly decreasing. And, finally, that the slope of v_k is no greater than the slope of the last line segment.

The line segments in question have direction $v_{i+1} - v_{i-1}$, so it must be shown that

$$\frac{q_{i+1} - q_{i-1}}{p_{i+1} - p_{i-1}} \geq \frac{q_{i+2} - q_i}{p_{i+2} - p_i}.$$

Cross-multiplying and using (4.2) shows this to be equivalent to $p_{i-1}q_{i+2} - p_{i+2}q_{i-1} \geq 1$, which follows from (4.4). For the last part of the claim, it must be shown that

$$\frac{q_k - q_{k-2}}{p_k - p_{k-2}} \geq \frac{p_k}{q_k},$$

which follows from (4.4), again after cross-multiplying. \square

We have shown that the vertices of $C + \text{Hull } S$ are all of the form $v_i + v_{i+1}$. (Although not all $v_i + v_{i+1}$ need be vertices: see Figure 1.) By (4.9), the localization of C at any such vertex can be taken by an element of $SL(2, \mathbb{Z})$ to a cone of the form $\mathbb{Q}_+\langle(1, 0), (p', q')\rangle$ for p, q' coprime and $0 \leq p' < q'$.

(4.13) Proposition. *Unless $p = q - 1$, every localized cone satisfies $q' < q$.*

Proof. There are two cases: the *internal case* where $0 < i < k - 1$, and the *external case* where $i = 0$ or $k - 1$.

In the internal case, the two edges of the localized cone are along $(v_{i-1} + v_i) - (v_i + v_{i+1}) = v_{i-1} - v_{i+1}$ and $(v_{i+1} + v_{i+2}) - (v_i + v_{i+1}) = v_{i+2} - v_i$. Hence $q' = \langle v_{i-1} - v_{i+1}, v_{i+2} - v_i \rangle = \langle v_{i-1}, v_{i+2} \rangle - 3$ by (4.2). By (4.4) $\langle v_{i-1}, v_{i+2} \rangle$ is the denominator of $[a_i, a_{i+1}, a_{i+2}]$, which by (4.8) is strictly less than q .

In the external case, consider first $i = 0$. The two edges of the localized cone are along v_0 and $(v_1 + v_2) - (v_0 + v_1) = v_2 - v_0$, so $q' = \langle v_0, v_2 - v_0 \rangle = \langle v_0, v_2 \rangle$, which is the denominator of $[a_1, a_2]$. Again by (4.8), this is strictly less than q unless $p/q = [a_1, a_2]$, so that $k = 2$. If so, the condition $0 \leq p < q$ implies $a_1 = 1$, so $p/q = (a_2 - 1)/a_2$ and $p = q - 1$.

Likewise, when $i = k - 1$, the two edges of the localized cone are along $v_{k-2} - v_k$ and v_k , so $q' = \langle v_{k-2} - v_k, v_k \rangle = \langle v_{k-2}, v_k \rangle$, which is the denominator of $[a_{k-1}, a_k]$. Again, this is strictly less than q unless $p/q = [a_{k-1}, a_k]$, so that $k = 2$. Hence $p = q - 1$ again. \square

We are now in a position to prove Gonzalez-Sprinberg's result [5, 6].

(4.14) Theorem. *Any toric surface is desingularized by a finite number of Nash blow-ups.*

Proof. The question is local, so it suffices to consider an affine toric surface corresponding to a cone $\mathbb{Q}_+\langle(1, 0), (p, q)\rangle$, with $0 \leq p < q$ and p, q coprime. This surface is smooth if

and only if $q = 1$, for only then will the Hilbert basis consist of exactly two elements. The previous proposition shows that q is strictly decreasing under Nash blow-ups except at external vertices for $p = q - 1$. In this case, a direct calculation shows that both external vertices have $p'/q' = (q - 2)/q$, so the denominator will strictly decrease at the next step. \square

5 A method for enumerating simplicial cones

In dimension > 2 , we have no general results on the resolution of toric varieties by Nash blow-ups. However, using a computer, we have carried out an extensive investigation of 3- and 4-dimensional examples. Our primary focus is on simplicial cones, which correspond in the toric dictionary to affine toric orbifolds. But, as we will see, more general cones appear in the Nash blow-ups of simplicial cones and must be treated as part of the recursions.

We shall begin, then, by explaining how the simplicial cones of a given dimension d , or rather their equivalence classes under the action of $GL(d, \mathbb{Z})$, can be systematically enumerated.

Any proper cone $C \subset \mathbb{Q}^d$ is defined by finitely many linear inequalities with integer coefficients, say $\sum_{j=1}^d a_{ij}x_j \geq 0$ for $1 \leq i \leq m$. Without loss of generality assume that (i) no inequality is *redundant* in that it follows from the others; and (ii) for each fixed i , the a_{ij} are coprime. The $m \times d$ integer matrix $A = (a_{ij})$ is then called a *presentation* of C . It is unique modulo the left action of the group S_m of permutation matrices. To classify cones modulo $GL(d, \mathbb{Z})$, then, is equivalent to classifying integer matrices A satisfying (i) and (ii) modulo $S_m \times GL(d, \mathbb{Z})$ acting on the left and right. This is accomplished in practice using the following invariant.

For a cone C with presentation A , let $\Lambda \subset \mathbb{Z}^d$ be the subgroup generated by the rows of A . Define the *index* $I(C) \in \mathbb{Z}_+$ to be the index of Λ as a subgroup of \mathbb{Z}^d . (This is the order of the orbifold group at the fixed point of the torus action.) Also, if $C^* = \{u \in \mathbb{Q}^d \mid \forall v \in C, u \cdot v \geq 0\}$ is the dual cone, define the *dual index* $I^*(C)$ to be $I(C^*)$. Clearly $I(C)$ and $I^*(C)$ are invariant under the $GL(d, \mathbb{Z})$ -action.

It is, of course, nettlesome to decide whether a given matrix satisfies the non-redundancy condition (i). But in the simplicial case it is easy: a presentation A of a simplicial cone is exactly a nonsingular square integer matrix satisfying (ii). As such, A can be taken by the right action of $GL(d, \mathbb{Z})$ into *Hermite normal form* [15, 4.1]. This means that there exists $B \in GL(d, \mathbb{Z})$ so that AB is lower triangular, with nonnegative entries, and each row has a unique greatest entry located on the diagonal. Furthermore, since the entries in any given row of A are coprime, the same is true of AB . These facts can be summarized as follows.

