Geometric category $\mathcal{O}$ and symplectic duality

The purpose of this proposal is to study algebraic symplectic varieties, which arise naturally in algebraic geometry (Hilbert schemes), representation theory (quiver varieties, Springer theory), combinatorics and polyhedral geometry (hypertoric varieties), and string theory (moduli spaces of gauge theories and of Higgs bundles).

Our primary interest will be a certain category of sheaves on these varieties, which we introduce in Section 1.3. The center of this category will be isomorphic to the cohomology ring of the variety, and its Grothendieck group will be isomorphic to a deformation of this cohomology ring. When our variety is a quiver variety, these cohomology groups carry geometrically-defined actions of Kac-Moody algebras, and these actions lift to actions on our category by endo-functors. Thus this proposal touches on both "geometric representation theory" (realizing representations as cohomology groups of varieties) and "higher representation theory" (actions of algebras on categories).

1 Geometric category $\mathcal{O}$

The category $\mathcal{O}$ of representations of a Lie algebra was introduced by Bernstein, Gelfand, and Gelfand [BGG], and has since been a central object of study in representation theory. One of the main breakthroughs in the field came in 1981, when Beilinson and Bernstein [BB] exhibited the blocks of this category as certain categories of twisted D-modules on the flag variety, which in turn may be interpreted as sheaves on the cotangent bundle of the flag variety. The purpose of this section is to define the appropriate analogue of this category when the cotangent bundle of the flag variety is replaced with another symplectic variety, which is not necessarily the cotangent bundle of anything. The resulting category is what we call **geometric category** $\mathcal{O}$.

1.1 Symplectic resolutions

Let M be a smooth, symplectic, algebraic variety over the complex numbers. By this we mean that M is equipped with a closed, nondegenerate, algebraic 2-form ω. Let $\mathbb{S}$ and $\mathbb{T}$ be two copies of the multiplicative group $\mathbb{C}^*$ acting on M. We assume that the action of $\mathbb{T}$ is hamiltonian, while the action of $\mathbb{S}$ has the property that $s \cdot \omega = s^m \omega$ for some integer $m \geq 1$. We also assume that $\mathbb{S}$ acts on the coordinate ring $\mathbb{C}[M]$ with only non-negative weights, and that the trivial weight space $\mathbb{C}[M]^{\mathbb{S}}$ is 1-dimensional, consisting only of the constant functions. Geometrically, this means that the affinization $N := \operatorname{Spec} \mathbb{C}[M]$ is a cone, and the $\mathbb{S}$-action contracts N to the cone point. Finally, we assume that the canonical map from M to N is a resolution of singularities. (That is, it must be projective and an isomorphism over the smooth locus of N.) Examples of such varieties include the following:

- M is a crepant resolution of $N = \mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $\mathrm{SL}(2;\mathbb{C})$.
- M is the Hilbert scheme of n points on the crepant resolution of $\mathbb{C}^2/\Gamma$, and N is the symmetric variety of n-tuples of unordered points on the singular space.
- $M = T^*(G/P)$ for a reductive algebraic group G and a Borel subgroup P, and N is the closure of a nilpotent orbit in the Lie algebra of G.
- M and N are Nakajima quiver varieties [N1, N2].
- M and N are hypertoric varieties [BD, P6].

The last two items are defined by a symplectic quotient construction. Let V be a linear representation of a reductive group G, and let $\mu : T^*V \to \mathfrak{g}^*$ be the moment map for the induced action. Let $\chi \in \mathrm{Hom}(G, \mathbb{C}^*)$ be a character of G. Let $M = \mu^{-1}(0)/\!\!/_{\chi} G$, and $N = \mu^{-1}(0)/\!\!/_{0} G$. Then N is affine, M maps surjectively and projectively to N, and, if α is generic enough, M is a smooth symplectic variety. The $\mathbb{T}$-action is induced by any action on V that commutes with the action of G, while the $\mathbb{S}$-action is induced by scaling the fibers of T^*V. If $V^G = 0$, then the $\mathbb{S}$-action will contract N to a single cone point, and will act on the symplectic form with weight 1. Hypertoric varieties are precisely those varieties obtained from this construction with G abelian. For quiver varieties, G is a product of general linear groups, one for each node in a quiver, and V is a representation of G determined by the arrows.

When the groups Γ and G of the first three examples have type ADE, then the first three examples are special cases of quiver varieties.[1] Note that all of these examples admit complete hyperkähler metrics, which can be seen by identifying the algebraic symplectic quotient construction above with the hyperkähler quotient construction of [HKLR].

1.2 The module category

A **quantization** of M is defined to be a sheaf $\mathcal{D}$ of free $\mathbb{C}[[h]]$-algebras on M along with a fixed isomorphism from $\mathcal{D}/h\mathcal{D}$ to the structure sheaf of M. Bezrukavnikov and Kaledin [BK] explain how to produce, for each $\lambda \in H^2(M; \mathbb{C})$, an $\mathbb{S} \times \mathbb{T}$-equivariant quantization $\mathcal{D}_\lambda$ of M. Consider the ring

$$A_\lambda := H^0(\mathcal{D}_\lambda[h^{-1}])^{\mathbb{S}}.$$

This ring inherits a $\mathbb{Z}$-grading from the $\mathbb{T}$-action, and a compatible $\mathbb{N}$-filtration $A^0_\lambda \subset A^1_\lambda \subset \ldots \subset A_\lambda$ given by putting $A^m_\lambda := \Gamma(h^{-m}\mathcal{D}_\lambda)^{\mathbb{S}}$. It is not hard to show that the associated graded ring $\operatorname{gr} A_\lambda$ is isomorphic as a $\mathbb{Z} \times \mathbb{N}$-graded ring to $\mathbb{C}[M]$. In many of the special cases discussed in the previous section, A_λ is a well-known ring of independent interest.

- If M is the Hilbert scheme of n points on a crepant resolution of $\mathbb{C}^2/\Gamma$, then A_λ is isomorphic to a specialization (depending on λ) of the spherical rational Cherednik algebra of the wreath product $S_n \wr \Gamma$.
- If $M = T^*(G/B)$, for a Borel subgroup $B \subset G$, then A_λ is isomorphic to a specialization (depending on λ) of the universal enveloping algebra $U(\mathfrak{g})$.
- If M is the resolution of a Slodowy slice to a nilpotent orbit in $\mathfrak{g}$, A_λ is isomorphic to a specialization (depending on λ) of a finite W-algebra.
- If M is a symplectic quotient of T^*V, then A_λ is a noncommutative hamiltonian reduction of the algebra of differential operators on V [CBEG].

