Q-spoiling and Directionality in Deformed Ring Cavities

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A ray-optics model is developed to describe the spoiling of the high-Q (whispering gallery) modes of ring-shaped cavities as they are deformed from perfect circularity. A sharp threshold is found for the onset of Q-spoiling as predicted by the KAM theorem of non-linear dynamics. Beyond the critical deformation, b_c, $Q \sim (b - b_c)^{-\alpha}$, $\alpha \approx 2.4 - 2.6$. The escaping light emerges in certain specific directions which may be predicted.

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Ring cavity resonators in which high Q-modes are created by total internal reflection of electromagnetic waves circulating around the perimeter are commonly used as components in lasers and sensitive detectors. These high Q modes are referred to as *whispering gallery* modes or *morphology-dependent resonances* (MDR's) and have been studied in optical fibers [1], liquid droplets [2], glass spheres [3] and in microdisk lasers [4]. An ideal resonator of this type consists of an axially or spherically symmetric dielectric with higher index of refraction than the surrounding medium. In such a geometry the "potential" which confines the light to the rim of the cavity arises from a combination of the effective potential due to angular momentum conservation and the index of refraction discontinuity at the interface [5]. The finiteness of Q in an ideal loss-less cavity arises only due to evanescent leakage ("tunneling") through the barrier created by the effective potential. $Q \geq 10^8$ is achieved in microspheres and droplets [2,3]; however in microdisk lasers the measured Q is much lower ($\sim 10^3$) perhaps due to surface roughness [4]. It is interesting to consider the effect of deformations from rotational symmetry on these high Q resonators for several reasons. First, in the context of MDR's, shape distortions are often induced by inertial forces, trapping electric fields or laser-induced electrostriction. Second, there is relatively little understanding of the robustness of these modes to shape imperfections, particularly when the deformations are large ($> 1\%$) and leave no residual symmetry. Third, ideal ring cavities emit light isotropically and there is practical interest in studying perturbations which might couple the light out of these modes in a preferred direction.

In this letter we present results from a model which combines ray optics with concepts from non-linear classical mechanics to describe the Q-factor and directional radiation from a highly deformed circular cavity. We show that due to the KAM theorem of classical mechanics [6] the Q factor will remain high up to a critical deformation and then decrease rapidly. In addition, beyond this critical deformation the light emission becomes highly directional.

We consider a deformable cavity with a uniform real index of refraction n which is greater than the index of the surrounding medium. For simplicity we treat the 2D scalar wave equation; we expect (due to the generality of the KAM theorem) that our qualitative conclusions apply to the 3D cases of interest (although the effects of Arnold diffusion in 3D needs further consideration [6]). The walls of the cavity are assumed to describe a smooth curve C in the plane which we parameterize by its arc length s and characterize by its local radius of curvature $\rho(s)$. The curves we will consider are everywhere convex, i.e. $\rho(s) > 0$ for all s, and are assumed to satisfy $\rho(s) > \lambda$ for all s (where λ is the wavelength of light in the cavity). With these conditions ray-optics applies everywhere within the cavity, and Snell's law determines the critical angle for escape, θ_c. When the cavity is deformed the angle of incidence of a light ray on the cavity walls varies with time and for a large starting angle (characteristic of whispering gallery modes) escape may still occur. We define the Q determined by escaping rays as $Q_R = \omega\tau$ where τ is the escape time and ω is the frequency of the light. The model neglects evanescent escape so $Q_R = \infty$ for a circular cavity. The ray optics of the model is equivalent to the hamiltonian dynamics of a point mass moving freely between specular reflections from the boundary [7], unless the angle of incidence $\theta \leq \theta_c$, in which case the particle escapes.

We now introduce a specific model deformable circular cavity (due to Robnik [8] and generalized by Berry and Robnik [9]). The cavity is described by a conformal transformation of the unit disk

$$w(z) = \frac{z + bz^2 + cz^3}{\sqrt{1 + 2b^2 + 3c^2}}, \qquad (1)$$

where b,and c are real parameters. The boundary of the cavity is given by letting $z = e^{i\phi}, 0 \leq \phi \leq 2\pi$; b and c must be chosen such that $|w'(z)| > 0$ for $|z| \leq 1$. When b is small and $c = 0$, the distortions described by this map are *primarily quadrupolar.* The deviation from circularity enters at order b^2, c^2, and typical values required for Q-spoiling are $b, c \approx 0.1$ corresponding to roughly 1% radial distortion.