(5.1) Proposition. *Every simplicial cone C is equivalent to one with a presentation A which is in Hermite normal form, and each of whose rows has coprime entries.* \square

(5.2) Corollary. *There are finitely many equivalence classes of simplicial cones of dimension d and index I .*

Proof. In the simplicial case $I(C) = |\det A|$, so if A is in Hermite normal form, its diagonal entries multiply to I . Hence there are only finitely many choices for the diagonal entries of A , and so for the subdiagonal entries as well. \square

I	$T_3(I)$	I	$T_3(I)$	I	$T_3(I)$	I	$T_3(I)$	I	$T_3(I)$	I	$T_3(I)$	I	$T_3(I)$
1	1	31	182	61	662	91	1679	121	2705	151	3902	181	5582
2	2	32	227	62	693	92	1643	122	2583	152	4591	182	6595
3	4	33	241	63	898	93	1715	123	2951	153	4872	183	6425
4	7	34	221	64	838	94	1551	124	2919	154	4777	184	6633
5	8	35	277	65	883	95	1825	125	3072	155	4717	185	6667
6	11	36	311	66	915	96	2051	126	3484	156	5298	186	6729
7	14	37	254	67	794	97	1634	127	2774	157	4214	187	6695
8	21	38	273	68	925	98	1846	128	3211	158	4293	188	6555
9	23	39	329	69	965	99	2110	129	3239	159	4875	189	7872
10	25	40	381	70	1057	100	2135	130	3445	160	5555	190	7177
11	28	41	308	71	888	101	1768	131	2948	161	5047	191	6208
12	43	42	393	72	1206	102	2099	132	3852	162	5283	192	7942
13	38	43	338	73	938	103	1838	133	3485	163	4538	193	6338
14	45	44	411	74	975	104	2227	134	3105	164	5021	194	6435
15	59	45	476	75	1254	105	2617	135	4114	165	6211	195	8569
16	66	46	391	76	1143	106	1961	136	3709	166	4731	196	7799
17	60	47	400	77	1219	107	1980	137	3220	167	4760	197	6600
18	76	48	546	78	1257	108	2561	138	3763	168	6589	198	8292
19	74	49	477	79	1094	109	2054	139	3314	169	5187	199	6734
20	101	50	508	80	1434	110	2499	140	4454	170	5783	200	8624
21	107	51	543	81	1350	111	2417	141	3853	171	6046	201	7727
22	99	52	561	82	1189	112	2702	142	3479	172	5511	202	6969
23	104	53	504	83	1204	113	2204	143	3985	173	5104	203	7933
24	153	54	610	84	1644	114	2601	144	4668	174	5907	204	8866
25	135	55	643	85	1473	115	2639	145	4141	175	6566	205	8153
26	135	56	703	86	1305	116	2565	146	3675	176	6370	206	7245
27	163	57	671	87	1507	117	2908	147	4584	177	6017	207	8762
28	183	58	609	88	1625	118	2419	148	4113	178	5429	208	8774
29	160	59	620	89	1380	119	2809	149	3800	179	5460	209	8311
30	211	60	878	90	1828	120	3483	150	4894	180	7712	210	10273

TABLE 1. Number of $GL(3, \mathbb{Z})$ -equivalence classes of simplicial cones in 3 dimensions.

I	$T_4(I)$	I	$T_4(I)$	I	$T_4(I)$	I	$T_4(I)$	I	$T_4(I)$
1	1	11	101	21	788	31	1550	41	3399
2	3	12	262	22	851	32	3083	42	7441
3	7	13	154	23	682	33	2622	43	3891
4	16	14	264	24	1778	34	2799	44	7172
5	18	15	337	25	1037	35	2969	45	7652
6	37	16	476	26	1338	36	5403	46	6552
7	36	17	305	27	1530	37	2544	47	5012
8	83	18	657	28	2123	38	3821	48	12605
9	85	19	409	29	1288	39	4155	49	6512
10	116	20	894	30	3006	40	6591	50	10047

TABLE 2. Number of $GL(4, \mathbb{Z})$ -equivalence classes of simplicial cones in 4 dimensions.

For a fixed value of I , it is now practical to enumerate the equivalence classes of cones C using (5.1). Indeed, two matrices A and A' are equivalent if and only if $SAT = A'$ for some $S \in S_d$ and $T \in GL(d, \mathbb{Z})$. Detecting this is a tractable problem for small d , as one can consider $A^{-1}SA'$ for all $S \in S_d$ and see whether any of them is an integer matrix. In this manner, the numbers $T_d(I)$ of equivalence classes of d -dimensional cones of index I were determined with a computer for small values of I . These numbers are presented in Table 1 for $d = 3$ and in Table 2 for $d = 4$. A list of explicit representatives for each of these equivalence classes, for the first few values of I , is given in Table 3 for $d = 3$ and in Table 4 for $d = 4$. Many cones are *reducible* to a direct sum of cones of lower dimension; if so, the direct sum in question is shown in the right-hand column of Tables 3 and 4. By remark (3), the Nash blow-up of a direct sum of cones is the direct sum of their Nash blow-ups, so only irreducible cones are interesting for our purposes.