Let A^+_λ be the non-negative degree part of A_λ with respect to the $\mathbb{Z}$-grading, and let $\mathcal{O}_\lambda$ be the category of finitely generated A_λ-modules that admit filtrations compatible with that of A_λ and are locally finite with respect to the action of A^+_λ. When $M = T^*(G/P)$, we recover a block of the classical BGG parabolic category $\mathcal{O}$. When M is a quiver variety, we obtain the category studied by Zheng in [Zh].

[1] If Γ is the trivial group, then the condition $V^G = 0$ will fail in the quiver construction of M, and it will not be possible to find an $\mathbb{S}$-action that acts on the symplectic form with weight 1. We can, however, take the action induced by scalar multiplication on $\mathbb{C}^2$, which acts on the symplectic form with weight 2. This is why we allow $m > 1$ in our definitions.

1.3 The geometric category

Assume for simplicity that $\mathbb{T}$ acts on M with finitely many fixed points $\{p_\alpha \mid \alpha \in \mathcal{I}\}$. For each $\alpha \in \mathcal{I}$, let $X_\alpha \subset M$ be the closure of the set $\{p \in M \mid \lim_{t\to 0} t \cdot p = p_\alpha\}$, and let $M^+ = \bigcup X_\alpha$. The fact that the action of $\mathbb{T}$ is hamiltonian tells us that each X_α is a lagrangian subvariety. If M is a crepant resolution of $\mathbb{C}^2/\Gamma$ with $\Gamma \cong \mathbb{Z}/k\mathbb{Z}$, the subvarieties $\{X_\alpha \mid \alpha \in \mathcal{I}\}$ will consist of a chain of $k-1$ projective lines, along with a copy of $\mathbb{C}$ at one end of the chain. If M is $T^*(G/P)$, they will be the conormal varieties to the Schubert strata of G/P. If M is a hypertoric variety, they will all be toric varieties. The cotangent bundle of $\mathbb{P}^1$ is a special case of all three of these examples; in this case we have two subvarieties: the zero section and one of the fibers.

Let $\mathcal{Q}_\lambda$ be the category of coherent $\mathbb{S}$-equivariant $\mathcal{D}_\lambda((h))$-modules that admit $\mathcal{D}_\lambda$-lattices and are set-theoretically supported on M^+. If M is a cotangent bundle, then then $\mathcal{Q}_\lambda$ is equivalent to the category of λ-twisted holonomic D-modules on the base, smooth with respect to the Białynicki-Birula stratification induced by the action of $\mathbb{T}$. If M is a symplectic quotient of T^*V by G, there is a functor to $\mathcal{Q}_\lambda$ from the category of λ-twisted holonomic D-modules on the stack $[V/G]$, smooth with respect to the Białynicki-Birula stratification. This functor is not an equivalence, as it kills any object whose characteristic variety in $T^*[V/G] \cong [\mu^{-1}(0)/G]$ is supported on the χ-unstable locus. We conjecture, however, that the functor is essentially surjective, thus allowing us to think about objects in our somewhat mysterious category in terms of D-modules, which are much more concrete. Conjecture 1 should be regarded as a quantized, sheafy version of Kirwan surjectivity.

Conjecture 1 *$\mathcal{Q}_\lambda$ is equivalent to a quotient of the category of λ-twisted holonomic D-modules on the stack $[V/G]$, smooth with respect to the Białynicki-Birula stratification.*

In [BLPW2], we define natural functors

$$\mathcal{O}_\lambda \xrightarrow{\mathrm{Loc}_\lambda} \mathcal{Q}_\lambda \xrightarrow{\mathrm{Sec}_\lambda} \mathcal{O}_\lambda$$

between these categories, and we show that the composition $\mathrm{Sec}_\lambda \circ \mathrm{Loc}_\lambda$ is always derived equivalent to the identity functor on the bounded derived category of $\mathcal{O}_\lambda$. When $M = T^*(G/P)$ and $\lambda = 0$, Sec_λ and Loc_λ are honest quasi-inverse equivalences of categories. This is the celebrated Beilinson-Bernstein localization theorem [BB] that has been used to prove, among other things, the Kazhdan-Lusztig conjecture [KL]. Similar results have been obtained for Hilbert schemes [GG, GS1, GS2, KR, GGS] and resolved Slodowy slices [Gi]. We conjecture that this will always be the case.

Conjecture 2 *There is a class $\rho \in H^2(M;\mathbb{R})$ such that, if $\lambda \in H^2(M;\mathbb{R})$ and $\rho + \lambda$ lies in the interior of the ample cone of M, then Sec_λ and Loc_λ are inverse equivalences of categories.*

Note that for any two parameters $\lambda, \lambda' \in H^2(M;\mathbb{C})$ that differ by an integral class, we have an equivalence between $\mathcal{Q}_\lambda$ and $\mathcal{Q}_{\lambda'}$ given by tensoring with the quantization of a line bundle with Chern class $\lambda' - \lambda$. Thus we will define $\mathcal{Q} := \mathcal{Q}_0$, and note that Conjecture 2 says that $\mathcal{Q}$ is equivalent to the module category $\mathcal{O}_\lambda$ for all "ample enough" λ.

1.4 Homological properties of $\mathcal{Q}$

We now describe a number of homological features that we expect our category to exhibit, all of which will underlie the conjectures in Section 2. Establishing these properties, either in general or in certain specific cases, is one of the main goals of this proposal.

Assume for simplicity that, for each $\alpha \in \mathcal{I}$, the weights of the $\mathbb{T}$-action on $T_{p_\alpha}M$ are relatively prime. This guarantees that the attracting set for X_α is a cell, and therefore that X_α has no local systems that will complicate the statements that follow. The following conjecture is well-known when $M = T^*(G/P)$ and $\mathcal{Q}$ is equivalent to a block of BGG parabolic category $\mathcal{O}$.

Conjecture 3 *The simple objects $L_\alpha \in \mathcal{Q}$ are naturally indexed by $\mathcal{I}$.*

Consider the partial order on $\mathcal{I}$ generated by the relation $\alpha \leq \beta$ if $p_\alpha \in X_\beta$. Assume that our category has enough projectives, and for each $\alpha \in \mathcal{I}$, let P_α be the indecomposable projective cover of L_α. Let V_α be the largest quotient of P_α whose composition series consists only of simple modules L_β with $\beta \leq \alpha$. This object is called a **standard object**, and generalizes the notion of Verma modules in BGG parabolic category $\mathcal{O}$. The category $\mathcal{Q}$ is said to be **quasi-hereditary** if two conditions are satisfied. First, for each $\alpha \in \mathcal{I}$, L_α appears only once in the composition series of V_α. Second, for each $\alpha \in \mathcal{I}$, P_α admits a filtration with subquotients in $\{V_\beta \mid \alpha \leq \beta\}$, with V_α appearing only once. Roughly speaking, these conditions say that the simple, standard, and projective modules each form bases for the Grothendieck group $K(\mathcal{Q})$, and that the transition matrix between the standard basis and either of the other two bases is upper triangular with ones on the diagonal.