As is typical in non-linear dynamics, the motion of a particle/ray in the cavity is described by phase-space coordinates and can be most easily studied through the

Poincaré surface of section (SOS) [6,10]. The SOS is a two-dimensional cut through 4D phase-space; the particle motion is represented by plotting the coordinates at each return to the surface. The natural (area-preserving) coordinates for the SOS here [10] are the arc-length at the *nth* bounce, s_n, and $\sin\theta_n$, where θ is the angle of incidence on the boundary. For small distortions the arc-length s may be replaced by the angular coordinate ϕ_n. $(\phi_{n+1}, \sin\theta_{n+1})$ can be obtained from $(\phi_n, \sin\theta_n)$ by the solution of a cubic equation derived from simple geometric considerations. This defines the *bounce map* of the 2D cavity. Figure 1 shows the SOS of the cavity and the real-space motion of the ray for $c = 0$ and three different values of b.

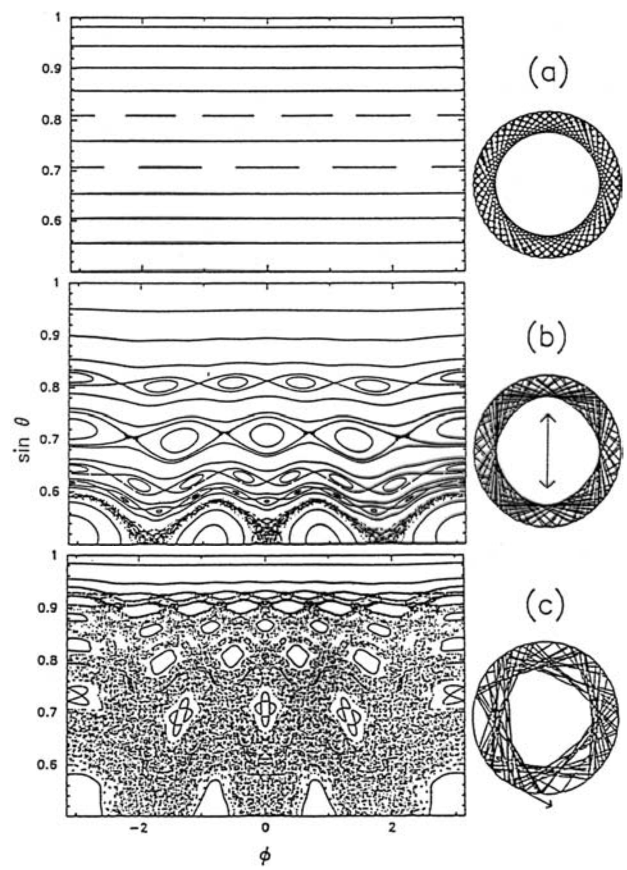

FIG. 1. Surfaces of section for three values of the deformation parameter, (a) $b = 0$, (b) $b = 0.11$, (c) $b = 0.15$. The real space trajectories shown to the right correspond to a ray started with $\sin\theta = 0.7$, near the four-bounce periodic orbit. The starting value of ϕ in the lower two plots was chosen in the stochastic region between the four-island chain. The vertical arrow in (b) indicates the direction in which the elongation of the disk occurs when $b > 0$.

Figure 1(a) shows the SOS for a circle ($b = 0$). Here, $\sin\theta$ is proportional to the angular momentum and hence is constant at each return to the SOS; thus typical phase space points trace out straight lines. The real-space trajectory is confined to an annulus by a circular caustic. Periodic orbits do not give a caustic but rather trace out an N-sided polygon, and appear as N discrete points on the SOS. The SOS for $b = 0.11$ shown in Fig. 1(b) is typical of systems undergoing a KAM transition to chaos. Stable islands have formed around the periodic orbits, and a stochastic layer indicating chaotic motion has formed in between the islands. However there still remain unbroken curves crossing the SOS. The KAM theorem implies that such curves (known as KAM tori) will exist over some finite range of the deformation parameter b [6], and will partition the phase-space into regions bounded by two such tori. Thus a particle (ray) with an initial value of $\sin\theta \geq \sin\theta_c$ can never escape as long as one such unbroken torus intervenes between $\sin\theta$ and $\sin\theta_c$. We infer that Q_R remains infinite for all b until the last intervening torus(LIT) is broken at a critical deformation b_c. Hence high-Q resonances should persist despite significant deformations of the cavity. The real-space trajectory shows that the ray remains confined to an annular region. However, now the rotational symmetry of the motion is explicitly broken and dense ray-tracing occurs near a four-bounce periodic orbit with a fixed orientation. This is interesting because previous work suggests that the underlying wave equation will have solutions with high field intensity [11] concentrated in these regions.

Fig. 1(c) shows the SOS for a high enough deformation ($b = 0.15 > b_c$) such that the LIT is broken and an initially trapped ray can now escape by phase-space diffusion to the critical angle, leading to a finite value of Q_R. Note that the ray bounces many times before escaping, so that Q_R can remain fairly large even beyond the threshold b_c at which Q-spoiling begins. Also as the diffusion in $\sin\theta$ is rather slow, escape typically occurs near the critical angle, implying that outside the cavity the ray is emitted almost tangentially as shown in Fig. 1c.