Name	I	I^*	Presentation	Reducibility	Name	I	I^*	Presentation	Reducibility
$C_{1,1}$	1	1	(e_1, e_2, e_3)	$A \oplus A \oplus A$	$C_{5,4}$	5	25	$(e_1, e_2, (1, 1, 5))$	
$C_{2,1}$	2	2	$(e_1, e_2, (0, 1, 2))$	$B_{2,1} \oplus A$	$C_{5,5}$	5	25	$(e_1, e_2, (1, 2, 5))$	
$C_{2,2}$	2	4	$(e_1, e_2, (1, 1, 2))$		$C_{5,6}$	5	25	$(e_1, e_2, (2, 2, 5))$	
$C_{3,1}$	3	3	$(e_1, e_2, (0, 1, 3))$	$B_{3,1} \oplus A$	$C_{5,7}$	5	25	$(e_1, e_2, (2, 4, 5))$	
$C_{3,2}$	3	3	$(e_1, e_2, (0, 2, 3))$	$B_{3,2} \oplus A$	$C_{5,8}$	5	25	$(e_1, e_2, (4, 4, 5))$	
$C_{3,3}$	3	9	$(e_1, e_2, (1, 1, 3))$		$C_{6,1}$	6	6	$(e_1, e_2, (0, 1, 6))$	$B_{6,1} \oplus A$
$C_{3,4}$	3	9	$(e_1, e_2, (2, 2, 3))$		$C_{6,2}$	6	6	$(e_1, e_2, (0, 5, 6))$	$B_{6,2} \oplus A$
$C_{4,1}$	4	4	$(e_1, e_2, (0, 1, 4))$	$B_{4,1} \oplus A$	$C_{6,3}$	6	36	$(e_1, e_2, (1, 1, 6))$	
$C_{4,2}$	4	4	$(e_1, e_2, (0, 3, 4))$	$B_{4,2} \oplus A$	$C_{6,4}$	6	18	$(e_1, e_2, (1, 2, 6))$	
$C_{4,3}$	4	16	$(e_1, e_2, (1, 1, 4))$		$C_{6,5}$	6	12	$(e_1, e_2, (1, 3, 6))$	
$C_{4,4}$	4	8	$(e_1, e_2, (1, 2, 4))$		$C_{6,6}$	6	6	$(e_1, e_2, (2, 3, 6))$	
$C_{4,5}$	4	8	$(e_1, e_2, (2, 3, 4))$		$C_{6,7}$	6	18	$(e_1, e_2, (2, 5, 6))$	
$C_{4,6}$	4	16	$(e_1, e_2, (3, 3, 4))$		$C_{6,8}$	6	6	$(e_1, e_2, (3, 4, 6))$	
$C_{4,7}$	4	2	$(e_1, (1, 2, 0), (1, 0, 2))$		$C_{6,9}$	6	12	$(e_1, e_2, (3, 5, 6))$	
$C_{5,1}$	5	5	$(e_1, e_2, (0, 1, 5))$	$B_{5,1} \oplus A$	$C_{6,10}$	6	18	$(e_1, e_2, (4, 5, 6))$	
$C_{5,2}$	5	5	$(e_1, e_2, (0, 2, 5))$	$B_{5,2} \oplus A$	$C_{6,11}$	6	36	$(e_1, e_2, (5, 5, 6))$	
$C_{5,3}$	5	5	$(e_1, e_2, (0, 4, 5))$	$B_{5,3} \oplus A$					

TABLE 3. Classification of simplicial cones in 3 dimensions.

6 Results of computer investigations

We are now in a position to describe the empirical data obtained with a computer. Our program, entitled `resolve`, was written in the language C++ and relied heavily on the Boost open-source software libraries for C++, especially the linear algebra library `uBLAS` of Joerg Walter and Mathias Koch [18]. Our source code, as well as extensive tables of output, are available at <http://www.math.columbia.edu/~thaddeus/nash.html>.

One function of the program is to enumerate the simplicial cones of a given dimension and index, as described in the previous section. However, the primary function of `resolve` is to implement the algorithm of §3 for carrying out the Nash blow-up and to perform it iteratively. The C++ program often invokes the external programs `4ti2` [9], `lrs` [1], and `qhull` [2], which perform isolated parts of the computation. Specifically, `4ti2` is used in Step

Name	I	I^*	Presentation	Reducibility
$D_{1,1}$	1	1	(e_1, e_2, e_3, e_4)	$4A$
$D_{2,1}$	2	2	$(e_1, e_2, e_3, (0, 0, 1, 2))$	$B_{2,1} \oplus 2A$
$D_{2,2}$	2	4	$(e_1, e_2, e_3, (0, 1, 1, 2))$	$C_{2,1} \oplus A$
$D_{2,3}$	2	8	$(e_1, e_2, e_3, (1, 1, 1, 2))$	
$D_{3,1}$	3	3	$(e_1, e_2, e_3, (0, 0, 1, 3))$	$B_{3,1} \oplus 2A$
$D_{3,2}$	3	3	$(e_1, e_2, e_3, (0, 0, 2, 3))$	$B_{3,2} \oplus 2A$
$D_{3,3}$	3	9	$(e_1, e_2, e_3, (0, 1, 1, 3))$	$C_{3,3} \oplus A$
$D_{3,4}$	3	9	$(e_1, e_2, e_3, (0, 2, 2, 3))$	$C_{3,4} \oplus A$
$D_{3,5}$	3	27	$(e_1, e_2, e_3, (1, 1, 1, 3))$	
$D_{3,6}$	3	27	$(e_1, e_2, e_3, (1, 1, 2, 3))$	
$D_{3,7}$	3	27	$(e_1, e_2, e_3, (2, 2, 2, 3))$	
$D_{4,1}$	4	4	$(e_1, e_2, e_3, (0, 0, 1, 4))$	$B_{4,1} \oplus 2A$
$D_{4,2}$	4	4	$(e_1, e_2, e_3, (0, 0, 3, 4))$	$B_{4,2} \oplus 2A$
$D_{4,3}$	4	16	$(e_1, e_2, e_3, (0, 1, 1, 4))$	$C_{4,3} \oplus A$
$D_{4,4}$	4	8	$(e_1, e_2, e_3, (0, 1, 2, 4))$	$C_{4,4} \oplus A$
$D_{4,5}$	4	8	$(e_1, e_2, e_3, (0, 2, 3, 4))$	$C_{4,5} \oplus A$
$D_{4,6}$	4	16	$(e_1, e_2, e_3, (0, 3, 3, 4))$	$C_{4,6} \oplus A$
$D_{4,7}$	4	64	$(e_1, e_2, e_3, (1, 1, 1, 4))$	
$D_{4,8}$	4	32	$(e_1, e_2, e_3, (1, 1, 2, 4))$	
$D_{4,9}$	4	64	$(e_1, e_2, e_3, (1, 1, 3, 4))$	
$D_{4,10}$	4	16	$(e_1, e_2, e_3, (1, 2, 2, 4))$	
$D_{4,11}$	4	16	$(e_1, e_2, e_3, (2, 2, 3, 4))$	
$D_{4,12}$	4	32	$(e_1, e_2, e_3, (2, 3, 3, 4))$	
$D_{4,13}$	4	64	$(e_1, e_2, e_3, (3, 3, 3, 4))$	
$D_{4,14}$	4	2	$(e_1, e_2, (0, 1, 2, 0), (0, 1, 0, 2))$	$C_{4,7} \oplus A$
$D_{4,15}$	4	4	$(e_1, e_2, (0, 1, 2, 0), (1, 0, 0, 2))$	$2B_{2,1}$
$D_{4,16}$	4	4	$(e_1, e_2, (0, 1, 2, 0), (1, 1, 0, 2))$	
$D_{5,1}$	5	5	$(e_1, e_2, e_3, (0, 0, 1, 5))$	$B_{5,1} \oplus 2A$
$D_{5,2}$	5	5	$(e_1, e_2, e_3, (0, 0, 2, 5))$	$B_{5,2} \oplus 2A$
$D_{5,3}$	5	5	$(e_1, e_2, e_3, (0, 0, 4, 5))$	$B_{5,3} \oplus 2A$
$D_{5,4}$	5	25	$(e_1, e_2, e_3, (0, 1, 1, 5))$	$C_{5,4} \oplus A$
$D_{5,5}$	5	25	$(e_1, e_2, e_3, (0, 1, 2, 5))$	$C_{5,5} \oplus A$
$D_{5,6}$	5	25	$(e_1, e_2, e_3, (0, 2, 2, 5))$	$C_{5,6} \oplus A$
$D_{5,7}$	5	25	$(e_1, e_2, e_3, (0, 2, 4, 5))$	$C_{5,7} \oplus A$
$D_{5,8}$	5	25	$(e_1, e_2, e_3, (0, 4, 4, 5))$	$C_{5,8} \oplus A$
$D_{5,9}$	5	125	$(e_1, e_2, e_3, (1, 1, 1, 5))$	
$D_{5,10}$	5	125	$(e_1, e_2, e_3, (1, 1, 2, 5))$	
$D_{5,11}$	5	125	$(e_1, e_2, e_3, (1, 1, 3, 5))$	
$D_{5,12}$	5	125	$(e_1, e_2, e_3, (1, 1, 4, 5))$	
$D_{5,13}$	5	125	$(e_1, e_2, e_3, (1, 2, 2, 5))$	
$D_{5,14}$	5	125	$(e_1, e_2, e_3, (1, 2, 3, 5))$	
$D_{5,15}$	5	125	$(e_1, e_2, e_3, (2, 2, 2, 5))$	
$D_{5,16}$	5	125	$(e_1, e_2, e_3, (2, 2, 4, 5))$	
$D_{5,17}$	5	125	$(e_1, e_2, e_3, (2, 4, 4, 5))$	
$D_{5,18}$	5	125	$(e_1, e_2, e_3, (4, 4, 4, 5))$	