Our category is called **Koszul** if it has a graded lift in which each simple object admits a linear projective resolution. It is called **standard Koszul** if it is quasi-hereditary and it has a graded lift in which each standard object admits a linear projective resolution. Standard Koszul categories are necessarily Koszul, and form in fact the largest class of simultaneously quasi-hereditary and Koszul categories that is closed under the operation of Koszul duality [ADL].

Conjecture 4 *The category $\mathcal{Q}$ is standard Koszul.*

We know that blocks of BGG parabolic category $\mathcal{O}$ are standard Koszul [Ba, RCW], as are the combinatorial categories introduced in [BLPW1]; all of these categories are conjectured to arise via our geometric construction. Conjecture 4 is open, however, for rational Cherednik algebras and finite W-algebras. The Koszulity of $\mathcal{Q}$ will be the basis for our definition of symplectic duality in Section 2.

Consider the cycle map $\kappa : K(\mathcal{Q}) \to H^{\dim M}_{\mathbb{T}}(M;\mathbb{Z})$ taking a sheaf to its support, with multiplicity. It is not hard to show that $\kappa([V_\alpha])$ restricts to a nonzero element of $H^{\dim M}_{\mathbb{T}}(p_\alpha)$, and to zero in $H^{\dim M}_{\mathbb{T}}(p_\beta)$ for all $\beta \neq \alpha$, and therefore that $\kappa_{\mathbb{C}} : K(\mathcal{Q})_{\mathbb{C}} \to H^{\dim M}_{\mathbb{T}}(M;\mathbb{C})$ is an isomorphism. When M is a quiver variety, the vector space $H^{\dim M}_{\mathbb{T}}(M;\mathbb{C})$ may be identified with a tensor product of irreducible representations of a Kac-Moody algebra [N4], and the algebra action categorifies to an action by endo-functors on $D^b(\mathcal{Q})$ [Zh].

The singular variety N admits a stratification by its symplectic leaves. For each $\alpha \in \mathcal{I}$, let $S_\alpha \subset N$ be the smallest leaf closure containing the image of X_α. It is tempting to guess that the simple object L_α is supported on X_α, as will be the case whenever X_α is smooth. This is known to fail, however, when $M = T^*(G/B)$.[2] We conjecture that the statement holds up to certain correction terms that are controlled by the inclusion relations between the subvarieties $S_\alpha \subset N$.

Conjecture 5 *For each $\alpha \in \mathcal{I}$,*

$$\kappa([L_\alpha]) = [X_\alpha] + \sum_{\beta \in \mathcal{I}} n_{\alpha\beta}[X_\beta],$$

where $n_{\alpha\beta} = 0$ unless $S_\beta \subsetneq S_\alpha$.

[2] When G is not simply laced, this fails immediately. In Type A, however, one needs to go up to $\mathrm{SL}(8;\mathbb{C})$ to find the first counter-example [KS].

Let $B = \operatorname{Ext}^*(\oplus L_\alpha, \oplus L_\alpha)$. This is a graded ring, and assuming that Conjecture 4 holds, its module category is Koszul dual to B. In particular, B is itself standard Koszul, with simple objects indexed by $\mathcal{I}$. The action of the constant sheaf $\underline{\mathbb{C}}$ on $\oplus L_\alpha$ induces a ring homomorphism from $H^*(M;\mathbb{C}) \cong \operatorname{Ext}^*(\underline{\mathbb{C}}, \underline{\mathbb{C}})$ to $Z(B)$, the center of B.

Conjecture 6 *The map $H^*(M;\mathbb{C}) \to Z(B)$ is an isomorphism.*

This conjecture is well-known to hold when $M = T^*(G/P)$. When M is a hypertoric variety, $\mathcal{Q}$ is conjectured to coincide with the category defined in [BLPW1], and we proved in [BLPW1] that the center of this category is isomorphic to the cohomology ring of M.

Let C be any standard Koszul algebra with simple objects indexed by $\mathcal{I}$. In [BLPPW], we define a canonical graded deformation $\tilde{C}$ of C. Thus $\tilde{C}$ and its center $Z(\tilde{C})$ are both graded algebras over a polynomial ring $\operatorname{Sym} U$, and setting all of the elements of U to zero recovers B and $Z(C)$, respectively. Furthermore, we produce a $\operatorname{Sym} U$-algebra homomorphism $h_\alpha : Z(\tilde{C}) \to \operatorname{Sym} U$ for each $\alpha \in \mathcal{I}$.

Let T be the unique maximal torus of the hamiltonian symplectomorphism group of M that contains $\mathbb{T}$. The following conjecture is motivated by the cases of $T^*(G/P)$ (in Type A) and hypertoric varieties, both of which are analyzed in [BLPPW].

Conjecture 7 *The vector space U is canonically isomorphic to $\mathfrak{t}^*$, and $Z(\tilde{C})$ is isomorphic to $H_T^*(M;\mathbb{C})$ as an algebra over $\operatorname{Sym}\mathfrak{t}^* \cong H_T^*(pt;\mathbb{C})$. Furthermore, the map $h_\alpha : Z(\tilde{C}) \to \operatorname{Sym}\mathfrak{t}^*$ coincides with the localization map from $H_T^*(M;\mathbb{C})$ to $H_T^*(p_\alpha;\mathbb{C})$.*

It is surprising that, despite the fact that the category $\mathcal{Q}$ is defined using only the one-dimensional torus $\mathbb{T}$, Conjecture 7 tells us that this category somehow "knows about" the larger torus T. The role of Conjecture 7 in the symplectic duality program will be explained in Section 2.2.

1.5 Twisting and shuffling functors

Suppose that $\mathbb{T}$ and $\mathbb{T}'$ are two different one-parameter subtori of a maximal torus T of the hamiltonian symplectomorphism group of M, giving rise to two different categories $\mathcal{Q}$ and $\mathcal{Q}'$. We will assume that $\mathbb{T}$ and $\mathbb{T}'$ are chosen generically enough so that they each fix finitely many points in M. Recall that $\mathcal{Q}$ and $\mathcal{Q}'$ are two different subcategories of the same category $\mathcal{D}_0((h)) - \operatorname{mod}$ of coherent $\mathbb{S}$-equivariant $\mathcal{D}_0((h))$-modules.