To calculate Q above the threshold we need to specify initial conditions on the phase-space distribution and compute the escape time. The appropriate initial conditions depend on the experimental system of interest but in all cases a certain set of modes will be most relevant. In the 2d case there are two mode indices, the radial index p and the azimuthal (or angular momentum) index m, which be related to a particular value of $\sin\theta$ by semiclassical quantization. The known result is [8]

$$\cot\theta + \theta = \pi[\frac{p + 3/4}{m} + \frac{1}{2}]. \tag{2}$$

If the important modes (p, m) are specified then Eq. (2) yields a minimum and maximum value of $\sin\theta$ corresponding to these modes. We then start a uniform density of phase space points in motion between these two values and compute the length of each escaping trajectory as a function of the deformation b. If the frequency of the light, the size of the cavity, and the index of refraction inside and outside the cavity is given, then the mean inverse length can be converted into an escape rate, $1/\tau$, such that $Q = \omega\tau$ (if we wished to calculate the mean time we would have to avoid starting points within stable islands which would remain trapped indefinitely).

Representative results are given in Fig. 2. *Note the sharp threshold for Q-spoiling* b_c as expected from our previous discussion. Beyond b_c, Q decreases as $(b-b_c)^{-\alpha}$, where numerically we find $\alpha \approx 2.4 - 2.6$. Numerical investigations indicate that this power law is insensitive to initial conditions or to the details of the deformation (e.g. setting $b = 0$ and varying c gives similar values of α). This non-trivial power law dependence of Q on the deformation parameter is very significant; such a result could never arise from perturbation theory. It is a signature of the well-known fact that phase-space diffusion in the KAM region is anomalous [6,12]. If such a power-law is measured experimentally it would constitute an unambiguous verification of the physics underlying our model for the Q-spoiling.

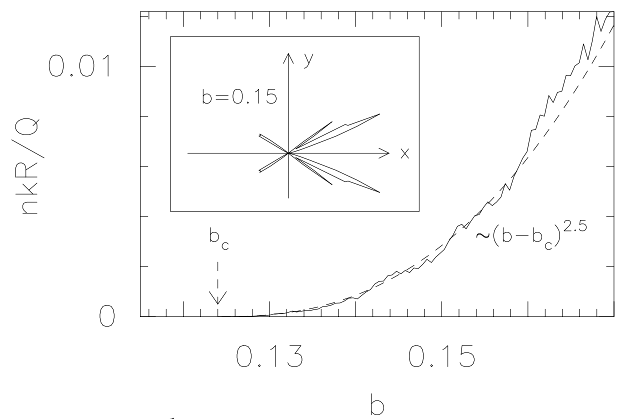

FIG. 2. Q^{-1} vs. deformation b obtained from average escape rate (see text). The unperturbed circle has radius R, k is the vacuum wavenumber and n the refractive index of the dielectric. Escaping rays are started in a strip with $0.66 \le \sin\theta \le 0.71$. Least-squares fit to the data gives a power law $nkR/Q = 23.7\,(b - b_c)^{\alpha}$ with $b_c = 0.124$ and $\alpha = 2.5$. Inset: Polar plot of the escape probability vs. angle in the far-field for $b = 0.15$ showing strong directionality of emission.

Finally, our model allows us to evaluate the directionality of the emitted light once Q-spoiling occurs simply by evaluating the escape rate as a function of the angle ϕ. Figure 1 suggests that the escape occurs non-uniformly; the phase-space structure funnels the escaping flux to certain angular intervals. A typical ray escapes near θ_c so it emerges along the tangent to the point of escape [see Fig. 1(c)]. The emitted intensity should be proportional to the escape rate in a given angular interval. The anisotropy of the escape rate then leads to the highly directional emission shown in the polar plot in Fig. 2 (inset). We note that the calculational technique used here should generalize to other types of deformations and to 3D without becoming computationally intractable; hence it may provide a practical method for predicting and controlling directional emission.

Several possible realizations of deformable ring cavities can be imagined using e.g. levitated droplets, microdisk lasers or optical fibers. The optimal experimental system would be deformable by means of a continuously tunable control parameter. An interesting application of this theory would be possible in a system which could be deformed so rapidly that Q could be spoiled during the lifetime of a given resonance. In such a case the deformation would operate as a Q-switch for dumping out the stored light. To achieve high sensitivity to small deformations the starting point should be a cavity deformed at or beyond the threshold b_c; then relatively small further deformations would lead to large changes in Q. It is possible that local modulation of the index of refraction within the cavity could simulate deformation of the cavity and have a similar effect.

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