TABLE 4. Classification of simplicial cones in 4 dimensions.

1 to find the Hilbert basis of $C \cap \mathbb{Z}^n$, while `lrs` is used in Steps 3 and 4 to determine the vertices of the polyhedron $C + \text{Hull } S$, and the localization at each vertex; `qhull` is also used in Step 3 to simplify the determination of the convex hull. Because of the intensive nature of the latter computation, Step 3 requires by far the most computing time.

We used `resolve` to find *Nash resolutions* (that is, finite resolution trees of Nash blow-ups) for all 1602 3-dimensional simplicial cones with $I \leq 27$ and all 201 4-dimensional simplicial cones with $I \leq 8$, following the classification. A few higher-dimensional cones were also resolved, but these required considerably more time. To improve efficiency, `resolve` ceases searching deeper in a resolution tree whenever it reaches a simplicial cone with I strictly less than the initial value, since this has been resolved already. However, many non-simplicial cones are encountered in the process of resolving simplicial cones. So are simplicial cones with equal or greater values of I .

Table 5 presents the Nash resolutions of all irreducible 3-dimensional simplicial cones of index $I \leq 4$. Likewise, Table 6 presents the Nash resolutions of all irreducible 4-dimensional simplicial cones of index $I \leq 4$. In both tables, each line displays the rows of a presentation of a single cone. The index I and dual index I^* are shown in brackets at right. The first line in each block of text represents the original cone being resolved. The singly indented lines below it show the cones appearing in the Nash blow-up of that cone. Subsequent to each of those, the doubly indented lines show the cones appearing in the Nash blow-ups of those cones.

Figure 2 depicts the resolution trees of all 3-dimensional irreducible simplicial cones of index $I \leq 6$. Likewise, Figure 3 depicts the resolution trees of all but one of the 4-dimensional irreducible simplicial cones of index $I \leq 5$. (One cone of index 5, namely $D_{5,14}$, has an enormous resolution tree and has been omitted.) To avoid redundancy, each tree has been pruned of subtrees sprouting from simplicial cones that appear elsewhere on the page. Also, identical subtrees sprouting from the same node have been shown only once, but with the multiplicity appearing as a coefficient of the first cone on the subtree. Furthermore, a multiple branch of the form $kC_{1,1}$ or $kD_{1,1}$ (k copies of the orthant) is denoted even more concisely by the number k inside a circle. Thus, for example, the notation for $C_{5,4}$ is meant to convey that a single Nash blow-up produces the 5 cones $C_{5,6}$, $C_{3,2}$, $C_{3,2}$, $C_{1,1}$, and $C_{1,1}$. By definition, all leaves of a resolution tree are orthants, but this is not immediately apparent from the diagram because of the pruning convention just mentioned.

The cones appearing in double-outlined boxes are non-simplicial cones, with the number of facets in parentheses. We did not classify these cones, so we continue their resolution trees until they reach simplicial cones encountered before. Evidently, non-simplicial cones are ubiquitous even in the resolution of simplicial cones. (A note about the grouping of cones by multiplicity in the figures: simplicial cones have been grouped if and only if they are equivalent, whereas non-simplicial cones are grouped if and only if they have identical resolution trees. This is a weaker condition; in some cases, such as the $4C(4)$ in the resolution tree of $C_{5,5}$, we know that the cones in question are not equivalent.)

What patterns can be observed in the data? Most obviously, all of the thousands of cones we have studied are eventually resolved by Nash blow-ups. This strongly supports an affirmative answer to Question **(1.2)**.

However, although the resolution seems always to exist, it also seems to obey neither rhyme nor reason. Almost every straightforward conjecture one might make about patterns in the

Nash resolution seems to be false. We have already seen, for example, that the resolution of a simplicial cone may involve non-simplicial cones. One might hope that the number of facets in the cone remains within some reasonable bound, but the resolutions of 4-dimensional simplicial cones can require cones with as many as 10 facets, with no end in sight.