Let $\iota_\mathbb{T} : D^b(\mathcal{Q}) \to D^b(\mathcal{D}_0((h)) - \operatorname{mod})$ be the inclusion functor, and let $\pi_\mathbb{T} : D^b(\mathcal{D}_0((h)) - \operatorname{mod}) \to D^b(\mathcal{Q})$ be its left adjoint, which one can think of as a "projection" onto $D^b(\mathcal{Q})$. Let

$$\Phi_{\mathbb{T},\mathbb{T}'} = \pi_{\mathbb{T}'} \circ \iota_\mathbb{T} : D^b(\mathcal{Q}) \to D^b(\mathcal{Q}').$$

Conjecture 8 *The functor $\Phi_{\mathbb{T},\mathbb{T}'}$ is an equivalence.*

Conjecture 8 is made more interesting by the fact that the equivalence is *not* canonical, that is, $\Phi_{\mathbb{T},\mathbb{T}'} \circ \Phi_{\mathbb{T}',\mathbb{T}}$ is not isomorphic to the identity functor. We define a **twisting functor** to be an auto-equivalence of $D^b(\mathcal{Q})$ of the form

$$\Phi_{\mathbb{T}_0,\mathbb{T}_1} \circ \Phi_{\mathbb{T}_1,\mathbb{T}_2} \circ \ldots \Phi_{\mathbb{T}_{n-1},\mathbb{T}_n},$$

with $\mathbb{T}_0 = \mathbb{T}_n = \mathbb{T}$. When $M = T^*(G/B)$ and $\mathcal{Q}$ is equivalent to a block of BGG category $\mathcal{O}$, these functors specialize to the twisting functors constructed by Arkhipov [Ar, KM]. When M is a hypertoric variety, these functors were constructed explicitly in [BLPW1]. When M is a quiver variety, the category $\mathcal{Q}$ admits a graded lift whose Grothendieck group may be identified with tensor products of representations of quantum groups [Zh]. In this case, we conjecture that the twisting functors categorify the braiding action by R-matrices.

It natural to attempt to identify the group of auto-equivalences of $D^b(\mathcal{Q})$ generated by the semigroup of twisting functors. The set of one-parameter subtori of T form a lattice in the Lie algebra $\mathfrak{t}$, set of $\mathbb{T}$ such that $|M^{\mathbb{T}}| < \infty$ is the complement of a finite collection of codimension 1 sublattices. Let X be the complement of the corresponding hyperplane arrangement in $\mathfrak{t}$. The combinatorics of compositions of twisting functors, along with Paris's work on the Deligne groupoid of a hyperplane arrangement [Pa], suggest the following conjecture.

Conjecture 9 *There is a natural map from the fundamental group of X to the group of auto-equivalences of $D^b(\mathcal{Q})$ generated by twisting functors.*

At present, Conjecture 9 is based on combinatorial data. In other words, our evidence for this conjecture is based on a combinatorial presentation of the fundamental group of X, rather than on some geometric structure lying over X. There is a second group action on our derived category that is harder to define, but more geometric in nature.

By the work of Kaledin [Ka], N admits a universal deformation $\mathcal{N}$ over the base $H^2(M;\mathbb{C})$, and we may endow $\mathcal{N}$ with a real 2-form η that restricts to a real symplectic form on the smooth part of each fiber. (If N is a symplectic quotient of T^*V, then $\mathcal{N}$ is obtained by varying the moment map parameter. The 2-form η is obtained by interpreting the symplectic quotient as a hyperkähler quotient.) Let $Y \subset H^2(M;\mathbb{C})$ be the locus over which the fibers of $\mathcal{N}$ are smooth; it is the complement of a hyperplane arrangement. For all $y \in Y$, the fiber (Y_y, η_y) is symplectomorphic to $(M, \mathrm{Re}\,\omega)$, where $\mathrm{Re}\,\omega$ is the real part of the algebraic symplectic form. If M is a cotangent bundle, then the work of Nadler and Zaslow [NZ] implies that our category $\mathcal{Q}$ embeds into the Fukaya category $\mathrm{Fuk}(M, \mathrm{Re}\,\omega) \simeq \mathrm{Fuk}(\mathcal{N}_y, \eta_y)$; we conjecture that this embedding exists more generally. Furthermore, we believe that we can define a monodromy action of the fundamental group of Y on $D^b(\mathcal{Q})$. We call the resulting auto-equivalences of $D^b(\mathcal{Q})$ **shuffling functors**, as we expect them to generalize the shuffling functors of [Ir] on the derived category of a block of BGG category $\mathcal{O}$.

We emphasize that X and Y are not isomorphic; they are complements of hyperplane arrangements in two different vector spaces of different dimensions. The conjectural relationship between twisting functors and shuffling functors is more subtle; we expect the twisting functors for M to coincide with the shuffling functors for a different symplectic variety $M^\vee$ with the property that $D^b(\mathcal{Q})$ is equivalent to $D^b(\mathcal{Q}^\vee)$. We elaborate on this expectation in the next section.

2 Symplectic duality

In this section we describe a conjectural relationship between geometric categories associated to certain pairs of symplectic varieties. In a number of special cases, we expect this relationship to provide links between previously studied geometric and categorical constructions, including two seemingly unrelated categorifications of representations of Kac-Moody algebras [CK, Zh] as well as two superficially different sets of link invariants [MS, SS].

Our relationship is defined at the categorical level, but it has as two very concrete cohomological consequences. The first (Section 2.2) is a phenomenon originally discovered in certain special cases by Goresky and MacPherson [GM]; by relating this phenomenon to symplectic duality we have generated new classes of examples and provided a satisfying explanation for those in [GM]. The second (Section 2.3) is a duality of vector spaces that explains previously known numerical identities in combinatorics [BLPW1] and illuminates the phenomena of Schur-Weyl duality and level-rank duality in representation theory.

At present, this relationship is defined in Conjecture 10 purely as an algebraic relationship between the two categories. Ultimately, one of our goals is to understand it as a consequence of some geometric relationship between the varieties, just as homological mirror symmetry is (conjecturally) a consequence of the Strominger-Yau-Zaslow formulation. At present, our best clue for how to do this comes from string theory: many known examples of related pairs arise from mirror dual quantum field theories (Section 2.1).

2.1 Definition and examples

Let M be a symplectic variety satisfying all of the assumptions of Section 1.

Conjecture 10 *There exists another symplectic variety $M^\vee$, which we call the* **symplectic dual** *of M, with the following properties.*

(1) The fixed point sets $M^{\mathbb{T}}$ and $(M^\vee)^{\mathbb{T}^\vee}$ are indexed by the same finite set $\mathcal{I}$.

(2) The categories $\mathcal{Q}$ and $\mathcal{Q}^\vee$ are Koszul dual in a matter compatible with the bijection in (1). In particular, we obtain an equivalence of bounded derived categories $D^b(\mathcal{Q}) \simeq D^b(\mathcal{Q}^\vee)$.