The behavior of the indices I and I^* is equally perplexing. A 2-dimensional cone $\mathbb{Q}_+\langle(1, 0), (p, q)\rangle$ with p coprime to q has $I(C) = I^*(C) = q$. As we saw in **(4.13)**, this is non-increasing under Nash blow-up (indeed, decreasing except for p odd and $q = p - 1$). But I and I^* can increase under Nash blow-ups, even in dimension 3 and even when the cones involved are simplicial. For example, $C_{6,5} = \mathbb{Q}_+\langle(1, 0, 0), (0, 1, 0), (1, 3, 6)\rangle$ with $I = 6$ gives rise, after a single Nash blow-up, to $\mathbb{Q}_+\langle(1, 3, 6), (1, 3, 3), (2, 3, 6)\rangle \cong C_{9,23}$ with $I = 9$. A glimmer of hope is offered by I^* . For, among the thousands of cones we have examined, there appears not one example of a simplicial cone giving rise, after a single Nash blow-up, to another simplicial cone with greater I^* . However, there are rare cases where, after two Nash blow-ups, one obtains a simplicial cone with greater I^* . For example, $C_{9,22} = \mathbb{Q}_+\langle(1, 0, 0), (1, 3, 0), (1, 0, 3)\rangle$ with $I^* = 3$ gives rise, after two Nash blow-ups, to $\mathbb{Q}_+\langle(1, 1, 0), (1, 0, 1), (4, 3, 3)\rangle$ and two other cones all with $I^* = 4$. Moreover, there are many cases where I^* increases when one of the cones is not simplicial. This can be seen, for example, in the resolution tree of $C_{7,6}$, where $\mathbb{Q}_+\langle(1, 0, 0), (0, 1, 0), (2, 4, 7), (1, 1, 2)\rangle$ with $I^* = 1$ gives rise, after a single Nash blow-up, to $\mathbb{Q}_+\langle(1, 0, 0), (0, 1, 0), (1, 2, 2)\rangle \cong C_{2,1}$ with $I^* = 2$.

The question is reminiscent of other famous iterative problems such as the notorious Collatz conjecture [11], but in some ways it is even worse behaved. A striking empirical feature is the existence of simplicial cones whose Nash resolution is vastly larger than those of other simplicial cones with the same index. In dimension 4, for example, the seemingly innocent $D_{5,14} = \mathbb{Q}_+\langle e_1, e_2, e_3, (1, 2, 3, 5)\rangle$, with $I = 5$, has a resolution tree with depth 8 and 14253 cones, while no other simplicial cone with $I = 5$ needs more than depth 3 and 108 cones. Likewise, $D_{7,24} = \mathbb{Q}_+\langle e_1, e_2, e_3, (1, 2, 5, 7)\rangle$, with $I = 7$, has a resolution tree with depth 11 and 35299 cones, while no other simplicial cone with $I = 7$ needs more than depth 7 and 5061 cones, and only one other needs more than depth 5 and 804 cones.

In conclusion, Question **(1.2)** remains wide open, but we have amassed considerable empirical evidence supporting an affirmative answer. In light of the 2-dimensional case, one might hope for a proof involving some kind of higher-dimensional analogue of continued fractions.

$C_{2,2}$:

(1,0,0),(0,1,0),(1,1,2) [2,4]
(1,1,2),(1,0,0),(1,1,1) [1,1]
(0,1,0),(1,0,0),(1,1,1) [1,1]
(0,1,0),(1,1,2),(1,1,1) [1,1]

$C_{3,3}$:

(1,0,0),(0,1,0),(1,1,3) [3,9]
(1,0,0),(2,2,3),(1,1,2) [1,1]
(0,1,0),(2,2,3),(1,1,2) [1,1]
(1,1,3),(1,0,0),(1,1,2) [1,1]
(0,1,0),(1,1,3),(1,1,2) [1,1]
(0,1,0),(1,0,0),(2,2,3) [3,9]
(1,0,0),(0,1,0),(1,1,1) [1,1]
(2,2,3),(0,1,0),(1,1,1) [1,1]
(2,2,3),(1,0,0),(1,1,1) [1,1]

$C_{3,4}$:

(1,0,0),(0,1,0),(2,2,3) [3,9]
(0,1,0),(1,1,1),(1,0,0) [1,1]
(2,2,3),(1,1,1),(1,0,0) [1,1]
(2,2,3),(0,1,0),(1,1,1) [1,1]

$C_{4,3}$:

(1,0,0),(0,1,0),(1,1,4) [4,16]
(1,0,0),(1,1,2),(1,1,3) [1,1]
(0,1,0),(1,1,2),(1,1,3) [1,1]
(1,1,4),(1,0,0),(1,1,3) [1,1]
(0,1,0),(1,1,4),(1,1,3) [1,1]
(0,1,0),(1,0,0),(1,1,2) [2,4]
(1,1,2),(1,0,0),(1,1,1) [1,1]
(0,1,0),(1,0,0),(1,1,1) [1,1]
(0,1,0),(1,1,2),(1,1,1) [1,1]

$C_{4,4}$:

(1,0,0),(0,1,0),(1,2,4) [4,8]
(0,1,0),(1,2,2),(1,0,0) [2,2]
(0,1,0),(1,2,2),(1,1,1) [1,1]
(0,1,0),(1,0,0),(1,1,1) [1,1]
(1,1,2),(1,2,2),(1,0,0) [2,2]
(1,1,2),(1,1,1),(1,2,2) [1,1]
(1,0,0),(1,1,2),(1,1,1) [1,1]
(1,1,2),(1,2,4),(1,2,2) [2,2]
(1,1,2),(1,2,4),(1,2,3) [1,1]
(1,1,2),(1,2,2),(1,2,3) [1,1]
(1,2,4),(0,1,0),(1,2,2) [2,2]
(0,1,0),(1,2,4),(1,2,3) [1,1]
(0,1,0),(1,2,2),(1,2,3) [1,1]

$C_{4,5}$:

(1,0,0),(0,1,0),(2,3,4) [4,8]
(0,1,0),(1,0,0),(1,1,1) [1,1]
(2,3,4),(1,0,0),(1,1,1) [1,1]
(0,1,0),(1,2,2),(1,1,1) [1,1]
(2,3,4),(1,2,2),(1,1,1) [1,1]

$C_{4,6}$:

(1,0,0),(0,1,0),(3,3,4) [4,16]
(0,1,0),(1,1,1),(1,0,0) [1,1]
(3,3,4),(1,1,1),(1,0,0) [1,1]
(3,3,4),(0,1,0),(1,1,1) [1,1]

$C_{4,7}$:

(1,0,0),(1,2,0),(1,0,2) [4,2]
(1,0,1),(1,1,0),(1,0,0) [1,1]
(1,0,1),(1,1,0),(1,1,1) [1,1]
(1,0,2),(1,0,1),(1,1,1) [1,1]
(1,2,0),(1,1,0),(1,1,1) [1,1]

TABLE 5. Nash resolutions of irreducible simplicial cones in 3 dimensions.

$D_{2,3}$: $(1,0,0,0), (0,1,0,0), (0,0,1,0), (1,1,1,2)$ [2,8] $(0,0,1,0), (1,1,1,2), (1,0,0,0), (1,1,1,1)$ [1,1] $(0,1,0,0), (1,1,1,2), (1,0,0,0), (1,1,1,1)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,0,0,0), (1,1,1,1)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,1,1,2), (1,1,1,1)$ [1,1]	$D_{4,7}$: $(1,0,0,0), (0,1,0,0), (0,0,1,0), (1,1,1,4)$ [4,64] $(0,0,1,0), (1,0,0,0), (1,1,1,2), (1,1,1,3)$ [1,1] $(0,1,0,0), (1,0,0,0), (1,1,1,2), (1,1,1,3)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,1,1,2), (1,1,1,3)$ [1,1] $(0,0,1,0), (1,1,1,4), (1,0,0,0), (1,1,1,3)$ [1,1] $(0,1,0,0), (1,1,1,4), (1,0,0,0), (1,1,1,3)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,1,1,4), (1,1,1,3)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,0,0,0), (1,1,1,2)$ [2,8] $(0,0,1,0), (1,1,1,2), (1,0,0,0), (1,1,1,1)$ [1,1] $(0,1,0,0), (1,1,1,2), (1,0,0,0), (1,1,1,1)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,0,0,0), (1,1,1,1)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,1,1,2), (1,1,1,1)$ [1,1]
$D_{3,5}$: $(1,0,0,0), (0,1,0,0), (0,0,1,0), (1,1,1,3)$ [3,27] $(0,0,1,0), (1,0,0,0), (2,2,2,3), (1,1,1,2)$ [1,1] $(0,1,0,0), (1,0,0,0), (2,2,2,3), (1,1,1,2)$ [1,1] $(0,1,0,0), (0,0,1,0), (2,2,2,3), (1,1,1,2)$ [1,1] $(0,0,1,0), (1,1,1,3), (1,0,0,0), (1,1,1,2)$ [1,1] $(0,1,0,0), (1,1,1,3), (1,0,0,0), (1,1,1,2)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,1,1,3), (1,1,1,2)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,0,0,0), (2,2,2,3)$ [3,27] $(2,2,2,3), (1,0,0,0), (0,0,1,0), (1,1,1,1)$ [1,1] $(0,1,0,0), (1,0,0,0), (0,0,1,0), (1,1,1,1)$ [1,1] $(0,1,0,0), (2,2,2,3), (0,0,1,0), (1,1,1,1)$ [1,1] $(0,1,0,0), (2,2,2,3), (1,0,0,0), (1,1,1,1)$ [1,1]	$D_{4,8}$: $(1,0,0,0), (0,1,0,0), (0,0,1,0), (1,1,2,4)$ [4,32] $(1,0,0,0), (1,1,1,2), (1,1,2,2), (1,1,2,3)$ [1,1] $(0,1,0,0), (1,1,1,2), (1,1,2,2), (1,1,2,3)$ [1,1] $(0,0,1,0), (1,0,0,0), (1,1,2,2), (1,1,2,3)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,1,2,2), (1,1,2,3)$ [1,1] $(1,1,2,4), (0,0,1,0), (1,0,0,0), (1,1,2,3)$ [1,1] $(0,1,0,0), (1,1,2,4), (0,0,1,0), (1,1,2,3)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,0,0,0), (1,1,2,2)$ [2,4] $(0,0,1,0), (1,1,2,2), (1,0,0,0), (1,1,1,1)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,0,0,0), (1,1,1,1)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,1,2,2), (1,1,1,1)$ [1,1] $(1,1,2,4), (1,0,0,0), (1,1,1,2), (1,1,2,3)$ [1,1] $(0,1,0,0), (1,1,2,4), (1,1,1,2), (1,1,2,3)$ [1,1] $(0,1,0,0), (1,0,0,0), (1,1,1,2), (1,1,2,2)$ [2,4] $(0,1,0,0), (1,1,1,2), (1,1,2,2), (1,1,1,1)$ [1,1] $(1,0,0,0), (1,1,1,2), (1,1,2,2), (1,1,1,1)$ [1,1] $(1,0,0,0), (0,1,0,0), (1,1,1,2), (1,1,1,1)$ [1,1]
$D_{3,6}$: $(1,0,0,0), (0,1,0,0), (0,0,1,0), (1,1,2,3)$ [3,27] $(0,0,1,0), (1,0,0,0), (2,2,3,3), (1,1,1,1)$ [1,1] $(1,1,2,3), (1,0,0,0), (2,2,3,3), (1,1,1,1)$ [1,1] $(0,1,0,0), (1,1,2,3), (2,2,3,3), (1,1,1,1)$ [1,1] $(0,1,0,0), (1,1,2,3), (1,0,0,0), (1,1,1,1)$ [1,1] $(0,1,0,0), (0,0,1,0), (2,2,3,3), (1,1,1,1)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,0,0,0), (1,1,1,1)$ [1,1] $(0,0,1,0), (1,1,2,2), (1,0,0,0), (2,2,3,3)$ [1,1] $(1,1,2,3), (1,1,2,2), (1,0,0,0), (2,2,3,3)$ [1,1] $(0,1,0,0), (1,1,2,3), (1,1,2,2), (2,2,3,3)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,1,2,2), (2,2,3,3)$ [1,1] $(1,1,2,3), (0,0,1,0), (1,1,2,2), (1,0,0,0)$ [1,1] $(0,1,0,0), (1,1,2,3), (0,0,1,0), (1,1,2,2)$ [1,1]	$D_{4,9}$: $(1,0,0,0), (0,1,0,0), (0,0,1,0), (1,1,3,4)$ [4,64] $(0,0,1,0), (1,0,0,0), (1,1,2,2), (1,1,3,3)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,1,2,2), (1,1,3,3)$ [1,1] $(1,1,3,4), (1,0,0,0), (1,1,2,2), (1,1,3,3)$ [1,1] $(0,1,0,0), (1,1,3,4), (1,1,2,2), (1,1,3,3)$ [1,1] $(1,1,3,4), (0,0,1,0), (1,0,0,0), (1,1,3,3)$ [1,1] $(0,1,0,0), (1,1,3,4), (0,0,1,0), (1,1,3,3)$ [1,1] $(0,0,1,0), (1,0,0,0), (1,1,1,1), (1,1,2,2)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,1,1,1), (1,1,2,2)$ [1,1] $(1,1,3,4), (1,0,0,0), (1,1,1,1), (1,1,2,2)$ [1,1] $(0,1,0,0), (1,1,3,4), (1,1,1,1), (1,1,2,2)$ [1,1] $(0,1,0,0), (1,1,3,4), (1,0,0,0), (1,1,1,1)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,0,0,0), (1,1,1,1)$ [1,1]
$D_{3,7}$: $(1,0,0,0), (0,1,0,0), (0,0,1,0), (2,2,2,3)$ [3,27] $(2,2,2,3), (0,0,1,0), (1,1,1,1), (1,0,0,0)$ [1,1] $(0,1,0,0), (0,0,1,0), (1,1,1,1), (1,0,0,0)$ [1,1] $(0,1,0,0), (2,2,2,3), (1,1,1,1), (1,0,0,0)$ [1,1] $(0,1,0,0), (2,2,2,3), (0,0,1,0), (1,1,1,1)$ [1,1]	