(3) We have isomorphisms $X^\vee \cong Y$ and $Y^\vee \cong X$. The twisting action of $\pi_1(X^\vee)$ coincides with the shuffling action of $\pi_1(Y)$, and similarly for $\pi_1(X)$ and $\pi_1(Y^\vee)$.

The following is a list of some conjectural examples of dual pairs of symplectic varieties.

- $T^*(G/B)$ is dual to $T^*(G^\vee/B^\vee)$, where $G^\vee$ is the Langlands dual of G. The corresponding Koszul duality theorem was proven in [BGS].
- $T^*(\mathrm{GL}(n)/P)$ is dual to a resolution of the Slodowy slice to a $\mathfrak{gl}(n)$ nilpotent orbit determined by the parabolic P. Through this example, we hope to use property (3) of Conjecture 10 to relate the knot invariants of [MS], defined using twisting functors on the category of D-modules on $\mathrm{GL}(n)/P$, to those of [SS], defined using shuffling functors on the Fukaya category of the resolved Slodowy slice.
- A hypertoric variety M is dual to the **Gale dual** hypertoric variety $M^\vee$, as explained in [BLPW1].
- Let $\mathcal{H}(k,n)$ be the Hilbert scheme of n points on a crepant resolution of $\mathbb{C}^2/\Gamma$, where $\Gamma = \mathbb{Z}/k\mathbb{Z}$. Let $\mathcal{M}(k,n)$ be the moduli space of torsion-free sheaves E on $\mathbb{P}^2$ with $\operatorname{rank} E = k$ and $c_2(E) = n$, along with a framing $\Phi : E|_{\mathbb{P}^1} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1}^{\oplus k}$. Then $\mathcal{H}(k,n)$ is dual to $\mathcal{M}(k,n)$. In particular, when $k = 1$, this says that $\mathrm{Hilb}^n \mathbb{C}^2$ is self-dual.
- More generally, the moduli space of G-instantons on a crepant resolution of $\mathbb{C}^2/\Gamma$ is dual to the moduli space of G'-instantons on a crepant resolution of $\mathbb{C}^2/\Gamma'$, where G is matched to Γ' and G' is matched to Γ via the MacKay correspondence.
- A quiver variety of affine type ADE is dual to a normal slice to one Schubert variety inside another one in the affine grassmannian for the Langlands dual group. Through this example, we hope to relate two different categorifications of representations of Kac-Moody algebras, one by Zheng [Zh], using quiver varieties, and the other by Cautis and Kamnitzer [CK], using the affine grassmannian.
- The Higgs branch of the moduli space of vacua for an $N = 4, d = 3$ supersymmetric quantum field theory is dual to the Higgs branch of the moduli space of the mirror dual theory, or (equivalently) the Coulomb branch of the moduli space for the same theory. The fact that cotangent bundles of Langlands dual flag varieties arise in this way is shown in [GW], while the fact that Gale dual hypertoric varieties arise in this way appears in [KSt].

The meaning of the relationship between symplectic duality in mathematics and mirror duality in physics remains mysterious; understanding it is one of the important goals of this proposal.

2.2 Goresky-MacPherson duality

In this section and the next, we discuss two cohomological consequences of symplectic duality. These cohomological relationships are much easier to check than properties (1)-(3) of Conjecture 10, and the existence of pairs of varieties satisfying these relationships should be taken as evidence for the conjecture. We begin with an example.

Let $M = T^*\mathbb{P}^2$, and let $M^\vee$ be the crepant resolution of $\mathbb{C}^2/\Gamma$, where $\Gamma \cong \mathbb{Z}/3Z$. These two varieties form a symplectic dual pair, and they in fact constitute a special case of six(!) of the examples listed in the previous section (all but the first one). The T^2-equivariant cohomology ring of M is isomorphic to $\mathbb{C}[x_1, x_2, x_3]/\langle x_1x_2x_3\rangle$, and its spectrum is isomorphic to the union of the three coordinate planes in $\mathbb{C}^3$. On the other hand, the T^1-equivariant cohomology ring of $M^\vee$ is isomorphic to $\mathbb{C}[y_1, y_2, y_3]/\langle y_1y_2, y_1y_3, y_2y_3\rangle$, and its spectrum is isomorphic to the union of the three coordinate lines in $\mathbb{C}^3$. The right way to understand this example is to regard the two copies of $\mathbb{C}^3$ as dual, in which case the three coordinate lines are the perpendicular spaces to the three coordinate planes.

More generally, let M be a T-space with M^T indexed by a finite set $\mathcal{I}$. For all $\alpha \in \mathcal{I}$, let $H_\alpha \in H_2^T(M;\mathbb{C})$ be the image of $\mathfrak{t} \cong H_2^T(p_\alpha;\mathbb{C})$ along the equivariant pushforward induced by the inclusion of p_α into M. Thus H_α is a linear subspace that projects isomorphically onto $\mathfrak{t}$ via the canonical projection $H_2^T(M;\mathbb{C}) \twoheadrightarrow \mathfrak{t}$. In the case where $M = T^*\mathbb{P}^2$, we obtain the three coordinate planes in $\mathbb{C}^3$.

Now let $T^\vee$ be a different torus, and let $M^\vee$ be a $T^\vee$-space with $(M^\vee)^{T^\vee}$ indexed by the same finite set $\mathcal{I}$. The following definition was made in [BLPPW], inspired by [GM].

Definition 11 We say that (M, T) is **Goresky-MacPherson dual** to $(M^\vee, T^\vee)$ if there exists a perfect pairing between $H_2^T(M;\mathbb{C})$ and $H_2^{T^\vee}(M^\vee;\mathbb{C})$ such that H_α and $H_\alpha^\vee$ are perpendicular to each other for all $\alpha \in \mathcal{I}$, as are the kernels of the two canonical projections.

The example at the beginning of this section motivates the following conjecture.

Conjecture 12 *Let (M, T) be as in Conjecture 7, and let $(M^\vee, T^\vee)$ be the symplectic dual variety and its corresponding torus. Then (M, T) is Goresky-MacPherson dual to $(M^\vee, T^\vee)$.*

Conjecture 12 was proven by Goresky and MacPherson [GM] for the second example of symplectic dual pairs in Section 2.1, and for the third example in [BLPPW]. Furthermore, we proved in [BLPPW] that the general case of Conjecture 12 is a consequence of Conjectures 3, 4, and 7.