TABLE 6. Nash resolutions of irreducible simplicial cones in 4 dimensions.
(continued on next page)

$D_{4,10}$:

(1,0,0,0),(0,1,0,0),(0,0,1,0),(1,2,2,4) [4,16]
(1,1,1,2),(0,1,0,0),(1,2,2,2),(1,0,0,0) [2,2]
(0,1,0,0),(1,1,1,2),(1,1,1,1),(1,2,2,2) [1,1]
(1,0,0,0),(0,1,0,0),(1,1,1,2),(1,1,1,1) [1,1]
(0,0,1,0),(0,1,0,0),(1,2,2,2),(1,0,0,0) [2,2]
(0,1,0,0),(0,0,1,0),(1,2,2,2),(1,1,1,1) [1,1]
(0,1,0,0),(0,0,1,0),(1,0,0,0),(1,1,1,1) [1,1]
(0,0,1,0),(1,1,1,2),(1,2,2,2),(1,0,0,0) [2,2]
(1,1,1,2),(0,0,1,0),(1,1,1,1),(1,2,2,2) [1,1]
(1,0,0,0),(1,1,1,2),(0,0,1,0),(1,1,1,1) [1,1]
(1,1,1,2),(1,2,2,4),(0,1,0,0),(1,2,2,2) [2,2]
(0,1,0,0),(1,1,1,2),(1,2,2,4),(1,2,2,3) [1,1]
(0,1,0,0),(1,1,1,2),(1,2,2,2),(1,2,2,3) [1,1]
(0,0,1,0),(1,2,2,4),(0,1,0,0),(1,2,2,2) [2,2]
(0,1,0,0),(0,0,1,0),(1,2,2,4),(1,2,2,3) [1,1]
(0,1,0,0),(0,0,1,0),(1,2,2,2),(1,2,2,3) [1,1]
(0,0,1,0),(1,1,1,2),(1,2,2,4),(1,2,2,2) [2,2]
(0,0,1,0),(1,1,1,2),(1,2,2,4),(1,2,2,3) [1,1]
(0,0,1,0),(1,1,1,2),(1,2,2,2),(1,2,2,3) [1,1]

$D_{4,11}$:

(1,0,0,0),(0,1,0,0),(0,0,1,0),(2,2,3,4) [4,16]
(1,1,1,1),(0,0,1,0),(1,1,2,2),(1,0,0,0) [1,1]
(2,2,3,4),(1,1,1,1),(1,1,2,2),(1,0,0,0) [1,1]
(0,1,0,0),(1,1,1,1),(0,0,1,0),(1,0,0,0) [1,1]
(2,2,3,4),(0,1,0,0),(1,1,1,1),(1,0,0,0) [1,1]
(0,1,0,0),(1,1,1,1),(0,0,1,0),(1,1,2,2) [1,1]
(2,2,3,4),(0,1,0,0),(1,1,1,1),(1,1,2,2) [1,1]

$D_{4,12}$:

(1,0,0,0),(0,1,0,0),(0,0,1,0),(2,3,3,4) [4,32]
(0,1,0,0),(0,0,1,0),(1,2,2,2),(1,1,1,1) [1,1]
(2,3,3,4),(0,0,1,0),(1,2,2,2),(1,1,1,1) [1,1]
(1,0,0,0),(2,3,3,4),(0,0,1,0),(1,1,1,1) [1,1]
(2,3,3,4),(0,1,0,0),(1,2,2,2),(1,1,1,1) [1,1]
(1,0,0,0),(2,3,3,4),(0,1,0,0),(1,1,1,1) [1,1]
(1,0,0,0),(0,1,0,0),(0,0,1,0),(1,1,1,1) [1,1]

$D_{4,13}$:

(1,0,0,0),(0,1,0,0),(0,0,1,0),(3,3,3,4) [4,64]
(0,1,0,0),(0,0,1,0),(1,1,1,1),(1,0,0,0) [1,1]
(3,3,3,4),(0,0,1,0),(1,1,1,1),(1,0,0,0) [1,1]
(3,3,3,4),(0,1,0,0),(1,1,1,1),(1,0,0,0) [1,1]
(3,3,3,4),(0,1,0,0),(0,0,1,0),(1,1,1,1) [1,1]

TABLE 6. Nash resolutions of irreducible simplicial cones in 4 dimensions.
(continued from previous page)

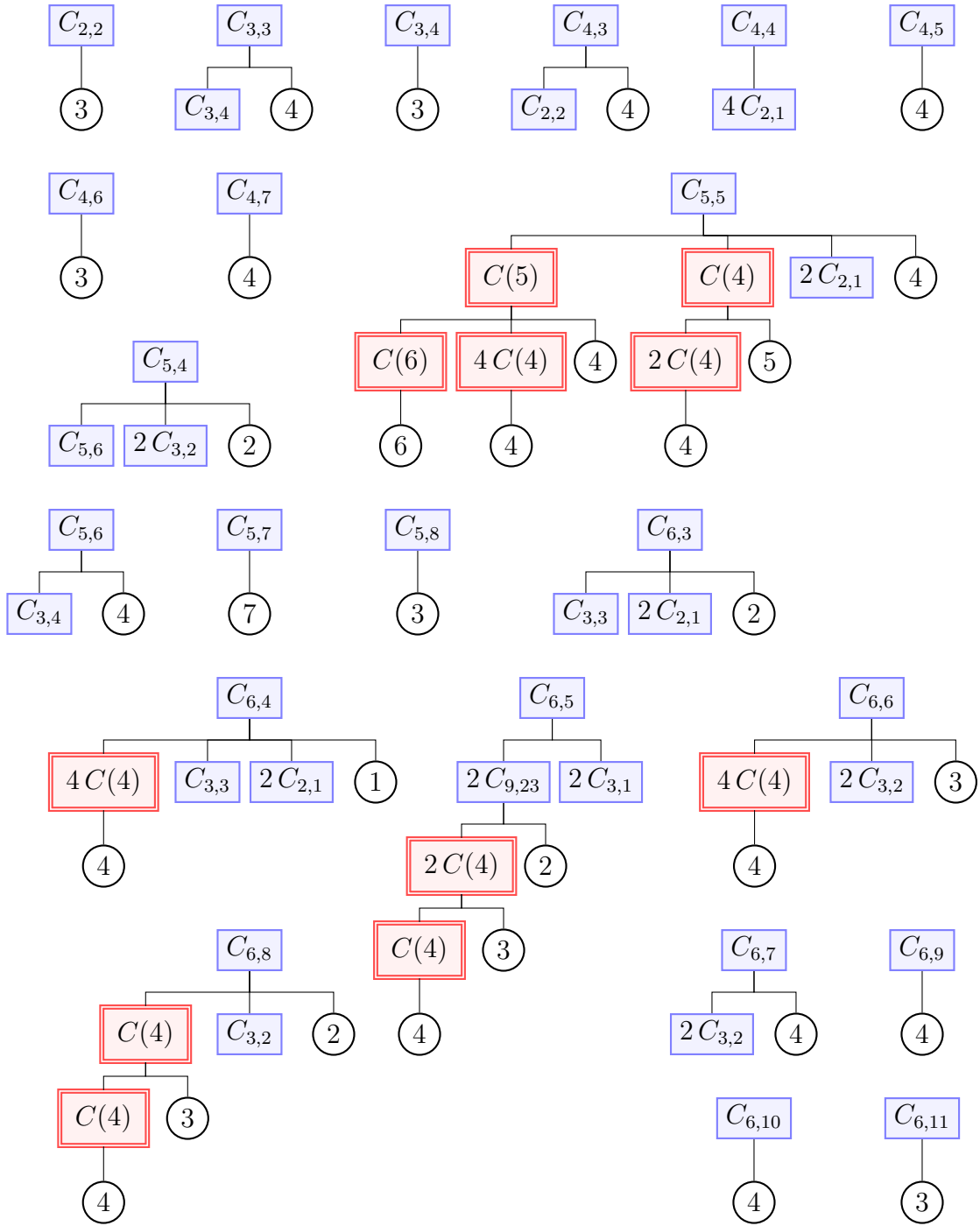


FIGURE 2. Resolution trees of irreducible simplicial cones in 3 dimensions.

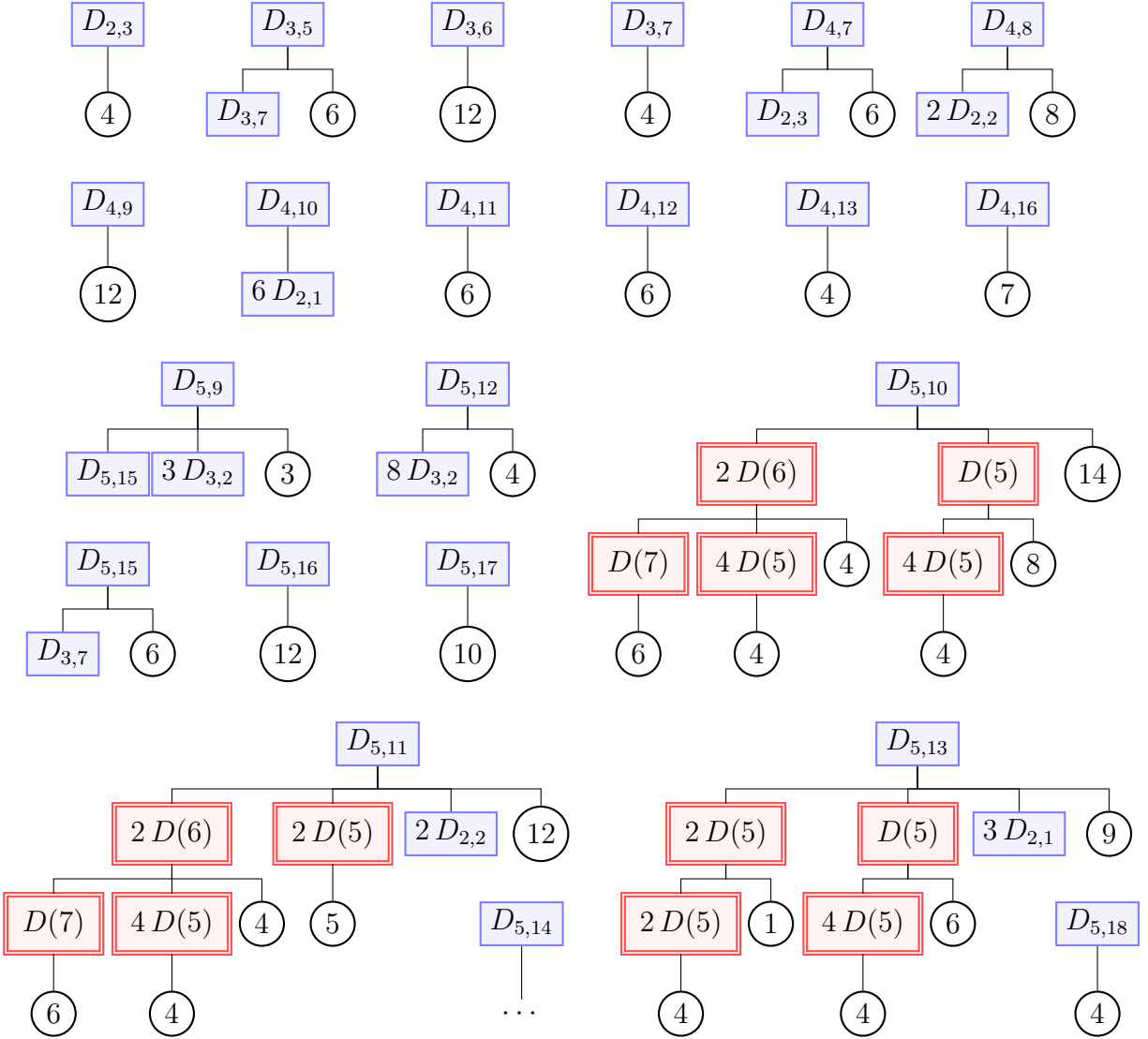


FIGURE 3. Resolution trees of irreducible simplicial cones in 4 dimensions.

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