2.3 Cohomological symplectic duality

In this section we discuss a second cohomological consequence of symplectic duality. For simplicity, we will add the extra hypothesis that, for every symplectic leaf of N, the Gauss-Manin connection on the top-degree cohomology groups of the fibers of π is trivial. This hypothesis is satisfied by all hypertoric varieties and quiver varieties [PW, N3]. By the decomposition theorem of [BBD], it implies that

$$R\pi_*\mathbb{C}_M \cong \bigoplus_{S\subset N} \mathrm{IC}_S \otimes H^{2\dim F_S}(F_S),$$

where the sum runs over symplectic leaf closures, and F_S is the fiber of π over an element of S. Taking $\mathbb{T}$-equivariant cohomology in degree $\dim M$, we obtain the identity

$$H^{\dim M}_{\mathbb{T}}(M;\mathbb{C}) \cong \bigoplus_{S\subset N} IH^{\dim S}_{\mathbb{T}}(S;\mathbb{C}) \otimes H^{2\dim F_S}(F_S). \tag{1}$$

For the purposes of this section, it is more natural to regard the direct sum decomposition in Equation (1) as a filtration. Thus, for each leaf closure $S \subset N$, we let

$$D_S = \bigoplus_{S\subset S'} IH^{\dim S'}_{\mathbb{T}}(S';\mathbb{C}) \otimes H^{2\dim F_{S'}}(F_{S'}) \quad \text{and} \quad E_S = \bigoplus_{S\subsetneq S'} IH^{\dim S'}_{\mathbb{T}}(S';\mathbb{C}) \otimes H^{2\dim F_{S'}}(F_{S'}).$$

Thus $H^{\dim M}_{\mathbb{T}}(M;\mathbb{C})$ is filtered by the vector subspaces D_S, and for each S, D_S/E_S is isomorphic to the summand of Equation (1) indexed by S.

Our first observation is that this filtration has a categorical interpretation. Recall the isomorphism $\kappa_{\mathbb{C}} : K(\mathcal{Q})_{\mathbb{C}} \to H^{\dim M}_{\mathbb{T}}(M;\mathbb{C})$ of Section 1.4. Also recall from Section 1.4 that, for all $\alpha \in \mathcal{I}$, S_α is defined to be the smallest symplectic leaf closure in N containing $\pi(X_\alpha)$.

Theorem 13 [BLPW2] *$\kappa_{\mathbb{C}}$ identifies D_S with $\mathbb{C}\{[P_\alpha] \mid S \subset S_\alpha\}$ and E_S with $\mathbb{C}\{[P_\alpha] \mid S \subsetneq S_\alpha\}$.*

Theorem 13 demonstrates that the filtered pieces D_S and E_S of $H^{\dim M}_{\mathbb{T}}(M;\mathbb{C})$, and even the summands of Equation (1), have canonical bases consisting of classes of projective objects. These bases do not have clear topological interpretations; rather they are reminiscent of Lusztig's canonical basis for a quantum group.

Assuming the existence of a symplectic dual variety as in Conjecture 10, let $S^\vee_\alpha$ be the smallest leaf closure in $N^\vee$ containing $\pi^\vee(X^\vee_\alpha)$.

Conjecture 14 *There is an inclusion-reversing bijection $S \leftrightarrow S^\vee$ between the sets of symplectic leaf closures for N and $N^\vee$. Furthermore, for all $\alpha \in \mathcal{I}$ we have $S^\vee_\alpha = (S_\alpha)^\vee$.*

Recall that property (2) of Conjecture 10 tells us that $D^b(\mathcal{Q}) \simeq D^b(\mathcal{Q}')$, and therefore that $K(\mathcal{Q})$ is isomorphic to (or, using the Euler form, dual to) $K(\mathcal{Q}')$. Theorem 13 and Conjecture 14 together imply the following result.

Conjecture 15 *For each symplectic leaf S, the duality between $K(\mathcal{Q})_{\mathbb{C}}$ and $K(\mathcal{Q}')_{\mathbb{C}}$ induces a duality between D_S/E_S and $D^\vee_{S^\vee}/E^\vee_{S^\vee}$.*

In the case where M and $M^\vee$ are hypertoric varieties, Conjecture 15 is proven in [BLPW2]. Taking dimensions recovers a well-known relationship between the h-numbers of a matroid and its Gale dual. In the case where M and $M^\vee$ are quiver varieties of affine type, Conjecture 15 relates to a phenomenon in representation theory known as **level-rank duality**, as we explain below.

Let $\widehat{\mathfrak{gl}(k)}$ be the affine Kac-Moody algebra associated to $\mathfrak{gl}(k)$. To any partition λ of k, one may associate an irreducible representation V_λ of $\widehat{\mathfrak{gl}(k)}$ on which the central parameter acts with eigenvalue $\ell = \lambda_1$. This parameter is known as the **level** of V_λ or of λ.

The transpose λ^t of λ defines a level k representation of $\widehat{\mathfrak{gl}(\ell)}$. The affine Lie algebra $\widehat{\mathfrak{gl}(k\ell)}$ admits a map from the product $\widehat{\mathfrak{gl}(k)} \times \widehat{\mathfrak{gl}(\ell)}$, which can be seen by writing $\mathbb{C}^{k\ell} = \mathbb{C}^k \otimes \mathbb{C}^\ell$. Frenkel [Fr] defines a level 1 representation $V^{k,\ell}_{\text{basic}}$ of $\widehat{\mathfrak{gl}(k\ell)}$ such that, when we decompose it under the action of $\widehat{\mathfrak{gl}(k)} \times \widehat{\mathfrak{gl}(\ell)}$, we obtain

$$V^{k,\ell}_{\text{basic}} \;\cong \bigoplus_{\substack{\lambda \text{ a partition}\\ \text{of length } k\\ \text{and level } \ell}} V_\lambda \otimes V_{\lambda^t}. \tag{2}$$

For simplicity, let us restrict our attention to the case where $\ell = 1$. Nakajima constructs an action of the Lie algebra $\widehat{\mathfrak{gl}(k)}$ on the direct sum of the cohomology groups $\bigoplus_n H^*(\mathcal{M}(k,n);\mathbb{C})$, which he shows is isomorphic as a $\widehat{\mathfrak{gl}(k)}$ representation to $V^{k,1}_{\text{basic}}$ [N3]. This construction can be modified to obtain an action on $\bigoplus_n H_{\mathbb{T}}^{\dim \mathcal{M}(k,n)}(\mathcal{M}(k,n);\mathbb{C})$, also isomorphic to the basic representation. Furthermore, the two direct sum decompositions of Equations (1) and (2) coincide.

Similarly, Nakajima constructs an action of the Lie algebra $\widehat{\mathfrak{gl}(1)}$ on the direct sum of the cohomology groups $\bigoplus_n H_{\mathbb{T}}^{\dim \mathcal{H}(k,n)}(\mathcal{H}(k,n);\mathbb{C})$, which he shows is isomorphic as a $\widehat{\mathfrak{gl}(1)}$ representation to $V^{k,1}_{\text{basic}}$ [N3]. Again, the two direct sum decompositions of Equations (1) and (2) coincide.

Thus Nakajima constructs the basic representation geometrically, but the vector space that he uses to see the $\widehat{\mathfrak{gl}(k)}$ action and the $\widehat{\mathfrak{gl}(1)}$ action are on the surface very different. The conjecture that $\mathcal{M}(k,n)$ is symplectic dual to $\mathcal{H}(k,n)$ tells us that these two vector spaces are canonically dual (or isomorphic, since the basic representation is self-dual). Furthermore, Conjecture 15 tells us that this duality (or isomorphism) is compatible with the decomposition (2) into isotypic components.

References

[ADL] I. Ágoston, V. Dlab, and E. Lukács. Quasi-hereditary extension algebras. *Algebras and Rep. Th.* 6 (2003), 97–117.

[Ar] S. Arkhipov. Semi-infinite cohomology of associative algebras and bar duality. *Internat. Math. Res. Notices* 1997, no. 17, 833–863.

[ALPT] A. Atanasov, C. Lopez, A. Perry, N. Proudfoot, and M. Thaddeus. Nash blow-ups of toric varieties. In preparation.

[Ba] E. Backelin. Koszul duality for parabolic and singular category $\mathcal{O}$. *Represent. Theory* 3 (1999), 139–152.

[BB] A. Beilinson and J. Bernstein. Localisation de g-modules. *C. R. Acad. Sci. Paris Sér. I Math.* 292 (1981), no. 1, 15–18.

[BBD] A. Beilinson, J. Bernstein, and P. Deligne. *Faisceaux Pervers*, in *Analysis and topology on singular spaces, I (Luminy, 1981)*, 5–171, Astérisque, 100, Soc. Math. France, Paris, 1982.

[BGG] J. Bernstein, I. Gelfand, and S. Gelfand. A certain category of $\mathfrak{g}$-modules. *Funkcional. Anal. i Priložen.* 10 (1976), no. 2, 1–8.

[BGS] A. Beilinson, V. Ginzburg, and W. Soergel. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.* 9 (1996), no. 2, 473–527.

[BP1] A. Bergman and N. Proudfoot. Moduli spaces for D-branes at the tip of a cone. *Journal of High Energy Physics* 03 (2006), 073.

[BP2] A. Bergman and N. Proudfoot. Moduli spaces for Bondal quivers. To appear in the *Pacific Journal of Mathematics*.

[BK] R. Bezrukavnikov and D. Kaledin. Fedosov quantization in algebraic context. *Mosc. Math. J.* 4 (2004), no. 3, 559–592, 782.

[BD] R. Bielawski and A. Dancer. The geometry and topology of toric hyperkähler manifolds. *Comm. Anal. Geom.* 8 (2000), 727–760.

[BJSST] T. Bogart, A. Jensen, D. Speyer, B. Sturmfels, and R. Thomas. Computing tropical varieties. *J. Symbolic Comput.* 42 (2007), no. 1-2, 54–73.

[BLPW1] T. Braden, A. Licata, N. Proudfoot, and B. Webster. Gale duality and Koszul duality. `arXiv:0806.3256v2`.

[BLPPW] T. Braden, A. Licata, N. Proudfoot, C. Phan, and B. Webster. Goresky-MacPherson duality and deformations of Koszul algebras. `arXiv:0905.1335v1`.

[BLPW2] T. Braden, A. Licata, N. Proudfoot, and B. Webster. Geometric category $\mathcal{O}$ and symplectic duality. In preparation.

[BP] T. Braden and N. Proudfoot, The hypertoric intersection cohomology ring. *Invent. Math.* 177 (2009), no. 2, 337–379.

[B1] T. Bridgeland. Stability conditions on triangulated categories. *Ann. of Math.* (2) 166 (2007), no. 2, 317–345.

[B2] T. Bridgeland. Stability conditions on $K3$ surfaces. *Duke Math. J.* 141 (2008), no. 2, 241–291.

[B3] T. Bridgeland. T-structures on some local Calabi-Yau varieties. *J. Algebra* 289 (2005), no. 2, 453–483.

[B4] T. Bridgeland. Stability conditions and Kleinian singularities. `arXiv:math/0508257v2`.

[B5] T. Bridgeland. Stability conditions on a non-compact Calabi-Yau threefold. *Comm. Math. Phys.* 266 (2006), no. 3, 715–733.

[B6] T. Bridgeland. Derived categories of coherent sheaves. *International Congress of Mathematicians.* Vol. II, 563–582, Eur. Math. Soc., Zürich, 2006.

[B7] T. Bridgeland. Spaces of stability conditions. `arXiv:math/0611510v1`.

[BTL] T. Bridgeland and V. Toledano-Laredo. Stability conditions and Stokes factors. `arXiv:0801.3974v1.`

[CK] S. Cautis and J. Kamnitzer. Knot homology via derived categories of coherent sheaves. II. $mathfraksl_m$ case. *Invent. Math.* 174 (2008), no. 1, 165–232.

[CBEG] W. Crawley-Boevey, P. Etingof, and V. Ginzburg. Noncommutative geometry and quiver algebras. *Adv. Math.* 209 (2007), no. 1, 274–336.

[FG] B. Fantechi and L. Göttsche. Orbifold cohomology for global quotients. *Duke Math. J.* 117 (2003), no. 2, 197–227.

[Fr] I. Frenkel. Representations of affine Lie algebras, Hecke modular forms and Korteweg-de Vries type equations. *Lie algebras and related topics* (New Brunswick, N.J., 1981), pp. 71–110, Lecture Notes in Math., 933, Springer, Berlin-New York, 1982.

[GG] W. Gan and V. Ginzburg. Almost-commuting variety, $\mathcal{D}$-modules, and Cherednik algebras. With an appendix by Ginzburg. *Int. Math. Res. Pap.* 2006, 26439, 1–54.

[GW] D. Giatto and E. Witten. S-duality of boundary conditions in $n = 4$ super-Yang Mills theory. `arXiv:0807.3720v1`.

[Gi] V. Ginzburg. Harish-Chandra bimodules for quantized Slodowy slices. To appear in *J. Representation Theory.* `arXiv:0807.0339v4`.

[GGS] V. Ginzburg, I. Gordon, and J. Stafford. Differential operators and Cherednik algebras. `arXiv:0803.3349v1`.

[GSS] C. Giusti, P. Salvatore, and D. Sinha. Mod-two cohomology of symmetric groups as a Hopf ring. In preparation.

[GS1] I. Gordon and J. Stafford. Rational Cherednik algebras and Hilbert schemes. *Adv. Math.* 198 (2005), no. 1, 222–274.

[GS2] I. Gordon and J. Stafford. Rational Cherednik algebras and Hilbert schemes. II. Representations and sheaves. *Duke Math. J. 132* (2006), no. 1, 73–135.

[GM] M. Goresky and R. MacPherson. On the spectrum of the equivariant cohomology ring. Preprint: `http://www.math.ias.edu/∼goresky/pdf/Spec6.pdf`.

[HP1] M. Harada and N. Proudfoot. Properties of the residual circle action on a hypertoric variety. *Pacific J. Math.* 214 (2004), no. 2, 263–284.

[HP2] M. Harada and N. Proudfoot. Hyperpolygon spaces and their cores. *Trans. A.M.S.* 357 (2005), 1445-1467.

[HP] T. Hausel and N. Proudfoot. Abelianization for hyperkähler quotients. *Topology* 44 (2005), 231-248.

[HRV] T. Hausel and F. Rodriguez-Villegas. Mixed Hodge polynomials of character varieties. With an appendix by Nicholas M. Katz. *Invent. Math.* 174 (2008), no. 3, 555–624.

[HS] T. Hausel and B. Sturmfels. Toric hyperkähler varieties. *Doc. Math.* 7 (2002), 495–534.

[HKLR] N. Hitchin, A. Karlhede, U. Lindström, M. Rocek. Hyperkähler metrics and supersymmetry. *Comm. Math. Phys.* 108 (1987), 535-589.

[Ir] R. Irving. Shuffled Verma modules and principal series modules over complex semisimple Lie algebras. *J. London Math. Soc.* (2) 48 (1993), no. 2, 263–277.

[Ka] D. Kaledin. Geometry and topology of symplectic resolutions. Algebraic geometry—Seattle 2005. Part 2, 595–628, *Proc. Sympos. Pure Math.*, 80, Part 2, Amer. Math. Soc., Providence, RI, 2009.

[KSt] A. Kapustin and M. Strassler. One mirror symmetry in three-dimsnional abelian gauge theories. *J. High Energy Phys.* (1999), no. 4, Paper 21, 26 pp.

[KS] M. Kashiwara and Y. Saito. Geometric construction of crystal bases. *Duke Math. J.* 89 (1997), no. 1, 9–36.

[KL] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.* 53 (1979), 165–184.

[KR] M. Kashiwara and R. Rouquier. Microlocalization of rational Cherednik algebras. *Duke Math. J.* 144 (2008), no. 3, 525–573.

[KM] O. Khomenko and V. Mazorchuk. On Arkhipov's and Enright's functors. *Math. Z.* 249 (2005), no. 2, 357–386.

[Ko] H. Konno. Cohomology rings of toric hyperkähler manifolds. *Int. J. of Math.* 11 (2000), no. 8, 1001–1026.

[LS] M. Lehn and C. Soerger. Symmetric groups and the cup product on the cohomology of Hilbert schemes. (English summary) *Duke Math. J.* 110 (2001), no. 2, 345–357.

[MS] V. Mazorchuk and C. Stroppel. Translation and shuffling of projectively presentable modules and a categorification of a parabolic Hecke module. *Trans. Amer. Math. Soc.* 357 (2005), no. 7, 2939–2973.

[NZ] D. Nadler and E. Zaslow. Constructible Sheaves and the Fukaya Category. `arXiv:math/0604379v4`.

[N1] H. Nakajima. Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras. *Duke Math. J.* 76 (1994) no. 2, 365–416.

[N2] H. Nakajima. Quiver varieties and Kac-Moody algebras. *Duke Math. J.* 91 (1998) no. 3, 515–560.

[N3] H. Nakajima. Quiver varieties and finite-dimensional representations of quantum affine algebras. *J. Amer. Math. Soc.* 14 (2001) no. 1, 145–238.

[N4] H. Nakajima. Quiver varieties and tensor products. *Invent. Math.* 146 (2001), no. 2, 399–449.

[Pa] L. Paris. On the fundamental group of the complement of a complex hyperplane arrangement. Arrangements—Tokyo 1998, 257–272, *Adv. Stud. Pure Math.*, 27, Kinokuniya, Tokyo, 2000.

[P1] N. Proudfoot. The equivariant Orlik-Solomon algebra. *J. Alg.* 305 (2006), 1186–1196.

[P2] N. Proudfoot. Hyperplane arrangements and K-theory. *Top. Appl.* 153 (2006), 2866–2875.

[P3] N. Proudfoot. Geometric invariant theory and projective toric varieties. *Snowbird Lectures in Algebraic Geometry*, Contemporary Mathematics 388, 2005.

[P4] N. Proudfoot. A nonhausdorff model for the complement of a complexified arrangement. *Proc. Amer. Math. Soc.* 135 (2007), 3989–3994.

[P5] N. Proudfoot. All the GIT quotients at once. To appear in *Trans. Amer. Math. Soc.* `arXiv:math/0510055v2`.

[P6] N. Proudfoot. A survey of hypertoric geometry and topology. *Toric Topology*, Contemporary Mathematics 460, 2008.

[PS] N. Proudfoot and D. Speyer. A broken circuit ring. *Beitrage zur Algebra und Geometrie* 47 (2006), no. 1, 161–166.

[PW] N. Proudfoot and B. Webster. Intersection cohomology of hypertoric varieties. *J. Alg. Geom.* 16 (2007) 39–63.

[RCW] A. Rocha-Caridi and N. Wallach. Projective modules over graded Lie algebras, I. *Math. Z.* 180 (1982), no. 2, 151–177.

[SS] P. Seidel and I. Smith. A link invariant from the symplectic geometry of nilpotent slices. *Duke Math. J.* 134 (2006), no. 3, 453–514.

[Si] C. Simpson. Nonabelian Hodge theory. *Proceedings of the International Congress of Mathematicians*, Vol. I, II (Kyoto, 1990), 747–756, Math. Soc. Japan, Tokyo, 1991.

[S1] D. Speyer. Tropical linear spaces. `arXiv:math/0410455v1`.

[S2] D. Speyer. Uniformizing Tropical Curves I: Genus Zero and One. `arXiv:0711.2677v1`.

[SS1] D. Speyer and B. Sturmfels. Tropical Mathematics. `arXiv:math/0408099v1`.

[SS2] D. Speyer and B. Sturmfels. The tropical Grassmannian. *Adv. Geom.* 4 (2004), no. 3, 389–411.

[SW] D. Speyer and L. Williams. The tropical totally positive Grassmannian. *J. Algebraic Combin.* 22 (2005), no. 2, 189–210.

[Zh] H. Zheng. Categorification of integrable representations of quantum groups. `arXiv:0803.3668v2`.